



Characteristics Neutrosophic Ideals For Neutrosophic Rings: On Review

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Abstract

The main objective of this paper is to present a review study with more information on the neutrosophic Ideal, Principle Ideal, Prim Ideal, Pseudo Neutrosophic Ideal, Quotient ring, and Pseudo Quotient ring. Neutrosophic ring theory is a branch of neutrosophic Algebra which introduced by Florentin Smarandache in 2006.

Keywords: Neutrosophic ring; Neutrosophic ideals; Types of Neutrosophic Ideals; Neutrosophic Quotient ring; Pseudo Neutrosophic Quotient Rings.

1. Introduction

In 2006, Kandasamy and Smarandache introduced neutrosophic algebra structure, N-algebra structure, and neutrosophic ring in their works. They studied the structure of Neutrosophic Groups, Neutrosophic N-Groups structure, Neutrosophic rings, and Neutrosophic group rings [10,12], Moreover, Kandasamy has written a book about Smarandache neutrosophic algebraic structure, he showed it through neutrosophic groups (semigroups), Bi-groups, and N-groups in [9]. Previous work has motivated other researchers such Agboola, Adeleke, and Oyebola, they introduced some elementary properties of neutrosophic rings with neutrosophic polynomial rings in [2]. In 2012, they studied ideals of neutrosophic rings and Neutrosophic quotient rings in [1]. Later on, in 2023, Al-Odhari presented a review of the neutrosophic ring and neutrosophic subring with properties of neutrosophic elements of the neutrosophic ring in [3] and some algebraic structure of neutrosophic matrices in [4]. In 2024, Al-Odhari presented a neutrosophic set from the classical set and studied more properties of neutrosophic set theory in [6,8,9] with some works in the structure of neutrosophic group theory in [5,7]. In this article, we present a review and expansion of the Neutrosophic Ideal, Types of Neutrosophic Ideal, Neutrosophic Quotient Rings, and Pseudo Neutrosophic Quotient Rings.

2. Preliminaries

In this section, we introduce a review of the neutrosophic ring, and neutrosophic subring, which will be used in the next sections.

Definition 2.1 [10] Let R be any ring. The neutrosophic ring is also a ring generated by R and I under the operations of R , and denoted by $N(R)$, such that:

$$N(R) = \langle R \cup I \rangle = \{a + bI : a, b \in R\}.$$

Theorem 2.1 [10] Let $\langle R \cup I \rangle$ be Neutrosophic ring. Then $\langle R \cup I \rangle$ is ring.

Definition 2.2 [3,10] Let R be a nonempty set and the triple $(R, +, \bullet)$ be a ring, and consider the neutrosophic set $N(R) = R[I] = \langle R \cup I \rangle = \{a + bI : a, b \in R\}$. then the neutrosophic algebra structure (NAS): $N(R) = \langle R \cup I \rangle = \langle R[I], +, \bullet \rangle$ is called the neutrosophic ring which is a generated by I and R under operations addition "+" and multiplications " \bullet " respectively, if satisfy the following axiomatic conditions of ring:

NR₁. For all x, y and $z \in N(R)$, $N(R) = \langle R[I], + \rangle$ is a neutrosophic an abelian group under addition;

NR₂. For all x, y and $z \in N(R^*)$, $N(R^*) = \langle R^*[I], \bullet \rangle$ is a mathematical associative neutrosophic system under multiplications, that is, $N(R^*) = \langle R^*[I], \bullet \rangle$ is neutrosophic semi group, and

NR₃: $x(y + z) = (x \cdot y) + (x \cdot z)$, and $(y + z)x = (y \cdot x) + (z \cdot x)$ "left and right distribution laws"

Theorem 2.2 [4] Let $\langle R \cup I \rangle$ be a neutrosophic ring and $x, y \in \langle R \cup I \rangle$. then:

1. $x \cdot 0 = 0 \cdot x = 0$.
2. $x \cdot (-y) = (-x) \cdot y = -(xy)$.
3. $(-x) \cdot (-y) = xy$.

Definition 2.3 [10] Let $\langle R \cup I \rangle$ be a neutrosophic ring. A proper subset P of $\langle R \cup I \rangle$ is said to be a neutrosophic subring if P itself is a neutrosophic ring under the operations of $\langle R \cup I \rangle$. It is essential that $P = \langle S \cup nI \rangle$, n a positive integer where S is a subring of R . i.e. $\{P$ is generated by the subring S with $nI, (n \in \mathbb{Z}^+)$. Note: Even if P is a ring and cannot be represented as $\langle S \cup nI \rangle$, where S is a subring of R . Then we do not call P a neutrosophic subring of $\langle R \cup I \rangle$.

Theorem 2.3 [3] Let $\langle R \cup I \rangle$ be a neutrosophic ring and $N(S) \subseteq N(R)$, then $N(S)$ is called a neutrosophic subring of $N(R)$ iff:

1. $\forall a, b \in N(S) \Rightarrow a - b \in N(S)$.
2. $\forall a, b \in N(S) \Rightarrow ab \in N(S)$.

3. Neutrosophic Ideals and Pseudo Neutrosophic Ideals

In this section, we introduce the neutrosophic ideal, neutrosophic principle ideal, neutrosophic prime ideal, neutrosophic maximal ideal, and pseudo neutrosophic ideal.

Definition 3.1 [10] Let $\langle R \cup I \rangle$ be any neutrosophic ring. A non-empty neutrosophic subset P of $\langle R \cup I \rangle$ is called a neutrosophic left ideal of $\langle R \cup I \rangle$, if the following conditions are satisfied

1. P is a neutrosophic subring $\langle R \cup I \rangle$, and
2. For every $p \in P$, and $r \in \langle R \cup I \rangle$, $rp \in P$. Also, P is called a neutrosophic right ideal of $\langle R \cup I \rangle$, if the following conditions are satisfied
 1. P is a neutrosophic subring $\langle R \cup I \rangle$, and
 2. For every $p \in P$, and $r \in \langle R \cup I \rangle$, $pr \in P$.

Definition 3.2 Let $\langle R \cup I \rangle$ be any neutrosophic ring. A none-empty neutrosophic subset P of $\langle R \cup I \rangle$ is defined to be a neutrosophic ideal (two sided) of $\langle R \cup I \rangle$ if the following conditions are satisfied;

1. P is a neutrosophic subring of $\langle R \cup I \rangle$.
2. For all $p \in P$ and $r \in \langle R \cup I \rangle$, then $rp \in P$ and $pr \in P$.

Example 3.1 We have $(\mathbb{Z}, +, \bullet)$ is integers ring, $(2\mathbb{Z}, +, \bullet)$ is subring of $(\mathbb{Z}, +, \bullet)$. $\langle \mathbb{Z} \cup I \rangle$ is neutrosophic ring of integers. clear that $\langle 2\mathbb{Z} \cup I \rangle$ is neutrosophic subring of $\langle \mathbb{Z} \cup I \rangle$. Also, for all $p = 2a + 2bI \in \langle 2\mathbb{Z} \cup I \rangle$ and $r = a + bI \in \langle \mathbb{Z} \cup I \rangle$, then

$$rp = (a + bI)(2a + 2bI)$$

$$\begin{aligned}
&= (2aa + (2ab + 2ba + 2bb)I) \\
&= (2aa + 2(ab + ba + bb)I), \text{ and} \\
pr &= (2a + 2bI)(a + bI) \\
&= (2aa + (2ab + 2ba + 2bb)I) \\
&= (2aa + 2(ab + ba + bb)I) \in \langle 2Z \cup I \rangle. \text{ So, } \langle 2Z \cup I \rangle \text{ is neutrosophic ideal of } \langle Z \cup I \rangle.
\end{aligned}$$

Proposition 3.1 Every neutrosophic ideal P of $\langle R \cup I \rangle$ is a neutrosophic subring of $\langle R \cup I \rangle$, but the converse may not be true.

