Neutrosophic Burr Distribution for Modeling Health Risk Factors

Fuad S. Alduais[1], Zahid Khan[2,*]

[1]Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj, 11942, Saudi Arabia

[2]Department of Quantitative Methods, Pannon Egyetem, Veszprem, H-8200, Hungary

Emails f.alduais@psau.edu.sa; zahidkhan@hu.edu.pk

Abstract

The Burr distribution is one of the most important and commonly used probability distribution in statistical analysis. In this study, a new class of univariate distribution based on the Burr random variable is proposed. Characteristics of the proposed neutrosophic Burr distribution (NBD) are discussed. The neutrosophic form of the proposed distribution is particularly advantageous for handling the imprecise and uncertain information commonly present in real-world problems. The statistical properties and the shapes of corresponding probability density and cumulative density functions are illustrated. Some important functions commonly utilized in survival studies are formulated within neutrosophic structures. General expressions for other distributional properties of the proposed NBD are developed under neutrosophic framework. The inverse cumulative method is used to find random numbers from the suggested model. Maximum likelihood method for estimating the model parameters is described, and the performance of estimated parameters are assessed using a Monte Carlo simulation experiment. Finally, the paper demonstrates the practical use of the proposed model through a real-world application of malaria cases per thousand population at risk.

Keywords: Burr model; Neutrosophic probability; Neutrosophic measures; Estimation; Simulation

1. Introduction

Probability distributions form the basis for many topics in statistics and data analysis, as they are mathematical models that describe the likelihood of various outcomes of uncertain processes [1]. They are useful in numerous data types, used for different purposes, such as decision-making, predictive modeling, and risk assessment. Probability distributions are also fundamental in healthcare domains including survival analysis, prediction of patient outcomes, and modeling disease progression [2]. There are specific probability distributions of great importance, such as normal distributions in handling physiological measurements, and exponential and Weibull distributions in modeling patient life and failure times of medical devices and Pareto distributions in risk assessment data are few examples [3]. These distributions are also harnessed in healthcare operations to determine reliable preparedness and allocation of resources as well as medical service forecasting and improving diagnostics. The Burr distribution is a very flexible distribution, as the number of parameters could take any desirable values [4] . Since the distribution is versatile and can take on a variety of shapes (skewed, heavy-tailed, unimodal, etc.), it becomes a very valuable mean of modeling many different sorts of real-world data. Due to its flexible nature with respect to numerous data types, the Burr distribution has been widely suggested in applications in reliability analysis, risk assessment, and survival studies [5] . Its parameters can control the shape and scale of the distribution accurately, making it suitable in environments with high uncertainty, including extreme events.

One of the most important issues in economics is the distribution of people by income [6]. Burr distribution describes the distribution of people by income. Many standard theoretical distributions, including the exponential, Weibull, logistic, Gompertz, generalized logistic normal, extreme value, and uniform distributions, are special cases or limiting cases of the Burr system of distributions [7]. The Burr distribution is also one of the essential distributions within the healthcare sector to solve the complicated problems of patient care, operational efficiencies, and medical research, etc. It is often used in survival analysis to model the time to death or the time

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in between certain medical events, like disease onset or recovery after treatment [8]. In healthcare datasets, one of the characterization of extreme values may be the outliers in response times for patients, the output of the Burr distribution is particularly useful with the heavy-tailed nature allowing for such extreme values [9]. In healthcare, Burr distribution is used for reliability analysis of pumps and equipment, which helps in their efficient and effective functioning. It minimizes the chances of failure of healthcare equipment. When applied to public health, it is utilized to simulate the propagation of diseases and evaluate the efficiency of intervention strategies. In addition, its relevance in risk management helps healthcare providers and insurers to predict financial risks and create sustainable health plans. The Burr distribution serves as a tool for operationalizing data-driven healthcare outcome management, enabling clinicians and health managers to leverage its characteristics to inform the decision-making process [9-10].

In fact, real data obtained commonly are not always precise. Indeterminacy is the common real-world phenomenon, which is related to a number of real-world problems [11]. To cover the indeterminate characteristic of the problems in real-world problem, Smarandache initiated neutrosophic logic [12]. This is a generalization of fuzzy and classical set theory that adds, apart from the degree of truth, degrees of indeterminacy, and falsity, to more adequately represent uncertainty, incomplete knowledge, and vagueness [13-14]. This is quite useful in fields heavily filled with ambiguities like medical diagnostics, machine learning, and survival studies [15]. Neutrosophic logic is also extended to statistical methods. In context of statistical methods, it is known as neutrosophic statistics. Neutrosophic statistics is a branch of neutrosophy that combines statistical methods and neutrosophic logic together corresponding to data, which can be inexact, vague or partial in some way [16-18]. Unlike classical statistical methods, which seek only to make it as exact as possible, neutrosophic statistical methods offer a grayed-in-and-in-its-own-expression-of truth, indeterminacy and falsity [19]. Such orientations are especially critical in some fields like risk management in healthcare, where the discrepancies between measures could result from measurement or technical errors, missing data or subjective judgments [20]. Neutrosophic statistics provide a framework that are able to moderate, make a decision and measure the uncertainty of acquired conclusion. It is frequently adopted in areas consisting of reliability testing, risk management, medical trials, survival analysis [21-23].

