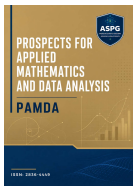
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REVIEW ARTICLE

Characteristics Neutrosophic Ideals for Neutrosophic Rings: On Review

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ABSTRACT

The main objective of this paper is to present a review study with more information on the neutrosophic ideal, principal ideal, prime ideal, pseudo neutrosophic ideal, quotient ring, and pseudo quotient ring. Neutrosophic ring theory is a branch of neutrosophic algebra which was introduced by Florentin Smarandache in 2006.

Keywords: Neutrosophic ring ▪ Neutrosophic ideals ▪ Types of neutrosophic ideals ▪ Neutrosophic quotient ring ▪ Pseudo neutrosophic quotient rings

1. INTRODUCTION

In 2006, Kandasamy and Smarandache introduced neutrosophic algebra structures, N-algebra structures, and neutrosophic rings. They studied the structure of neutrosophic groups, neutrosophic N-groups, neutrosophic rings, and neutrosophic group rings [10, 12]. Moreover, their work on Smarandache neutrosophic algebraic structures presented neutrosophic groups (semigroups), bi-groups, and N-groups [11].

Previous work motivated other researchers. Agboola, Akinola, and Oyebola introduced elementary properties of neutrosophic rings together with neutrosophic polynomial rings [2]. In 2012, Agboola, Adeleke, and Akinleye studied ideals of neutrosophic rings and neutrosophic quotient rings [1]. Later, in 2023, Al-Odhari presented a review of neutrosophic rings and neutrosophic subrings, with properties of the neutrosophic elements of a ring [3], and investigated algebraic structures of neutrosophic matrices [4].

Between 2020 and 2023, subsequent studies extended this theory through refined neutrosophic rings and their homomorphisms [5, 6], nil ideals [7], groups of units [8], and neutrosophic sub-rings [9]. In this article, we present a review and an expansion of neutrosophic ideals, types of neutrosophic ideals, neutrosophic quotient rings, and pseudo neutrosophic quotient rings.

2. PRELIMINARIES

In this section, we introduce a review of the neutrosophic ring and the neutrosophic subring, which will be used in the following sections.

Definition 2.1 ([10]). Let R be any ring. The neutrosophic ring is a ring generated by R and I under the operations of R, and is denoted by $N(R)$, where

$$N(R) = \langle R \cup I \rangle = \{a + bI : a, b \in R\}. \tag{1}$$

Theorem 2.1 ([10]). Let $\langle R \cup I \rangle$ be a neutrosophic ring. Then $\langle R \cup I \rangle$ is a ring.

Definition 2.2 ([3, 10]). Let R be a nonempty set and let the triple $(R, +, \cdot)$ be a ring. Consider

$$N(R) = R[I] = \langle R \cup I \rangle = \{a + bI : a, b \in R\}.$$

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The neutrosophic algebra structure

$$N(R) = \langle R \cup I \rangle = \langle R[I], +, \cdot \rangle$$

is called a neutrosophic ring generated by I and R under addition and multiplication, respectively, if it satisfies the following ring axioms:

NR1. For all $x, y, z \in N(R)$, $\langle R[I], + \rangle$ is a neutrosophic abelian group under addition.

NR2. For all $x, y, z \in N(R^*)$, $\langle R^*[I], \cdot \rangle$ is a mathematical associative neutrosophic system under multiplication; that is, it is a neutrosophic semigroup.

NR3. The left and right distributive laws hold:

$$x(y+z) = xy + xz, \qquad (y+z)x = yx + zx.$$

Theorem 2.2 ([4])**.** Let $\langle R \cup I \rangle$ be a neutrosophic ring and let $x, y \in \langle R \cup I \rangle$. Then:

1. $x \cdot 0 = 0 \cdot x = 0$;
2. $x(-y) = (-x)y = -(xy)$;
3. $(-x)(-y) = xy$.

Definition 2.3 ([10])**.** Let $\langle R \cup I \rangle$ be a neutrosophic ring. A proper subset P of $\langle R \cup I \rangle$ is said to be a neutrosophic subring if P is itself a neutrosophic ring under the operations of $\langle R \cup I \rangle$. It is essential that

$$P = \langle S \cup I \rangle \quad \text{or, more generally,} \quad P = \langle S \cup nI \rangle,$$

where n is a positive integer and S is a subring of R. Thus, P is generated by the subring S together with nI, where $n \in \mathbb{Z}^+$. Even if P is a ring, if it cannot be represented in this form, it is not called a neutrosophic subring of $\langle R \cup I \rangle$.

Theorem 2.3 ([3])**.** Let $\langle R \cup I \rangle$ be a neutrosophic ring and $N(S) \subseteq N(R)$. Then $N(S)$ is a neutrosophic subring of $N(R)$ if and only if:

1. $a, b \in N(S)$ implies $a - b \in N(S)$;
2. $a, b \in N(S)$ implies $ab \in N(S)$.

3. NEUTROSOPHIC IDEALS AND PSEUDO NEUTROSOPHIC IDEALS

In this section, we introduce the neutrosophic ideal, neutrosophic principal ideal, neutrosophic prime ideal, neutrosophic maximal ideal, and pseudo neutrosophic ideal.

Definition 3.1 ([10])**.** Let $\langle R \cup I \rangle$ be any neutrosophic ring. A nonempty neutrosophic subset P of $\langle R \cup I \rangle$ is called a neutrosophic left ideal if:

1. P is a neutrosophic subring of $\langle R \cup I \rangle$;
2. for every $p \in P$ and $r \in \langle R \cup I \rangle$, $rp \in P$.

It is called a neutrosophic right ideal if the first condition holds and $pr \in P$ for every $p \in P$ and $r \in \langle R \cup I \rangle$.

