Modeling Extreme Healthcare Costs Using the Neutrosophic Cauchy Distribution

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Abstract

Real data modelling of extreme events, such as rainfall, temperature, financial costs is very important in neutrosophic statistical methods. The Cauchy distribution is one of statistical models used for modelling such extreme events in natural processes. In cases of imprecise data which most often involve vague, incomplete and ambiguous information, standard statistical methods cannot fully describe the spectrum of uncertainty. In this study, we have considered a new Cauchy distribution under neutrosophic context to deal with uncertain data. The proposed neutrosophic Cauchy distribution (NCD) may analysis extreme events data involving incomplete observations. We provide basic mathematical characteristics and important statistical functions of the Cauchy model under neutrosophic framework. A complete procedure of random numbers generation using neutrosophic quantile function is discussed. The unknown parameters of the proposed are estimated using the maximum likelihood approach. Numerical results show that the proposed model adequately fits the data involving extreme and imprecise values. The performance and flexibility of the model are also supported by an application to a real data set.

Keywords: Neutrosophic probability; Neutrosophic density; Cauchy model; Simulation; Estimation

1. Introduction

The Cauchy distribution, also known as the Lorentz distribution in physics, is a continuous probability distribution with distinct characteristics that set it apart from other common distributions [1]. The Cauchy distribution is named after the French mathematician Augustin-Louis Cauchy; it is widely recognized for its "heavy-tailed" nature, which means that it has higher probabilities of extreme values compared to distributions like the normal distribution [2]. This feature makes the Cauchy distribution particularly interesting in statistical theory and applications. A key characteristic of the Cauchy distribution is its lack of defined moments, such as the mean and variance. Unlike the normal distribution, where the mean and variance provide useful summaries of the data, these measures are undefined for the Cauchy distribution because the integral required to compute them diverges. Consequently, the median and mode are used as central measures instead of the mean. The median and mode of the Cauchy distribution both coincide with the location parameter [3]. Another striking property of the Cauchy distribution is its invariance under linear transformations. Specifically, the sum of two independent Cauchy random variables also follows a Cauchy distribution, which is a unique feature not shared by many distributions. This property makes the Cauchy distribution a fundamental example of a "stable distribution [4]. The Cauchy distribution finds applications in various fields due to its heavy-tailed behavior. In physics, it is used to model resonance phenomena and the spectral line shapes in spectroscopy. In finance, the distribution is occasionally employed to model extreme market movements or heavy-tailed returns. Moreover, the distribution has relevance in the field of signal processing, particularly in the analysis of noise with impulsive characteristics. The Cauchy distribution serves as

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a counter example to several statistical principles. For instance, the mean of a sample drawn from a Cauchy distribution does not converge to the true location parameter as the sample size increases, defying the law of large numbers [5]. This makes the distribution an important teaching tool for illustrating the limitations of standard statistical methods.

The Cauchy distribution is a mathematically intriguing and practically useful distribution that challenges conventional statistical assumptions. Its heavy-tailed nature and lack of defined moments require specialized methods for analysis, making it a vital component in the study of robust statistics and extreme value phenomena [6-7]. The Cauchy distribution may not be as widely known as the normal (bell curve) distribution, but it plays an important role in several real-world situations [8]. This distribution has a unique shape: it has a tall peak in the centre and very long "tails," meaning it allows extreme values to occur more frequently than in a normal distribution.

Neutrosophic statistics extends classical and fuzzy statistics by incorporating indeterminacy alongside certainty and uncertainty, making it essential for analysing complex, ambiguous, and incomplete data [9-11]. Unlike classical statistics, which relies on precise probabilities, neutrosophic statistics allows for flexibility by representing data and outcomes as intervals or ranges of truth, falsity, and indeterminacy [12-16]. This approach is particularly important in real-world scenarios where data is noisy, conflicting, or lacks complete reliability, such as in medical diagnoses, engineering reliability studies, and socio-economic research. By capturing uncertainty and vagueness more comprehensively, neutrosophic statistics enables robust decision-making in environments characterized by high complexity [17]. Furthermore, it offers innovative tools for understanding systems that traditional statistical methods struggle with, including multi-criteria decision-making, risk assessment, and modelling dynamic, imprecise phenomena [18-19]. The integration of neutrosophic principles in statistics is crucial for advancing analytical frameworks to meet the challenges of real-world data-driven problems.