This study continues this line of work providing a new perspective of classical Burr model by adopting the neutrosophic to increase the flexibility of classical model. Neutrosophic structure of the Burr model improves its efficacy to consider vagueness, uncertainty, and imprecision in empirical datasets. Such a framework can be especially beneficial for modelling complex phenomena as in health care, reliability analysis; risk management, etc. where partial information is sometimes available due to personal judgements, lack of accuracy of measurements devices or small sample sizes under studies. The neutrosophic Burr distribution serves as a valuable tool for tackling issues in different areas, such as survival analysis, reliability studies, and medical diagnostics, by facilitating the incorporation of the versatility of the Burr distribution along with a comprehensive uncertainty representation provided by neutrosophic theory.

The structure of the paper is organized as follows. In Section 2 we provide a general theoretical explanation for the classical Burr distribution. In Section 3, the proposed NBD with some basic characteristics is presented. Section4 illustrates the random number generation process of the proposed model. In Section 5 estimation of parameters of model, using sampling data is described. In Section 6 real-world application of the model is provided. Finally, major findings of the work are concluded in Section 7.

2. Preliminaries

Burr distribution is the member of Burr family of distributions. It is continuous distribution with non-negative values of Burr variable. It is one of the versatile probabilistic models that widely used to analysis data commonly exist in fields of reliability analysis, hydrology and finance. Due to its diverse nature, Burr distribution has the ability to represent many data sets with various shapes of kurtosis and skewness. The probability density function (PDF) of the Burr model with shape and scale parameter is given by:

$$f(x;c,k)=\frac{ckx^{c-1}}{(1+x^{c})^{k+1}},\quad x>0,\quad \text{and}\quad f(x;c,k)=0,\quad x\leq 0. \tag{1}$$

$x>0$ is the random variable,

$c>0$ is the shape parameter controlling the tail behavior,

$k>0$ is the shape parameter controlling the peak and spread.

The PDF function is probability theory is one of the important concepts for presenting the probabilistic behavior of natural phenomena. It explains the behavior of the random variable by assuming different non-negative values. In other words, it describes the likelihood of variable taking specific given range values. Total area under the PDF curve is usually equal to one. Furthermore, PDF is often used in decision-making fields such as machine learning,

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engineering, economics etc. It plays a vital role in understanding the behavior of random variable for variety of natural phenomena. The behavior of PDF curves for different values of shape and location parameters is shown in Figure 1.

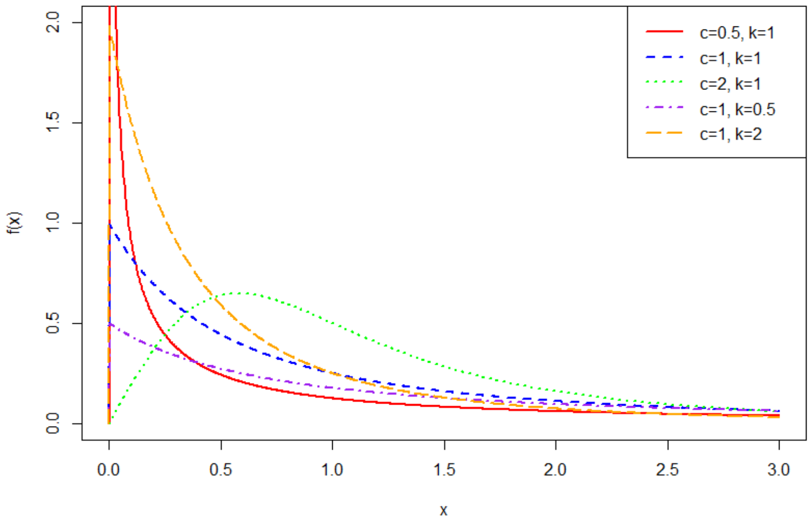

Figure 1. PDF curves of the Burr distribution with different precise values of parameters

The PDF of the Burr distribution for selected values of shape parameters c and k can be seen in Figure 1 shows that how the peak and tail behavior on this distribution change, reflecting the influence of these parameters over the data's distribution. Here we are assuming that location and shape parameters are exact numbers and does not involve any type of uncertainty.

The other function related to PDF is the cumulative distribution function (CDF). The CDF for the Burr model can be represented as:

$F(x;c,k) = 1-(1+x^c)^{-k}, x > 0$, and $F(x;c,k) = 0, x \leq 0.$ (2)

The CDF is very useful in probability theory and statistics, it is the probability that a random variable X will take a value less than or equal to x. It allows us a total look at the distribution, capturing every possibility up to any value of interest. It is important for comparing distributions by determining quantiles and evaluating probabilities over ranges (see, the CDF). This is particularly important for reliability analysis, statistical inference (hypothesis testing), and stochastic modeling both in the explanation of underlying ideas (practical satisfiability, etc.) and exact computational work on any continuous and discrete random variables. The CDF curves for different values of c and k are given in Figure 2.