Definition 3.2. Let $\langle R \cup I \rangle$ be any neutrosophic ring. A nonempty neutrosophic subset P of $\langle R \cup I \rangle$ is a neutrosophic ideal (two-sided) if:

1. P is a neutrosophic subring of $\langle R \cup I \rangle$;
2. for all $p \in P$ and $r \in \langle R \cup I \rangle$, both $rp \in P$ and $pr \in P$.

Example 3.1. The triple $(\mathbb{Z}, +, \cdot)$ is the integer ring and $(2\mathbb{Z}, +, \cdot)$ is a subring. Hence $\langle \mathbb{Z} \cup I \rangle$ is a neutrosophic ring and $\langle 2\mathbb{Z} \cup I \rangle$ is a neutrosophic subring. For

$$p = 2a + 2bI \in \langle 2\mathbb{Z} \cup I \rangle, \qquad r = a + bI \in \langle \mathbb{Z} \cup I \rangle,$$

we have

$$\begin{aligned} rp &= (a+bI)(2a+2bI) \\ &= 2aa + (2ab + 2ba + 2bb)I \\ &= 2aa + 2(ab + ba + bb)I, \\ pr &= (2a+2bI)(a+bI) \\ &= 2aa + (2ab + 2ba + 2bb)I \\ &= 2aa + 2(ab + ba + bb)I \in \langle 2\mathbb{Z} \cup I \rangle. \end{aligned}$$

Thus, $\langle 2\mathbb{Z} \cup I \rangle$ is a neutrosophic ideal of $\langle \mathbb{Z} \cup I \rangle$.

Proposition 3.1. Every neutrosophic ideal P of $\langle R \cup I \rangle$ is a neutrosophic subring of $\langle R \cup I \rangle$, but the converse need not be true.

Proposition 3.2. In the integer ring $\mathbb{Z}$, every $n\mathbb{Z}$ is an ideal for $n \in \mathbb{Z}^+$. Then every $\langle n\mathbb{Z} \cup I \rangle$ is a neutrosophic ideal of $\langle \mathbb{Z} \cup I \rangle$.

Proof. The set $n\mathbb{Z}$ is a subring of $\mathbb{Z}$, so $\langle n\mathbb{Z} \cup I \rangle$ is a neutrosophic subring of $\langle \mathbb{Z} \cup I \rangle$. Let

$$x = a_1 + b_1 I \in \langle n\mathbb{Z} \cup I \rangle, \qquad r = a + bI \in \langle \mathbb{Z} \cup I \rangle,$$

where $a_1, b_1 \in n\mathbb{Z}$ and $a, b \in \mathbb{Z}$. Then

$$rx = aa_1 + (ab_1 + ba_1 + bb_1)I.$$

Since $n\mathbb{Z}$ is an ideal of $\mathbb{Z}$, both coefficients belong to $n\mathbb{Z}$, and hence $rx \in \langle n\mathbb{Z} \cup I \rangle$. Similarly, $xr \in \langle n\mathbb{Z} \cup I \rangle$. □

Example 3.2. In $(\mathbb{Z}, +, \cdot)$, the set $5\mathbb{Z}$ is an ideal; therefore $\langle 5\mathbb{Z} \cup I \rangle$ is a neutrosophic ideal of $\langle \mathbb{Z} \cup I \rangle$. For $a, b, a', b' \in \mathbb{Z}$,

$$\begin{aligned} (5a+5bI)(a'+b'I) &= 5aa' + (5ab' + 5ba' + 5bb')I \\ &= 5aa' + 5(ab' + ba' + bb')I \\ &\in \langle 5\mathbb{Z} \cup I \rangle, \end{aligned}$$

and the product in the reverse order also belongs to $\langle 5\mathbb{Z} \cup I \rangle$.

Theorem 3.1. Let R be any ring and let S be an ideal of R. If $\langle R \cup I \rangle$ is a neutrosophic ring, then $\langle S \cup I \rangle$ is a neutrosophic ideal of $\langle R \cup I \rangle$.

Proof. Since S is an ideal of R, it is a subring, and $P = \langle S \cup I \rangle$ is a neutrosophic subring of $\langle R \cup I \rangle$. For $p = a_1 + b_1 I \in P$, where $a_1, b_1 \in S$, and $r = a + bI \in \langle R \cup I \rangle$, where $a, b \in R$,

$$\begin{aligned} rp &= (a+bI)(a_1 + b_1 I) \\ &= aa_1 + (ba_1 + ab_1 + bb_1)I. \end{aligned}$$

Both coefficients lie in S, because S is an ideal of R. Hence $rp \in P$. The same argument gives $pr \in P$. □

Example 3.3. Let $(\mathbb{Z}_4, \oplus_4, \odot_4)$ be the ring of integers modulo 4 and let $H = \{0,2\}$ be an ideal. Then $\langle \mathbb{Z}_4 \cup I \rangle$ is a neutrosophic ring and

$$\langle H \cup I \rangle = \{0, 2, 2I, 2+2I\}$$

is a neutrosophic subring. For $(a+bI) \in \langle \mathbb{Z}_4 \cup I \rangle$ and $(a_1 + b_1 I) \in \langle H \cup I \rangle$,

$$(a+bI)(a_1+b_1I) = aa_1 + (ab_1 + ba_1 + bb_1)I \in \langle H \cup I \rangle,$$

and the reverse product also belongs to $\langle H \cup I \rangle$.

Proposition 3.3. If H is a neutrosophic left, right, or two-sided ideal of $\langle R \cup I \rangle$, and J is an ideal of the corresponding type in H, then J is not necessarily an ideal of the same type in $\langle R \cup I \rangle$.