Proposition 3.2 In a ring of integers Z , every nZ is ideal, $\forall n \in Z^+$. Then every $\langle nZ \cup I \rangle$ is a neutrosophic ideal of $\langle Z \cup I \rangle$, $\forall n \in Z^+$.

Proof. Firstly, nZ is subring of Z for all $n \in Z^+$. So, $\langle nZ \cup I \rangle$ is a neutrosophic subring of $\langle Z \cup I \rangle$. Now, consider $x \in \langle nZ \cup I \rangle$ and $r \in \langle Z \cup I \rangle$, then there are $a_1, b_1 \in nZ$, $a, b \in Z$ such that $x = a_1 + b_1I$, $r = a + bI$, then $rx = (a + bI)(a_1 + b_1I)$
 $= (aa_1 + (ab_1 + ba_1 + bb_1)I)$. It is clear that, $aa_1, (ab_1 + ba_1 + bb_1) \in nZ$ (because nZ ideal of Z), implies that $(aa_1 + (ab_1 + ba_1 + bb_1)I) \in \langle nZ \cup I \rangle$. Also, by the similar way, $xr = (a_1 + b_1I)(a + bI) \in \langle nZ \cup I \rangle$. $\langle nZ \cup I \rangle$ is neutrosophic ideal of $\langle Z \cup I \rangle$.

Example 3.2 In a ring of integers $(Z, +, \cdot)$. $(5Z, +, \cdot)$ is an ideal, then $\langle 5Z \cup I \rangle$ is a neutrosophic ideal of $\langle Z \cup I \rangle$, where $\langle 5Z \cup I \rangle$ is a neutrosophic subring of $\langle Z \cup I \rangle$, and for all $(5a + 5bI) \in \langle 5Z \cup I \rangle$, $(a' + b'I) \in \langle Z \cup I \rangle$, $(a, b, a', b' \in Z)$ is $(5a + 5bI)(a' + b'I) = (5aa' + (5ab' + 5ba' + 5bb')I)$
 $= (5aa' + 5(ab' + ba' + bb')I) \in \langle 5Z \cup I \rangle$, also, $(a' + b'I)(5a + 5bI) \in \langle 5Z \cup I \rangle$.

Theorem 3.1 Let R be any ring, S is ideal of R . If $\langle R \cup I \rangle$ neutrosophic ring. Then $\langle S \cup I \rangle$ is a neutrosophic ideal of $\langle R \cup I \rangle$.

Proof. Firstly, since S is ideal of R , then S is a subring of R , hence $P = \langle S \cup I \rangle$ is a neutrosophic subring of $\langle R \cup I \rangle$. Now, For any $p \in P$ there is $a_1, b_1 \in S$, such that $p = a_1 + b_1I$, and for all $r \in \langle R \cup I \rangle$ there is $a, b \in R$, such that $r = a + bI$. Then $rp = r(a_1 + b_1I) = r(a_1) + r(b_1I) = (a + bI)a_1 + (a + bI)b_1I = aa_1 + (ba_1 + ab_1 + bb_1)I$, but $aa_1, (ba_1 + ab_1 + bb_1) \in S$ (because S is an ideal of R), then $aa_1 + (ba_1 + ab_1 + bb_1)I \in P$, this mean that, $rp \in P$. Also by similar method, we have $pr \in P$.

Example 3.3 Let (Z_4, \oplus, \odot) be a ring of integers modulo n , $H = \{0, 2\}$ is ideal of (Z_4, \oplus, \odot) . Then $\langle Z_4 \cup I \rangle$ is a neutrosophic ring, and $\langle H \cup I \rangle = \{0, 2, 2I, 2 + 2I\}$ is a neutrosophic subring of $\langle Z_4 \cup I \rangle$, and for all $(a + bI) \in \langle Z_4 \cup I \rangle$, $(a_1 + b_1I) \in \langle H \cup I \rangle$, then $(a + bI)(a_1 + b_1I) = (aa_1 + (ab_1 + ba_1 + bb_1)I) \in \langle H \cup I \rangle$, also, $(a_1 + b_1I)(a + bI) \in \langle H \cup I \rangle$.

Proposition 3.3 If H is a neutrosophic ideal (neutrosophic left ideal/ neutrosophic right ideal/ neutrosophic- two –sided) of $\langle R \cup I \rangle$ and J is a neutrosophic ideal (neutrosophic left ideal/ neutrosophic right ideal/ neutrosophic- two –sided) of H . Then not necessarily J is an ideal (neutrosophic left ideal/ neutrosophic right ideal/ neutrosophic- two –sided) of $\langle R \cup I \rangle$.

Definition 3.3 Let P be neutrosophic ideal of $\langle R \cup I \rangle$, P is generated by an element $r = a + bI \in \langle R \cup I \rangle$; $a, b \neq 0$. Then P is called a neutrosophic principal ideal of $\langle R \cup I \rangle$ denoted by $\langle r \rangle$.

Definition 3.4 Let R be any commutative ring, $a \in R$. An ideal $H = aR = \{ar : r \in R\}$ is called a principal ideal of R . A neutrosophic ideal $P = a\langle R \cup I \rangle = \{ax : x \in \langle R \cup I \rangle\}$ is called neutrosophic principal ideal of $\langle R \cup I \rangle$.

Proposition 3.4 Let $\langle Z \cup I \rangle$ be a neutrosophic ring of integers, if S is an ideal of integers ring Z . Then $P = \langle S \cup I \rangle$ is a principal ideal of $\langle Z \cup I \rangle$.

Proof. Since S is an ideal of Z , then S is a principal ideal of Z , and $S = nZ$ ($n \in Z^+$). Firstly, If $P = \{0\} \Rightarrow S = \{0\} = 0Z \Rightarrow n = 0$, So $P = \langle 0Z \cup I \rangle = \{0\}$, this mean that, P is a principal ideal, $P = \{0r : r \in \langle Z \cup I \rangle\}$, $P = \langle 0 \rangle$. Secondly, If $P \neq \{0\} \Rightarrow S \neq \{0\} \Rightarrow n \neq 0$, so $S = nZ$, $n \in Z^+$ and $P = \langle nZ \cup I \rangle$. Now, suppose that n is smallest positive number in S , then $n, nI, n + nI \in P$. Also, for all $m \in S$, we have $m, mI, m + mI \in P$ such that $m = qn$; $q \in Z$, then $n(q), n(qI), n(q + qI) \in P$, such that $q, qI, q + qI \in \langle Z \cup I \rangle$, so $P = \{nr : r \in \langle Z \cup I \rangle\}$.

Proposition 3.5 Let R be any ring, H is a principal ideal of R . Then $P = \langle H \cup I \rangle$ is a neutrosophic principal ideal of $\langle R \cup I \rangle$.

Definition 3.5 Let P be a neutrosophic ideal of $\langle R \cup I \rangle$. if for any neutrosophic ideal J of $\langle R \cup I \rangle$ is either $J \subseteq P$ or $J = P$, then P is called a neutrosophic maximal ideal.

Example 3.4 $\langle 3Z \cup I \rangle$ is a neutrosophic maximal ideal of $\langle Z \cup I \rangle$.