The Cauchy distribution is crucial in modelling phenomena with heavy tails and extreme outliers, such as resonance in physics, financial market fluctuations, and impulsive noise in signal processing. However, real-world data often involves imprecision, ambiguity, or incomplete information, which classical Cauchy models cannot capture effectively. A neutrosophic version of the Cauchy distribution addresses this limitation by incorporating truth, indeterminacy, and falsity components into its framework. This allows the distribution to handle uncertain, inconsistent, or vague data more robustly, making it ideal for complex systems where both randomness and indeterminacy coexist, such as in medical diagnostics, risk assessment, or decision-making.

This paper is organized as follows: Section 2 reviews the key properties of the classical Cauchy distribution. Section 3 presents the proposed NCD, extending the classical model to include indeterminacy. Section 4 discusses parameter estimation techniques for the neutrosophic Cauchy distribution. Section 5 explores its real-world applications and properties, while Section 6 concludes the study, summarizing key findings and highlighting its significance.

2. Preliminary

The Cauchy distribution is continuous (heavy-tail and sharp peak) in mathematical structure. It is used in statistical physics, as well as in signal processing and an example of a distribution that has mean and variance undefined. The Cauchy distribution has the following probability density function (pdf) and cumulative distribution function (cdf):

$$f(x; x_0, \gamma) = \frac{1}{\pi\gamma} \cdot \frac{\gamma^2}{(x-x_0)^2+\gamma^2} \tag{1}$$

$$F(x; x_0, \gamma) = \frac{1}{\pi} \arctan\left(\frac{x-x_0}{\gamma}\right) + \frac{1}{2} \tag{2}$$

Where the location parameter x_0 and scale γ parameter determines the shape and position of a probability distribution. The location parameter moves the distribution along the horizontal axis, thus representing a central value of the distribution, often its mode or median. In the case of a Cauchy distribution, x_0 the peak of the distribution since it is symmetric. The scale parameter, on the contrary, dictates how spread or how wide your distribution is. That dictates how tall the peak is and how "fat" are the tails. With larger γ, the distribution becomes wider and flatter, whereas with smaller values a sharper peak forms. Available parameters for the underlying distribution offer flexibility to model data by shifting and scaling to fit the real world scenarios. The classical structures of pdf and cdf are depicted in Figure 1.

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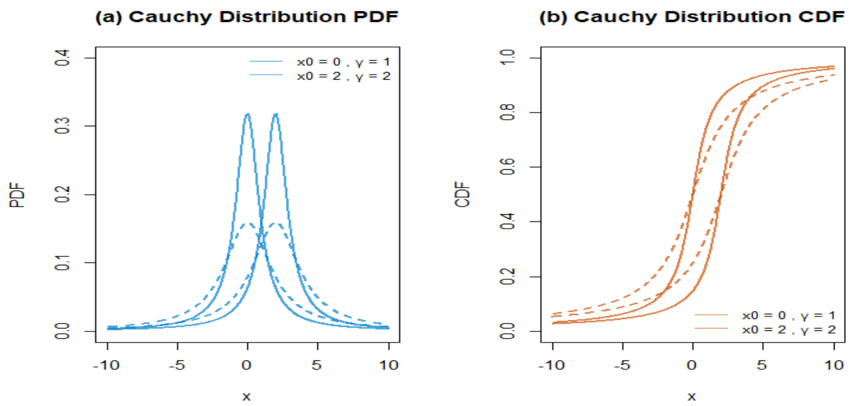

Figure 1. The pdf and cdf curves of the classical Cauchy distribution

Figure 1 shows the pdf and cdf curves of the classical model of the Cauchy distribution. Understanding the pdf and cdf of Cauchy distribution helps you to grasp how this distribution behaves. It describes the shape of your distribution, with a very sharp peak located at the location parameter and heavy tails involved in scale parameter. Where location parameter is determining the center of distribution. The scale parameter is affecting spread fatness tails and how quickly it tapers off. Though the pdf does convey information about how likely various values are, we cannot take it to mean that some outcome is normally more or less probable than another since neither finite mean nor variance exist. Conversely, the cdf indicates a cumulative probability: it specifies the likelihood that the random variable will take a value less than or equal to some given value. It confirms the symmetry of distribution with respect to location parameter and control the steepness of curve by passing scale parameter. The cdf provides a tool for calculating probabilities and quantiles, which will be useful for us. Hence, we can get a better idea of the Cauchy distribution by looking at both pdf and cdf along with its properties through parameters. The unique characteristics of the Cauchy distribution is known from its undefined mean and variance characteristics. Due to this property, the higher moments (e.g., skewness, kurtosis) are undefined as well. If we express the distribution so that the location parameter is x_0, then the mode and median are equal to this value.