Definition 3.3. Let P be a neutrosophic ideal of $\langle R \cup I \rangle$. If P is generated by an element $r = a + bI \in \langle R \cup I \rangle$, where $a, b \neq 0$, then P is called a neutrosophic principal ideal of $\langle R \cup I \rangle$, denoted by (r).

Definition 3.4. Let R be a commutative ring and $a \in R$. An ideal

$$H = aR = \{ar : r \in R\}$$

is called a principal ideal of R. A neutrosophic ideal

$$P = a\langle R \cup I \rangle = \{ax : x \in \langle R \cup I \rangle\}$$

is called a neutrosophic principal ideal of $\langle R \cup I \rangle$.

Proposition 3.4. Let $\langle \mathbb{Z} \cup I \rangle$ be the neutrosophic ring of integers. If S is an ideal of $\mathbb{Z}$, then $P = \langle S \cup I \rangle$ is a principal ideal of $\langle \mathbb{Z} \cup I \rangle$.

Proof. Since S is an ideal of $\mathbb{Z}$, it is principal and $S = n\mathbb{Z}$ for some $n \in \mathbb{Z}^+$, allowing $n = 0$ for the zero ideal. If $P = \{0\}$, then $S = 0\mathbb{Z}$ and

$$P = \{0r : r \in \langle \mathbb{Z} \cup I \rangle\} = (0).$$

If $P \neq \{0\}$, let n be the smallest positive integer in S. For every $m \in S$, $m = qn$ for some $q \in \mathbb{Z}$. Consequently m, mI, and $m+mI$ are obtained by multiplying n by q, qI, and $q+qI$, respectively. Hence

$$P = \{nr : r \in \langle \mathbb{Z} \cup I \rangle\}.$$

□

Proposition 3.5. Let R be any ring and let H be a principal ideal of R. Then $P = \langle H \cup I \rangle$ is a neutrosophic principal ideal of $\langle R \cup I \rangle$.

Definition 3.5. Let P be a neutrosophic ideal of $\langle R \cup I \rangle$. If every neutrosophic ideal J of $\langle R \cup I \rangle$ satisfying $P \subseteq J$ has either $J = P$ or $J = \langle R \cup I \rangle$, then P is called a neutrosophic maximal ideal.

Example 3.4. The ideal $\langle 3\mathbb{Z} \cup I \rangle$ is a neutrosophic maximal ideal of $\langle \mathbb{Z} \cup I \rangle$.

Example 3.5. Consider the ring $(\mathbb{Z}_4, \oplus_4, \odot_4)$ and its ideal $H = \{0,2\}$. The neutrosophic ring is

$$\langle \mathbb{Z}_4 \cup I \rangle = \{0,1,2,3,I,2I,3I,1+I,1+2I,1+3I, \\ 2+I,2+2I,2+3I,3+I,3+2I,3+3I\}.$$

The set $\langle H \cup I \rangle = \{0,2,2I,2+2I\}$ is a neutrosophic ideal. Since there is no proper neutrosophic ideal of $\langle \mathbb{Z}_4 \cup I \rangle$ strictly containing $\langle H \cup I \rangle$, it is a neutrosophic maximal ideal.

Proposition 3.6. Let R be any ring and let S be a maximal ideal of R. Then $P = \langle S \cup I \rangle$ is a neutrosophic maximal ideal of $\langle R \cup I \rangle$.

Definition 3.6. Let P be a neutrosophic ideal of $\langle R \cup I \rangle$. If, for any two neutrosophic ideals J, Q of $\langle R \cup I \rangle$, the inclusion $JQ \subseteq P$ implies either $J \subseteq P$ or $Q \subseteq P$, then P is called a neutrosophic prime ideal.

Definition 3.7. Let P be a neutrosophic ideal of $\langle R \cup I \rangle$. If, for any $x, y \in \langle R \cup I \rangle$, $xy \in P$ implies either $x \in P$ or $y \in P$, then P is called a neutrosophic prime ideal.

Example 3.7. Consider the ring $(\mathbb{Z}_6, \oplus_6, \odot_6)$ and its ideal $H = \{0,2,4\}$. The neutrosophic ring is

$$\langle \mathbb{Z}_6 \cup I \rangle = \left\{ \begin{matrix} 0,1,2,3,4,5,I,2I,3I,4I,5I, \\ 1+I,1+2I,1+3I,1+4I,1+5I, \\ 2+I,2+2I,2+3I,2+4I,2+5I, \\ 3+I,3+2I,3+3I,3+4I,3+5I, \\ 4+I,4+2I,4+3I,4+4I,4+5I, \\ 5+I,5+2I,5+3I,5+4I,5+5I \end{matrix} \right\}.$$

Furthermore,

$$\langle H \cup I \rangle = \{0,2,4,2I,4I,2+2I,2+4I,4+2I,4+4I\}.$$

For $x, y \in \langle \mathbb{Z}_6 \cup I \rangle$, if $xy \in \langle H \cup I \rangle$, then either $x \in \langle H \cup I \rangle$ or $y \in \langle H \cup I \rangle$. Thus $\langle H \cup I \rangle$ is a neutrosophic prime ideal of $\langle \mathbb{Z}_6 \cup I \rangle$.

Proposition 3.7. Let R be any ring and let S be a prime ideal of R. Then $P = \langle S \cup I \rangle$ is a neutrosophic prime ideal of $\langle R \cup I \rangle$.

Theorem 3.2. Let $\langle R \cup I \rangle$ be a neutrosophic ring and let P be a subset of $\langle R \cup I \rangle$. Then P is an ideal of $\langle R \cup I \rangle$ if and only if:

1. $0 \in P$;
2. $x - y \in P$ for all $x, y \in P$;
3. $rp, pr \in P$ for all $r \in R$ and $p \in P$.