Example 3.5 Consider (Z_4, \oplus, \odot) is a ring, $H = \{0, 2\}$ is ideal of (Z_4, \oplus, \odot) . The neutrosophic ring $\langle Z_4 \cup I \rangle = \{0, 1, 2, 3, I, 2I, 3I, 1 + I, 1 + 2I, 1 + 3I, 2 + I, 2 + 2I, 2 + 3I, 3 + I, 3 + 2I, 3 + 3I\}$. $\langle H \cup I \rangle$ is a neutrosophic ideal of $\langle Z_4 \cup I \rangle$, where $\langle H \cup I \rangle = \{0, 2, 2I, 2 + 2I\}$, we note that, there is no neutrosophic ideal P of $\langle Z_4 \cup I \rangle$ contains $\langle H \cup I \rangle$, implies that $\langle H \cup I \rangle$ is a neutrosophic maximal ideal of $\langle Z_4 \cup I \rangle$.

Proposition 3.6 Let R be any ring, S is a maximal ideal of R , then $P = \langle S \cup I \rangle$ is a neutrosophic maximal ideal of $\langle R \cup I \rangle$.

Definition 3.6 Let P be neutrosophic ideal of $\langle R \cup I \rangle$. If for any two neutrosophic ideals J, Q of $\langle R \cup I \rangle$ such that $JQ \subseteq P$ is either $J \subseteq P$ or $Q \subseteq P$. Then P is called a neutrosophic prime ideal.

Definition 3.7 Let P be neutrosophic ideal of $\langle R \cup I \rangle$. If for any $x, y \in \langle R \cup I \rangle$, $xy \in P$ is either $x \in P$ or $y \in P$. Then P is called a neutrosophic prime ideal.

Example 3.7 Consider (Z_6, \oplus, \odot) is ring, $H = \{0, 2, 4\}$ is ideal of (Z_6, \oplus, \odot) , $\langle Z_6 \cup I \rangle$ is a neutrosophic ring, where

$$\langle Z_6 \cup I \rangle = \left\{ \begin{array}{l} 0, 1, 2, 3, 4, 5, I, 2I, 3I, 4I, 5I, \\ 1 + I, 1 + 2I, 1 + 3I, 1 + 4I, 1 + 5I, \\ 2 + I, 2 + 2I, 2 + 3I, 2 + 4I, 2 + 5I, \\ 3 + I, 3 + 2I, 3 + 3I, 3 + 4I, 3 + 5I, \\ 4 + I, 4 + 2I, 4 + 3I, 4 + 4I, 4 + 5I, \\ 5 + I, 5 + 2I, 5 + 3I, 5 + 4I, 5 + 5I, \end{array} \right\}$$

$\langle H \cup I \rangle$ is neutrosophic ideal of $\langle Z_6 \cup I \rangle$, where

$$\langle H \cup I \rangle = \left\{ \begin{array}{l} 0, 2, 4, 2I, 4I, \\ 2 + 2I, 2 + 4I, \\ 4 + 2I, 4 + 4I \end{array} \right\}, \text{ and}$$

for any $x, y \in \langle Z_6 \cup I \rangle$ such that $xy \in \langle H \cup I \rangle$ is either $x \in \langle H \cup I \rangle$ or $y \in \langle H \cup I \rangle$. This means that $\langle H \cup I \rangle$ is neutrosophic prime ideal of $\langle Z_6 \cup I \rangle$.

Proposition 3.7 Let R be any ring, S is a prime ideal of R . Then $P = \langle S \cup I \rangle$ is a neutrosophic prime ideal of $\langle R \cup I \rangle$.

Theorem 3.2 Let $\langle R \cup I \rangle$ be a neutrosophic ring, let P be a subset of $\langle R \cup I \rangle$. Then P is an ideal of $\langle R \cup I \rangle$, if and only if:

1. $0 \in P$,
2. $x - y \in P$ for all $x, y \in P$.
3. $rp, pr \in P$ for all $r \in R, p \in P$.

Proof. Firstly, if P is ideal of $\langle R \cup I \rangle$, since P is ideal. Then P is subring of $\langle R \cup I \rangle$, so $0 \in P$ and for all $x, y \in P$ is $x - y, xy \in P$. Let $r \in R, p \in P$. Then $rp, pr \in P$ (because P ideal). Secondly, we assume that all three conditions are satisfies, then from (3) it is clear that P is ideal of $\langle R \cup I \rangle$.

Proposition 3.8 Let P and J are neutrosophic ideals of a neutrosophic ring $\langle R \cup I \rangle$. Then:

1. $P + J$ is a neutrosophic ideal of a neutrosophic ring $\langle R \cup I \rangle$.
2. PJ is a neutrosophic ideal of a neutrosophic ring $\langle R \cup I \rangle$.
3. $P \cap J$ is a neutrosophic ideal of a neutrosophic ring $\langle R \cup I \rangle$.

Proof.

1. We define $P + J$ as following $P + J = \{u = x + y : x \in P, y \in J\}$, so $P + J \neq \emptyset$.

Since $0 \in P, 0 \in J$, So $0 + 0 = 0 \in P + J$, also Let $u, v \in P + J$. Then there are $x_1 = a_1 + b_1I, x_2 = a_2 + b_2I \in P$ and $y_1 = c_1 + d_1I, y_2 = c_2 + d_2I \in J$, So

$u = x_1 + y_1 = (a_1 + b_1I) + (c_1 + d_1I)$, and $v = x_2 + y_2 = (a_2 + b_2I) + (c_2 + d_2I)$, then

$u - v = ((a_1 + b_1I) + (c_1 + d_1I)) - ((a_2 + b_2I) + (c_2 + d_2I))$

$= ((a_1 + b_1I) - (a_2 + b_2I)) + ((c_1 + d_1I) - (c_2 + d_2I))$. Note that, $(a_1 + b_1I) - (a_2 + b_2I) \in P$ (because P is an ideal), also $(c_1 + d_1I) - (c_2 + d_2I) \in J$ (because J ideal), then $((a_1 + b_1I) + (c_1 + d_1I)) - ((a_2 + b_2I) + (c_2 + d_2I)) \in P + J$, implies that

$u - v \in P + J$. In addition, let $r = a + bI \in \langle R \cup I \rangle, u \in P + J$, then

$ru = (a + bI)((a_1 + b_1I) + (c_1 + d_1I))$

$= ((a + bI)(a_1 + b_1I)) + ((a + bI)(c_1 + d_1I))$, since $(a + bI)(a_1 + b_1I) \in P$ (because P is an ideal), and $(a + bI)(c_1 + d_1I) \in J$ (because J is an ideal), then

$ru = (a + bI)((a_1 + b_1I) + (c_1 + d_1I)) \in P + J$, by similar way

$ur = ((a_1 + b_1I) + (c_1 + d_1I))(a + bI) \in P + J$. This means that $P + J$ neutrosophic ideal of $\langle R \cup I \rangle$.

