Neutrosophic Ideal of a Near Algebra

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Abstract

This article introduces the idea of neutrosophic ideal of a near algebra and provides a definition and example. A few fundamental features related to this approach are also explored. We also present the topics neutrosophic near algebra homomorphism, kernel of a neutrosophic near algebra and coset of a neutrosophic ideal of a near algebra. It is been briefed with the appropriate definitions and theorems on it. It is been proved that sum of the right neutrosophic ideal of a near algebra is also a right neutrosophic ideal of a near algebra over a neutrosophic field.

Keywords: Field; Neutrosophic set; Near algebra; Ideal; homomorphism

1. Introduction

G. Pilz [11] introduced the idea of a near ring. Additionally known as a near ring is the algebraic system with two binary operations gratifying the axioms of the ring and the suitable anomaly of a single distributive law. Field is admitted as a right operator domain by a near ring, which is called a near algebra. H. Brown [6] introduced near algebra. In P. Jordan's quantum mechanical theory, the operators only exactly form a near algebra. In this study of near algebra, it is so fascinating as an axiomatic question and for concrete reasons. L.A. Zadeh [15] first conceptualized fuzzy sets in 1965. Neutrosophic set is the supplement of fuzzy set theory. The new branch of philosophy called neutrosophy is devoted to the study of neutralities and their nature, extent, and interactions with extra ideational spectrum. By giving full credit to the notions of neutrosophic logic and neutrosophic set (NS) in 1995, F. Smarandache [12] paved the way for the entire world. F. Smarandache embodied all sets, which included interval fuzzy sets, intuitionist fuzzy sets, normal fuzzy sets, and classical sets respectively. The fuzzy logic extension familiar as neutrosophic logic incorporates indeterminacy. Neutrosophic logic awaits the consumption of all three quantities: the rate of falsity in a subset (F), the degree of uncertainty in a subset (I), and the amount of authenticity in a set of which all the essentials are enclosed in an alternative set (T). Here, T, I, and F are authenticated or non-authenticate subsets of the non-standard unit interlude]- 0,1+ [. Neutrosophic logic is used in many fields, including operations research, engineering, science, game theory, information technology, law, and politics. Kandaswamy and F. Smarandache, in 2006, introduced themselves to neutrosophic algebraic structures. Kandaswamy and F. Smarandache [13,14] were investigated fuzzy algebra and basic neutrosophic algebraic structures and their applications to fuzzy and neutrosophic models. A prearranged algebraic syntax $(Y,*)$ and a innovative algebraic syntax $(Y(I),*) =< Y, I >$ were fashioned by carrying the indeterminate piece I with the elements of the pre-arranged syntax. Smarandache and Kandaswamy jointly constructed few of the neutrosophic algebraic syntaxes because of the indeterminant quality I, which is that $I.I = I^2 = I$. A.A.A. Agboola [1,2,3,4] investigated the concepts of neutrosophic groups (NG) and semigroups, neutrosophic ring (NR), neutrosophic field (NF), neutrosophic linear space (NLS), and neutrosophic near ring. Neutrosophic sets have numerous applications in quantum mechanics, neural networks, developmental programming, artificial intelligence, and neutrosophic

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evolving systems. There are numerous uses for neutrosophic logic in the fields of science, engineering, IT, law, politics, and economics, among others. T. Nagaiah et al. [10] were investigated neutrosophic algebra. Bhurgula Harika et al. [7,8] were investigated neutrosophic near algebra and hybrid near algebra. P. Narasimha Swamy et al. [9] were developed hybrid coset of a near ring. L. Bhaskar et al. [5] were developed Vague Γ-Near Algebra over Vague Field. The idea of a neutrosophic ideal of a near algebra over a neutrosophic field is included in the current inquiry. Complete paper, Y means a (right) near algebra (NA) over a field X.

2. Preliminaries

We revisit few of the fundamental definitions of NS in this part.

Definition 2.1: [2] Let $(G,*)$ be any group. The NG is generated by I and G under $*$ defined by

$N(G) = \{\langle G \cup I\rangle, *\}$. It is being noted that in general, a NG $(N(G),*)$ is not a group. Yet, every additive NG $(N(G),+)$ is a group.

Definition 2.2: [2] Let $(X,+,.)$ be any field and let $X(I) = \langle X \cup I\rangle$ be a NS created by X and I.