Proof. If P is an ideal of $\langle R \cup I \rangle$, then P is a subring; hence $0 \in P$ and $x - y, xy \in P$ for all $x, y \in P$. Also, $rp, pr \in P$ for $r \in R$ and $p \in P$. Conversely, if the three conditions hold, the ideal conditions follow directly. □

Proposition 3.8. Let P and J be neutrosophic ideals of the neutrosophic ring $\langle R \cup I \rangle$. Then:

1. $P+J$ is a neutrosophic ideal of $\langle R\cup I\rangle$;
2. PJ is a neutrosophic ideal of $\langle R\cup I\rangle$;
3. $P\cap J$ is a neutrosophic ideal of $\langle R\cup I\rangle$.

Proof. For the first assertion, define

$$P+J=\{u=x+y : x\in P,\ y\in J\}.$$

Since $0\in P$ and $0\in J$, $P+J$ is nonempty. Let

$$\begin{aligned} x_1&=a_1+b_1I, \quad x_2=a_2+b_2I\in P,\\ y_1&=c_1+d_1I, \quad y_2=c_2+d_2I\in J. \end{aligned}$$

For $u=x_1+y_1$ and $v=x_2+y_2$,

$$\begin{aligned} u-v=&\big((a_1+b_1I)-(a_2+b_2I)\big)\\ &+\big((c_1+d_1I)-(c_2+d_2I)\big)\in P+J. \end{aligned}$$

For $r=a+bI\in\langle R\cup I\rangle$,

$$\begin{aligned} ru&=(a+bI)\big((a_1+b_1I)+(c_1+d_1I)\big)\\ &=(a+bI)(a_1+b_1I)+(a+bI)(c_1+d_1I)\in P+J. \end{aligned}$$

Similarly, $ur\in P+J$.

For the second assertion, define

$$\begin{aligned} PJ&=\sum_{i=1}^{n}(x_iy_i)\\ &=\{x_1y_1+x_2y_2+\cdots+x_ny_n : x_i\in P[I],\ y_i\in J[I],\\ &\qquad 1\le i\le n,\ n\in\mathbb{N}\}. \end{aligned}$$

Write $x_i=x_i'+x_i''I$ and $y_i=y_i'+y_i''I$. As displayed in the source, this expands to

$$PJ=\textstyle\sum_{i=1}^{n}\left((x_i'+y_i')+(x_i'y_i''+x_i''y_i'+x_i''y_i'')I\right).$$

Let

$$p_1=\sum_{i=1}^{n}x_iy_i, \qquad p_2=\sum_{j=1}^{n}z_jw_j,$$

where $x_i,z_j\in P$ and $y_i,w_j\in J$. Then

$$\begin{aligned} p_1-p_2&=\sum_{i=1}^{n}x_iy_i-\sum_{j=1}^{n}z_jw_j\\ &=(x_1y_1+\cdots+x_ny_n)-(z_1w_1+\cdots+z_nw_n). \end{aligned}$$

Writing $z_i=z_i'+z_i''I$ and $w_i=w_i'+w_i''I$, the elementwise expansion in the source is

$$\begin{aligned} p_1-p_2=&\sum_{i=1}^{n}(x_i'-z_i')+\sum_{i=1}^{n}(y_i'-w_i')\\ &+\sum_{i=1}^{n}\Big[(x_i'y_i''-z_i'w_i'')+(x_i''y_i'-z_i''w_i')+(x_i''y_i''-z_i''w_i'')\Big]I\in PJ. \end{aligned}$$

If $p=\sum_{i=1}^{n}x_iy_i\in PJ$ and $r\in\langle R\cup I\rangle$, then

$$\begin{aligned} rp&=r\sum_{i=1}^{n}(x_iy_i)=r(x_1y_1+x_2y_2+\cdots+x_ny_n)\\ &=\sum_{i=1}^{n}\left((rx_i'+ry_i')+(rx_i'y_i''+rx_i''y_i'+rx_i''y_i'')I\right)\in PJ. \end{aligned}$$

The same method gives $pr\in PJ$.

For the third assertion, define

$$P\cap J=\{x=a+bI : x\in P,\ x\in J\}.$$

Clearly $0\in P\cap J$. If $a_1+b_1I, a_2+b_2I\in P\cap J$, then

$$(a_1+b_1I)-(a_2+b_2I)\in P\cap J.$$

For $a+bI\in\langle R\cup I\rangle$ and $a_1+b_1I\in P\cap J$, both

$$(a+bI)(a_1+b_1I), \qquad (a_1+b_1I)(a+bI)$$

belong to $P\cap J$. Hence $P\cap J$ is a neutrosophic ideal of $\langle R\cup I\rangle$. □

Example 3.8. Let $\langle M_{2\times2}(\mathbb{R})\cup I\rangle$ be a neutrosophic matrix ring with matrix addition and multiplication, and let

$$M=\left\{\begin{bmatrix}x & y\\ 0 & z\end{bmatrix} : x,y,z\in\langle R\cup I\rangle\right\}.$$

Define

$$\begin{aligned} P&=\left\{\begin{bmatrix}a+bI & c+dI\\ 0 & 0\end{bmatrix} : a,b,c,d\in\mathbb{R}\right\},\\ J&=\left\{\begin{bmatrix}0 & e+fI\\ 0 & h+gI\end{bmatrix} : e,f,h,g\in\mathbb{R}\right\}. \end{aligned}$$

These are two neutrosophic ideals of M. For

$$p=\begin{bmatrix}a+bI & c+dI\\ 0 & 0\end{bmatrix}\in P, \quad j=\begin{bmatrix}0 & e+fI\\ 0 & h+gI\end{bmatrix}\in J,$$

we obtain

$$\begin{aligned} p+j&=\begin{bmatrix}a+bI & c+dI\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & e+fI\\ 0 & h+gI\end{bmatrix}\\ &\in\left\{\begin{bmatrix}a+bI & k+rI\\ 0 & h+gI\end{bmatrix} : a,b,k,r,h,g\in\mathbb{R}\right\}. \end{aligned}$$