The Cauchy distribution also have very interesting properties. For example if $X_1 \sim \text{Cauchy}(x_1, \gamma_1)$ and $X_2 \sim \text{Cauchy}(x_2, \gamma_2)$ then, $X_1 + X_2 \sim \text{Cauchy}(x_1 + x_2, \gamma_1 + \gamma_2)$. This property is called closure under addition. Likewise, Cauchy distribution has scale invariance property i.e., $X \sim \text{Cauchy}(x_0, \gamma)$ then for any constants a and b, $aX + b \sim \text{Cauchy}(ax_0 + b, |a|\gamma)$. Cauchy distribution is also known as stable distribution with the characteristic function $\phi(t) = \exp(itx_0 - \gamma|t|)$.

3. Proposed Model

This section presents mathematical notions of the proposed model and organizes it into a logical structure. The subsequent notions establish a relationship between the proposed model, and it uses in applied statistical methodology. A random variable $\mathcal{X}_N$ has a NCD with parameters λ_n and δ_n if its probability density function (pdf) is defined as;

$$f_n(\mathcal{X}) = \begin{cases} \frac{\delta_n}{\pi} \cdot \left(\frac{1}{\delta_n^2 + (x - \lambda_n)^2}\right), -\infty < x < \infty, -\infty < \lambda_n < \infty, \delta_n > 0 \\ 0, \qquad Otherwise \end{cases} \tag{3}$$

The parameters $\lambda_n = [\lambda_l, \lambda_u]$ and $\delta_n = [\delta_l, \delta_u]$ are neutrosophic location and scale parameters of NCD respectively. It should be highlighted that the proposed model is different from the current cauchy model framework, where the parameters are clearly specified. When the indeterminate component of the proposed model is zero, i.e., the suggested model equals the classical model. In statistics and probability theory, the probability density function (PDF) is an important function that gives the relative likelihood of a continuous random variable to take on a particular value. This allows us to get insights on how the data is distributed, gives us the ability to calculate probabilities over the intervals or supports modeling of any physical phenomena that aids decision-making and statistical inference. That really is a basic function of any distribution. The pdf function of the proposed model with different parameter setting is shown in Figure 2.

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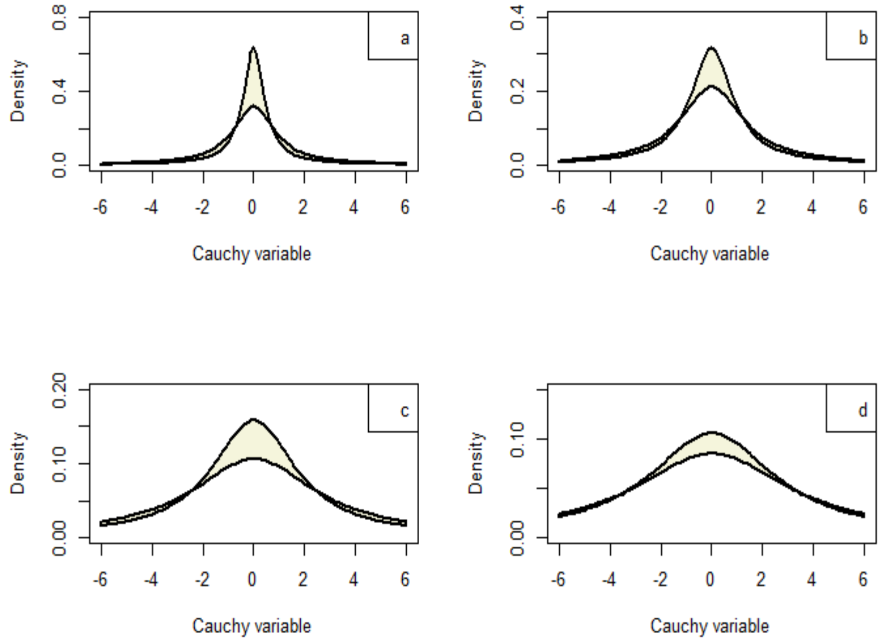

Figure 2. PDF plots of the proposed NCD with different neutrosophic scale parameter (a) $\delta_n = [0.5, 1]$ (b) $\delta_n = [1, 1.5]$ (c) $\delta_n = [2, 3]$ and (d) $\delta_n = [3, 3.75]$