2. We define PJ as following,

$PJ = \sum_{i=1}^n (x_i y_i)$ (where $x_i \in P, y_i \in J, 1 \leq i \leq n, n \in \mathbb{N}$)

$= \{x_1 y_1 + x_2 y_2 + \dots + x_n y_n : x_i \in P[I], y_i \in J[I], 1 \leq i \leq n, n \in \mathbb{N}\}$

$$= \left\{ \begin{array}{l} (x'_1 + x''_1 I)(y'_1 + y''_1 I) \\ + \\ (x'_2 + x''_2 I)(y'_2 + y''_2 I) \\ + \\ \vdots \\ + \\ (x'_n + x''_n I)(y'_n + y''_n I) \end{array} \right\},$$

$$= \left\{ \begin{array}{l} (x'_1 + y'_1) + (x'_1 y'_1 + x''_1 y'_1 + x'_1 y''_1) I \\ + \\ (x'_2 + y'_2) + (x'_2 y'_2 + x''_2 y'_2 + x'_2 y''_2) I \\ + \\ \vdots \\ + \\ (x'_n + y'_n) + (x'_n y'_n + x''_n y'_n + x'_n y''_n) I \end{array} \right\}$$

$= \sum_{i=1}^n ((x'_i + y'_i) + (x'_i y'_i + x''_i y'_i + x'_i y''_i) I)$, (where $x'_i, x''_i \in P, y'_i, y''_i \in J, 1 \leq i \leq n, n \in \mathbb{N}$), then Let $p_1, p_2 \in PJ$, then we have

$p_1 = \sum_{i=1}^n (x_i y_i)$, where $x_i \in P, y_i \in J, 1 \leq i \leq n, n \in \mathbb{N}$, and $p_2 = \sum_{j=1}^n (z_j w_j)$, where $z_j \in P, w_j \in J, 1 \leq j \leq n, n \in \mathbb{N}$. Then

$$\begin{aligned} p_1 - p_2 &= \sum_{i=1}^n (x_i y_i) - \sum_{j=1}^n (z_j w_j) \\ &= (x_1 y_1 + x_2 y_2 + \dots + x_n y_n) - (z_1 w_1 + z_2 w_2 + \dots + z_n w_n) \end{aligned}$$

$$\begin{aligned}
 &= \left\{ \begin{array}{c} (x'_1 + x''_1 I)(y'_1 + y''_1 I) \\ + \\ (x'_2 + x''_2 I)(y'_2 + y''_2 I) \\ \vdots \\ + \\ (x'_n + x''_n I)(y'_n + y''_n I) \end{array} \right\} - \left\{ \begin{array}{c} (z'_1 + z''_1 I)(w'_1 + w''_1 I) \\ + \\ (z'_2 + z''_2 I)(w'_2 + w''_2 I) \\ \vdots \\ + \\ (z'_n + z''_n I)(w'_n + w''_n I) \end{array} \right\}, \\
 &= \left\{ \begin{array}{c} (x'_1 + y'_1) + (x'_1 y''_1 + x''_1 y'_1 + x''_1 y''_1) I \\ + \\ (x'_2 + y'_2) + (x'_2 y''_2 + x''_2 y'_2 + x''_2 y''_2) I \\ \vdots \\ + \\ (x'_n + y'_n) + (x'_n y''_n + x''_n y'_n + x''_n y''_n) I \end{array} \right\} - \left\{ \begin{array}{c} (z'_1 + w'_1) + (z'_1 w''_1 + z''_1 w'_1 + z''_1 w''_1) I \\ + \\ (z'_2 + w'_2) + (z'_2 w''_2 + z''_2 w'_2 + z''_2 w''_2) I \\ \vdots \\ + \\ (z'_n + w'_n) + (z'_n w''_n + z''_n w'_n + z''_n w''_n) I \end{array} \right\} \\
 &= \left\{ \begin{array}{c} ((x'_1 + y'_1) - (z'_1 + w'_1)) + ((x'_1 y''_1 + x''_1 y'_1 + x''_1 y''_1) - (z'_1 w''_1 + z''_1 w'_1 + z''_1 w''_1)) I \\ + \\ ((x'_2 + y'_2) - (z'_2 + w'_2)) + ((x'_2 y''_2 + x''_2 y'_2 + x''_2 y''_2) - (z'_2 w''_2 + z''_2 w'_2 + z''_2 w''_2)) I \\ \vdots \\ + \\ ((x'_n + y'_n) - (z'_n + w'_n)) + ((x'_n y''_n + x''_n y'_n + x''_n y''_n) - (z'_n w''_n + z''_n w'_n + z''_n w''_n)) I \end{array} \right\} \\
 &= \left\{ \begin{array}{c} ((x'_1 - z'_1) + (y'_1 - w'_1)) + ((x'_1 y''_1 - z'_1 w''_1) + (x''_1 y'_1 - z''_1 w'_1) + (x''_1 y''_1 - z''_1 w''_1)) I \\ + \\ ((x'_2 - z'_2) + (y'_2 - w'_2)) + ((x'_2 y''_2 - z'_2 w''_2) + (x''_2 y'_2 - z''_2 w'_2) + (x''_2 y''_2 - z''_2 w''_2)) I \\ \vdots \\ + \\ ((x'_n - z'_n) + (y'_n - w'_n)) + ((x'_n y''_n - z'_n w''_n) + (x''_n y'_n - z''_n w'_n) + (x''_n y''_n - z''_n w''_n)) I \end{array} \right\} \\
 &= \left\{ \begin{array}{c} \sum_{i=1}^n (x'_i - z'_i) \\ + \\ \sum_{j=1}^n (y'_j - w'_j) \\ + \\ \sum_{i,j=1}^n ((x'_i y''_j - z'_i w''_j) + (x''_i y'_j - z''_i w'_j) + (x''_i y''_j - z''_i w''_j)) I \in P J. \end{array} \right\}
 \end{aligned}$$

If $p = \sum_{i=1}^n (x_i y_i)$, where $x_i \in P, y_i \in J, 1 \leq i \leq n, n \in \mathbb{N}$ and $r \in \langle R \cup I \rangle$. Then $rp = r \sum_{i=1}^n (x_i y_i) = r(x_1 y_1 + x_2 y_2 + \dots + x_n y_n)$

$$= r \left\{ \begin{array}{c} (x'_1 + x''_1 I)(y'_1 + y''_1 I) \\ + \\ (x'_2 + x''_2 I)(y'_2 + y''_2 I) \\ \vdots \\ + \\ (x'_n + x''_n I)(y'_n + y''_n I) \end{array} \right\}$$

$$\begin{aligned}
 &= r \left\{ \begin{array}{l} (x'_1 + y'_1) + (x'_1 y''_1 + x''_1 y'_1 + x''_1 y''_1)I \\ + \\ (x'_2 + y'_2) + (x'_2 y''_2 + x''_2 y'_2 + x''_2 y''_2)I \\ + \\ \vdots \\ + \\ (x'_n + y'_n) + (x'_n y''_n + x''_n y'_n + x''_n y''_n)I \end{array} \right\} \\
 &= \left\{ \begin{array}{l} (rx'_1 + ry'_1) + (rx'_1 y''_1 + rx''_1 y'_1 + rx''_1 y''_1)I \\ + \\ (rx'_2 + ry'_2) + (rx'_2 y''_2 + rx''_2 y'_2 + rx''_2 y''_2)I \\ + \\ \vdots \\ + \\ (rx'_n + ry'_n) + (rx'_n y''_n + rx''_n y'_n + rx''_n y''_n)I \end{array} \right\}
 \end{aligned}$$

$= \sum_{i=1}^n ((rx'_i + ry'_i) + (rx'_i y''_i + rx''_i y'_i + rx''_i y''_i)I)$, where $x'_i, x''_i \in P, y'_i, y''_i \in J, 1 \leq i \leq n, n \in \mathbb{N}$ and indeterminacy I . Hence $rp \in PJ$. By the similar method $pr \in PJ$. This means that PJ is a neutrosophic ideal of $\langle R \cup I \rangle$.

3. we define $P \cap J$ as following $P \cap J = \{x = (a + bI) : x \in P, x \in J\}$, so, $0 \in P \cap J$.

