Subclass of uniformly starlike functions associated with a linear operator whose coefficients are the reciprocal Gamma function

Jamal Salah[1,*]

[1]College of Applied and Health Sciences, A'Sharqiyah University, Post Box No. 42, Post Code No. 400 Ibra, Sultanate of Oman

Email: damous73@yahoo.com

Abstract

This study, aims to consider the coefficients of the reciprocal Gamma function in order introduce a linear operator by the means of Hadamard product. Thus, we define a new subclass of uniformly starlike functions of order α, $\Gamma^{-1}(\alpha)$. Further, we obtain coefficient estimates, distortion theorems, convex linear combinations and radii of close-to-convexity, starlikeness and convexity for functions $f \in \Gamma^{-1}(\alpha)$. In addition, we investigate the inclusion conditions for the Hadamard product and the Integral transform.

Keywords: Uniformly starlike functions; Gamma function; reciprocal Gamma function; Hadamard product

1. Introduction

In the early phases of the theory of univalent functions, the significance of the coefficients provided by the power series $f(t) = t + \sum_{k=2}^{\infty} a_k t^k$became apparent. Bieberbach demonstrated earlier in 1916 that the second coefficient$|a_2| \leq 2$, and that equality holds if and only if f is a rotation of the Köebe's function, which is defined as follows: $f(t) = t + \sum_{k=2}^{\infty} a_k t^k$. Furthermore, The universal coefficient bound that $|a_k| \leq 2 = k, k \geq 2$was conjectured by Bieberbach in the same study. The equality is only valid if and only if f is a rotation of the Köebe's function. This conjecture—known as the renowned Bieberbach conjecture—was difficult to verify for roughly 70 years until Louis de Branges did it in 1985, at which point the conclusion became known as de Branges's theorem. This preserves the function's univalence but geometrically entails reshaping and perhaps decreasing its domain and rotating it. As a result, this raises many queries for the connection between a given coefficient sequence $a_1, a_2, a_3, \dots a_k$, and the basic geometric properties of $f(\mathbb{D})$. Consequently, New ideas, such as starlike, convex, spiral-like, and uniformly starlike (convex), were added to the theory of univalent functions in order to address the aforementioned queries.

The Gamma function is the well-known special function given by

$$\Gamma(t) = \int_0^{\infty} e^{-s} s^{t-1} ds, \quad \Re(t) > 0.$$

It has come to play a pivotal role in almost every branch of pure and applied mathematics, statistics, physics, chemistry and engineering. It is certainly one of the major special functions [1-3]. [4] give the Taylor series expansion of the reciprocal Gamma function

DOI: https://doi.org/10.54216/IJNS.250429

$$\frac{1}{\Gamma(t)} = t + \sum_{k=2}^{\infty} b_k t^k = t + \gamma t^2 + \left(\frac{\gamma^2}{2} - \frac{\pi^2}{12}\right) t^3 + \cdots, t \in \mathbb{C}, \tag{1}$$

where $\gamma = 0.577216\ldots$is the Euler-Mascheroni constant.

Since $\Gamma(t)$ is a non-zero function in the complex plane $\mathbb{C}$, then its reciprocal is an entire function in $\mathbb{C}$. Also, the coefficients b_k in (1) have the following integral representation (See [5])

$$b_k = \frac{(-1)^k}{\pi k!} \int_0^{\infty} e^{-s} \mathrm{Im}[(\log t - i\pi)^k] ds\,. \tag{2}$$

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(t) = t + \sum_{k=2}^{\infty} a_k t^k, \quad t \in \mathbb{D} := \{t \in \mathbb{C} : |t| < 1\}, \tag{3}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent (or one-to-one) functions on $\mathbb{D}$. Let T be the subclass of $\mathcal{S}$ consisting of functions of the form

$$f(t) = t - \sum_{k=2}^{\infty} a_k t^k, \qquad a_k \geq 0. \tag{4}$$

The focus of this research is to introduce a linear operator to define a new subclass of analytic functions of order α such that $0 \leq \alpha < 1$. First, it is necessary to recall the two well-known subclasses of starlike and convex functions of order α, as given below:

$$\mathcal{ST}(\alpha) = \left\{ f \in \mathcal{S} : \Re\left(\frac{tf'(t)}{f(t)}\right) > \alpha, \qquad t \in \mathbb{D} \right\}$$

and

$$\mathcal{C}(\alpha) = \left\{ f \in \mathcal{S} : \Re\left(1 + \frac{tf''(t)}{f'(t)}\right) > \alpha, \quad t \in \mathbb{D} \right\}.$$

Selectively, when $\alpha = 0$, the above classes are reduced to their standard definition and are simply called the starlike and convex functions.