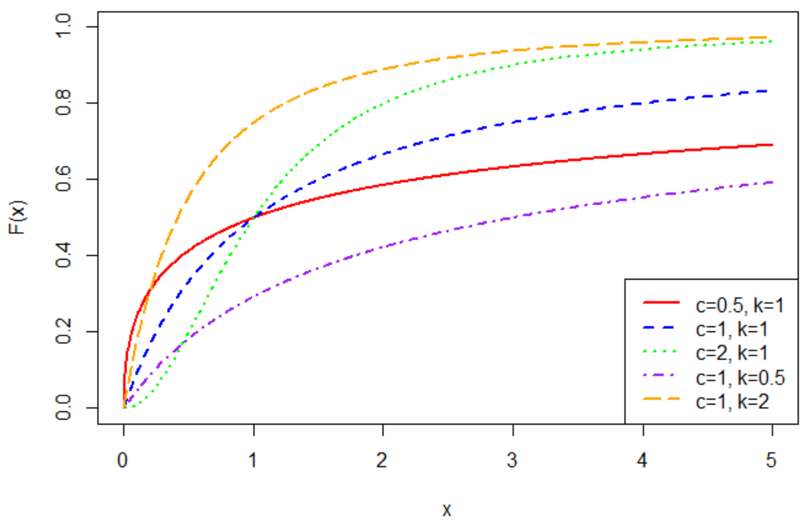

Figure 2. PDF curves of the Burr distribution with different precise values of parameters

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Figure 2 shows the CDF of the classical Burr distributed data: The function, also termed probability accumulation, graphically depicts how rapidly the cumulative probability increases over time for different combinations of shape parameters (c and k), displaying how higher or lower values affect distribution spread and tail behavior. Each CDF curve ranges from 0 to 1.

There are also many other vital functions related to Burr model. The hazard function of the Burr model can be obtained as:

$$h(x;c,k) = \frac{f(x;c,k)}{1-F(x;c,k)} = \frac{\frac{ckx^{c-1}}{(1+x^c)^{k+1}}}{(1+x^c)^{-k}},\ x > 0 \quad (3)$$

The definition of the hazard function is an important concept in survival analysis and reliability engineering, which describes the instantaneous failure rate of a system or event. It gives some estimates of the probability of failure, conditional on survival to a point. Understanding the hazard function is also crucial to understand time-to-event data, and therefore design maintenance policies, risk assessment and reliability strategies. The other related function to hazard model is survival function. The survival function of the proposed model is given by:

$$S(x;c,k) = 1 - F(x;c,k) = (1+x^c)^{-k}, x > 0 \quad (4)$$

In survival analysis and reliability engineering, the survival function is a mathematical function that models the probability of success over time. This offers a contrasting with the more conventional CDF, which underscores failure rather than longevity. It has important implications for the analysis of time-to-event data, such as estimating life expectancy and assessing reliability, and it is commonly used in medicine, actuarial science, engineering and even history.

Random number generation is an important characteristic of any probability distribution. Quantile function of the Burr distribution can be expressed as:

$$Q(p;c,k) = \left[(1-p)^{-1/k} - 1\right]^{1/c}, \quad 0 < p < 1 \quad (5)$$

The rth moment of the Burr distribution can be determined by solving the equation:

$$\mu_r' = E[X^r] = \int_0^\infty x^r f(x;c,k)\ dx = \int_0^\infty x^r \frac{ckx^{c-1}}{(1+x^c)^{k+1}}\ dx \quad (6)$$

The Burr is a flexible distribution that finds applications in many fields. It models a range of data from symmetric data to highly skewed by allowing the PDF and CDF take random shapes. It is useful in reliability engineering and survival analysis for lifetimes, failure rates and event occurrence in clinical studies. Throughout its moments, mean or variance properties lead to Burr model useful in statistical analysis in the field of finance, medical and survival studies for modeling income and risk data. Furthermore, its heavy tailed nature and the generality that it includes other distributions make it suitable in econometrics, environmental modeling and as a general shock model. Now in the following section, we proposed the neutrosophic version of the Burr model.

3. Proposed Neutrosophic Structure

Neutrosophic probability distribution is an extension of classical probabilistic distribution by enrichment of its parameters with imprecision, indeterminacy and inconsistency; therefore, neutrosophic version is suitable for modeling uncertainty in reality world. In neutrosophic Burr distribution, some or all parameters can represent neutrosophic numbers, meaning they can take on values that are not necessarily precise or fully determined. The ability to work with imprecise data and reflect uncertainty makes it valuable across many fields where precise values are often not available or change over time, like reliability analysis, finance, clinical studies, healthcare studies, and environmental studies. The neutrosophic framework is aimed at enhancing the distribution properties, which handle vague, incomplete or inconsistent data to bring it into uncertain systems. The PDF of the proposed NBD is given by:

$$f_N(x) = \frac{d\mathcal{F}_N(x)}{dx} = \beta_n \alpha_n x^{\alpha_n - 1}(1 + x^{\alpha_n})^{-(\beta_n+1)},\ \ x > 0. \quad (7)$$

where

$\mathcal{F}_N(x)$ is the CDF of the proposed model

The CDF of the proposed model is defined by:

$$\mathcal{F}_N(x) = \mathcal{P}(\mathcal{X} \le x) = 1 - (1 + x^{\alpha_n})^{-\beta_n},\ \ x \ge 0 \quad (8).$$