Figure 2 illustrates the graphic display of neutrosophic form of the proposed model. The neutrosophic pdf has three components truth, indeterminacy, and falsity, which are not traditional in a graphical structure. This uncertainty, this imprecision of the data can be visualized as several overlapping regions. Since this is all about relationships between the components, each curve represents ranges, not specific values, allowing us to naturally build complex interactions, which are also important in practice for modeling real world data. Similarly, the cumulative function is another important function of the NCD. This is used to determine the likelihood that the number of operational items will fail inside or at the designated time. The following is the expression for the cumulative distribution function of the NCD.

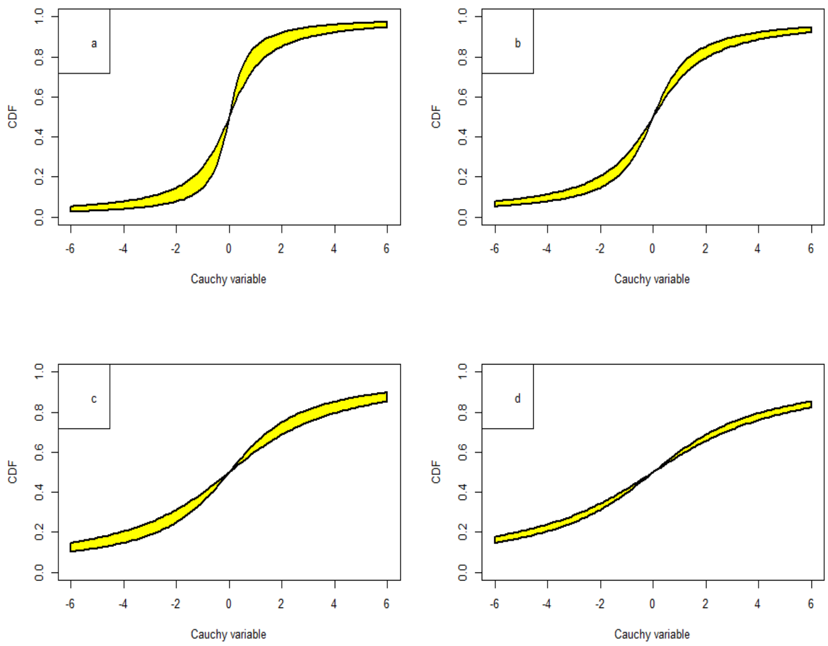

Figure 3. CDF plots of the proposed NCD with different neutrosophic scale parameter (a) $\delta_n = [0.5, 1]$ (b) $\delta_n = [1, 1.5]$ (c) $\delta_n = [2, 3]$ and (d) $\delta_n = [3, 3.75]$

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The other related function with pdf is cdf which can be written as:

$$\mathcal{F}_n(\mathcal{X}) = \frac{1}{\pi} arctan\left(\frac{x-\lambda_n}{\delta_n}\right) + \frac{1}{2} \quad (4)$$

Figure 3 shows the cdf plot of the proposed distribution with different parameters setting. The cdf plot gives the probability of a random variable being less than or equal to a given point. This gives a complete picture of the distribution allowing for probability calculations, data set comparisons, and statistical methods like hypothesis testing. Shaded region in the plot represents the neutrosophic area.

The major statistical properties of the proposed model can be established in forms of the following theorems:

Definition 1. The median of the CD_N is λ_n

Proof: If $\mathcal{M}$ represents the median of CD_N, then

$$\int_{-\infty}^{\mathcal{M}} f_n(\mathcal{X})dx = \frac{1}{2} \quad (5)$$

Let $\frac{x-\lambda_n}{\delta_n} = Z \Rightarrow dZ = \delta_n,$

$x = -\infty \Rightarrow \mathrm{Z} = -\infty$ and $x = \mathcal{M} \Rightarrow \mathrm{Z} = \frac{\mathcal{M}-\lambda_n}{\delta_n}$ we have,

$$\frac{1}{\pi}\int_{-\infty}^{\frac{\mathcal{M}-\lambda_n}{\delta_n}} \frac{dZ}{1+Z^2} = \frac{1}{2}$$

$$\Rightarrow \frac{1}{\pi}[\tan^{-1}\mathrm{z}]_{-\infty}^{\frac{\mathcal{M}-\lambda_n}{\delta_n}} = \frac{1}{2}$$