Definition 1. A function $f \in \mathcal{A}$ of the form (3) is β-uniformly starlike of order α, if

$$\Re\left\{\frac{tf'(t)}{f(t)} - \alpha\right\} \geq \beta \left|\frac{tf'(t)}{f(t)} - 1\right|, \qquad (0 \leq \alpha < 1, \beta \geq 0; t \in \mathbb{D}).$$

We denote by $\mathcal{UST}(\alpha, \beta)$ the class of all such functions.

Definition 2. A function $f \in \mathcal{A}$ of the form (3) is β-uniformly convex of order α, if

$$\Re\left\{1 + \frac{tf''(t)}{f'(t)} - \alpha\right\} \geq \beta \left|\frac{tf''(t)}{f'(t)}\right|, (0 \leq \alpha < 1, \beta \geq 0; t \in \mathbb{D}).$$

We denote by $\mathcal{UCV}(\alpha, \beta)$, the class of all such functions.

In particular, the classes $\mathcal{UCV} \coloneqq \mathcal{UCV}(1,0)$, $\mathcal{UST} := \mathcal{UST}(1,0)$ and $\beta - \mathcal{UCV} \coloneqq \mathcal{UCV}(\beta, 0)$, $\beta - \mathcal{UST} \coloneqq \mathcal{UST}(\beta, 0)$ are introduced by Goodman [6,7] (see also, William Ma and David Minda [8].

DOI: https://doi.org/10.54216/IJNS.250429

Obviously, $f(t) \in \mathcal{UCV}(\alpha,\beta) \Leftrightarrow tf'(t) \in \mathcal{UST}(\alpha,\beta)$.

By following the same pattern, this study attempts to apply the Lambert series to the theory of univalent functions. Consequently, this may lead to relevant studies if one considers extending the Lambert series to other subclasses of analytic functions. Hence, we can investigate various topics such as Hankel determinants, subordination properties, and Fekete-Szegö inequalities. Furthermore, these results can be extended to multivalent functions and meromorphic functions.

We also recall the definition of the Hadamard product (convolution): For a given function $f \in \mathcal{A}$ of the form (3) and $g \in \mathcal{A}$ of the form

$$g(t) = t + \sum_{k=2}^{\infty} c_k t^k, \qquad t \in \mathbb{D}, \tag{5}$$

the convolution $(*)$ of the two functions f and g is obtained as follows:

$$(f * g)(t) \coloneqq t + \sum_{k=2}^{\infty} a_k c_k t^k, \quad t \in \mathbb{D}. \tag{6}$$

The q-shifted factorial [See (9)] is defined for $\lambda, q \in \mathbb{C}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ as follows:

$$(\lambda;q)_k = \begin{cases} 1, & \text{for } k = 0 \\ (1-\lambda)(1-\lambda q)\dots(1-\lambda q^{k-1}) & \text{for } k \in \mathbb{N} \end{cases}$$

By using the q-gamma function $\Gamma_q(t)$, we get

$$\left(q^{\lambda};q\right)_k = \frac{(1-q)^k \Gamma_q(\lambda+k)}{\Gamma_q(\lambda)}, \qquad (k \in \mathbb{N}_0),$$

where

$$\Gamma_q(t) = (1-q)^{1-t} \frac{(q;q)_\infty}{(q^t;q)_\infty} \; (|q| < 1)$$

Also, we note that

$$(\lambda;q)_\infty = \prod_{k=0}^{\infty} (1-\lambda q^k) \; (|q| < 1)$$

and, the q-gamma function $\Gamma_q(t)$ is known

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t),$$

where $[k]_q$ denotes the basic q-number defined as follows:

$$[k]_q := \begin{cases} \dfrac{1-q^k}{1-q}, & k \in \mathbb{C}, \\ 1 + \displaystyle\sum_{j=1}^{k-1} q^j, & k \in \mathbb{N}. \end{cases} \tag{7}$$

From (7), we get

(a) For any non-negative integer k, the q-shifted factorial is given by

DOI: https://doi.org/10.54216/IJNS.250429

$$[k]_q! := \begin{cases} 1, & \text{if } k = 0, \\ \prod_{n=1}^{k} [n]_q, & \text{if } k \in \mathbb{N}. \end{cases}$$