In fact, the neutrosophic PDF and CDF are essential tools for exploring data with fuzzy, vague, or contradictory elements, thus providing a more comprehensive view than classical distributions. Neutrosophic PDF for examples of neutrosophic Burr distribution shows the PDF of the different parameters under uncertainty cases, providing

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the probability of observable events and values under indeterminacy, and neutrosophic phenomena, representing the cumulative probabilities. Such functions are useful for applications in fields such as reliability analysis, risk assessment and decision-making, where precise values of parameters are not available. Their ability to reproduce uncertainty and vagueness makes them indispensable in practical life scenarios such as financial prediction, environmental allocation, healthcare risk assessment and survival evaluation, where there is substantial data volatility and vagueness.

The PDF and CDF of the proposed Burr model are given in Figure 3 and Figure 4 respectively.

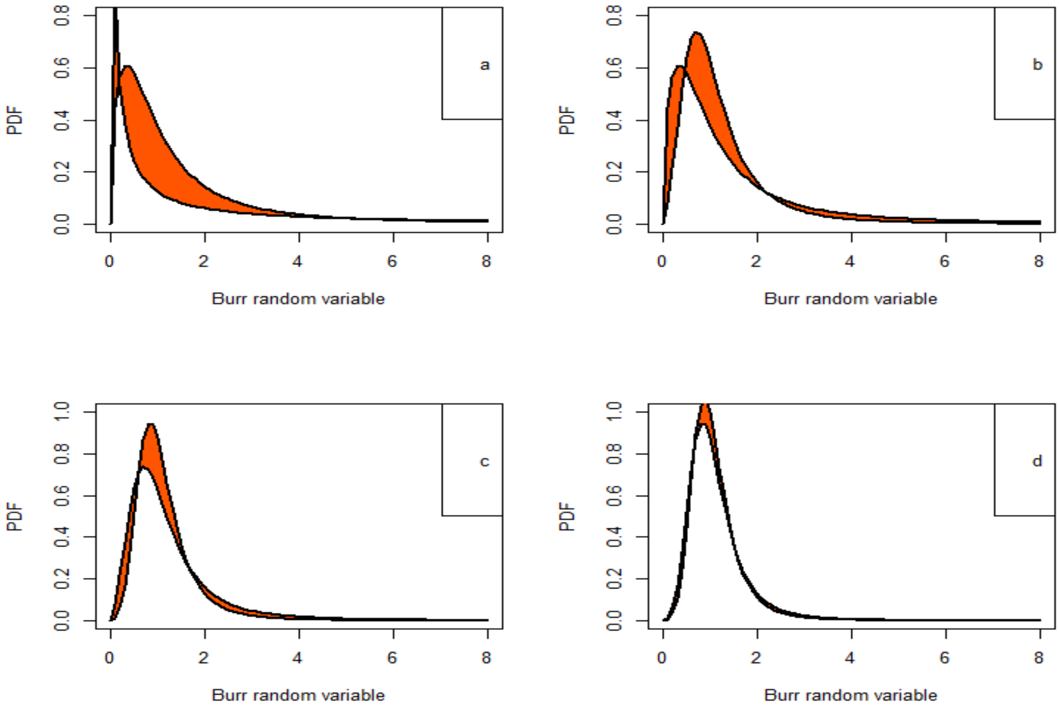

Figure 3. PDF plot of the neutrosophic Burr model

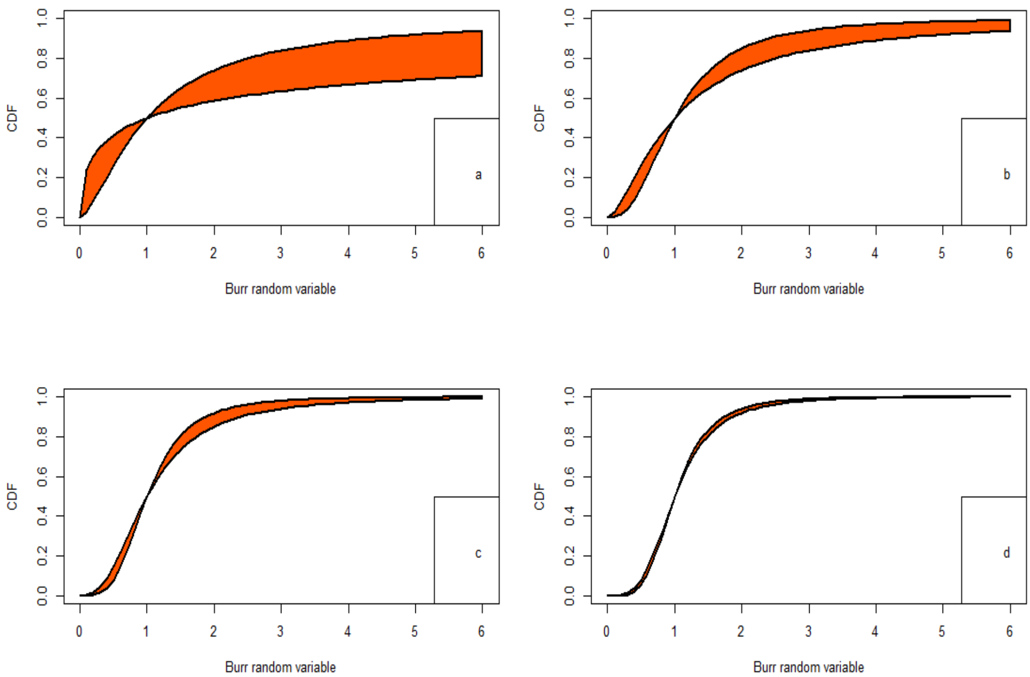

Figure 4. CDF plot of the neutrosophic Burr model

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The neutrosophic PDF of the Burr distribution for uncertain values for location and shape parameters is demonstrated in Figure 3. The shaded areas indicate the uncertainty of parameter values, emphasizing how the probability of endpoints depends on imprecise parameters.

The neutrosophic CDF of the Burr distribution is given in Figure 4 for the same ranges of parameters. CDF curves exhibit cumulative probabilities under uncertainty, and display how indeterminacy manifest in the distribution, rendering insights on how imprecise data appears.

Several other supporting properties of the NBD may be expressed as in the following versions of mathematical expressions:

If we assume that $\mathcal{F}_N(x)$ indicates the likelihood that an item will fail before or at time x, where x is the item's failure time random variable, then the function $\mathcal{S}_N(x) = \mathcal{P}(\mathcal{X} > x)$represents the likelihood that the item will fail after time x. The neutrosophic survival function of x, denoted by $\mathcal{S}_N$, is defined by

$$\mathcal{S}_N(x) = 1 - \mathcal{F}_N(x) = (1 + x^{\alpha_n})^{-\beta_n}, \ \ x \geq 0 \tag{9}$$

It is seen Eq (9) that the neutrosophic survival function of the Burr distribution represents the probability that the process survives beyond a given point when considered with uncertain parametric values. This captures the uncertainty over the shape and location parameters of the distribution, which better reflects the best effort at modeling for systems with vague/incomplete information. The application of neutrosophic survival function significantly contributes to the treatment of vague lifetimes or reliability with implications on risk theory, reliability engineering, and healthcare where exact parameter values are often unknown or subject to slight alteration. Its flexibility incorporates a wider variety of real-world scenarios than classical survival function.

If the component does not fail until time x, its instantaneous failure rate is indicated by $\hbar_N(x)$. The function $\hbar_N$ is the neutrosophic hazard rate function of the Burr random variable $\mathcal{X}$, defined as:

$$\hbar_N(x) = \frac{f_N(x)}{\mathcal{S}_N(x)} = \frac{\beta_n\alpha_n x^{\alpha_n-1}(1+x^{\alpha_n})^{-(\beta_n+1)}}{(1+x^{\alpha_n})^{-\beta_n}} = \frac{\beta_n\alpha_n x^{\alpha_n-1}}{1+x^{\alpha_n}}, x > 0. \tag{10}$$

The function $\hbar_N(x)$ is called the hazard function of the proposed model.

Similarly, if we the moments of the proposed model under the neutrosophic framework can be obtained as follows: $\mathrm{E}(\mathcal{X}^m) = \int_{-\infty}^{\infty} x^m f_N(x)dx = \beta_n \int_0^{\infty} x^m [(1 + x^{\alpha_n})^{-1}]^{\beta_n}.\, \alpha_n x^{\alpha_n - 1}\, dx.$(11)

Let $u = (1 + x^{\alpha_n})^{-1}$ or $x = [(1-u)/u]^{1/\alpha_n}$, then $du = -\alpha_n x^{\alpha_n-1}(1 + x^{\alpha_n})^{-2}dx \ = -\alpha_n x^{\alpha_n-1}u^2 dx$ or $\alpha_n x^{\alpha_n-1}dx = -u^2 du$.