Let $(a_1 + b_1I), (a_2 + b_2I) \in P \cap J$, then

$(a_1 + b_1I), (a_2 + b_2I) \in P, (a_1 + b_1I)$, and $(a_1 + b_1I), (a_2 + b_2I) \in J$. So

$(a_1 + b_1I) - (a_2 + b_2I) \in P$, and $(a_1 + b_1I) - (a_2 + b_2I) \in J$, then

$(a_1 + b_1I) - (a_2 + b_2I) \in P \cap J$. Let $(a + bI) \in \langle R \cup I \rangle, (a_1 + b_1I) \in P \cap J$, then

$(a + bI)(a_1 + b_1I) \in P \cap J$ (because $(a + bI)(a_1 + b_1I) \in P, (a + bI)(a_1 + b_1I) \in J$), and

$(a_1 + b_1I)(a + bI) \in P \cap J$. (because $(a_1 + b_1I)(a + bI) \in P, (a_1 + b_1I)(a + bI) \in J$),

Hence $P \cap J$ is a neutrosophic ideal of $\langle R \cup I \rangle$.

Example 3.8 Let $\langle M_{2 \times 2}(\mathbb{R}) \cup I \rangle$ be a neutrosophic matrix ring of real numbers with addition, multiplication of matrix, let M be a neutrosophic ring has the form:

$$M = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} : x, y, z \in \langle R \cup I \rangle \right\}. \text{ If } P = \left\{ \begin{bmatrix} a + bI & c + dI \\ 0 & 0 \end{bmatrix} : a, b, c, d \text{ are real numbers} \right\}$$

$$\text{and } J = \left\{ \begin{bmatrix} 0 & e + fI \\ 0 & h + gI \end{bmatrix} : e, f, h, g \text{ are real numbers} \right\} \text{ are two neutrosophic ideals of } M. \text{ Then for}$$

$$\text{all } p = \begin{bmatrix} a + bI & c + dI \\ 0 & 0 \end{bmatrix} \in P \text{ and } j = \begin{bmatrix} 0 & e + fI \\ 0 & h + gI \end{bmatrix} \in J, \text{ we get,}$$

$$\begin{aligned}
 p + j &= \begin{bmatrix} a + bI & c + dI \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & e + fI \\ 0 & h + gI \end{bmatrix} \\
 &= \left\{ \begin{bmatrix} a + bI & k + rI \\ 0 & h + gI \end{bmatrix} : a, b, k, r, h, g \text{ are real numbers} \right\},
 \end{aligned}$$

hence $P + J$ is a neutrosophic ideal of M . Furthermore,

$$P \cap J = \left\{ \begin{bmatrix} 0 & k + rI \\ 0 & 0 \end{bmatrix} : k, r \text{ are real numbers} \right\}, \text{ it is clear that } P \cap J \text{ is a neutrosophic subring, and}$$

$$\text{for any } x \in P \cap J, m \in M \text{ we have, } xm = \left\{ \begin{bmatrix} 0 & s + tI \\ 0 & 0 \end{bmatrix} : s, t \text{ are real numbers} \right\} \text{ and}$$

$$xm = \left\{ \begin{bmatrix} 0 & n + lI \\ 0 & 0 \end{bmatrix} : n, l \text{ are real numbers} \right\}, \text{ therefore } P \cap J \text{ is a neutrosophic ideal of } M.$$

Definition 3.8 Let $\langle R \cup I \rangle$ be a neutrosophic ring, P is a pseudo neutrosophic subring of $\langle R \cup I \rangle$. If for all $p \in P$ and $r \in \langle R \cup I \rangle$ are rp and $pr \in P$. Then P is called a pseudo neutrosophic ideal of $\langle R \cup I \rangle$.

Example 3.9 In a neutrosophic ring $(\langle Z_2 \cup I \rangle, \oplus_2, \odot_2) = \{0, 1, I, 1 + I\}$. $P = \{0, I\}$ is a pseudo neutrosophic subring of $\langle Z_2 \cup I \rangle$, to construct the neutrosophic algebra structure (NAS): (P, \oplus_2) by the visualizing table as shown in table.3.1, and, to construct the neutrosophic algebra structure (NAS): (P, \odot_2) by the visualizing table as shown in table.3.2 respectively,

Table 3.1 for (P, \oplus_2) .

\oplus_2	0	I
0	0	I
I	I	0

Table 3.2 for (P, \otimes_2)

\otimes_2	0	I
0	0	0
I	0	I

Also, $1 \odot_2 0 = 0 \in P$, $1 \odot_2 I = I \in P$, and $(1 + I) \odot_2 0 = 0 \in P$, $(1 + I) \odot_2 I = 0 \in P$, P is a pseudo neutrosophic ideal of $\langle Z_2 \cup I \rangle$.

Example 3.10 Let $(T, +, \bullet)$ be a pseudo neutrosophic ring, where

$$T = \{\dots, -4I, -3I, -2I, -I, 0, I, 2I, 3I, 4I, \dots\}.$$

Then $P = \{\dots, 6I, -4I, -2I, 0, 2I, 4I, 6I, \dots\}$ is pseudo neutrosophic ideal of T .

Example 3.11 Let $(T, +, \bullet)$ be a pseudo neutrosophic ring, where

$$T = \{\dots, -8I, -6I, -4I, -2I, 0, 2I, 4I, 6I, 8I, \dots\}.$$

Then $P = \{\dots, -16I, -12I, -8I, -4I, 0, 4I, 8I, 12I, 16I, \dots\}$ is a pseudo neutrosophic ideal of T .

Definition 3.9 Let P be a pseudo neutrosophic ideal of $\langle R \cup I \rangle$, if P is generated by an element $r = aI \in \langle R \cup I \rangle, a \neq 0$. Then P is called a pseudo neutrosophic principal ideal, denoted by (r) .

Example 3.12 Let $(T, +, \bullet)$ be a pseudo neutrosophic ring, where

$$T = \{\dots, -4I, -3I, -2I, -I, 0, I, 2I, 3I, 4I, \dots\}.$$

Then $P = \{\dots, -6I, -4I, -2I, 0, 2I, 4I, 6I, \dots\}$ is a pseudo neutrosophic ideal generated by element $2I$, so P is a pseudo neutrosophic principal ideal of T .

Example 3.13 Let $(T, +, \bullet)$ be a pseudo neutrosophic ring, where

$$T = \{\dots, -8I, -6I, -4I, -2I, 0, 2I, 4I, 6I, 8I, \dots\}.$$

Then $P = \{\dots, -16I, -12I, -8I, -4I, 0, 4I, 8I, 12I, 16I, \dots\}$ is a pseudo neutrosophic ideal generated by element $2I$, so P is a pseudo neutrosophic principal ideal of T .

Definition 3.10 Let P be pseudo neutrosophic ideal of $\langle R \cup I \rangle$. If for any pseudo neutrosophic ideal J of $\langle R \cup I \rangle$, is either $J \subseteq P$ or $J = P$, then P is called a pseudo neutrosophic maximal ideal.

Example 3.14 In example 4.3 P is a pseudo neutrosophic maximal ideal of T .

Definition 3.11 Let P be a pseudo neutrosophic ideal of $\langle R \cup I \rangle$. If for any two pseudos' neutrosophic ideals J, Q of $\langle R \cup I \rangle, JQ \subseteq P$ is either $J \subseteq P$ or $Q \subseteq P$. Then P is called a pseudo neutrosophic prime ideal.

Example 3.15 Let $(T, +, \bullet)$ be a pseudo neutrosophic ring, where

$$T = \{\dots, -4I, -3I, -2I, -I, 0, I, 2I, 3I, 4I, \dots\}.$$

Then $P = \{\dots, -6I, -4I, -2I, 0, 2I, 4I, 6I, \dots\}$ is a pseudo neutrosophic prime ideal of T .

4. Neutrosophic Quotient Rings and Pseudo Neutrosophic Quotient Rings

In this section, we introduce a neutrosophic quotient ring and pseudo neutrosophic with some examples.