(b) For any positive number r, the q-generalized Pochhammer symbol is defined by

$$[r]_{q,k} := \begin{cases} 1, & \text{if } k = 0, \\ \prod_{n=r}^{r+k-1} [n]_q, & \text{if } k \in \mathbb{N}. \end{cases}$$

We also know that

$$\Gamma_q(t) \to \Gamma(t) \ \text{ as } q \to 1^-.$$

Also, we obtain

$$\lim_{q \to 1^-} \left\{ \frac{(q^\lambda; q)_k}{(1-q)^k} \right\} = (\lambda)_k.$$

For $0 < q < 1$, we consider the q-derivative operator [see (10)]

$$\begin{aligned} D_q\left(\frac{1}{\Gamma(t)}\right) &:= \frac{\frac{1}{\Gamma(t)} - \frac{1}{\Gamma(qt)})}{t(1-q)} \\ &= 1 + \sum_{k=2}^{\infty} [k]_q b_k t^{k-1} \quad , t \in \mathbb{D}, \end{aligned}$$

where

$$[k]_q := \frac{1-q^k}{1-q} = 1 + \sum_{j=1}^{k-1} q^j, \ [0, q] := 0.$$

Now, we define a linear operator $\Gamma^{-1} f: \mathcal{A} \to \mathcal{A}$ as follows:

$$\Gamma^{-1} f(t) \quad = \mathrm{t}\, D_q\left(\frac{1}{\Gamma(t)}\right) * f(t) = \mathrm{t} + \sum_{k=2}^{\infty} [k]_q b_k a_k t^k, \quad t \in \mathbb{D},$$

from (2), we obtain

$$\Gamma^{-1} f(t) = \mathrm{t} + \sum_{k=2}^{\infty} \left\{ \frac{(-1)^k}{\pi k!} \int_0^\infty e^{-s} \mathrm{Im}[(\log t - i\pi)^k] ds \right\} [k]_q a_k t^k, \qquad t \in \mathbb{D}.$$

For brevity, we let

$$\omega(\mathrm{k}) = (-1)^k \frac{[k]_q}{\pi k!} \int_0^\infty e^{-s} \mathrm{Im}[(\log t - i\pi)^k] ds. \tag{8}$$

The above linear operator leads us to propose a definition in the following manner.

Definition 3. A function $f \in \mathcal{A}$ of the form (3) is said to be in the class $\mathcal{UST}\Gamma^{-1}(\alpha)$ if the function f satisfies the following condition:

DOI: https://doi.org/10.54216/IJNS.250429

$$\Re\left\{\frac{t\left(\Gamma^{-1}f(t)\right)'}{\Gamma^{-1}f(t)}-\alpha\right\}\geq\left|\frac{t\left(\Gamma^{-1}f(t)\right)'}{\Gamma^{-1}f(t)}-1\right|,\qquad 0\leq\alpha<1,\qquad t\in\mathbb{D}.$$

(9)

Finally, we consider functions with negative coefficients $f\in\mathrm{T}$, similarly to (9), and we simply write $\Gamma^{-1}(\alpha)=\mathcal{UST}\Gamma^{-1}(\alpha)\cap\mathrm{T}$. Based on Definition 3 and the subclass $\Gamma^{-1}(\alpha)$, the analytic characterization of the function f reduces to the following definition.

Definition 4. A function $f\in\mathcal{A}$ of the form (4) is said to be in the class $\Gamma^{-1}(\alpha)$ if the function f satisfies the condition (9).

New subclasses of analytic functions have been introduced for various applications, such as fractional calculus and quantum calculus, by involving some special functions, such as the Mittag-Leffler and Faber polynomial functions [11–17]. The most common concern in such studies is the inclusion conditions. Alternatively, it means that for a given new subclass, $\mathcal{H}$, we seek a set of useful conditions on the sequence $\{a_n\}$ that are both necessary and sufficient for $f(z)$ to be a member of $\mathcal{H}$.

Now, the question arises: under what conditions does an analytic function belong to our new subclass? Additionally, what are the inclusion conditions for the Hadamard product of two analytic functions? What is the integral transform of a function in the new subclass?

2. Characterization Property

In this section, we discuss the characterization properties of the members that belong to the new family of analytic functions. The characterization properties include a couple of theorems related to the inclusion of functions, consequent corollaries, and a closure theorem.