It is clear that $u \longrightarrow 1$ as $x \longrightarrow 0$, and $u \longrightarrow 0$ as $x \longrightarrow \infty$. therefore, Eq (11) becomes

$$\mathrm{E}(\mathcal{X}^m) = \beta_n \int_1^0 \left[\left(\frac{1-u}{u}\right)^{1/\alpha_n}\right]^m u^{\beta_n+1}(-u^{-2})du$$

$$= \beta_n \int_0^1 u^{(\beta_n - m/\alpha_n)^{-1}}(1-u)^{(1+m/\alpha_n)^{-1}} du. \tag{12}$$

When $\beta_n - m/\alpha_n > 0$ and $1 + m/\alpha_n > 0$ (or when $m < \beta_n\alpha_n$), the integral part in Eq (12) exists and is equal to $\mathcal{B}(\beta_n - m/\alpha_n, 1 + m/\alpha_n)$, where $\mathcal{B}(\because)$ is beta function. Because $\mathcal{B}(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ for any $a, b > 0$, where $\Gamma(\cdot)$ is gamma function, the m^{th} raw moment of $\mathcal{X}$ can be written as

$$\mathrm{E}(\mathcal{X}^m) = \beta_n\mathcal{B}(\beta_n - m/\alpha_n, 1 + m/\alpha_n) = \frac{\Gamma(\beta_n - m/\alpha_n)\Gamma(1+m/\alpha_n)}{\Gamma(\beta_n)}, \ \ m < \beta_n\alpha_n. \tag{13}$$

From Eq (13), for $m = 1$, we obtain the first raw moment which is called mean and denoted by μ.

Therefore, the mean of $\mathcal{X}$ is defined by

$$\mu = \mathrm{E}(\mathcal{X}) = \frac{\Gamma(\beta_n - 1/\alpha_n)\Gamma(1+1/\alpha_n)}{\Gamma(\beta_n)}, \ \ \beta_n\alpha_n > 1. \tag{14}$$

For $m = 2$, we obtain the second raw moment $\mathrm{E}(\mathcal{X}^2)$. on the other hands, the second central moment or varivance can be defined as $\sigma^2 = [(\mathcal{X} - \mu)^2] = \mathrm{E}(\mathcal{X}^2) - \mu^2$. hence,we obtain

$$\sigma^2 = \frac{\Gamma(\beta_n - 1/\alpha_n)\Gamma(1+1/\alpha_n)\Gamma(\beta_n) - \Gamma^2(\beta_n - 1/\alpha_n)\Gamma^2(1+1/\alpha_n)}{\Gamma^2(\beta_n)}, \ \ \beta_n\alpha_n > 2. \tag{15}$$

The aforementioned properties are basic statistical characteristics under neutrosophic framework. Now in the next sections we will discuss the random number generation process and sample estimation procedure of the proposed model.

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4. Simulation

In this section, we have used the inverse CDF method to generate random numbers. We can use the quantile function of the proposed model to generate random numbers. The quantile function is defined as the inverse of the cumulative distribution function (CDF) of a probability distribution. This is a very important function when we are looking to generate random numbers from a distribution. The quantile function is then applied to the uniformly distributed random numbers drawn from uniform distribution to convert them into the random numbers of the desired probability distribution. The inverse transform method is one of the most commonly used methods for generating random samples, especially when it comes to simulations, statistical modeling, and Monte Carlo methods, it helps in generating the random data that matches the theoretical data values based on the specific distribution.

The quantile function of the proposed model is given by:

$$Q_N = \left[(1-p)^{-1/\alpha_n} - 1\right]^{1/\beta_n} \tag{16}$$

The Burr distribution, in terms of the netruosophic quantile function, is an extension of the classical quantile function that includes uncertainty or imprecision in its parameters. Neutrosophic quantile function provides a way to generate random values considering the uncertainty in location and shape parameters. This allows strong modeling in the presence of uncertainty or incomplete information. The neutrosophic quantile function from neutrosophic Burr distribution can be utilized in Monte Carlo simulation by converting random uniform variable into neutrosophic random numbers including uncertainty in the parameters. We suppose that 40 random numbers are generated from the proposed NBD with parameter $\alpha_n = [1, 1.5]$ and $\beta_n = [2, 2]$. Numerical results from this simulated experiment are shown in Table 1.

Table 1: Random numbers generated from the proposed NBD

Random numbers				
[0.185, 0.324]	[1.113, 1.173]	[0.301, 0.449]	[1.547, 1.924]	[2.125, 3.098]
[0.024, 0.082]	[0.456, 0.592]	[1.613, 2.049]	[0.493, 0.624]	[0.357, 0.503]
[2.441, 3.813]	[0.353, 0.499]	[0.761, 0.834]	[0.530, 0.655]	[0.056, 0.146]
[1.671, 2.160]	[0.152, 0.284]	[0.022, 0.078]	[0.220, 0.364]	[2.387, 3.688]
[1.592, 2.009]	[0.804, 0.865]	[0.668, 0.764]	[5.303, 12.210]	[0.704, 0.792]
[0.852, 0.899]	[0.481, 0.614]	[0.570, 0.687]	[0.186, 0.326]	[0.083, 0.190]
[2.603, 4.200]	[1.691, 2.199]	[0.798, 0.860]	[1.136, 1.211]	[0.013, 0.054]
[0.384, 0.528]	[1.023, 1.035]	[0.130, 0.256]	[0.211, 0.354]	[0.141, 0.271]

Table 1 shows the desired random samples from the NBD with parametric values $\alpha_n = [1, 1.5]$ and $\beta_n = [2, 2]$. These neutrosophic numbers can be used to establish statistical properties of the proposed distribution. Due to imprecision in the study parameters, each interval has a range of values, reflecting inherent imprecision in the distribution. These ranges can be large or small depending on parametric values.