Definition 4.1 Let $\langle R \cup I \rangle$ be a neutrosophic ring, let P be a neutrosophic ideal of $\langle R \cup I \rangle$. The neutrosophic quotient ring is defined to be the following:

$$\langle R \cup I \rangle / P = \{r + P : r \in \langle R \cup I \rangle\}, \text{ such that for all } r_1 + P, r_2 + P \in \langle R \cup I \rangle / P \text{ is } (r_1 + P) + (r_2 + P) = (r_1 + r_2) + P, \text{ and } (r_1 + P)(r_2 + P) = (r_1 r_2) + P. .$$

Example 4.1 In ring $(Z_4, \oplus_4, \otimes_4) = \{0, 1, 2, 3\}$, we have $H = \{0, 2\}$ is ideal of Z_4 and $P = \langle H \cup I \rangle = \{0, 2, 2I, 2 + 2I\}$ is neutrosophic ideal of $\langle Z_4 \cup I \rangle$, where

$$\langle Z_4 \cup I \rangle = \left\{ \begin{matrix} 0, 1, 2, 3, I, 2I, 3I, \\ 1 + I, 1 + 2I, 1 + 3I, \\ 2 + I, 2 + 2I, 2 + 3I, \\ 3 + I, 3 + 2I, 3 + 3I, \end{matrix} \right\}. \text{ We define neutrosophic quotient ring on } P \text{ as following:}$$

$$\langle Z_4 \cup I \rangle / \langle H \cup I \rangle = \langle Z_4 \cup I \rangle / P = \left\{ \begin{matrix} 0 + P, 1 + P, 2 + P, 3 + P, \\ I + P, 2I + P, 3I + P, \\ (1 + I) + P, (1 + 2I) + P, (1 + 3I) + P, \\ (2 + I) + P, (2 + 2I) + P, (3 + 3I) + P, \\ (3 + I) + P, (3 + 2I) + P, (3 + 3I) + P, \end{matrix} \right\}. \text{ Note that}$$

$$0 + P = 2 + P = 2I + P = (2 + 2I) + P = P \text{ (because } 0, 2, 2I, (2 + 2I) \in P \text{), so}$$

$$\langle Z_4 \cup I \rangle / P = \left\{ \begin{matrix} P, 1 + P, 3 + P, I + P, 3I + P, \\ (1 + I) + P, (1 + 2I) + P, (1 + 3I) + P, \\ (2 + I) + P, (2 + 3I) + P, (3 + I) + P, \\ (3 + I) + P, (3 + 2I) + P, (3 + 3I) + P \end{matrix} \right\}, \text{ and the zero element is } P, \text{ to}$$

construct then the neutrosophic algebra structure (NAS): $(\langle Z_4 \cup I \rangle / P, \oplus_4)$ by the visualizing table as shown in Table.4.1.

Table 4.1 for $(\langle Z_4 \cup I \rangle / P, \oplus_4)$

\oplus_4	P	$1 + P$	$3 + P$	$I + P$	$3I + P$	$(1+I) + P$	$(1+2I) + P$	$(1+3I) + P$	$(2+I) + P$	$(2+3I) + P$	$(3+I) + P$	$(3+2I) + P$	$(3+3I) + P$
P	P	$1+P$	$3+P$	$I+P$	$3I+P$	$(1+I) + P$	$(1+2I) + P$	$(1+3I) + P$	$(2+I) + P$	$(2+3I) + P$	$(3+I) + P$	$(3+2I) + P$	$(3+3I) + P$
$1 + P$	$1+P$	$2+P$	P	$(1+I) + P$	$(1+3I) + P$	$(2+I) + P$	$(2+2I) + P$	$(2+3I) + P$	$(3+I) + P$	$(3+3I) + P$	$I+P$	$2I+P$	$3I+P$
$3 + P$	$3+P$	P	$2+P$	$(3+I) + P$	$(3+3I) + P$	$I+P$	$2I+P$	$3I+P$	$(1+I) + P$	$(1+3I) + P$	$(2+I) + P$	$2I+P$	$(2+3I) + P$
$I + P$	$I+P$	$(1+I) + P$	$(3+I) + P$	$2I+P$	P	$(1+2I) + P$	$(1+3I) + P$	$1+P$	$(2+2I) + P$	$2+P$	$(3+2I) + P$	$(3+3I) + P$	$3+P$
$3I + P$	$3I+P$	$(1+3I) + P$	$(3+3I) + P$	P	$2I+P$	$1+P$	$(1+I) + P$	$(1+2I) + P$	$2+P$	$(2+2I) + P$	$3+P$	$(3+I) + P$	$(3+2I) + P$
$(1+I) + P$	$(1+I) + P$	$(2+I) + P$	$I+P$	$(1+2I) + P$	$1+P$	$(2+2I) + P$	$(2+3I) + P$	$2+P$	$(3+2I) + P$	$3+P$	$2I+P$	$3I+P$	P
$(1+2I) + P$	$(1+2I) + P$	$(2+2I) + P$	$2I+P$	$(1+3I) + P$	$(1+I) + P$	$(2+3I) + P$	$2+P$	$(2+I) + P$	$(3+3I) + P$	$(3+I) + P$	$3I+P$	P	$I+P$
$(1+3I) + P$	$(1+3I) + P$	$(2+3I) + P$	$2I+P$	$1+P$	$(1+2I) + P$	$2+P$	$(2+I) + P$	$(2+2I) + P$	$3+P$	$(3+2I) + P$	P	$I+P$	$2I+P$
$(2+I) + P$	$(2+I) + P$	$(3+I) + P$	$(1+I) + P$	$(2+2I) + P$	$2+P$	$(3+2I) + P$	$(3+3I) + P$	$3+P$	$2I+P$	P	$(1+2I) + P$	$(1+3I) + P$	$1+P$
$(2+3I) + P$	$(2+3I) + P$	$(3+3I) + P$	$(1+3I) + P$	$2+P$	$(2+2I) + P$	$3+P$	$(3+I) + P$	$(3+2I) + P$	P	$2I+P$	$1+P$	$(1+I) + P$	$(1+2I) + P$
$(3+I) + P$	$(3+I) + P$	$I+P$	$(2+I) + P$	$(3+2I) + P$	$3+P$	$2I+P$	$3I+P$	P	$(1+I) + P$	$1+P$	$(2+2I) + P$	$(2+3I) + P$	$2+P$
$(3+2I) + P$	$(3+2I) + P$	$2I+P$	$(2+2I) + P$	$(3+3I) + P$	$(3+I) + P$	$3I+P$	P	$I+P$	$(1+3I) + P$	$(1+I) + P$	$(2+3I) + P$	$2+P$	$(2+I) + P$
$(3+3I) + P$	$(3+3I) + P$	$3I+P$	$(2+3I) + P$	$3+P$	$(3+2I) + P$	P	$I+P$	$2I+P$	$1+P$	$(1+2I) + P$	$2+P$	$(2+I) + P$	$(2+2I) + P$

Table 4.2 of inverse element.

g	P	1+P	3+P	I+P	3I+P	(1+I)+P	(1+2I)+P	(1+3I)+P	(2+I)+P	(2+3I)+P	(3+I)+P	(3+2I)+P	(3+3I)+P
-g	P	3+P	1+P	3I+P	I+P	(3+3I)+P	(3+2I)+P	(3+I)+P	(2+3I)+P	(2+I)+P	(1+3I)+P	(1+2I)+P	(1+I)+P

And, to construct then the neutrosophic algebra structure (NAS) $(\langle Z_4 \cup I \rangle / P, \otimes_4)$ by the visualizing table as shown in Table.4.3

