On Modules Related to Homomorphism Their Kernel Equal Zero in Neutrosophic Theory

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Abstract

Neutrosophic set is a modern branch as a generalization of fuzzy concept. Zadeh in 1965 presented fuzzy concept and later he introduced more applications in more subjects of mathematics. On of the type branch of mathematics is fuzzy algebra. In this work, we present and clarify several results of several modules, which has zero-kernel, and zero homomorphism in neutrosophic theory. The aim modules are mnonoform and small monoform modules. Several concepts have been studied in this paper like Quasi-dedekind and uniform modules. We proved that if ($\mathcal{M}(\mathcal{T})$) is a module over neutrosophic ring $\mathfrak{R}(\mathcal{T})$. If $\mathcal{M}(\boldsymbol{\mathcal{T}})$ is a directed sum of simple submodules an $\mathcal{SM}(\mathcal{T})$ is monoform, then $\mathcal{M}(\mathcal{T})$ is monoform module. Also, if $\mathfrak{R}(\mathcal{T})$ is a semi simple ring and $\mathcal{M}(\mathcal{T})$ is a $\mathfrak{R}(\mathcal{T})$-module, so $\mathcal{M}(\mathcal{T})$ is small and satisfies all conditions of monoform with Q-dedekind property. On the other hand, let $(\mathcal{M},+,.)$be an R-module. $\mathcal{M}(\mathcal{T})$ is a neutrosophic modules and generated by $\mathcal{M}$ and $\mathcal{T}$. So, $\mathcal{M}(\mathcal{T})$ is a weak neutrosophic. Finally, we presented more results, examples and properties about the topic with new results in neutrosophic algebra.

Keywords: Neutrosophic module; Neutrosophic ring; Uniform module; Quasi-dedekind module; Small module

1. Introduction

In this work, we deal with commutative rings has 1 and all modules have unity. We denoted to neutrosophic ring and module by $\mathcal{M}(\mathcal{T}), \mathfrak{R}(\mathcal{T})$. A submodule A of$\mathcal{M}$ is considered small in $\mathcal{M}$ if there exists another proper submodule, specifically B; A+B$\neq \mathcal{M}$ [1]. We denoted i:A$\rightarrow \mathcal{M}$ to inclusion homomorphism; $i(a) = a$. If $A \leq \mathcal{M}$ and $f: A \rightarrow \mathcal{M}$ equal zero with kernel of this homomorphism also is equal zero, so this concept is called monoform module [2]. $\mathcal{M}$ is a prime in [3]. More details about annihilator concept in [4]. $\mathcal{M}$ is a uniform if for all $A \leq \mathcal{M}$ is essential [5]. Also, compressible module with several propertiesand closed-CS-module in [6]. Neutrosophic concept is one of important application in algebra and we can refer to neutrosophic concept by three cases, the first is true case (T), the second is Indeterminacy ($\mathcal{T}$) and the third is false (F). Samarndach presented several concepts as a generalization of fuzzy concept in 1995. In addition, neutrosophic set, neutrosophic ring and neutrosophic module in [7, 8, 9]. $\mathfrak{R}$ is called neutrosophic ring if the union of the ring$\mathfrak{R}$ with indeterminacy$\mathfrak{R}$ ($\mathcal{T}$) [10]. Neutrosophic module in [11]. A ring with faithful polyform modules in [12]. The author in [13], presented neutrosophic multiplication module and several properties of BCK-algebra in [14]. More details about ring concepts in [15, 16, 17].

The set Z is not a semi-ring with respect to addition and multiplication but Z forms a semi-ring, where =Z The vital reason for the development results of rings, bemiring is a generalization rings, semi-groups, so-called BCK-algebras are inspired by a BCK logic. We have BCK-algebra and BCK logic, BCI algebra and BCI logic, implicative BCK-algebra and implicative logic, implicative BCK-algebra and implicative logic**.** If $\mathcal{FM}(\mathcal{T})$ is a free $Z(\mathcal{T})$-module, so $\mathcal{SM}(\mathcal{T})$ is small and monoform iff $\mathcal{M}(\mathcal{T})$ is a monoform**.** If $\mathfrak{R}(\mathcal{T})$ is a semi simple ring and $\mathcal{M}(\mathcal{T})$ is a $\mathfrak{R}(\mathcal{T})$-module, so $\mathcal{M}(\mathcal{T})$ is small and satisfies all conditions of monoform with Q-dedekind property. The main contributions in this paper are:

- New results on the isomorphism nucleus.
- Important additions to the concept of the nitrosophyc.