$$\Rightarrow \left[\left(\frac{1}{\pi}[\tan^{-1}\mathrm{z}]_{-\infty}^{\frac{\mathcal{M}-\lambda_l}{\delta_l}} = \frac{1}{2}\right), \left(\frac{1}{\pi}[\tan^{-1}\mathrm{z}]_{-\infty}^{\frac{\mathcal{M}-\lambda_u}{\delta_u}} = \frac{1}{2}\right)\right]$$

$$\Rightarrow \frac{1}{\pi}\left[\tan^{-1}\frac{\mathcal{M}-\lambda_n}{\delta_n}\right] = 0$$

$$\Rightarrow \frac{\mathcal{M}-\lambda_n}{\delta_n} = 0 \Rightarrow \mathcal{M} = \lambda_n \quad (6)$$

Question (4) further yeilded

$$\frac{\mathcal{M}-\lambda_l}{\delta_l} = 0 \Rightarrow \mathcal{M} = \lambda_l,$$

And

$$\frac{\mathcal{M}-\lambda_u}{\delta_u} = 0 \Rightarrow \mathcal{M} = \lambda_u$$

So,

$[\lambda_l, , \lambda_u,] = \lambda_n$, hence proved.

Definition 2. The mode of CD_N is λ_n.

Proof: By definition the Cauchy random variable probability density function is:

$$f_n(\mathcal{X}; \lambda_n, \delta_n) = \frac{\delta_n}{\pi}.\left(\frac{1}{\delta_n^2+(x-\lambda_n)^2}\right) \quad (7)$$

Taking log on both side (5), we have

$$\log f_n(\mathcal{X}; \lambda_n, \delta_n) = log\left(\frac{\delta_n}{\pi}\right) - log(\delta_n^2 + (x-\lambda_n)^2) \quad (8)$$

Differentiate w.r.t x, and equating to zero, we get

$$\frac{\partial \log f_n(\mathcal{X}; \lambda_n, \delta_n)}{\partial x} = \frac{-2(x-\lambda_n)}{{\delta_n}^2+(x-\lambda_n)^2} = 0$$

$$\Rightarrow (x - \lambda_n) = 0$$

$$\Rightarrow x = \lambda_n$$

Where

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$\lambda_n = [\lambda_l , \lambda_u]$, hence proved.

Definition 3: The quartiles and quartile derivation of CD_N are δ_n.

Proof: Assuming that Q_1represents the distribution first quartile, then

$$\mathcal{P}(\mathcal{X} \leq \mathrm{Q}_1) = \int_{-\infty}^{\mathrm{Q}_1} f_n(\mathcal{X})dx = \frac{1}{4}. \quad (9)$$

Let $\frac{x-\lambda_n}{\delta_n} = Z \Rightarrow dZ = \delta_n$,

$x = -\infty \Rightarrow \mathrm{Z} = -\infty$ and $x = \mathrm{Q}_1 \Rightarrow \mathrm{Z} = \frac{\mathrm{Q}_1-\lambda_n}{\delta_n}$. we have,

$$\int_{-\infty}^{\mathrm{Q}_1} f_n(\mathcal{X})dx = \frac{1}{4} \Rightarrow \frac{1}{\pi}\int_{-\infty}^{\frac{\mathrm{Q}_1-\lambda_n}{\delta_n}} \frac{dZ}{1+Z^2} = \frac{1}{4}$$

$$\Rightarrow \left[\left(\frac{1}{\pi}[\tan^{-1}\mathrm{z}]_{-\infty}^{\frac{\mathrm{Q}_1-\lambda_l}{\delta_l}} = \frac{1}{4}\right), \left(\frac{1}{\pi}[\tan^{-1}\mathrm{z}]_{-\infty}^{\frac{\mathrm{Q}_1-\lambda_u}{\delta_u}} = \frac{1}{4}\right)\right]$$

$$\Rightarrow \tan^{-1}\frac{\mathrm{Q}_1-\lambda_n}{\delta_n} = -\frac{\pi}{4}$$

$$\Rightarrow \frac{\mathrm{Q}_1-\lambda_n}{\delta_n} = tan\left(-\frac{\pi}{4}\right) = -1$$

$$\Rightarrow \mathrm{Q}_1 = \lambda_n - \delta_n.$$

Therefore, for the CD_N, first quartile is $\mathrm{Q}_1 = [\lambda_n - \delta_n] = [(\lambda_l - \delta_l), (\lambda_u - \delta_u)]$