Theorem 1. A function $f\in\mathcal{A}$ of the form (3) is said to be in the class $\Gamma^{-1}(\alpha)$if and only if

$$\sum_{k=2}^{\infty}\left(\frac{2k-1-\alpha}{1-\alpha}\right)|\omega(k)||a_k|\leq 1, \tag{10}$$

Proof. To prove the assertion (10), it is sufficient to show that

$$\left|\frac{t(\Gamma^{-1}f(t))'}{\Gamma^{-1}f(t)}-1\right|\leq\Re\left\{\frac{t(\Gamma^{-1}f(t))'}{\Gamma^{-1}f(t)}-\alpha\right\}.$$

After adding and subtracting 1 from the right-hand side, we obtain

$$\left|\frac{t(\Gamma^{-1}f(t))'}{\Gamma^{-1}f(t)}-1\right|\leq\Re\left\{\frac{t(\Gamma^{-1}f(t))'}{\Gamma^{-1}f(t)}-1\right\}+(1-\alpha),$$

that is

$$\begin{aligned}\left|\frac{t(\Gamma^{-1}f(t))'}{\Gamma^{-1}f(t)}-1\right|-\Re\left\{\frac{t(\Gamma^{-1}f(t))'}{\Gamma^{-1}f(t)}-1\right\} &\leq 2\left|\frac{t(\Gamma^{-1}f(t))'}{\Gamma^{-1}f(t)}-1\right|\\ &\leq\frac{\sum_{k=2}^{\infty}(k-1)\,|\omega(k).a_k|}{1-\sum_{k=2}^{\infty}|\omega(k).a_k|}.\end{aligned}$$

The above expression is bounded by $(1-\alpha)$, thus proving our assertion.

Conversely, let us assume that $f\ \in\Gamma^{-1}(\alpha)$, then (9) yields

$$\frac{1-\sum_{k=2}^{\infty}k\,|\omega(k).a_k|t^{k-1}}{1-\sum_{k=2}^{\infty}|\omega(k).a_k|t^{k-1}}-\alpha\geq\frac{1-\sum_{k=2}^{\infty}(k-1)\,|\omega(k).a_k|t^{k-1}}{1-\sum_{k=2}^{\infty}|\omega(k).a_k|t^{k-1}}.$$

Letting $z \to 1$ along the real axis results in the inequality

$$\sum_{k=2}^{\infty}(2k-1-\alpha)\,|\omega(k).a_k| \le 1-\alpha.$$

Corollary 1. Let a function f defined (4) belong to the class $\Gamma^{-1}(\alpha)$, then,

$$|a_k| \le \frac{1}{|\omega(k)|}\cdot\frac{1-\alpha}{2k-1-\alpha}, \qquad k \ge 2.$$

Theorem 2. Let the function $f \in \mathcal{A}$ of the form (3) be in the class $\Gamma^{-1}(\alpha)$. Then for $|t| = r < 1$, we have

$$r - \frac{1}{|\omega(2)|}\cdot\frac{1-\alpha}{3-\alpha}\,r^2 \le |f(t)| \le r + \frac{1}{|\omega(2)|}\cdot\frac{1-\alpha}{3-\alpha}\,r^2 \tag{11}$$

the equality (11) is attained for the function

$$f(t) = t - \frac{1}{|\omega(2)|}\cdot\frac{1-\alpha}{3-\alpha}\,t^2 \tag{12}$$

at $t = r$ and $t = re^{i(2n+1)\pi},\ n \in \mathbb{N}$.

Proof. Since for $k \ge 2$,

$$(3-\alpha)|\omega(2)|.|a_k| \ \le (2k-\alpha-1)|\omega(k)||a_k|,$$

using Theorem 1, we have

$$(3-\alpha)|\omega(2)|\sum_{k=2}^{\infty} a_k \le \sum_{k=2}^{\infty}(2k-1-\alpha)\,|\omega(k).a_k| \le (1-\alpha)\sum_{k=2}^{\infty} a_k \ \le \frac{1-\alpha}{(3-\alpha)|\omega(2)|} \tag{13}$$

From (3) and (13), we have

$$|f(t)| \ge r - r^2\sum_{k=2}^{\infty} a_k \ge r - \frac{1-\alpha}{(3-\alpha)|\omega(2)|},$$

and

$$|f(t)| \le r + r^2\sum_{k=2}^{\infty} a_k \le r + \frac{1-\alpha}{(3-\alpha)|\omega(2)|}.$$

This completes the Proof. □

Corollary 2. Let the function $f \in \mathcal{A}$ of the form (3) be in the class $\Gamma^{-1}(\alpha)$. Then for $|t| = r < 1$, we have

$$1 - \frac{2}{|\omega(2)|}\cdot\frac{1-\alpha}{3-\alpha} r \le |f'(t)| \le 1 + \frac{2}{|\omega(2)|}\cdot\frac{1-\alpha}{3-\alpha}\,r. \tag{14}$$

The result is sharp for function given by (12).