5. Estimation Procedure

In this section, we have used the method of maximum likelihood (ML) to estimate the parameters of the proposed model. The ML approach is one of the basic concepts of statistics to estimate the parameters of a probability distribution. It determines the parameter values that correspond to the highest likelihood of achieving the observed data, hence maximizing the model's goodness of fit. This method is widely used by researchers because, under regular conditions, it has many favorable features including consistency, efficiency, and asymptotic normality of estimators. This becomes especially potent when dealing with complex models, such as those with multiple parameters/dependencies. The maximum likelihood method is a fundamental tool in statistical modeling and applied data analysis.

Let we assume that $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$ is a random sample of size n of the Burr random variable from the proposed model. The joint probability density function $\mathcal{L}$ of $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$ is given as:

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$$\mathcal{L}(\beta_n, \alpha_n;\ x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i; \beta_n, \alpha_n) = \beta_n{}^{n} \alpha_n{}^{n} \left(\prod_{i=1}^{n} x_i{}^{\alpha_n - 1}\right)\left[\prod_{i=1}^{n} (1 + x_i{}^{\alpha_n})^{-(\beta_n+1)}\right] \quad (17)$$

The probability function of the random sample $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$ is another name for the function $\mathcal{L}$. To acquire accurate estimates for β_n and α_n, determine the values that maximise the likelihood $\mathcal{L}(\beta_n, \alpha_n;\ x_1, x_2, \ldots, x_n)$. To maximise the probability, we equate the likelihood function to zero and solve the resultant equations. We know that at the same values of β_n and α_n, the likelihood function $\mathcal{L}$ and its logarithm, or the log-likelihood function $\ln(\mathcal{L})$, are maximised. As a result, it will be simpler to solve the equations $\partial \ln[\mathcal{L}(\beta_n, \alpha_n;\ x_1, x_2, \ldots, x_n)]/\partial\beta_n = 0$ and $\partial \ln[\mathcal{L}(\beta_n, \alpha_n;\ x_1, x_2, \ldots, x_n)]/\partial\alpha_n = 0$.

$$\ln[\mathcal{L}(\beta_n, \alpha_n;\ x_1, x_2, \ldots, x_n)] = \ln(\beta_n{}^{n}) + \ln(\alpha_n{}^{n}) + \ln\left(\prod_{i=1}^{n} x_i{}^{\alpha_n - 1}\right)$$

$$+\ln\left[\prod_{i=1}^{n} (1 + x_i{}^{\alpha_n})^{-(\beta_n+1)}\right]$$

$$= n\ln(\beta_n) + n\ln(\alpha_n) + (\alpha_n - 1)\sum_{i=1}^{n} \ln x_i - (\beta_n + 1)\sum_{i=1}^{n} \ln(1 + x_i{}^{\alpha_n}). \quad (18)$$

Let $\widehat{\beta_n}$ and $\widehat{\alpha_n}$ be the maximum values for the likelihood and log-likelihood functions, respectively.

When we evaluate $\partial \ln[\mathcal{L}(\beta_n, \alpha_n;\ x_1, x_2, \ldots, x_n)]/\partial\beta_n = 0$ and $\partial \ln[\mathcal{L}(\beta_n, \alpha_n;\ x_1, x_2, \ldots, x_n)]/\partial\alpha_n = 0$ at $\widehat{\beta_n}$ and $\widehat{\alpha_n}$.

we have

$$0 = \frac{\partial \ln[\mathcal{L}(\beta_n, \alpha_n;\ x_1, x_2, \ldots, x_n)]}{\partial\beta_n}\Big|_{\beta_n = \widehat{\beta_n}, \alpha_n = \widehat{\alpha_n}} = \frac{n}{\widehat{\beta_n}} - \sum_{i=1}^{n} \ln\left(1 + x_i^{\widehat{\alpha_n}}\right) \quad (19)$$

$$0 = \frac{\partial \ln[\mathcal{L}(\beta_n, \alpha_n;\ x_1, x_2, \ldots, x_n)]}{\partial\alpha_n}\Big|_{\beta_n = \widehat{\beta_n}, \alpha_n = \widehat{\alpha_n}} = \frac{n}{\widehat{\alpha_n}} + \sum_{i=1}^{n} \ln(x_i) - \left(\widehat{\beta_n} + 1\right)\sum_{i=1}^{n} \frac{x_i^{\widehat{\alpha_n}} \ln(x_i)}{1 + x_i^{\widehat{\alpha_n}}} \quad (14)$$

and we obtain

$$\widehat{\beta_n} = \frac{n}{\sum_{i=1}^{n} \ln\left(1 + x_i^{\widehat{\alpha_n}}\right)} \quad (20)$$

$$\frac{n}{\widehat{\alpha_n}} + \sum_{i=1}^{n} \ln(x_i) - \left[\frac{n}{\sum_{i=1}^{n} \ln\left(1 + x_i^{\widehat{\alpha_n}}\right)} + 1\right]\sum_{i=1}^{n} \frac{x_i^{\widehat{\alpha_n}} \ln(x_i)}{1 + x_i^{\widehat{\alpha_n}}} \quad (21)$$

The highest likelihood estimates for the parameters β_n and α_n are found in the solutions of $\widehat{\beta_n}$ and $\widehat{\alpha_n}$ from Eq (20) and Eq (21), respectively. Nevertheless, there is no analytical solution to these equations. It is more practical to solve these equations numerically using a method like the Newton-Raphson method and it is can easily be implemented using R software. Using the random samples from the proposed model with $\alpha_n = [1, 1.5]$ and $\beta_n = [2, 2]$, estimates results are shown in Table 2.