Then $(X(I),+,.)$ is entitled a NF. The zero element $0 \in X$ is denoted by $0 + 0I$ in $X(I)$ and $1 \in X$ is denoted by $1 + 0I$ in $X(I)$.

Definition 2.3: [1] Let $(N,+,.)$ be any near ring. The triple $(N(I),+,.)$ is called a NNR.

For all $\varrho + p = (g\,, lI) + (r, mI) = (g + r, (l + m)I)$

$$-\varrho = -(g\,, lI) = (-g, -lI)$$

$$\varrho.p = (g\,, lI).(r, mI) = (gr\,, (gm + lr + lm)I).$$

Definition 2.4: [2] Let V be any linear space over a field X and let $V(I) = \langle V \cup I\rangle$ be a NS generated by V and I. The set $V(I)$ is called a weak NLS over a field X. If $V(I)$ is a NLS over a NF $X(I)$, then $V(I)$ is called a strong neutrosophic linear space. The elements of $V(I)$ are called neutrosophic vectors and elements of $X(I)$ are called neutrosophic scalars.

If $\chi = h + lI, \varsigma = r + mI$ where h, l, r, m are vectors in V and $d = d_1 + d_2 I$ where d_1, d_2 are scalars in X, and $\chi + \varsigma = (g + lI) + (r + mI) = (\,g + r) + (l + m)I$,

$$d\varsigma = (d_1 + d_2 I)(\,g + lI) = d_1 g + (d_1 l + d_2 g + d_2 l)I.$$

Definition 2.5: A NA Y over a field X is a linear space Y over X on which multiplication is defined as

(i) $(Y\,,.)$ is a semi group

(ii) $(s + l).v = s.v + l.v\ \forall\, s, l, v \in Y$

(iii) $\lambda(sl) = (\lambda s)l\ \ \forall\, \lambda \in X, s, l \in Y.$

Definition 2.6: [7] A NNA $Y(I)$ over a N F $X(I)$ is a linear space $Y(I)$ over NF $X(I)$ on which multiplication is defined as

(i) $(Y(I)\,,.)$ is a semi group
$(ii)$$(s + l).v = s.v + l.v\ \forall\, s, l, v \in Y(I)$
$(iii)$$\lambda(sl) = (\lambda s)l\ \ \forall\, \lambda \in X(I),\ s, l \in Y(I).$

The elements of $Y(I)$ are called neutrosophic vectors and the elements of $X(I)$ are called neutrosophic scalars.

3. Main Results

Neutrosophic ideal of a near algebra (NINA), neutrosophic near algebra homomorphism (NNAH) and Kernal of a NNA are revealed in this section.

Definition 3.1: A non-empty subset $\mathfrak{S}(I)$ of a NNA $Y(I)$ over a NF $X(I)$ is called NINA if

(i) $\mathfrak{S}(I)$ is a NLS of $Y(I)$,

(ii) $p\xi \in \mathfrak{S}(I)\ \forall\, p \in \mathfrak{S}(I), \xi \in Y(I)$,

(iii) $\varrho(\xi + p) - \varrho\xi \in \mathfrak{S}(I)\ \forall\, \xi, \varrho \in Y(I), p \in \mathfrak{S}(I)$.

If $\mathfrak{S}(I)$ satisfies (i) and (ii) then $\mathfrak{S}(I)$ is called a right NINA over a NF $X(I)$.

If $\mathfrak{S}(I)$ satisfies (i) and (iii) then $\mathfrak{S}(I)$ is called a left NINA over a NF $X(I)$.

Example 3.2: Let $X(I) = \{0,1\}_{\oplus_2, \otimes_2}$ be a NF and let $Y(I) = \{0, r_\alpha, s_\alpha, t_\alpha\}$ be a set with two binary operations $' + '$ and $' . '$ as follows:

$+$	0	r_α	s_α	t_α
0	0	r_α	s_α	t_α
r_α	r_α	0	t_α	s_α
s_α	s_α	t_α	0	r_α
t_α	t_α	s_α	r_α	0

$.$	0	r_α	s_α	t_α
0	0	0	0	0
r_α	r_α	r_α	r_α	r_α
s_α	s_α	s_α	s_α	s_α
t_α	t_α	t_α	t_α	t_α

Clearly, $Y(I)$ forms a NNA over the NF $X(I)$. Let $\mathfrak{S}(I) = \{0, r_\alpha\} \subseteq Y(I)$. This implies that $\mathfrak{S}(I)$ is NINA of $Y(I)$ over the NF.