Theorem 3. Let a function f defined by (3) and $g(t) = t - \sum_{k=2}^{\infty} b_k\, t^k$ be in the class $\Gamma^{-1}(\alpha)$ then, the function h states that

DOI: https://doi.org/10.54216/IJNS.250429

$$h(t) = (1-\beta)f(t) + \beta g(t) = t - \sum_{k=2}^{\infty} c_k\, t^k\,, \tag{15}$$

where $c_k = (1-\beta)a_k + \beta b_k,\quad 0 \leq \beta \leq 1$ also belongs to the class $\Gamma^{-1}(\alpha)$

Proof. The result follows easily upon using (10) and (15). □

Next, we define the following functions $f_i(t),\ (i = 1,2,3,\ldots m)$ of the form

$$f_i(t) = t - \sum_{k=2}^{\infty} a_{k,i}\, t^k, \qquad a_{k,i} \geq 0, \qquad t \in \mathbb{D}. \tag{16}$$

Theorem 4. Let the functions $f_i(t), (i = 1,2,3,\ldots m)$ defined by (1) be in the classes$\Gamma^{-1}(\alpha_i)$, then, the function h defined by

$$h(t) = t - \frac{1}{m}\sum_{k=2}^{\infty}\left(\sum_{i=1}^{m} a_{k,i}\right)t^k$$

belongs to the class $\Gamma^{-1}(\alpha)$ for $\alpha = \min_{1\leq i\leq m}\{\alpha_i\}$, with $0 \leq \alpha_i < 1$.

Proof. Since $f_i \in \Gamma^{-1}(\alpha_i),\ i = 1,2,3,\ldots m)$, by applying Theorem 1, we observe that

$$\sum_{k=2}^{\infty}|\omega(k)|\,(2k-1-\alpha)\left(\frac{1}{m}\sum_{i=1}^{m} a_{k,i}\right) = \frac{1}{m}\sum_{i=1}^{m}\left(\sum_{k=2}^{\infty}|\omega(k)|(2k-1-\alpha)a_{k,i}\right).$$

Theorem 1 again entails that h is a member of $\Gamma^{-1}(\alpha)$. □

3. Results Involving Convolution

This section discusses the convolutional results of two functions, $f_1(t) \in \Gamma^{-1}(\alpha)$ and $f_2(t) \in \Gamma^{-1}(\beta)$. Apart from presenting several theorems, some useful corollaries are also deduced.

Theorem 5. For two functions $f_i(t), (i = 1,2)$ defined by (16), let $f_1(t) \in \Gamma^{-1}(\alpha)$ and $f_2(t) \in \Gamma^{-1}(\beta)$. Then, $f_1 * f_2 \in \Gamma^{-1}(\xi)$where

$$\xi \leq 1 - \frac{2(k-1)(1-\alpha)(1-\beta)}{(2k-1-\alpha)(2k-1-\beta)|\omega(k)| - (1-\alpha)(1-\beta)}, \qquad k \geq 2. \tag{17}$$

Proof. In view of Theorem 1, it suffices to prove that

$$\sum_{k=2}^{\infty}\frac{2k-1-\xi}{1-\xi}\omega(k)a_{k,1}a_{k,2} \leq 1. \qquad k\ \geq 2$$

It follows from Theorem 1 and the Cauchy–Schwarz inequality that

$$\sum_{k=2}^{\infty}\frac{\sqrt{2k-1-\alpha}\cdot\sqrt{2k-1-\beta}}{\sqrt{(1-\alpha)(1-\beta)}}\sqrt{|\omega(k)|}\sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{18}$$

Thus, it suffices find ξ such that

$$\sum_{k=2}^{\infty}\frac{2k-1-\xi}{1-\xi}\sqrt{|\omega(k)|a_{k,1}a_{k,2}} \leq \sum_{k=2}^{\infty}\frac{\sqrt{2k-1-\alpha}\cdot\sqrt{2k-1-\beta}}{\sqrt{(1-\alpha)(1-\beta)}}\sqrt{|\omega(k)|}\sqrt{a_{k,1}a_{k,2}} \leq 1,$$