Let Q_3 represent the second quartile of the CD_N.

$$\mathcal{P}(\mathcal{X} \leq \mathrm{Q}_3) = \int_{-\infty}^{\mathrm{Q}_3} f_n(\mathcal{X})dx = \frac{3}{4}. \quad (10)$$

Let $\frac{x-\lambda_n}{\delta_n} = Z \Rightarrow dZ = \delta_n$,

$x = -\infty \Rightarrow \mathrm{Z} = -\infty$ and $x = \mathrm{Q}_3 \Rightarrow \mathrm{Z} = \frac{\mathrm{Q}_3-\lambda_n}{\delta_n}$. we have,

$$\int_{-\infty}^{\mathrm{Q}_3} f_n(\mathcal{X})dx = \frac{3}{4} \Rightarrow \frac{1}{\pi}\int_{-\infty}^{\frac{\mathrm{Q}_3-\lambda_n}{\delta_n}} \frac{dZ}{1+Z^2} = \frac{3}{4}$$

$$\Rightarrow \left[\left(\frac{1}{\pi}[\tan^{-1}\mathrm{z}]_{-\infty}^{\frac{\mathrm{Q}_3-\lambda_l}{\delta_l}} = \frac{3}{4}\right), \left(\frac{1}{\pi}[\tan^{-1}\mathrm{z}]_{-\infty}^{\frac{\mathrm{Q}_3-\lambda_u}{\delta_u}} = \frac{3}{4}\right)\right]$$

$$\Rightarrow \tan^{-1}\frac{\mathrm{Q}_3-\lambda_n}{\delta_n} = \frac{\pi}{4}$$

$$\Rightarrow \frac{\mathrm{Q}_1-\lambda_n}{\delta_n} = tan\left(\frac{\pi}{4}\right) = 1$$

$$\Rightarrow \mathrm{Q}_3 = \lambda_n + \delta_n.$$

Therefore, for the CD_N, third quartile is $\mathrm{Q}_3 = [\lambda_n + \delta_n] = [(\lambda_l + \delta_l), (\lambda_u + \delta_u)]$

Thus, the Quartile Deviation of the CD_N is,

$$Q.\mathrm{D} = \frac{\mathrm{Q}_3-\mathrm{Q}_1}{2} = \frac{\lambda_n+\delta_n-(\lambda_n-\delta_n)}{2} = \delta_n$$

Therefore, $\delta_n = [\delta_l , \delta_u]$. Hence proved.

Definition 4: The neutrosophic Cauchy random variable distribution function is $\frac{1}{\pi}tan^{-1}\left(\frac{\mathcal{X}-\lambda_n}{\delta_n}\right) + \frac{1}{2}$

Proof: By definition the neurosophic Cauchy random variable function is:

$$\mathcal{F}(\mathcal{X}) = \mathcal{P}(\mathcal{X} \leq x) = \int_{-\infty}^{x} f_n(\mathcal{X})dx. \quad (11)$$

$$\mathcal{F}(\mathcal{X}) = \mathcal{P}(\mathcal{X} \leq x) = \int_{-\infty}^{x} \frac{\delta_n}{\pi}.\left(\frac{1}{\delta_n^2+(x-\lambda_n)^2}\right)dx.$$

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Let $\frac{x-\lambda_n}{\delta_n} = Z \Rightarrow dZ = \delta_n$,

$x = -\infty \Rightarrow \mathrm{Z} = -\infty$ and $x = \mathcal{M} \Rightarrow \mathrm{Z} = \frac{\mathcal{M}-\lambda_n}{\delta_n}$ we have,

The CD_N function is

$$\mathcal{F}(\mathcal{X}) = \frac{1}{\pi} tan^{-1}\left(\frac{\mathcal{X}-\lambda_n}{\delta_n}\right) + \frac{1}{2}$$

Where $\frac{1}{\pi} tan^{-1}\left(\frac{\mathcal{X}-\lambda_n}{\delta_n}\right) + \frac{1}{2}$

$\frac{1}{\pi} tan^{-1}\left(\frac{\mathcal{X}-\lambda_n}{\delta_n}\right) + \frac{1}{2} = \left[\frac{1}{\pi} tan^{-1}\left(\frac{\mathcal{X}-\lambda_l}{\delta_l}\right) + \frac{1}{2}, \frac{1}{\pi} tan^{-1}\left(\frac{\mathcal{X}-\lambda_u}{\delta_u}\right) + \frac{1}{2}\right]$. Hence proved.