Theorem 3.3: If $\mathfrak{S}_1(I)$ and $\mathfrak{S}_2(I)$ are NINA $Y(I)$ over a NF $X(I)$, then $\mathfrak{S}_1(I) \cap \mathfrak{S}_2(I)$ is also a NINA $Y(I)$ over the NF $X(I)$.

Proof: Given $\mathfrak{S}_1(I)$ and $\mathfrak{S}_2(I)$ are NINA $Y(I)$ over the NF $X(I)$. Then $\mathfrak{S}_1(I)$ and $\mathfrak{S}_2(I)$ are subspaces of the NLS of $Y(I)$. So $\mathfrak{S}_1(I) \cap \mathfrak{S}_2(I)$ is a NLS of $Y(I)$ over the NF $X(I)$. Let $\xi, \varrho \in Y(I)$ and $p \in \mathfrak{S}_1(I) \cap \mathfrak{S}_2(I)$. Then $p \in \mathfrak{S}_1(I)$ and $p \in \mathfrak{S}_2(I)$. Since $\mathfrak{S}_1(I)$and $\mathfrak{S}_2(I)$ are NINA of $Y(I)$, we get $p\xi \in \mathfrak{S}_1(I), \varrho(\xi + p) - \varrho\xi \in \mathfrak{S}_1(I)$ and $p\xi \in \mathfrak{S}_2(I), \varrho(\xi + p) - \varrho\xi \in \mathfrak{S}_2(I)$. Thus, $p\xi \in \mathfrak{S}_1(I) \cap \mathfrak{S}_2(I)$ and $\varrho(\xi + p) - \varrho\xi \in \mathfrak{S}_1(I) \cap \mathfrak{S}_2(I)$. Hence $\mathfrak{S}_1(I) \cap \mathfrak{S}_2(I)$ is a NINA $Y(I)$ over the NF $X(I)$.

Theorem 3.4: Let $\mathfrak{S}_1(I)$ and $\mathfrak{S}_2(I)$ be right NINAs $Y(I)$ over a NF $X(I)$. Then $\mathfrak{S}_1(I) + \mathfrak{S}_2(I)$ is a right NINA of $Y(I)$.

Proof: Let $\mathfrak{S}_1(I)$ and $\mathfrak{S}_2(I)$ be right NINAs $Y(I)$ over the NF $X(I)$. Then $\mathfrak{S}_1(I) + \mathfrak{S}_2(I) \subseteq Y(I)$. Let $p, n \in \mathfrak{S}_1(I) + \mathfrak{S}_2(I)$. Then $p = p_1 + p_2$ and $n = n_1 + n_2$ for every $p_1, n_1 \in \mathfrak{S}_1(I), p_2, n_2 \in \mathfrak{S}_2(I)$. If $g, d \in X(I)$, then $gp_1 + dn_1 \in \mathfrak{S}_1(I)$ and $gp_2 + dn_2 \in \mathfrak{S}_2(I)$. Now $gp + dn = g(p_1 + p_2) + d(n_1 + n_2) = (gp_1 + dn_1) + (gp_2 + dn_2) \in \mathfrak{S}_1(I) + \mathfrak{S}_2(I)$. Therefore for every $g, d \in X(I)$ and $p, n \in \mathfrak{S}_1(I) + \mathfrak{S}_2(I)$, we have $gp + dn \in \mathfrak{S}_1(I) + \mathfrak{S}_2(I)$. Thus $\mathfrak{S}_1(I) + \mathfrak{S}_2(I)$ is a neutrosophic linear subspace of $Y(I)$ over the NF $X(I)$. Let $\xi \in Y(I), p \in \mathfrak{S}_1(I) + \mathfrak{S}_2(I)$. Then for $p = p_1 + p_2$, where $p_1 \in \mathfrak{S}_1(I), p_2 \in \mathfrak{S}_2(I)$, we have $p\xi = (p_1 + p_2)\xi = p_1\xi + p_2\xi \in \mathfrak{S}_1(I) + \mathfrak{S}_2(I)$. Thus $\mathfrak{S}_1(I) + \mathfrak{S}_2(I)$ is a right NINA of $Y(I)$.