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- Study of isomorphism's using the nitrosophyc theory

In this paper, we presented and explain more results of monoform modules in neutrosophic theory. Also, If$\boldsymbol{\mathcal{M}}(\mathcal{T})$ be a non-singular, then S$\boldsymbol{\mathcal{M}}(\mathcal{T})$ is small and hence $\boldsymbol{\mathcal{M}}(\mathcal{T})$ is Q-Dedekind module. Note that we utilize from condition nonsingular in the next result, because for example the module Z_4 over the ring Z is small, monoform, and not Q-dedekind. On the other hand, Z_4 not be nonsingular module over Z because Z (Z_4) not equal zero.

2. Preliminaries

In this section, we introduce various tools like definitions, remarks and some properties of Neutrosophic concept.

Definition 2.1. **[15]** A notion of module $\mathcal{M}$ is a comm.gp; with a *: $\mathfrak{R} \times \mathcal{M} \longrightarrow \mathcal{M}: f\ (r_1 a) = r\ a\ \forall\ r$ in $\mathfrak{R}$, a in$\mathcal{M}$;

1) $r\ (a + b) = ra + rb\ \forall\ a, b \in \mathcal{M}, r \in \mathfrak{R}$.

2) $(r_1 + r_2)a = r_1 a + r_2 a\ \forall\ a \in \mathcal{M}, r_1 \in \mathfrak{R}, r_2 \in \mathfrak{R}$.

3) $r_1\ (r_2 a) = (r_1 r_2)\ a$

4)$1.a = a.1 = a$

Definition 2.2. If $\emptyset \neq A$ is a set, then the set generated by A with named $\mathcal{T}$ refer to neutrosophic A($\mathcal{T}$).

Remark 2.3. We refer to real and subset of [0,1] by Neutrosophic of $\mathcal{B}$ in S, $T_{\mathcal{B}}$(x), $\mathcal{T}_{\mathcal{B}}$(x), and $F_{\mathcal{B}}(x)$, were $\mathcal{B} \subseteq S$ and Neutrosophic of $\mathcal{B}$ in S,

$\mathcal{B} = \{< y, T_{\mathcal{B}}(y), \mathcal{T}_{\mathcal{B}}(y), F_{\mathcal{B}}(y), > : y \in U, T(y), \mathcal{T}(y), F(y)$ belong to the interval [0,1].

Note: $0 \leq T_{\mathcal{B}}(y) + \mathcal{T}_{\mathcal{B}}(y) + F_{\mathcal{B}}(y) \leq 3$.

Definition 2.4. A fuzzy of the set $A = \{<a, \mu A(a)>|\ a \in A\}$ is describe by a function:

$\mu A: A \to [0,1], \mu A(a), a \in A$.

Remark2.5.

1) $\boldsymbol{\mathcal{T}}^2 = \boldsymbol{\mathcal{T}}, (0.\boldsymbol{\mathcal{T}}) = (0.\boldsymbol{\mathcal{T}}^{-1})$.

2) $\nexists\ \boldsymbol{\mathcal{T}}^{-1}$.

Definition 2.6. [ ]. Let $(\mathcal{M}, +, .)$be an R-module. $\mathcal{M}(\mathcal{T})$ is a neutrosophic modules and generated by $\mathcal{M}$ and $\mathcal{T}$. So, $\mathcal{M}(\mathcal{T})$ is a weak neutrosophic.

Definition (2.7). [11]. If $\mathcal{M}(\mathcal{T})$is a neutrosophic over $\mathfrak{R}(\mathcal{T})$, so K($\mathcal{T}$) is a neutrosophic submodule if:

1) $K(\boldsymbol{\mathcal{T}}) \neq A$.

2) $K(\mathcal{T}) \subseteq \mathcal{M}\ (\mathcal{T})$

3) $K(\mathcal{T})$is an $\mathfrak{R}\ (\mathcal{T})$-module.

2. Main Results

Several new results of Neutrosophic monoform modules with some examples and properties are presented in this paper.

Definition 3.1. A module $\mathcal{M}\ (\mathcal{T})$ is said to be small monform (briefly, $S\mathcal{M}(\mathcal{T})\ if\ \forall\ \mathcal{K}(\mathcal{T}) \neq \emptyset\ with\ 0 \neq f \in (Hom\ \mathcal{K}\ (\mathcal{T}), \mathcal{M}\ (\mathcal{T}), so\ Ker\ (f) \ll \mathcal{K}(\mathcal{T})$.