DOI: https://doi.org/10.54216/IJNS.250429

or

$$\sqrt{a_{k,1}a_{k,2}} \le \frac{\sqrt{2k-1-\alpha}\cdot\sqrt{2k-1-\beta}}{\sqrt{(1-\alpha)(1-\beta)}}\cdot\frac{1-\xi}{2k-1-\xi}.$$

By virtue of (18), it suffices to find ξ such that

$$\frac{\sqrt{(1-\alpha)(1-\beta)}}{\sqrt{2n-1-\alpha}\sqrt{2k-1-\beta}\sqrt{|\omega(k)|}} \le \frac{\sqrt{2k-1-\alpha}\sqrt{2k-1-\beta}}{\sqrt{(1-\alpha)(1-\beta)}}\cdot\frac{1-\xi}{2k-1-\xi},$$

which concedes the assertion of our theorem. □

Corollary 3. Let the functions $f_j(z), (j=1,2)$ defined by (16) belong to the class $\Gamma^{-1}(\alpha)$. Then, $(f_1 * f_2)(z) \in \Gamma^{-1}(\rho)$, where

$$\rho \le 1 - \frac{2(k-1)(1-\alpha)^2}{(2k-1-\alpha)^2|\omega(k)| - (1-\alpha)^2}, \quad k \ge 2.$$

Proof. The result is established if we replace $\beta = \alpha$ in Theorem 5. □

Theorem 5. Let the function f defined by (3) belong to the class $\Gamma^{-1}(\alpha)$, and let $g(t) = t - \sum_{k=2}^{\infty} b_k t^k$, for $|b_k| \le 1$. Then, $(f * g)(t) \in \Gamma^{-1}(\alpha)$.

Proof. Using the convolution property and the concept defined by the left-hand side of (10), we construct the following relation:

$$\begin{aligned}\sum_{k=2}^{\infty}(2k-1-\alpha)|\omega(k)||a_k b_k| &= \sum_{k=2}^{\infty}(2k-1-\alpha)|\omega(k)||a_k||b_k| \\ &\le \sum_{k=2}^{\infty}(2k-1-\alpha)|\omega(k)||a_k| \le 1-\alpha.\end{aligned}$$

Hence, it follows that $(f * g)(t) \in \Gamma^{-1}(\alpha)$.. □

Corollary 4. Let the function f defined by (3) belong to the class $\Gamma^{-1}(\alpha)$.Furthermore, let $g(t) = t - \sum_{k=2}^{\infty} b_k t^k$ for $0 \le b_k \le 1$, then, $(f * g)(t) \in \Gamma^{-1}(\alpha)$..

Now, we consider the following:

Theorem 6. Let the functions $f_j(t), (j=1,2)$ defined by (16) belong to the class $\Gamma^{-1}(\alpha)$. Then, the function h, defined by $h(t) = t - \sum_{t=2}^{\infty}\left(a_{k,1}^2 + a_{k,2}^2\right)t^k$, belongs to the class $\Gamma^{-1}(\phi)$, where

$$\phi \le 1 - \frac{4(1-\alpha)^2}{(2k-1-\alpha)^2|\omega(k)| - 2(1-\alpha)^2}, \qquad k \ge 2. \tag{19}$$

Proof. In view of Theorem 1, it suffices to show that

$$\sum_{k=2}^{\infty}\frac{2k-1-\phi}{1-\phi}|\omega(k)|\left(a_{k,1}^2 + a_{k,2}^2\right) \le 1. \tag{20}$$

From (10) and Theorem 1, we find that

$$\sum_{k=2}^{\infty}\left[\frac{2k-1-\alpha}{1-\alpha}\,|\omega(k)|\right]^2 a_{k,j}^2 \leq \left[\sum_{k=2}^{\infty}\frac{2k-1-\alpha}{1-\alpha}\,|\omega(k)|a_{k,j}\right]^2, \tag{21}$$

which yields

$$\sum_{k=2}^{\infty}\frac{1}{2}\left[\frac{2k-1-\alpha}{1-\alpha}\,|\omega(k)|\right]^2\left(a_{k,1}^2+a_{k,2}^2\right)\leq 1. \tag{22}$$

Upon comparing (21) and (22), it is evident that the inequality (19) is satisfied if

$$\frac{2k-1-\phi}{1-\phi}\,|\omega(k)| \leq \frac{1}{2}\left[\frac{2k-1-\alpha}{1-\alpha}\,|\omega(k)|\right]^2, \quad k\geq 2,$$

that is, if

$$\phi \leq 1-\frac{4(1-\alpha)^2}{(2k-1-\alpha)^2|\omega(k)|-2(1-\alpha)^2}.$$

This completes the proof. □

4. Integral Transform of Class $\Gamma^{-1}(\alpha)$

To convert class $\Gamma^{-1}(\alpha)$ into integral form, we define the following integral transform:

$$V_\mu(f)(t)=\int_0^1 \mu(z)\frac{f(tz)}{z}\,dz,$$

where $\mu(z)$ is a real valued, non-negative, and normalized weight function such that $\int_0^1 \mu(z)dz=1$.