Hence $P+J$ is a neutrosophic ideal of M. Furthermore,

$$P\cap J=\left\{\begin{bmatrix}0 & k+rI\\ 0 & 0\end{bmatrix} : k,r\in\mathbb{R}\right\}.$$

For $x\in P\cap J$ and $m\in M$, the products have the forms

$$xm=\begin{bmatrix}0 & s+tI\\ 0 & 0\end{bmatrix}, \qquad mx=\begin{bmatrix}0 & n+lI\\ 0 & 0\end{bmatrix},$$

for real s,t,n,l. Therefore $P\cap J$ is a neutrosophic ideal of M.

Definition 3.8. Let $\langle R\cup I\rangle$ be a neutrosophic ring and let P be a pseudo neutrosophic subring of $\langle R\cup I\rangle$. If $rp, pr\in P$ for every $p\in P$ and $r\in\langle R\cup I\rangle$, then P is called a pseudo neutrosophic ideal of $\langle R\cup I\rangle$.

Example 3.9. In the neutrosophic ring

$$(\langle\mathbb{Z}_2\cup I\rangle,\oplus_2,\odot_2)=\{0,1,I,1+I\},$$

the set $P=\{0,I\}$ is a pseudo neutrosophic subring. Its addition and multiplication structures are shown in Tables 3.1 and 3.2.

Table 3.1. The operation $(P,\oplus_2)$.

$\oplus_2$	0	I
0	0	I
I	I	0

Table 3.2. The operation $(P,\otimes_2)$.

$\otimes_2$	0	I
0	0	0
I	0	I

Also,

$$1\odot_2 0=0,\quad 1\odot_2 I=I,$$

and

$$(1+I)\odot_2 0=0,\quad (1+I)\odot_2 I=0.$$

Thus P is a pseudo neutrosophic ideal of $\langle\mathbb{Z}_2\cup I\rangle$.

Example 3.10. Let $(T,+,\cdot)$ be the pseudo neutrosophic ring

$$T=\{\ldots,-4I,-3I,-2I,-I,0,I,2I,3I,4I,\ldots\}.$$

Then

$$P=\{\ldots,-6I,-4I,-2I,0,2I,4I,6I,\ldots\}$$

is a pseudo neutrosophic ideal of T.

Example 3.11. Let

$$T=\{\ldots,-8I,-6I,-4I,-2I,0,2I,4I,6I,8I,\ldots\}.$$

Then

$$P=\{\ldots,-16I,-12I,-8I,-4I,0,4I,8I,12I,16I,\ldots\}$$

is a pseudo neutrosophic ideal of T.

Definition 3.9. Let P be a pseudo neutrosophic ideal of $\langle R\cup I\rangle$. If P is generated by an element $r=aI\in\langle R\cup I\rangle$, where $a\neq 0$, then P is called a pseudo neutrosophic principal ideal, denoted by (r).

Example 3.12. For

$$T=\{\ldots,-4I,-3I,-2I,-I,0,I,2I,3I,4I,\ldots\},$$

the ideal

$$P=\{\ldots,-6I,-4I,-2I,0,2I,4I,6I,\ldots\}$$

is generated by $2I$. Hence P is a pseudo neutrosophic principal ideal of T.

Example 3.13. For

$$T=\{\ldots,-8I,-6I,-4I,-2I,0,2I,4I,6I,8I,\ldots\},$$

the ideal

$$P=\{\ldots,-16I,-12I,-8I,-4I,0,4I,8I,12I,16I,\ldots\}$$

is generated by $2I$ and is a pseudo neutrosophic principal ideal of T.

Definition 3.10. Let P be a pseudo neutrosophic ideal of $\langle R\cup I\rangle$. If every pseudo neutrosophic ideal J of $\langle R\cup I\rangle$ containing P is equal to either P or $\langle R\cup I\rangle$, then P is called a pseudo neutrosophic maximal ideal.

Example 3.14. The ideal P in Example 3.13 is a pseudo neutrosophic maximal ideal of T.

Definition 3.11. Let P be a pseudo neutrosophic ideal of $\langle R\cup I\rangle$. If, for any two pseudo neutrosophic ideals J,Q of $\langle R\cup I\rangle$, $JQ\subseteq P$ implies either $J\subseteq P$ or $Q\subseteq P$, then P is called a pseudo neutrosophic prime ideal.

Example 3.15. Let

$$T=\{\ldots,-4I,-3I,-2I,-I,0,I,2I,3I,4I,\ldots\}.$$

Then

$$P=\{\ldots,-6I,-4I,-2I,0,2I,4I,6I,\ldots\}$$

is a pseudo neutrosophic prime ideal of T.

4. NEUTROSOPHIC QUOTIENT RINGS AND PSEUDO NEUTROSOPHIC QUOTIENT RINGS

In this section, we introduce neutrosophic quotient rings and pseudo neutrosophic quotient rings, together with examples.