Remark 3.2. Every $\mathcal{M}\ (\mathcal{T})$ is $S\mathcal{M}(\mathcal{T})$. But the convers is not true. See the following example;

Example 3.3. f: $Z_4(\mathcal{T})$→ $Z_4(\mathcal{T})$∋f ($\bar{x}(\mathcal{T})$)=2($\bar{x}(\mathcal{T})$, $\bar{x}(\mathcal{T})$∈ $Z_4(\mathcal{T})$ with $Ker\ (f) = <\bar{2}> \neq <\bar{0}>$.

Example 3.4. Every $C\mathcal{M}(\mathcal{T})$ is $S\mathcal{M}(\mathcal{T})$, where $C\mathcal{M}(\mathcal{T})$ is chain module which means the lattic of submodule $A(\mathcal{T})$ is linearly ordered, $A(\mathcal{T}) \leq \mathcal{M}(\mathcal{T})$.

Remark 3.5. An epimorphic image of $S\mathcal{M}(\mathcal{T})$ is not $S\mathcal{M}(\mathcal{T})$.

Let $Z(\mathcal{T})$ be a uniform module $U\mathcal{M}(\mathcal{T})$ with prime property. Let π: Z($\mathcal{T}$)→ $\frac{Z}{12Z}$ ($\mathcal{T}$) $\cong$ $Z_{12}(\mathcal{T})$, π is a natural projection. Note that $Z_{12}(\mathcal{T})$ is not $SZ_{12}(\mathcal{T})$ because if:

$$A(\mathcal{T}) = <\bar{2}>\ with\ f: A(\mathcal{T}) \to Z_{12}(\mathcal{T})$$

defined by:

$$f\,(\bar{x}(\boldsymbol{\mathcal{T}})) = 2(\bar{x}(\boldsymbol{\mathcal{T}}));\ \bar{x}(\boldsymbol{\mathcal{T}}) \in\ A(\boldsymbol{\mathcal{T}})\ \boldsymbol{and}\ Ker\,(f) = \{\,\bar{0},\bar{6}\} \subseteq A(\boldsymbol{\mathcal{T}}).$$

$$\{\,\bar{0},\bar{6}\} + \{\,\bar{0},\bar{4},\bar{8}\} = A(\boldsymbol{\mathcal{T}}).Ker\,(f)\ \ll\ A(\boldsymbol{\mathcal{T}}).$$

Proposition 3.6. If $\mathcal{M}(\boldsymbol{\mathcal{T}})$ is module. Then $S\mathcal{M}(\mathcal{T})$ has homo. With kernel equal zero on R($\mathcal{T}$)-module iff $S\mathcal{M}(\mathcal{T})$ has $ker\ (f) = 0$, if is homo. on $\bar{\mathfrak{R}}(\boldsymbol{\mathcal{T}})$-module.

Proof: If $\boldsymbol{\mathcal{K}}(\mathcal{T}) \leq \mathcal{M}(\mathcal{T})$ and $f: \boldsymbol{\mathcal{K}}(\mathcal{T}) \to \mathcal{M}(\mathcal{T}) \ni\ f \neq 0$ is a $\bar{\mathfrak{R}}(\mathcal{T})$-homomorphism.So, $\mathcal{K}(\mathcal{T})$ is R($\mathcal{T}$)-submodule of$\mathcal{M}(\mathcal{T})$. To show f is $\mathfrak{R}(\mathcal{T})$_homomorphism?. $(r\mathcal{T}) \in\ \mathfrak{R}(\mathcal{T})$. Hence

$$f[(r\mathcal{T})\,x\mathcal{T}) = f[(r\mathcal{T}) + ann(\mathcal{M}(\mathcal{T})x\mathcal{T})]$$

$$=\ (r\mathcal{T}) + ann(\mathcal{M}(\mathcal{T}) +\ f(x\mathcal{T})\ (\,f\ is\ a\ \bar{R}(\mathcal{T}) - \text{homomorphism})$$

$$= (r\mathcal{T})f(x\mathcal{T}).$$

Hence f is a $\mathfrak{R}(\mathcal{T})$-homomorphism. But $S\mathcal{M}(\mathcal{T})$ has $ker\ (f) = 0$. So $ker\ (f) \ll \mathcal{K}(\mathcal{T})$. So, $ker\ (f) \ll\ A(\mathcal{T})$. The conversely is similarly.

Corollary 3.7. If $\mathcal{M}(\boldsymbol{\mathcal{T}})$ is a directed sum of simple submodules an $S\mathcal{M}(\mathcal{T})$ is monoform, then $\mathcal{M}(\mathcal{T})$ is monoform module.