Table 2: Statistical characteristics of the NBD using simulated data

Statistical characteristics	Estimated values
Shape Parameter	[1.341, 1.665]
Location Parameter	[1.719, 2.478]
Neutrosophic Mean	[0.719, 0.924]
Neutrosophic Variance	[0.903,1.211]
Neutrosophic Mode	[0.159, 0.347]
Neutrosophic median	[0.468, 0.656]

Results in Table 2 show that all statistical characteristics are in interval forms due to indeterminacy assumed in the distributional parameters. Range of indeterminacy in these findings depends on values of imprecision in distributional parameters. If we generate the samples from model with zero indeterminacy in the distributional parameters, then results from the proposed model will assemble with classical model.

6. Real Data Example

In this section, we have implemented our proposed model on malaria cases per 1000 population for Saudi Arabia for the time period ranges from 2005 to 2020. Malaria incident data are publicly available and organized for different countries at [24]. These numbers provide status updates on the prevalence, trend, and hotspots of malaria, which helps governments and health organizations allocate resources effectively. Monitoring malaria cases is important for assessing the impact of control measures such as insecticide-treated nets and antimalarial drugs, and for prioritizing resources in areas where elimination requires intensified intervention. Furthermore, these data are crucial to reaching global health targets such as reducing disease transmission and deaths from malaria, which

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helps to improve public health outcomes globally. Cases of malaria in the kingdom are notable because the Kingdom is particularly vigilant in the management and prevention of the disease. Malaria was endemic in large parts of the country historically. Yet, huge control initiatives emphasizing vector management, community awareness, and robust healthcare infrastructure have resulted in a steep decline in native cases. The majority of malaria cases now reported in Saudi Arabia are imported from high malaria transmission areas. The reported cases for Saudi Arabia are exact numbers; however, we have intentionally created neutrosophic data by using methodology described in [25]. The neutrosophic data are presented in Table 3.

Table 3: Malaria incident per 1000 population at risk for Saudi Arabia

[1.01, 1.585]	[0.288, 1.86]	[0.285, 1.103]	[0.397, 1.368]	[0.535, 1.345]
[0.423, 0.515]	[0.364, 0.692]	[0.871, 0.913]	[0.532, 0.570]	[0.447, 0.465]
[0.935, 0.978]	[0.428, 0.478]	[0.667, 0.687]	[0.563, 0.581]	[0.079, 0.126]
[0.885, 1.037]	[0.746, 0.842]	[0.841, 0.874]	[0.430, 0.449]	[0.888, 0.931]

More naturally, the malaria data involving uncertainties cannot be analyzed using the conventional Burr distribution. The statistical description of the Malaria data using the using the proposed model is provided in Table 4.

Table 4: Statistical analysis of malaria cases per 1000 population at risk

Descriptive measure	Estimated value
$\hat{\alpha}_n$	[2.718, 2.887]
$\hat{\beta}_n$	[1.697, 3.856]
Estimated mean	[0.581, 0.875]
Estimated variance	[0.071, 0.231]
Estimated mode	[0.497, 0.674]
Estimated median	[0.549, 0.789]

Table 4 provides the estimated values in interval forms for commonly used descriptive measures of malaria cases for the period 2001 to 2020. Indeterminacies in the estimated values are due to assumed vagueness in the processing data. Thus, the proposed model can effectively analysis the data when uncertainties are involved.

7. Conclusion

This work has proposed the neutrosophic Burr distribution (NBD), which is an extension of the classical Burr distribution by incorporating imprecision, indeterminacy, and inconsistency to its parameters. This proposed model is able to capture the uncertainty associated with real-world data, as shown under theoretical aspects, parameter estimation, and simulation studies. The ML approach has been used for estimating unknown parameters. Simulation study has performed to for generating the random samples and their subsequent used to find unknown parameters. Simulated results showed that neutrosophic paradigm enhances statistical data interpretation, allowing for interval interpretations of important statistics like mean, variance, mode, and median, thereby tackling the challenge of vague and incomplete information. The flexibility of the new model is illustrated with an application to a real data set. The empirical application to malaria incidence data in Saudi Arabia illustrates the practical utility of NBD in health-related datasets that are often subject to uncertainties. Numerical results reveal that NBD is robust enough to deal with imprecise data such as public health decision-making and resource allocation, unlike classical distributions. In essence, this study provides sufficient evidence of high versatility and reliability of the NBD as a tool for modeling complex phenomena that are common in biological, clinical, reliability or risk management studies.

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