Likewise, other statistical properties under the neutrosophic framework can be established.

4. Quantile Function and Sample Estimation

The inverse cdf method, or quantile function method, is a basic random number generation technique from some probability distribution. Therefore, there are two steps: first we generate from a uniform random variable uniform distribution, and then applying the inverse of the cdf to transform that into a random variable following any distribution. It is a common approach because it applies broadly: many distributions have a well-defined quantile function. Neutrosophic theory is a recent development which includes uncertainty, indeterminacy and imprecision and the neutrosophic quantile function thus can be viewed as an extension of classical quantile function. It specifies ranges on the quantiles, which means that for any specific probability point we could have multiple possible values due to uncertainty of the data or the model. When the generation of random isolated neutrosophic numbers using a neutrosophic quantile function requires uniform random variables in [0, 1] to be sampled and substituted into the neutrosophic quantile function; intervals or distributions of potential values are then obtained. It enables the generation of random numbers that are more representative of real-world uncertainty, and specifically suited for complex systems and decision-making problems. In context of neutrosophic framework inverse cdf of the proposed model can be written as:

$$Q_n(p) = \lambda_n + \delta_n \tan\left[\pi\left(p - \frac{1}{2}\right)\right] \qquad (12)$$

where $p \in [0, 1]$.

We can generate random numbers from the proposed model by providing different values of neutrosophic parameters. They can produce random numbers to identify unique statistical characteristics of proposed distributions. These random numbers could also be used to simulate different scenarios and evaluate the ability of the estimators when faced with new situations. In order to obtain the random numbers of proposed model, we consider that all neutrosophic parameters assuming different values. The above random samples have been created using a program in R. The neutrosophic parameters $\lambda_n = [0.5, 1]$ and $\delta_n = [1, 1]$ have been used to generate random numbers which are presented in Table 1.

Table 1: Random numbers generation from the proposed model

Random numbers				
[-0.287, 0.212]	[1.775, 2.275]	[0.205, 0.705]	[3.097, 3.597]	[5.784, 6.284]
[-6.439, -5.939]	[0.588, 1.088]	[3.345, 3.845]	[0.663, 1.16]	[0.362, 0.862]
[7.828, 8.328]	[0.352, 0.852]	[1.123, 1.623]	[0.732,1.232]	[-2.484, -1.984]

As we can see in Table 1, a number of sixteen neutrosophic numbers are randomly taken out from the proposed model, along with their lower and upper bounds. These indeterminacies are present because the value for the location parameter is imprecise. Similarly, if we take the proposed model with crisp values i.e, when the location and scale parameters have both upper and lower values that are equal, our results correspond to the classical model. Maximum likelihood estimation is a common technique used for estimating parameters of a distribution. The likelihood function of the neutrosophic Cauchy distribution for independent and identically distributed (i. i. d) random neutrosophic samples is defined as:

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$$\log L\,(\delta_n, \lambda_n) = -n \log \pi - n \log \lambda_n - \sum_{i=1}^{n} \log\left(1 + \left(\frac{x_i - \lambda_n}{\delta_n}\right)^2\right) \quad (13)$$

The log-likelihood function will be maximized to provide MLE estimates of λ_n and δ_n This usually means computing partial derivatives with respect to λ_n and δ_n, setting them equal to zero, and solving the resulting equations.

The location parameter λ_n is usually taken to be the median of the data, because the Cauchy distribution is symmetric around its location parameter.

The scale parameter δ_n can be estimated using an IQR-based method or as the mean absolute deviation about the median.

In fact, the Cauchy distribution does not have finite moments; MLE will often be practically intractable from an analytic standpoint. Thus, one must turn to numerical optimization methods to estimate these parameters. In some cases, other techniques like method of moments or even robust estimation approaches are implicated in determining Cauchy distribution parameters.Now use the neutrosophic random numbers we can estimate the parameters of the proposed model. Estimated location and scale parameters of the model using different random samples the NCD with $\lambda_n = [0.5, 0.5]$ and $\delta_n = [2, 3]$ are shown in Table 2.

Table 2: Estimated neutrosophic parameters from the proposed model

Random samples	Estimated λ_n	Estimated δ_n
50	[0.59, 0.64]	[1.94, 2.91]
100	[0.58, 0.62]	[2.05, 3.08]
1000	[0.52, 0.54]	[2.03, 3.04]
10000	[0.50, 0.51]	[2.04, 3.06]

Results in Table 2 indicate that due to indeterminacy in the assumed parameter values, estimated values are also imprecise. Additionally, as random sample size grows, estimation becomes more reliable. Results that are so more reliable are expected with larger sample sizes.