Proof: If $\boldsymbol{\mathcal{K}}(\boldsymbol{\mathcal{T}}) \ll \mathcal{M}(\boldsymbol{\mathcal{T}})$. and $f: \mathcal{K}(\mathcal{T}) \to\ \mathcal{M}(\mathcal{T})$ with $f \neq 0$. Note that $S\mathcal{M}(\mathcal{T})$ is small. Then

$Ker\ (f) \ll\ \mathcal{K}(\mathcal{T})$. But $\mathcal{M}(\mathcal{T})$ is semi simple. Hence $\mathcal{K}(\mathcal{T})$ is semi simple. Therefore $S\mathcal{K}(\mathcal{T})$ is submodule and namely $(0\mathcal{T})$. Hence $Ker\ (f)\ =\ 0\mathcal{T}$. Thus $\mathcal{M}(\mathcal{T})$ is a monoform.

Proposition 3.8. If $\mathcal{M}(\mathcal{T})$ is a semisimple and $\mathcal{M}(\mathcal{T})$ is monoform, so $\mathcal{M}\ (\mathcal{T})$ is a simple.

Proof: Let (x$\mathcal{T}$)∈ $\mathcal{M}(\mathcal{T})$ such that x$\mathcal{T}$≠0 a element. $\mathcal{M}(\mathcal{T})$ is a semi simple. So, $< x\mathcal{T} > \oplus K(\mathcal{T}) = \mathcal{M}(\mathcal{T}), K(\mathcal{T}) \leq\ \mathcal{M}(\mathcal{T})$.

Also $\mathcal{M}(\mathcal{T})$ is monoform. Then $f: < x\mathcal{T} > \to\ \mathcal{M}(\mathcal{T}), f \neq 0$ and $Ker\ (f) = 0\mathcal{T}$. Now

$$g: \mathcal{M}(\mathcal{T}) \to\ \mathcal{M}(\mathcal{T})\ by: g((r\mathcal{T})x\mathcal{T} + K(\mathcal{T}) = f((r\mathcal{T})x\mathcal{T}).$$

To prove that g is well-define:

Let $(\,r_1\mathcal{T})(x\mathcal{T}) + k_1\mathcal{T} = (r_2\mathcal{T})(x\mathcal{T}) + (k_2\mathcal{T})\ ;\ r_1\mathcal{T}, r_2\mathcal{T} \in R(\mathcal{T}), k_1\mathcal{T}, k_2\mathcal{T} \in K(\mathcal{T})$. Note that

$$(r_1\mathcal{T} - r_2\mathcal{T})(x\mathcal{T}) = k_2\mathcal{T} - k_1\mathcal{T} \in < x\mathcal{T} > \cap\ K(\mathcal{T}) = (0\mathcal{T}).$$

Henc

$$(r_1\mathcal{T} - r_2\mathcal{T})x\mathcal{T} = 0\mathcal{T} = k_2\mathcal{T} - k_1\mathcal{T}.$$

So,

$$(r_1\mathcal{T} - r_2\mathcal{T})\ = k_2\mathcal{T} - k_1\mathcal{T} \text{ and } k_1\mathcal{T} = k_2\mathcal{T}.$$

Thus,

$$f(r_1\mathcal{T}x\mathcal{T}) =\ f(\,r_2\mathcal{T}x\mathcal{T})$$

Also,

$$g(\,r_1\mathcal{T}x\mathcal{T} + k_1\mathcal{T}) = g(\,r_2\mathcal{T}x\mathcal{T} + k_2\mathcal{T}).$$

Let $(r\mathcal{T})x\mathcal{T} + K\mathcal{T} \in Ker(g)$. Hence

$$g((r\mathcal{T})x\mathcal{T} + K\mathcal{T}) = f((r\mathcal{T})x\mathcal{T}) = 0\mathcal{T}.$$

Therefore,

$$Ker\ (g) = Ker\ (f\ ) \oplus K(\mathcal{T})$$

$$= 0\mathcal{T} \oplus K)\mathcal{T})$$

$$=\ K(\mathcal{T}).$$

But $Ker\ (g) = 0$. Thus $K(\mathcal{T}) = 0$. Hence $< x\mathcal{T} >= \mathcal{M}(\mathcal{T})$. So $\mathcal{M}(\mathcal{T})$ is a simple.

Definition 3.9. If $\mathcal{M}(\mathcal{T})$ is a $\mathfrak{R}(\mathcal{T})$-module; $F\mathcal{M}(\mathcal{T})$ is a free if it has basis.