Table 4.3

\otimes_4	1+P	3+P	I+P	3I+P	(1+I)+P	(1+2I)+P	(1+3I)+P	(2+I)+P	(2+3I)+P	(3+I)+P	(3+2I)+P	(3+3I)+P
1+P	1+P	3+P	I+P	3I+P	(1+I)+P	(1+2I)+P	(1+3I)+P	(2+I)+P	(2+3I)+P	(3+I)+P	(3+2I)+P	(3+3I)+P
3+P	3+P	1+P	3I+P	I+P	(3+3I)+P	(3+2I)+P	(3+I)+P	(2+3I)+P	(2+I)+P	(1+3I)+P	(1+2I)+P	(1+I)+P
I+P	I+P	3I+P	I+P	3I+P	2I+P	3I+P	P	3I+P	I+P	P	I+P	2I+P
3I+P	3I+P	I+P	3I+P	I+P	I+P	3I+P	P	I+P	3I+P	P	3I+P	2+P
(1+I)+P	(1+I)+P	(3+3I)+P	2I+P	2I+P	(1+3I)+P	(1+I)+P	(1+3I)+P	2+P	2+P	(3+I)+P	(3+3I)+P	(3+I)+P
(1+2I)+P	(1+2I)+P	(3+2I)+P	3I+P	3I+P	(1+I)+P	1+P	(1+3I)+P	(2+3I)+P	(2+I)+P	(3+I)+P	(3+3I)+P	(3+3I)+P
(1+3I)+P	(1+3I)+P	(3+I)+P	P	P	(1+3I)+P	(1+3I)+P	(1+3I)+P	(2+2I)+P	(2+2I)+P	(3+I)+P	(3+I)+P	(3+I)+P
(2+I)+P	(2+I)+P	(2+3I)+P	3I+P	I+P	2+P	(2+3I)+P	(2+2I)+P	I+P	3I+P	(2+2I)+P	(2+I)+P	2+P
(2+3I)+P	(2+3I)+P	(2+I)+P	I+P	3I+P	2+P	(2+I)+P	(2+2I)+P	3I+P	I+P	(2+2I)+P	(2+I)+P	2+P
(3+I)+P	(3+I)+P	(1+3I)+P	P	P	(3+I)+P	(3+I)+P	(3+I)+P	(2+2I)+P	(2+2I)+P	(1+3I)+P	(1+3I)+P	(1+3I)+P
(3+2I)+P	(3+2I)+P	(1+2I)+P	I+P	3I+P	(3+3I)+P	3+P	(3+I)+P	(2+I)+P	(2+3I)+P	(1+3I)+P	1+P	(1+I)+P
(3+3I)+P	(3+3I)+P	2I+P	2I+P	2I+P	(3+I)+P	(3+3I)+P	(3+I)+P	2+P	2+P	(1+3I)+P	(1+I)+P	(1+3I)+P

Example 4.2 From example 2.2.1 $\langle R \cup I \rangle = \langle Z \cup I \rangle$ and $P = \langle 2Z \cup I \rangle$. Then:
 $\langle R \cup I \rangle / P = \langle Z \cup I \rangle / \langle 2Z \cup I \rangle = \{r + \langle 2Z \cup I \rangle : r \in \langle Z \cup I \rangle\}$.

Example 4.3 Consider a neutrosophic ring

$$\langle Z_6 \cup I \rangle = \left\{ \begin{array}{l} 0, 1, 2, 3, 4, 5, I, 2I, 3I, 4I, 5I, \\ 1 + I, 1 + 2I, 1 + 3I, 1 + 4I, 1 + 5I, \\ 2 + I, 2 + 2I, 2 + 3I, 2 + 4I, 2 + 5I, \\ 3 + I, 3 + 2I, 3 + 3I, 3 + 4I, 3 + 5I, \\ 4 + I, 4 + 2I, 4 + 3I, 4 + 4I, 4 + 5I, \\ 5 + I, 5 + 2I, 5 + 3I, 5 + 4I, 5 + 5I \end{array} \right\}, \text{ and}$$

$P = \{0, 2, 4, 2I, 4I, 2 + 2I, 2 + 4I, 4 + 2I, 4 + 4I\}$ is a neutrosophic ideal of $\langle Z_6 \cup I \rangle$, then

$$\langle Z_6 \cup I \rangle / P = \left\{ \begin{array}{l} 0 + P, 1 + P, 2 + P, 3 + P, 4 + P, 5 + P, \\ I + P, 2I + P, 3I + P, 4I + P, 5I + P, \\ (1 + I) + P, (1 + 2I) + P, (1 + 3I) + P, (1 + 4I) + P, (1 + 5I) + P, \\ (2 + I) + P, (2 + 2I) + P, (2 + 3I) + P, (2 + 4I) + P, (2 + 5I) + P, \\ (3 + I) + P, (3 + 2I) + P, (3 + 3I) + P, (3 + 4I) + P, (3 + 5I) + P, \\ (4 + I) + P, (4 + 2I) + P, (4 + 3I) + P, (4 + 4I) + P, (4 + 5I) + P, \\ (5 + I) + P, (5 + 2I) + P, (5 + 3I) + P, (5 + 4I) + P, (5 + 5I) + P, \end{array} \right\}$$

$$= \left\{ \begin{array}{l} P, 1 + P, 3 + P, 5 + P, \\ I + P, 3I + P, 5I + P, \\ (1 + I) + P, (1 + 2I) + P, (1 + 3I) + P, (1 + 4I) + P, (1 + 5I) + P, \\ (2 + I) + P, (2 + 3I) + P, (2 + 5I) + P, \\ (3 + I) + P, (3 + 2I) + P, (3 + 3I) + P, (3 + 4I) + P, (3 + 5I) + P, \\ (4 + I) + P, (4 + 3I) + P, (4 + 5I) + P, \\ (5 + I) + P, (5 + 2I) + P, (5 + 3I) + P, (5 + 4I) + P, (5 + 5I) + P \end{array} \right\}$$

is a neutrosophic quotient ring. Also $P = \{0, 3, 3I, 3 + 3I\}$ is a neutrosophic ideal of $\langle Z_6 \cup I \rangle$. So

$$\langle Z_6 \cup I \rangle / P = \left\{ \begin{array}{l} 0 + P, 1 + P, 2 + P, 3 + P, 4 + P, 5 + P, \\ I + P, 2I + P, 3I + P, 4I + P, 5I + P, \\ (1 + I) + P, (1 + 2I) + P, (1 + 3I) + P, (1 + 4I) + P, (1 + 5I) + P \\ (2 + I) + P, (2 + 2I) + P, (2 + 3I) + P, (2 + 4I) + P, (2 + 5I) + P \\ (3 + I) + P, (3 + 2I) + P, (3 + 3I) + P, (3 + 4I) + P, (3 + 5I) + P \\ (5 + I) + P, (5 + 2I) + P, (5 + 3I) + P, (5 + 4I) + P, (5 + 5I) + P \end{array} \right\}$$

$$= \left\{ \begin{array}{l} P, 1 + P, 2 + P, 4 + P, 5 + P, \\ I + P, 2I + P, 4I + P, 5I + P, \\ (1 + I) + P, (1 + 2I) + P, (1 + 3I) + P, (1 + 4I) + P, (1 + 5I) + P, \\ (2 + I) + P, (2 + 2I) + P, (2 + 3I) + P, (2 + 4I) + P, (2 + 5I) + P, \\ (3 + I) + P, (3 + 2I) + P, (2 + 4I) + P, (3 + 5I) + P, \\ (4 + I) + P, (4 + 2I) + P, (4 + 3I) + P, (4 + 4I) + P, (4 + 5I) + P, \\ (4 + I) + P, (4 + 2I) + P, (4 + 3I) + P, (4 + 4I) + P, (4 + 5I) + P \end{array} \right\}$$

is a neutrosophic quotient ring.