5. Real Data Application

In this section, we employ our proposed model to real data set of tuberculosis (TB) index collected for Saudia Arabia at specific year per 100000 population. This index is an important epidemiological statistic organized and monitored by World Health Organization (WHO) [20] to compare and monitor TB cases across the countries. This epidemiological rate of tuberculosis is important public health indicators for assessing trends in disease burden, epidemiological spread, and the impact of TB control programs. The data yield insights that are important for policymakers in understanding the social groups and regions that are vulnerable to shocks and in designing targeted interventions and better allocating resources. When rates rise, this index acts as an early warning system, drawing attention to gaps in the underlying health system. TB affects national GDP both directly through healthcare costs and indirectly through loss of productivity, the working population at work and the increased dependency. High TB burden countries have higher economic strain whilst healthy workforces and low public health expense have better contributions to economic stability in high TB control strategies. TB data provides health policy-makers essential baseline information for devising relevant health plans; investments in TB provide significant economic benefits through improvements in productivity and national income generation whilst decreasing poverty. Sustained efforts in disease control are critical for population health and economic development (TB incidence vs GDP. The TB incidence rates are available as precise values. However, exact rates may not reflect the true figures due to reporting delays, errors in data collection, or modeling assumptions. Therefore, these values should be considered approximate. For example, if the TB incidence rate is reported as 5 cases per 100,000 population, the true value could be represented as an interval, such as [4.5,5.5][4.5, 5.5] or [5.5,6][5.5, 6]. The lower bound of the interval reflects the minimum estimated rate, while the upper bound accounts for possible hidden or under-diagnosed cases. The neutrosophic TB incidence rate for Saudi Arabia is presented in Table 3.

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Table 3: TB incidence rate for Saudia Arabia for a specific period 2000-2015

[23, 26]	[21, 24]	[21, 23]	[19, 20]	[19,21]
[19, 221]	[19,21]	[9, 21]	[18,20]	[11, 12]
[17, 18]	[15, 17]	[13, 15]	[12, 14]	[12, 13]

Results in Table 3 indicate that the TB incidence rate is not so high. This reflects the kingdom's ongoing efforts in TB treatments and control programs and effective public health initiatives. However, the classical model of the Cauchy distribution cannot be employed to analysis such types of neutrosophic form data. In contrast, the proposed can easily be implemented to such data and its results converge to classical form under zero indeterminacy. The estimated neutrosophic scale and location parameters of the data using proposed model are provided in Table 4.

Table 4: Estimated values for neutrosophic TB data using proposed model

Parameter	Estimated values
Scale	[2.14, 2.46]
Location	[18.18, 20.31]
Median	[18.18, 20.31]
Mode	[18.18, 20.31]

Table 5 shows that the parameter estimates in interval forms for all results, indicating the vagueness of underlying data. Such approximate estimation offers a relaxed framework for defining the statistical properties of the TB data. In addition, certain values such as the median and mode coincide due to the identical results observed within the neutrosophic framework. The neutrosophic model has been shown to be more general and can be tailored more easily than existing Cauchy model by accommodating uncertainty.

6. Conclusions

The Neutrosophic Cauchy Distribution (NCD) for handling neutrosophic data has been proposed. The proposed model overcomes the modeling problems dealing with extreme events in uncertain situations with ambiguous information or incomplete data. The NCD broadens the use of classical Cauchy model using neutrosophic principles so can be applied to real-life applications of extreme events. Statistical characteristics of the NCD have been developed under the neutrosophic framework. Neutrosophic random number generation procedure based on the neutrosophic quantile function has been provided. The MLE approach has been adopted to determine the unknown parameters of the proposed model. Numerical results indicated that larger sample sizes offer reliable estimates for unknown parameters. Numerical simulations and an illustrative application to a real dataset of tuberculosis disease further confirmed the flexibility of the proposed model and its ability to fit data with extremes and imprecise values. The numerical findings implied that the proposed model can be used for analyzing uncertain extreme events effectively.

This work provides the groundwork for future research to investigate the adaptation of the NCD to more generic settings, as well as to compare its performance with other neutrosophic probabilistic models for fitting healthcare datasets with high levels of uncertainty and complexity.

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