In the next theorem, we explain the relationship between small monoform and monoform concepts.

Theorem 3.10. If $F\mathcal{M}(\mathcal{T})$ is a free $Z(\mathcal{T})$-module, so $S\mathcal{M}(\mathcal{T})$ is small and monoform iff $\mathcal{M}(\mathcal{T})$ is a monoform.

Proof: If $\mathcal{K}(\mathcal{T}) \leq \mathcal{M}(\mathcal{T})$ and let $f: \mathcal{K}(\mathcal{T}) \to \mathcal{M}(\mathcal{T}), f \neq 0$, so is a $S\mathcal{M}(\mathcal{T})$ is a small. Then $Ker\ (f) \ll \mathcal{K}(\mathcal{T})$. However, $F\mathcal{M}(\mathcal{T})$ is a free over integer ring Z($\mathcal{T}$). So $\mathcal{K}(\mathcal{T})$ has $0(\mathcal{T})$ is small. Hence $Ker(f) = 0\mathcal{T}$. Thus $\mathcal{M}(\mathcal{T})$ has $Ker\ (f) = 0$. The conversely is it is clear, by Remark3.2.

The next proposition explain the relationship between neutrosophic small monoform and the neutrosophic module $\mathcal{M}(\mathcal{T})$ has neutrosophic finitely generated submodule (neutrosophic Noetherian).

Proposition 3.11. If $\mathcal{M}(\mathcal{T})$ be Noetherian neutrosophic, so $S\mathcal{M}(\mathcal{T})$ is a small monoform iff

3-generted submodule of $\mathcal{M}(\mathcal{T})$ is also small.

Proof: $\Longrightarrow$ clear.

$\Longrightarrow$ Let $0 \neq$ 3-generted submodules of $\mathcal{M}(\mathcal{T})$ is Neutrosophic small mono-form. Suppose that anon-zero $f \in Hom(\mathcal{K}(\mathcal{T}), \mathcal{M}(\mathcal{T})$ and $\mathcal{K}(\mathcal{T}) \leq \mathcal{M}(\mathcal{T}) \ni 0 \neq \mathcal{K}(\mathcal{T})$ .We must show that $Ker\ (f) \ll \mathcal{K}(\mathcal{T})$. Assume that

$Ker\ (f) \neq 0$ and $(x\mathcal{T}) \neq 0\ with\ (x\mathcal{T}) \in Ker\ (f), (y\mathcal{T}) \in \mathcal{K}(\mathcal{T}), f(y\mathcal{T}) = Z(\mathcal{T})$.

Take $L(\mathcal{T})$ is 3-generted such that $L(\mathcal{T}) =< x\mathcal{T}, x\mathcal{T}, z\mathcal{T} > \leq \mathcal{M}(\mathcal{T})$. But $L(\mathcal{T}) \ll \mathcal{M}(\mathcal{T})$ is small monoform. Let $H(\mathcal{T}) =< x\mathcal{T}, y\mathcal{T} >$ and $g = f|_{H(\mathcal{T})} : H(\mathcal{T}) \to L(\mathcal{T})$. So, $Ker(g) \ll H(\boldsymbol{\mathcal{T}}) \leq \boldsymbol{\mathcal{K}}(\boldsymbol{\mathcal{T}}),$ because $L(\boldsymbol{\mathcal{T}})$ is a small. Hence

$Ker(g) \ll \boldsymbol{\mathcal{K}}(\boldsymbol{\mathcal{T}}). (x\boldsymbol{\mathcal{T}}) \in Ker(g)$ and then

$< x\boldsymbol{\mathcal{T}} > \subseteq Ker(g) \ll \mathcal{K}(\mathcal{T})$. So, $< x\mathcal{T} > \ll \mathcal{K}(\mathcal{T})$. But $\mathcal{M}$(I) is Noetherian. So, $Ker\ (f)$ is f −generated. Therefore $Ker\ (f) = \mathfrak{N}(\mathcal{T})\ x_1\mathcal{T} + \mathfrak{N}(\mathcal{T})\ x_2\mathcal{T} + \cdots \dots \dots + R(\mathcal{T})\ x_n\mathcal{T} =< x_1\mathcal{T}, x_2\mathcal{T} + \cdots \dots \dots, x_n\mathcal{T} >, x_1\mathcal{T}, \dots \dots \dots, x_n \in \mathcal{M}(\mathcal{T})$. But $< x_i\mathcal{T} > \ll \mathcal{K}(\mathcal{T}); i = 1, \dots, n$. Hence

$$Ker\ f = \sum_{i=1}^{n} R(\mathcal{T}) x_i\mathcal{T} \ll \mathcal{K}(\mathcal{T}).$$