Remark 3.5: Let $\mathfrak{S}_1(I)$ and $\mathfrak{S}_2(I)$ be right NINA $Y(I)$ over a NF $X(I)$. If $p_1 \in \mathfrak{S}_1(I)$ and $0 \in \mathfrak{S}_2(I)$, then $p_1 + 0 = p_1 \in \mathfrak{S}_1(I) + \mathfrak{S}_2(I)$. Therefore $\mathfrak{S}_1(I) \subseteq \mathfrak{S}_1(I) + \mathfrak{S}_2(I)$. Similarly $\mathfrak{S}_2(I) \subseteq \mathfrak{S}_1(I) + \mathfrak{S}_2(I)$. Hence $\mathfrak{S}_1(I) \cup \mathfrak{S}_2(I) \subseteq \mathfrak{S}_1(I) + \mathfrak{S}_2(I)$.

Theorem 3.6: Let $T(\mathfrak{S}(I))$ be the set such that $T(\mathfrak{S}(I)) = \{\xi \in Y(I) |\, \xi I = 0\}$ for any non-empty subset $\mathfrak{S}(I)$ of a NNA $Y(I)$ over a NF $X(I)$. Then $T(\mathfrak{S}(I))$ is a left NINA $Y(I)$ over the NF $X(I)$.

Proof: Given that $T(\mathfrak{S}(I)) = \{\xi \in Y(I) |\, \xi I = 0\} = \{\xi \in Y(I) |\, \xi p = 0, \text{for any } p \in \mathfrak{S}(I)\}$. Let $g, d \in X(I)$ and $\xi, \varrho \in T(\mathfrak{S}(I))$. Then $\xi p = 0$ and $\varrho p = 0$, for every $p \in \mathfrak{S}(I)$. Since $Y(I)$ itself is a NLS, we get $(g\xi + d\varrho) \in Y(I)$. For each $p \in \mathfrak{S}(I)$, consider $(g\xi + d\varrho)p = (g\xi)p + (d\varrho)p = g(\xi p) + d(\varrho p) = g(0) + d(0) = 0 + 0 = 0$. Therefore $(g\xi + d\varrho) \in T(\mathfrak{S}(I))$. Thus $T(\mathfrak{S}(I))$ is a NLS of $Y(I)$. Let $\wp, \beth \in Y(I)$ and $\xi \in T(\mathfrak{S}(I))$. Then $\xi p = 0$ for every $p \in \mathfrak{S}(I)$. It is clear that $\beth(\wp + \xi) - \beth\wp \in \mathfrak{S}(I)$. For each $p \in \mathfrak{S}(I)$, $(\beth(\wp + \xi) - \beth\wp)p = (\beth(\wp + \xi)p - (\beth\wp)p = \beth(\wp + \xi)p - (\beth\wp)p = \beth(\wp p + \xi p) - \beth\wp p = \beth(\wp p + 0) - \beth\wp p = \beth(\wp p) - \beth\wp p = (\beth\wp)p - \beth\wp p = 0$. So $(\wp + \xi) - \beth\wp \in T(\mathfrak{S}(I))$. Thus $T(\mathfrak{S}(I))$ is a left NINA $Y(I)$ over the NF $X(I)$.

Remark 3.7: In the above theorem, it is clear that $T(\mathfrak{S}(I))$ is not a right NINA $Y(I)$ over the NF $X(I)$. In this $T(\mathfrak{S}(I))$ is satisfying only left NINA. It will not satisfy the right NINA. Because, in the set itself we are defined $\xi p = 0$.

Definition 3.8: Let $Y(I)$ and $Y'(I)$ be two NNAs over a NF $X(I)$. A mapping $\mathfrak{T} : Y(I) \to Y'(I)$ is called a neutrosophic near algebra homomorphism (NNAH) if

(i) $\mathfrak{T}(\wp + \beth) = \mathfrak{T}(\wp) + \mathfrak{T}(\beth)$,
(ii) $\mathfrak{T}(\wp\beth) = \mathfrak{T}(\wp)\mathfrak{T}(\beth)$,

$(iii)\ \mathfrak{T}(\lambda\wp) = \lambda\mathfrak{T}(\wp)\ \forall\ \wp, \beth \in Y(I)$ and $\lambda \in\ X(I)$.