Definition 4.1. Let $\langle R\cup I\rangle$ be a neutrosophic ring and let P be a neutrosophic ideal of $\langle R\cup I\rangle$. The neutrosophic quotient ring is

$$\langle R\cup I\rangle/P=\{r+P:r\in\langle R\cup I\rangle\},\tag{2}$$

where, for $r_1+P,r_2+P\in\langle R\cup I\rangle/P$,

$$(r_1+P)+(r_2+P)=(r_1+r_2)+P,$$
$$(r_1+P)(r_2+P)=(r_1r_2)+P.$$

Example 4.1. In the ring $(\mathbb{Z}_4,\oplus_4,\otimes_4)=\{0,1,2,3\}$, let $H=\{0,2\}$ and

$$P=\langle H\cup I\rangle=\{0,2,2I,2+2I\}.$$

The neutrosophic ring $\langle\mathbb{Z}_4\cup I\rangle$ is the 16-element set displayed in Example 3.5. Its quotient by P is initially written as

$$\langle\mathbb{Z}_4\cup I\rangle/P=\{r+P:r\in\langle\mathbb{Z}_4\cup I\rangle\}.$$

Since

$$0+P=2+P=2I+P=(2+2I)+P=P,$$

the source lists the representatives

$$\langle\mathbb{Z}_4\cup I\rangle/P=\left\{\begin{matrix}P,1+P,3+P,I+P,3I+P,\\(1+I)+P,(1+2I)+P,(1+3I)+P,\\(2+I)+P,(2+3I)+P,(3+I)+P,\\(3+2I)+P,(3+3I)+P\end{matrix}\right\}.$$

The zero element is P. The addition structure $(\langle\mathbb{Z}_4\cup I\rangle/P,\oplus_4)$ is reproduced in Table 4.1.

The multiplication structure $(\langle\mathbb{Z}_4\cup I\rangle/P,\otimes_4)$ is reproduced in Table 4.3.

Example 4.2. From Example 2.1, let $\langle R\cup I\rangle=\langle\mathbb{Z}\cup I\rangle$ and $P=\langle 2\mathbb{Z}\cup I\rangle$. Then

$$\langle R\cup I\rangle/P=\langle\mathbb{Z}\cup I\rangle/\langle 2\mathbb{Z}\cup I\rangle=\{r+\langle 2\mathbb{Z}\cup I\rangle : r\in\langle\mathbb{Z}\cup I\rangle\}.$$

Example 4.3. Consider the neutrosophic ring

$$\langle\mathbb{Z}_6\cup I\rangle=\left\{\begin{matrix}0,1,2,3,4,5,I,2I,3I,4I,5I,\\1+I,1+2I,1+3I,1+4I,1+5I,\\2+I,2+2I,2+3I,2+4I,2+5I,\\3+I,3+2I,3+3I,3+4I,3+5I,\\4+I,4+2I,4+3I,4+4I,4+5I,\\5+I,5+2I,5+3I,5+4I,5+5I\end{matrix}\right\}.$$

Let

$$P=\{0,2,4,2I,4I,2+2I,2+4I,4+2I,4+4I\}.$$

The quotient is the complete coset set

$$\langle\mathbb{Z}_6\cup I\rangle/P=\left\{\begin{matrix}0+P,1+P,2+P,3+P,4+P,5+P,\\I+P,2I+P,3I+P,4I+P,5I+P,\\(1+I)+P,(1+2I)+P,(1+3I)+P,(1+4I)+P,(1+5I)+P,\\(2+I)+P,(2+2I)+P,(2+3I)+P,(2+4I)+P,(2+5I)+P,\\(3+I)+P,(3+2I)+P,(3+3I)+P,(3+4I)+P,(3+5I)+P,\\(4+I)+P,(4+2I)+P,(4+3I)+P,(4+4I)+P,(4+5I)+P,\\(5+I)+P,(5+2I)+P,(5+3I)+P,(5+4I)+P,(5+5I)+P\end{matrix}\right\}.$$

After identifying equal cosets, the source records

$$\left\{\begin{matrix}P,1+P,3+P,5+P,I+P,3I+P,5I+P,\\(1+I)+P,(1+2I)+P,(1+3I)+P,(1+4I)+P,(1+5I)+P,\\(2+I)+P,(2+3I)+P,(2+5I)+P,\\(3+I)+P,(3+2I)+P,(3+3I)+P,(3+4I)+P,(3+5I)+P,\\(4+I)+P,(4+3I)+P,(4+5I)+P,\\(5+I)+P,(5+2I)+P,(5+3I)+P,(5+4I)+P,(5+5I)+P\end{matrix}\right\},$$

which is a neutrosophic quotient ring.

Also, $P=\{0,3,3I,3+3I\}$ is a neutrosophic ideal of $\langle\mathbb{Z}_6\cup I\rangle$. The corresponding quotient is

$$\langle\mathbb{Z}_6\cup I\rangle/P=\left\{\begin{matrix}0+P,1+P,2+P,3+P,4+P,5+P,\\I+P,2I+P,3I+P,4I+P,5I+P,\\(1+I)+P,(1+2I)+P,(1+3I)+P,(1+4I)+P,(1+5I)+P,\\(2+I)+P,(2+2I)+P,(2+3I)+P,(2+4I)+P,(2+5I)+P,\\(3+I)+P,(3+2I)+P,(3+3I)+P,(3+4I)+P,(3+5I)+P,\\(4+I)+P,(4+2I)+P,(4+3I)+P,(4+4I)+P,(4+5I)+P,\\(5+I)+P,(5+2I)+P,(5+3I)+P,(5+4I)+P,(5+5I)+P\end{matrix}\right\}.$$

The source then reduces this display to

$$\left\{\begin{matrix}P,1+P,2+P,4+P,5+P,I+P,2I+P,4I+P,5I+P,\\(1+I)+P,(1+2I)+P,(1+3I)+P,(1+4I)+P,(1+5I)+P,\\(2+I)+P,(2+2I)+P,(2+3I)+P,(2+4I)+P,(2+5I)+P,\\(3+I)+P,(3+2I)+P,(3+4I)+P,(3+5I)+P,\\(4+I)+P,(4+2I)+P,(4+3I)+P,(4+4I)+P,(4+5I)+P,\\(5+I)+P,(5+2I)+P,(5+3I)+P,(5+4I)+P,(5+5I)+P\end{matrix}\right\},$$

again a neutrosophic quotient ring.