Definition 4.2 Let $\langle R \cup I \rangle$ be a neutrosophic ring, P be a pseudo neutrosophic ideal of $\langle R \cup I \rangle$. The pseudo neutrosophic quotient ring is defined to be the following:

$$\langle R \cup I \rangle / P = \{ r + P : r \in \langle R \cup I \rangle \}.$$

Example 4.4 In a pseudo neutrosophic ring T where $T = \{ \dots, -4I, -3I, -2I, -I, 0, I, 2I, 3I, 4I, \dots \}$, and $P = \{ \dots, -6I, -4I, -2I, 0, 2I, 4I, 6I, \dots \}$ is a pseudo neutrosophic ideal of T . then $T/P = \{ \dots, -4I + P, -3I + P, -2I + P, -I + P, P, I + P, 2I + P, 3I + P, 4I + P, \dots \}$ is a pseudo neutrosophic quotient ring.

Example 4.5 Consider a pseudo neutrosophic ring T where $T = \{ \dots, -8I, -6I, -4I, -2I, 0, 2I, 4I, 6I, 8I, \dots \}$, and $P = \{ \dots, -16I, -12I, -8I, -4I, 0, 4I, 8I, 12I, 16I, \dots \}$ is a pseudo neutrosophic ideal of T . So $T/P = \{ \dots, -8I + P, -6I + P, -4I + P, -2I + P, P, 2I + P, 4I + P, 6I + P, 8I + P, \dots \}$ is a pseudo neutrosophic quotient ring.

Example 4.6 In a neutrosophic ring $\langle Z_2 \cup I \rangle = \{ 0, 1, I, 1 + I \}$. $P = \{ 0, I \}$ is pseudo neutrosophic ideal of $\langle Z_6 \cup I \rangle$. Then $\langle Z_2 \cup I \rangle / P = \{ 0 + P, 1 + P, I + P, (1 + I) + P \} = \{ P, 1 + P, (1 + I) + P \}$ is a pseudo neutrosophic quotient ring.

Example 4.7 Let $\langle Z_4 \cup I \rangle$ be a neutrosophic ring, where $\langle Z_4 \cup I \rangle = \left\{ \begin{array}{l} 0, 1, 2, 3, \\ I, 2I, 3I, \\ 1 + I, 1 + 2I, 1 + 3I, \\ 2 + I, 2 + 2I, 2 + 3I, \\ 3 + I, 3 + 2I, 3 + 3I \end{array} \right\}$,

the neutrosophic subset $P = \{ 0, 2I \}$ of $\langle Z_4 \cup I \rangle$ is a pseudo neutrosophic ideal, hence

$$\langle Z_4 \cup I \rangle / P = \left\{ \begin{array}{l} 0 + P, 1 + P, 2 + P, 3 + P, \\ I + P, 2I + P, 3I + P, \\ (1 + I) + P, (1 + 2I) + P, (1 + 3I) + P, \\ (2 + I) + P, (2 + 2I) + P, (2 + 3I) + P, \\ (3 + I) + P, (3 + 2I) + P, (3 + 3I) + P \end{array} \right\}$$

$$\langle Z_4 \cup I \rangle / P = \left\{ \begin{array}{l} P, 1 + P, 2 + P, 3 + P, \\ I + P, 3I + P, \\ (1 + I) + P, (1 + 2I) + P, (1 + 3I) + P, \\ (2 + I) + P, (2 + 2I) + P, (2 + 3I) + P, \\ (3 + I) + P, (3 + 2I) + P, (3 + 3I) + P \end{array} \right\}$$

is a pseudo neutrosophic quotient ring.

Example 4.8 From a neutrosophic ring $\langle Z_6 \cup I \rangle$, the neutrosophic subset $P = \{0, 3I\}$ is a pseudo neutrosophic ideal of $\langle Z_6 \cup I \rangle$. So

$$\langle Z_6 \cup I \rangle / P = \left\{ \begin{array}{l} P, 1 + P, 2 + P, 3 + P, 4 + P, 5 + P, \\ I + P, 2I + P, 4I + P, 5I + P, \\ (1 + I) + P, (1 + 2I) + P, (1 + 3I) + P, (1 + 4I) + P, (1 + 5I) + P, \\ (2 + I) + P, (2 + 2I) + P, (2 + 3I) + P, (2 + 4I) + P, (2 + 5I) + P, \\ (3 + I) + P, (3 + 2I) + P, (3 + 3I) + P, (3 + 4I) + P, (3 + 5I) + P, \\ (4 + I) + P, (4 + 2I) + P, (4 + 3I) + P, (4 + 4I) + P, (4 + 5I) + P, \\ (5 + I) + P, (5 + 2I) + P, (5 + 3I) + P, (5 + 4I) + P, (5 + 5I) + P \end{array} \right\}$$

is a pseudo neutrosophic quotient ring.

References

- [1] Agboola, A. A., Adeleke, E. O., & Akinleye, A. A. (2012). Neutrosophic Rings II. *International J.Math. Combin*, 2, 1-8.
- [2] Agboola, A. A., Akinola, A. D., & Oyebola, O. Y. (2011). Neutrosophic Rings I. *International J.Math. Combin*, 4, 1-14.
- [3] Al-Odhari, A. M. (2023). A Review Study on Some Properties of The Structure of Neutrosophic Ring. *Neutrosophic Sets and Systems*, 54, 139-156. doi:DOI:10.5281/zenodo.7817736
- [4] Al-Odhari, A. M. (2023). Some Algebraic Structure of Neutrosophic Matrices. *Prospects for Applied Mathematics and data Analysis (PAMDA)*, 01(02), 37-44. doi:https://doi.org/10.54216/PAMDA.010204
- [5] Al-Odhari, A. M. (2024). Axiomatic of Neutrosophic Groups. *Sana'a University Journal of Applied Sciences and Technology*, 2(2), 205-214. doi:https://doi.org/10.59628/jast.v2i2.793
- [6] Al-Odhari, A. M. (2024). Basic Introduction of Neutrosophic Set Theory. *Plithogenic Logic and Computation*, 2, 20-28. doi:10.61356/j.plc.2024.2327
- [7] Al-Odhari, A. M. (2024). Characteristics Neutrosophic Subgroups of Axiomatic Neutrosophic Groups. *Neutrosophic Optimization and Intelligent Systems*, 3, 32-40. doi:10.61356/j.nois.2024.3265
- [8] Al-Odhari, A. M. (2024). On the Generalization of Neutrosophic Set Operations: Testing Proofs by Examples. *HyperSoft Set Methods in Engineering*, 2, 72-82. doi:10.61356/j.hsse.2024.2321
- [9] Al-Odhari, A. M. (2024). Some Aspects of Neutrosophic Set Theory. *to appear*.
- [10] Kandasamy, V. W., & Smarandache, F. (2006). *Neutrosophic Rings*. Hexis, Phoenix, Arizona.

- [11] Kandasamy, W. B. (2006). *SMARANDACHE NEUTROSOPHIC ALGEBRAIC STRUCTURES*. Retrieved from <http://mat.iitm.ac.in/~wbv>
- [12] Kandasamy, W. B., & Smarandache, F. (2006). *Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures*. Hexis, Phoenix, Arizona. Retrieved from <https://arxiv.org/ftp/math/papers/0603/0603581.pdf>
- [13] Rotman, J. J. (1995). *An Introduction to Theory of Group Theory* (Fourth Edition ed.). Springer-Verlag New York. NC.
- [14] Rotman, J. J. (2002). *Advanced Modern Algebra*. Pearson Education.
- [15] Smarandache, F. (2002). Neutrosophy, A New Branch of Philosophy, in Multiple-Valued Logic. *An International Journal*, 8(3), 297-384.