So, $S\mathcal{M}(\mathcal{T})$ is monoform module. Not that a module $\mathcal{M}(\mathcal{T})$ is a uniform if each non-zero submodules is essential. Also, $\mathcal{M}(\mathcal{T})$ is called Q-Dedekind module if $\mathcal{K}(\mathcal{T}) \leq \mathcal{M}(\mathcal{T}), \mathcal{K}(\mathcal{T}) \neq 0$, so $Hom(\mathcal{M}(\mathcal{T})/\mathcal{K}(\mathcal{T}), \mathcal{M}(\mathcal{T})) = 0\mathcal{T}$ [ ] and $\mathcal{K}(\mathcal{T}) \leq \mathcal{M}(\mathcal{T})$ is Q-invrtbile. Finally, $\mathcal{M}(\mathcal{T})$ is called small Q-Dedekind:

3. $\forall f \in End\ \mathcal{M}(\mathcal{T}), f \neq 0, imply\ Ker\ (f) \ll \mathcal{M}(\mathcal{T})$

Theorem 3.12. If $S\mathcal{M}(\mathcal{T})$ is monoform, so $\mathcal{M}(\mathcal{T})$ is a uniform and Q-Dedekind.

Proof: It is evident that$\mathcal{M}(\mathcal{T})$ is small Q-Dedekind. Let $\mathcal{K}(\mathcal{T}) \leq \mathcal{M}(\mathcal{T})$ with $\mathcal{K}(\mathcal{T}) \neq (0\mathcal{T})$. If $\mathcal{K}(\mathcal{T}) \leq_{ess} \mathcal{M}(\mathcal{T})$, nothing to prove. Assume that N$\nleq_{ess} \mathcal{M}(\mathcal{T})$, so $\exists\ H(\mathcal{T}) \leq \mathcal{M}(\mathcal{T}) \ni H(\mathcal{T})$ is relative complement of$\mathcal{K}(\mathcal{T})$[$\mathcal{K}(\mathcal{T}) \leq \mathcal{M}(\mathcal{T})$ is relative complement for $\boldsymbol{\mathcal{K}}(\mathcal{T})$ in $\mathcal{M}(\mathcal{T})$ means $K(\mathcal{T}) \leq \mathcal{M}(\mathcal{T})$ is a maximal w.r.s $\mathcal{K}(\mathcal{T}) \cap K(\mathcal{T}) = 0\mathcal{T}$]. Therefore $\mathcal{K}(\mathcal{T}) \oplus$H($\mathcal{T}$)$\leq_{ess} \mathcal{M}(\mathcal{T})$. Define:

$$f: \mathcal{K}(\mathcal{T}) \oplus H(\mathcal{T}) \to \mathcal{M}(\mathcal{T}) : f((x\mathcal{T}) + y\mathcal{T}) = (xi) \oplus (x\mathcal{T}) + (y\mathcal{T}) \in \mathcal{K}(\mathcal{T}) \oplus H(\mathcal{T}).$$

So,

$Ker\ (f) = (0\mathcal{T}) + H(\mathcal{T})$. But $S\mathcal{M}(\mathcal{T})$ is small.

So,

$$Ker\ (f) = (0\mathcal{T}) \oplus H(\mathcal{T}) \ll \mathcal{K}(\mathcal{T}) \oplus H(\mathcal{T}).$$

Hence $H(\mathcal{T}) \ll H(\mathcal{T})$ C!. This is true when $H(\mathcal{T}) = (0\mathcal{T})$. Then $\mathcal{K}(\mathcal{T}) \leq_{ess} \mathcal{M}(\mathcal{T})$. $\mathcal{M}(\mathcal{T})$ is a uniform module.

Corollary 3.13. Let $\mathcal{M}(\mathcal{T})$ be an $\mathfrak{N}(\mathcal{T})$-module. If $S\mathcal{M}(\mathcal{T})$ is a small, then $\mathcal{M}(\mathcal{T})$ is a uniform module with ann($\mathcal{M}(\mathcal{T})$=ann($\boldsymbol{\mathcal{K}}(\mathcal{T})$ for each $\boldsymbol{\mathcal{K}}(\mathcal{T})$ is not small in $\mathcal{M}(\mathcal{T})$.