Definition 4.2. Let $\langle R\cup I\rangle$ be a neutrosophic ring and let P be a pseudo neutrosophic ideal of $\langle R\cup I\rangle$. The pseudo neutrosophic quotient ring is defined by

$$\langle R\cup I\rangle/P=\{r+P : r\in\langle R\cup I\rangle\}. \tag{3}$$

Example 4.4. In the pseudo neutrosophic ring

$$T=\{\ldots,-4I,-3I,-2I,-I,0,I,2I,3I,4I,\ldots\},$$

let

$$P=\{\ldots,-6I,-4I,-2I,0,2I,4I,6I,\ldots\}.$$

Then

$$T/P=\{\ldots,-4I+P,-3I+P,-2I+P,-I+P,P,I+P,2I+P,3I+P,4I+P,\ldots\}$$

is a pseudo neutrosophic quotient ring.

Example 4.5. Consider

$$T=\{\ldots,-8I,-6I,-4I,-2I,0,2I,4I,6I,8I,\ldots\}$$

and its pseudo neutrosophic ideal

$$P=\{\ldots,-16I,-12I,-8I,-4I,0,4I,8I,12I,16I,\ldots\}.$$

Then

$$T/P=\{\ldots,-8I+P,-6I+P,-4I+P,-2I+P,P,2I+P,4I+P,6I+P,8I+P,\ldots\}$$

is a pseudo neutrosophic quotient ring.

Example 4.6. In $\langle\mathbb{Z}_2\cup I\rangle=\{0,1,I,1+I\}$, the set $P=\{0,I\}$ is a pseudo neutrosophic ideal. Hence

$$\begin{aligned}\langle\mathbb{Z}_2\cup I\rangle/P&=\{0+P,1+P,I+P,(1+I)+P\}\\&=\{P,1+P,(1+I)+P\}\end{aligned}$$

is a pseudo neutrosophic quotient ring.

Example 4.7. Let

$$\begin{aligned}\langle\mathbb{Z}_4\cup I\rangle=\{&0,1,2,3,I,2I,3I,1+I,1+2I,1+3I,\\&2+I,2+2I,2+3I,3+I,3+2I,3+3I\}.\end{aligned}$$

The neutrosophic subset $P=\{0,2I\}$ is a pseudo neutrosophic ideal. Thus

$$\langle\mathbb{Z}_4\cup I\rangle/P=\left\{\begin{matrix}0+P,1+P,2+P,3+P,I+P,2I+P,3I+P,\\(1+I)+P,(1+2I)+P,(1+3I)+P,\\(2+I)+P,(2+2I)+P,(2+3I)+P,\\(3+I)+P,(3+2I)+P,(3+3I)+P\end{matrix}\right\}$$

and, after identifying $0+P=2I+P=P$,

$$\langle\mathbb{Z}_4\cup I\rangle/P=\left\{\begin{matrix}P,1+P,2+P,3+P,I+P,3I+P,\\(1+I)+P,(1+2I)+P,(1+3I)+P,\\(2+I)+P,(2+2I)+P,(2+3I)+P,\\(3+I)+P,(3+2I)+P,(3+3I)+P\end{matrix}\right\}.$$

This is a pseudo neutrosophic quotient ring.

Table 4.1. Addition table for $(\langle \mathbb{Z}_4 \cup I \rangle / P, \oplus_4)$, reproduced from the source.

$\oplus_4$	P	$1+P$	$3+P$	$I+P$	$3I+P$	(1+I)+P	(1+2I)+P	(1+3I)+P	(2+I)+P	(2+3I)+P	(3+I)+P	(3+2I)+P	(3+3I)+P
P	P	1+P	3+P	I+P	3I+P	(1+I_)+P	(1+2I)+P	(1+3I)+P	(2+I)+P	(2+3I)+P	(3+I)+P	(3+2I)+P	(3+3I)+P
$1+P$	1+P	2+P	P	(1+I)+P	(1+3I)+P	(2+I)+P	(2+2I)+P	(2+3I)+P	(3+I)+P	(3+3I)+P	I+P	2I+P	3I+P
$3+P$	3+P	P	2+P	(3+I)+P	(3+3I)+P	I+P	2I+P	3I+P	(1+I)+P	(1+3I)+P	(2+I)+P	2I+P	(2+3I)+P
$I+P$	I+P	(1+I)+P	(3+I)+P	2I+P	P	(1+2I)+P	(1+3I)+P	1+P	(2+2I)+P	2+P	(3+2I)+P	(3+3I)+P	3+P
$3I+P$	3I+P	(1+3I)+P	(3+3I)+P	P	2I+P	1+P	(1+I)+P	(1+2I)+P	2+P	(2+2I)+P	3+P	(3+I)+P	(3+2I)+P
(1+I)+P	(1+I)+P	(2+I)+P	I+P	(1+2I)+P	1+P	(2+2I)+P	(2+3I)+P	2+P	(3+2I)+P	3+P	2I+P	3I+P	P
(1+2I)+P	(1+2I)+P	(2+2I)+P	2I+P	(1+3I)+P	(1+I)+P	(2+3I)+P	2+P	(2+I)+P	(3+3I)+P	(3+I)+P	3I+P	P	I+P
(1+3I)+P	(1+3I)+P	(2+3I)+P	2I+P	1+P	(1+2I)+P	2+P	(2+I)+P	(2+2I)+P	3+P	(3+2I)+P	P	I+P	2I+P
(2+I)+P	(2+I)+ P	(3+I)+P	(1+I)+P	(2+2I)+P	2+P	(3+2I)+P	(3+3I)+P	3+P	2I+P	P	(1+2I)+P	(1+3I)+P	1+P
(2+3I)+P	(2+3I)+P	(3+3I)+P	(1+3I)+P	2+P	(2+2I)+P	3+P	(3+I)+P	(3+2I)+P	P	2I+P	1+P	(1+I)+P	(1+2I)+P
(3+I)+P	(3+I)+P	I+P	(2+I)+P	(3+2I)+P	3+P	2I+P	3I+P	P	(1+I)+P	1+P	(2+2I)+P	(2+3I)+P	2+P
(3+2I)+P	(3+2I)+P	2I+P	(2+2I)+P	(3+3I)+P	(3+I)+P	3I+P	P	I+P	(1+3I)+P	(1+I)+P	(2+3I)+P	2+P	(2+I)+P
(3+3I)+P	(3+3I)+P	3I+P	(2+3I)+P	3+P	(3+2I)+P	P	I+P	2I+P	1+P	(1+2I)+P	2+P	(2+I)+P	(2+2I+P