The special case of $\mu(z)$ is $\mu(z)=\frac{(c+1)^\delta}{\mu(\delta)}z^c\left(\log\frac{1}{z}\right)^{\delta-1}$, $c>-1, \delta\geq 0$, which yields the Komatu operator.

Theorem 7. Let $f\in\Gamma^{-1}(\alpha)$ then, $V_\mu(f)\in\Gamma^{-1}(\alpha)$

Proof. By definition, we have,

$$\begin{aligned}V_\mu(f)&=\frac{(c+1)^\delta}{\mu(\delta)}\int_0^1(-1)^{\delta-1}z^c(\log z)^{\delta-1}\left(t-\sum_{k=2}^{\infty}a_kt^kz^{k-1}\right)dz\\&=\frac{(-1)^{\delta-1}(c+1)^\delta}{\mu(\delta)}\lim_{r\to 0^+}\left[\int_r^1 z^c(\log z)^{\delta-1}\left(t-\sum_{k=2}^{\infty}a_kt^kz^{k-1}\right)dz\right].\end{aligned}$$

By applying basic mathematical principles, we derive the following expression:

$$V_\mu(f)(t)=t-\sum_{k=2}^{\infty}\left(\frac{c+1}{c+k}\right)^\delta a_kt^k.$$

We need to prove that

$$\sum_{k=2}^{\infty}\frac{2k-1-\alpha}{1-\alpha}\,|\omega(k)|\left(\frac{c+1}{c+k}\right)^\delta |a_k| \leq 1. \tag{23}$$

Conversely, $f\in\Gamma^{-1}(\alpha)$ if and only if,

DOI: https://doi.org/10.54216/IJNS.250429

$$\sum_{k=2}^{\infty}\frac{2k-1-\alpha}{1-\alpha}\ |\omega(k)||a_k|\ \leq 1.$$

This shows that $\frac{c+1}{c+k}<1$, and hence, (23) holds. Thus, the proof is evident. □

Next, we derive the radii of starlikeness and convexity of $V_\mu(f)$

Theorem 8. Let $f\in\Gamma^{-1}(\alpha)$, then, $V_\mu(f)$ is starlike of order $0\leq\xi<1$ in $|t|<R_1$, where

$$R_1=\inf_k\left[\left(\frac{c+k}{c+1}\right)^{\delta}\cdot\frac{(1-\xi)(2k-1-\alpha)}{(k-\xi)(1-\alpha)}\ |\omega(k)|\right]^{\frac{1}{k-1}}.$$

Proof. It is sufficiently fair to confirm that $\left|\frac{t\left(V_\mu(f)(t)\right)'}{V_\mu(f)(t)}-1\right|<1-\xi$.

Considering the left-hand side of the above inequality, we write

$$\left|\frac{t\left(V_\mu(f)(t)\right)'}{V_\mu(f)(t)}-1\right| = \left|\frac{\sum_{k=2}^{\infty}(1-k)\left(\frac{c+1}{c+k}\right)^{\delta}\ |a_k|t^{k-1}}{1-\sum_{k=2}^{\infty}\left(\frac{c+1}{c+k}\right)^{\delta}\ |a_k|t^{k-1}}\right|$$

$$\leq \frac{\sum_{k=2}^{\infty}(k-1)\left(\frac{c+1}{c+k}\right)^{\delta}\ |a_k||t|^{k-1}}{1-\sum_{k=2}^{\infty}\left(\frac{c+1}{c+k}\right)^{\delta}\ |a_k||t|^{k-1}}.$$

The last expression is less than $1-\xi$ as

$$|t|^{k-1}<\left(\frac{c+k}{c+1}\right)^{\delta}\frac{(1-\xi)(2k-1-\alpha)}{(k-\xi)(1-\alpha)}\ |\omega(k)|.$$

This completes the proof. □

Theorem 9. If $f\in\Gamma^{-1}(\alpha)$, then, $V_\mu(f)$ is convex of order $0\leq\gamma<1$, in $|t|<R_2$, where

$$R_2=\inf_k\left[\left(\frac{c+k}{c+1}\right)^{\delta}\frac{(1-\gamma)(2k-1-\alpha)}{k(k-\gamma)(1-\alpha)}|\omega(k)|\right]^{\frac{1}{k-1}}.$$

Proof. The proof is evident from the fact that $f(t)$ is convex if and only if $tf'(t)$ is starlike.