Proof: From Theorem 3.12; $\mathcal{M}(\mathcal{T})$ is a uniform. Also, $S\mathcal{M}(\mathcal{T})$ is a small Q-Dedekind. Hence for each $\boldsymbol{\mathcal{K}}(\mathcal{T})$ is not small in $\mathcal{M}(\mathcal{T})$, we have $\boldsymbol{\mathcal{K}}(\mathcal{T})$ is quasi-invertible. Therefore,

$ann_{R(\mathcal{T})}\mathcal{M}(\mathcal{T}) = ann_{R(\mathcal{T})}\boldsymbol{\mathcal{K}}(\mathcal{T})$.

Note that we utilize from condition nonsingular in the next result, because for example the module Z_4 over the ring Z is small, monoform, and not Q-dedekind. On the other hand, Z_4 not be nonsingular module over Z because Z(Z_4) not equal zero.

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Definition 3.14. If $\boldsymbol{\mathcal{M}}(\mathcal{T})$ be a non-singular, then S$\boldsymbol{\mathcal{M}}(\mathcal{T})$ is small and hence $\boldsymbol{\mathcal{M}}(\mathcal{T})$ is Q-Dedekind module.

Proof: Suppose that $\boldsymbol{\mathcal{K}}(\mathcal{T}) \ll \mathcal{M}(\mathcal{T}). S\mathcal{M}(\mathcal{T})$ is small, so $\mathcal{M}(\mathcal{T})$ is a neutrosophic uniform (BY Th 3.12). Hence $\boldsymbol{\mathcal{K}}(\mathcal{T}) \leq_{ess} \mathcal{M}(\mathcal{T})$. But $\mathcal{M}(\mathcal{T})$ is a non-singular. Therefore $\frac{\mathcal{M}(\mathcal{T})}{\boldsymbol{\mathcal{K}}(\mathcal{T})}$ is a singular. Hence Hom($\frac{\mathcal{M}(\mathcal{T})}{\boldsymbol{\mathcal{K}}(\mathcal{T})}$, $\mathcal{M}(\mathcal{T}))z = 0\mathcal{T}$. Then, $\boldsymbol{\mathcal{K}}(\mathcal{T})$ is a Q-invertible. Thus $\mathcal{M}(\mathcal{T})$ is Q-Dedekind.

Recall that any ring $\mathfrak{N}(\mathcal{T})$ is semi-simple and this imply $\mathfrak{N}(\mathcal{T})$-module is a nonsingular, so we present the next result:

Corollary 3.15. If $\mathfrak{N}(\mathcal{T})$ is a semi simple ring and $\mathcal{M}(\mathcal{T})$ is a $\mathfrak{N}(\mathcal{T})$-module, so $\mathcal{M}(\mathcal{T})$ is small and satisfies all conditions of monoform with Q-dedekind property.

Proposition 3.16. Let $\mathcal{M}(\mathcal{T})$ be a fully retractable R($\mathcal{T}$)-module. If 0≠ A($\mathcal{T}$) ≤$\mathcal{M}(\mathcal{T})$ with

A($\mathcal{T}$) is a Q- Dedekind, so $\mathfrak{N}(\mathcal{T})$ satisfy the conditions of small and monoform.

Proof: Suppose that $A(\mathcal{T}) \leq \mathcal{M}(\mathcal{T}), f: A(\mathcal{T})\ be \rightarrow \mathcal{M}(\mathcal{T}) \ni f \neq 0$. But $\mathcal{M}(\mathcal{T})$ is fully retractable:

$$\exists\ 0 \neq h: \mathcal{M}(\mathcal{T}) \rightarrow A(\mathcal{T}) \text{ with } A(\mathcal{T}) \rightarrow f \rightarrow \mathcal{M}(\mathcal{T}) \rightarrow h \rightarrow A(\mathcal{T}).$$

Hence $h \circ f \neq 0$. Also from $A(\mathcal{T})$ is small and Q-dedekind this means $Ker(h \circ f) \ll A(\mathcal{T})$. Since $Ker(f)$ subset of $Ker(h \circ f)$ is small in $A(\mathcal{T})$, then $Ker(f)$ is small in $A(\mathcal{T})$. Thus, $\mathcal{M}(\mathcal{T})$ satisfy small property with monoform condition.