Table 4.2. Inverse elements in $\langle \mathbb{Z}_4 \cup I \rangle / P$, reproduced from the source.

g	P	1+P	3+P	I+P	3I+P	(1+I)+P	(1+2I)+P	(1+3I)+P	(2+I)+P	(2+3I)+P	(3+I)+P	(3+2I)+P	(3+3I)+P
-g	P	3+P	1+P	3I+P	I+P	(31+3I)+P	(3+2I)+P	(3+I)+P	(2+3I)+P	(2+I)+P	(1+3I)+P	(1+2I)+P	(1+I)+P

Table 4.3. Multiplication table for $(\langle \mathbb{Z}_4 \cup I \rangle / P, \otimes_4)$, reproduced from the source.

$\otimes_4$	1+P	3+P	I+P	3I+P	(1+I)+P	(1+2I)+P	(1+3I)+P	(2+I)+P	(2+3I)+P	(3+I)+P	(3+2I)+P	(3+3I)+P
1+P	1+P	3+P	I+P	3I+P	(1+I)+P	(1+2I)+P	(1+3I)+P	(2+I)+P	(2+3I)+P	(3+I)+P	(3+2I)+P	(3+3I)+P
3+P	3+P	1+P	3I+P	I+P	(3+3I)+P	(3+2I)+P	(3+I)+P	(2+3I)+P	(2+I)+P	(1+3I)+P	(1+2I)+P	(1+I)+P
I+P	I+P	3I+P	I+P	3I+P	2I+P	3I+P	P	3I+P	I+P	P	I+P	2I+P
3I+P	3I+P	I+P	3I+P	I+P	I+P	3I+P	P	I+P	3I+P	P	3I+P	2+P
(1+I)+P	(1+I)+P	(3+3I)+P	2I+P	2I+P	(1+3I)+P	(1+I)+P	(1+3I)+P	2+P	2+P	(3+I)+P	(3+3I)+P	(3+I)+P
(1+2I)+P	(1+2I)+P	(3+2I)+P	3I+P	3I+P	(1+I)+P	1+P	(1+3I)+P	(2+3I)+P	(2+I)+P	(3+I)+P	(3+3I)P	(3+3I)+P
(1+3I)+P	(1+3I)+P	(3+I)+P	P	P	(1+3I)+P	(1+3I)+P	(1+3I)+P	(2+2I)+P	(2+2I)+P	(3+I)+P	(3+I)+P	(3+I)+P
(2+I)+P	(2+I)+P	(2+3I)+P	3I+P	I+P	2+P	(2+3I)+P	(2+2I)+P	I+P	3I+P	(2+2I)+P	(2+I)+P	2+P
(2+3I)+P	(2+3I)+P	(2+I)+P	I+P	3I+P	2+P	(2+I)+P	(2+2I)+P	3I+P	I+P	(2+2I)+P	(2+I)+P	2+P
(3+I)+P	(3+I)+P	(1+3I)+P	P	P	(3+I)+P	(3+I)+P	(3+I)+P	(2+2I)+P	(2+2I)+P	(1+3I)+P	(1+3I)+P	(1+3I)+P
(3+2I)+P	(3+2I)+P	(1+2I)+P	I+P	3I+P	(3+3I)+P	3+P	(3+I)+P	(2+I)+P	(2+3I)+P	(1+3I)+P	1+P	(1+I)+P
(3+3I)+P	(3+3I)+P	2I+P	2I+P	2I+P	(3+I)+P	(3+3I)+P	(3+I)+P	2+P	2+P	(1+3I)+P	(1+I)+P	(1+3I)+P

Example 4.8. From the neutrosophic ring $\langle \mathbb{Z}_6 \cup I \rangle$, the neutrosophic subset $P = \{0, 3I\}$ is a pseudo neutrosophic ideal.

Therefore

$$\langle \mathbb{Z}_6 \cup I \rangle / P = \left\{ \begin{array}{l} P, 1+P, 2+P, 3+P, 4+P, 5+P, I+P, 2I+P, 4I+P, 5I+P, \\ (1+I)+P, (1+2I)+P, (1+3I)+P, (1+4I)+P, (1+5I)+P, \\ (2+I)+P, (2+2I)+P, (2+3I)+P, (2+4I)+P, (2+5I)+P, \\ (3+I)+P, (3+2I)+P, (3+3I)+P, (3+4I)+P, (3+5I)+P, \\ (4+I)+P, (4+2I)+P, (4+3I)+P, (4+4I)+P, (4+5I)+P, \\ (5+I)+P, (5+2I)+P, (5+3I)+P, (5+4I)+P, (5+5I)+P \end{array} \right\}$$

is a pseudo neutrosophic quotient ring.

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