5. Conclusion

In this article, we introduce a new subclass of uniformly starlike functions by utilizing the reciprocal Gamma Function Consequently; we explore the characteristics of the proposed subclass. Furthermore, we discussed several relevant topics, including the Hadamard product, integral transform, and radii of starlikeness and convexity. Thus, applying the reciprocal Gamma function to additional subclasses of analytic functions may lead to significant research outcomes. Consequently, we can conduct research on various subjects, including Fekete-Szegö inequalities, subordination characteristics, and the Hankel determinant. Furthermore, multivalent functions and meromorphic functions can be included in the scope of these conclusions.

Acknowledgements

The author is very grateful to experts for their appropriate and constructive suggestions to improve this template.

Mathematics Subject Classification: 30C45, 30C50.

References

[1] J. M. Borwein and R. M. Corless, "Gamma and factorial in the monthly," *Amer. Math. Monthly*, vol. 125, no. 5, pp. 400–424, 2018.

[2] P. J. Davis, "Leonhard Euler's integral: A historical profile of the gamma function: In memoriam: Milton Abramowitz," *Amer. Math. Monthly*, vol. 66, no. 10, pp. 849–869, 1959.

[3] J. Y. Salah, "A note on gamma function," *Int. J. Modern Sci. Eng. Technol. (IJMSET)*, vol. 2, no. 8, pp. 58–64, 2015.

[4] J. W. Jr. Wrench, "Concerning two series for the gamma function," *Math. Compute*, vol. 22, pp. 617–626, 1968.

[5] L. Fekih-Ahmed, "On the power series expansion of the reciprocal gamma function," 2014, [Online]. Available: hal-01029331v1.

[6] A. W. Goodman, "On uniformly convex functions," *Ann. Polon. Math.*, vol. 56, pp. 87–92, 1991.

[7] A. W. Goodman, "On uniformly starlike functions," *J. Math. Anal. Appl.*, vol. 155, pp. 364–370, 1991.

[8] W. Ma and D. Minda, "Convex functions," *Ann. Polon. Math.*, pp. 165–175, 1992.

[9] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its Applications, vol. 35, Cambridge University Press, Cambridge, 1990.

[10] F. H. Jackson, "On q-definite integrals," *Quart. J. Pure Appl. Math.*, vol. 41, pp. 193–203, 1910.

[11] G. Murugusundaramoorthy and N. Magesh, "Linear operators associated with a subclass of uniformly convex functions," *IJPAMS*, vol. 3, no. 1, pp. 113–125, 2006.

[12] B. A. Frasin, "On certain subclasses of analytic functions associated with Poisson distribution series," *Acta Univ. Sapientiae, Mathematica*, vol. 11, no. 1, pp. 78–86, 2019.

[13] R. Shah and A. Kumar, "Subclasses of analytic functions associated with generalized Pascal distribution," *Math. Methods Appl. Sci.*, vol. 44, no. 12, pp. 9530–9546, 2021, doi: 10.1002/mma.7345.

[14] A. Ali and M. Hossain, "Investigation of starlike and convex functions associated with modified Pascal distribution series," *Complex Anal. Oper. Theory*, vol. 17, no. 2, pp. 155–170, 2023, doi: 10.1007/s11785-023-01432-w.

[15] A. Amourah, B. Frasin, and G. Murugusundaramoorthy, "On certain subclasses of uniformly spirallike functions associated with struve functions," *J. Math. Comput. Sci.*, vol. 11, pp. 4586–4598, 2021.

[16] S. M. El-Deeb and G. Murugusundaramoorthy, "Applications on a subclass of β-uniformly starlike functions connected with q-Borel distribution," *Asian-European J. Math.*, vol. 2250158, 2023.

[17] J. Salah, H. U. Rehman, I. Al Buwaiqi, A. Al Azab, and M. Al Hashmi, "Subclasses of spiral-like functions associated with the modified Caputo's derivative operator," *AIMS Math.*, vol. 8, no. 8, pp. 18474–18490, 2023, doi: 10.3934/math.2023939.