4. Conclusion

In this work, we presented several new results of module, which has zero homomorphism with zero kernel in neutrosophic set. Corollary 3.13. showed that if $\mathcal{M}(\mathcal{T})$ is a $\mathfrak{N}(\mathcal{T})$-module and $S\mathcal{M}(\mathcal{T})$ is a small momoform, this means $\mathcal{M}(\mathcal{T})$ is a uniform; $ann(\mathcal{M}(\mathcal{T})) = ann(\mathcal{K}(\mathcal{T})) \ \forall\ \mathcal{K}(\mathcal{T})$ is not small in $\mathcal{M}(\mathcal{T})$. Also, in Theorem 3.12, we proved that if $\mathcal{M}(\mathcal{T})$ is a $\mathfrak{N}(\mathcal{T})$-module. If SM($\mathcal{T}$) is monoform, $\mathcal{M}(\mathcal{T})$ is uniform and Q-Dedekind. Finally, Corollary 3.15, explain that if we have $\mathfrak{N}(\mathcal{T})$ is a semi simple and $\mathcal{M}(\mathcal{T})$ is $\mathfrak{N}(\mathcal{T})$-module, so $\mathcal{M}(\mathcal{T})$ is small and satisfies all conditions of monoform together with Q-dedekind property.

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References

[1] M. M. Abed, "On Indeterminacy (Neutrosophic) of Hollow Modules," *Iraqi Journal of Science*, vol. 63, no. 6, pp. 2650-2655, 2022.

[2] J. M. Zelmanowitze, "An Extension of the Jacobson Density Theorem," *Bull. Amer. Math. Soc*., vol. 82, no. 4, pp. 551-553, 1976.

[3] Y. Tiraş and M. Alkan, "Prime Modules and Submodules," *Commun. Algebra*, vol. 31, no. 11, pp. 5253–5261, 2003.

[4] Kamal and A. M. Menshawy, "Annihilator Conditions CS-modules," *Int. J. Math. Math. Sci*., vol. 50, pp. 3195-3202, 2003.

[5] Faith, Lectures on Injective Modules and Quotient Rings, *Springer-Verlag, Berlin, Heidelberg*, New York, 1967.

[6] M. M. Abed, "A new view of closed-CS-module," *Italian Journal of Pure and Applied Mathematics*, no. 43, pp. 65-72, 2020.

[7] Y. B. Jun, F. Smarandache, and H. Bordbar, "Neutrosophic N-structures applied to BCK/BCI-algebras," *Information*, vol. 8, no. 4, pp. 128, 2017.

[8] M. Abed, D. A. Jarwan, and M. A. Salih, "On Neutrosophic Relations in Group Theory," *Int. J. Math., Stat., and Comput. Sci.*, vol. 1, pp. 48–52, 2023.

[9] V. Cetkin, B. P. Varol, and H. Aygün, "On neutrosophic submodules of a module," *Hacettepe Journal of Mathematics and Statistics*, vol. 46, no. 5, pp. 791-799, 2017.

[10] M. Abodala, "Small Homomorphism and Algebraic Relations between Strong Refined Neutrosophic Modules and Strong Neutrosophic Submodules," *Neutrosophic Sets and Systems*, vol. 39, pp. 1-8, 2021.

[11] V. Cetkin, B. P. Varol, and H. Aygün, "On neutrosophic submodules of a module," *Hacettepe Journal of Mathematics and Statistics*, vol. 46, no. 5, pp. 791-799, 2017.

[12] A. R. Ali and S. H. Zail, "On the Structure of Neutrosophic Modules and Their Applications," *Journal of Mathematical Structures and Applications*, vol. 10, no. 4, pp. 1-12, 2021.

[13] A. Tuganbaev, "Multiplication Modules," *Journal of Mathematical Sciences*, vol. 123, no. 2, pp. 3837-3860, 2004.

[14] S. H. Zail, M. M. Abed, and A. S. Faisal, "On neutrosophic BCK-algebra and Ω-BCK-algebra," *Int. J. Neutrosophic Sci.*, vol. 19, no. 3, pp. 8-15, 2022.

[15] F. F. Kareem and M. M. Abed, "Generalizations of Fuzzy k-ideals in a KU-algebra with semigroup," *Journal of Physics: Conference Series*, vol. 1879, pp. 022108, 2021.

[16] E. A. Naim and A. R. Mehdi, "On gs-pseudo-injective modules," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 25, no. 5, pp. 1535–1545, 2022. [Online]. Available: https://doi.org/10.1080/09720529.2022.2082724.

[17] M. Fujita and M. Kageyama, "Quasi-quadratic modules in pseudo-valuation domain," *arXiv preprint arXiv:* 2310.04116, Oct. 2023. [Online]. Available: https://arxiv.org/abs/2310.04116.

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