If $\mathfrak{T}$ is one-to-one, onto and homomorphism, we say that $\mathfrak{T}$ is a NNA isomorphism.

Definition 3.9: Let $Y(I)$ and $Y'(I)$ be two NNAs over a NF $X(I)$. Let $\mathfrak{T}: Y(I) \to Y'(I)$ be a NNAH. Then the Kernel of $\mathfrak{T}$ is denoted by Ker $\mathfrak{T}$ and is defined by Ker $\mathfrak{T} = \{\xi \in Y(I)/\mathfrak{T}(\xi) = 0', 0'\,\textit{is the additive identity in}\,Y'(I)\}$.

Theorem 3.10: If $\mathfrak{T}: Y(I) \to Y'(I)$ is a NNAH, then Ker $\mathfrak{T}$ is an ideal of $Y(I)$ over a NF $X(I)$.

Proof: Given $\mathfrak{T}: Y(I) \to Y'(I)$ be a NNAH. Since $\mathfrak{T}(0) = 0'$, we get $0 \in$ Ker $\mathfrak{T}$. Thus Ker $\mathfrak{T}$ is a non-empty subset of $Y(I)$. Let $g, d \in X(I), \xi, \varrho \in Y(I), p, n \in$ Ker $\mathfrak{T}$. Then $p, n \in Y(I)$ and $\mathfrak{T}(p) = 0'$,

$\mathfrak{T}(n) = 0'$. Since $Y(I)$ itself is a NLS, then $gp + dn \in\ Y(I)$. Consider $\mathfrak{T}(gp + dn) = \mathfrak{T}(gp) + \mathfrak{T}(dn)$

$= g\mathfrak{T}(p) + d\mathfrak{T}(n) = g0' + d0' = 0' + 0' = 0'$. Therefore $gp + dn \in$ Ker $\mathfrak{T}$. Thus Ker $\mathfrak{T}$ is a NLS of $Y(I)$. Now $p \in$ Ker $\mathfrak{T}$ implies $\mathfrak{T}(p) = 0', p \in$ Y(I). Then for any $\xi \in Y(I)$ and $\mathfrak{T}(p\xi) = \mathfrak{T}(p)\mathfrak{T}(\xi) = 0'\mathfrak{T}(\xi) = 0'$. Therefore $p\xi\ \in$ Ker $\mathfrak{T}$. Thus Ker $\mathfrak{T}$ is a right NINA of $Y(I)$.

Let $\xi, \varrho \in\ Y(I)$ and $Y(I)$ is a NNA, we get $\varrho(\xi + p), \varrho\xi \in Y(I)$. This implies that $\varrho(\xi + p) - \varrho\xi \in Y(I)$. Consider, $\mathfrak{T}(\varrho(\xi + p) - \varrho\xi) =\ \mathfrak{T}\big(\varrho(\xi + p)\big) - \mathfrak{T}(\varrho\xi)$

$$= \mathfrak{T}(\varrho)\mathfrak{T}(\xi + p) - \mathfrak{T}(\varrho)\mathfrak{T}(\xi)$$

$$=\ \mathfrak{T}(\varrho)\big(\mathfrak{T}(\xi) + \mathfrak{T}(p)\big) -\ \mathfrak{T}(\varrho)\mathfrak{T}(\xi)$$

$$= \mathfrak{T}(\varrho)(\mathfrak{T}(\xi) + 0') -\ \mathfrak{T}(\varrho)\mathfrak{T}(\xi)$$

$$=\ \mathfrak{T}(\varrho)\mathfrak{T}(\xi) + \mathfrak{T}(\varrho)\ 0' -\ \mathfrak{T}(\varrho)\mathfrak{T}(\xi)$$

$$=\ \mathfrak{T}(\varrho)\mathfrak{T}(\xi) +\ 0' -\ \mathfrak{T}(\varrho)\mathfrak{T}(\xi)$$

$$=\ \mathfrak{T}(\varrho)\mathfrak{T}(\xi) -\ \mathfrak{T}(\varrho)\mathfrak{T}(\xi) =\ 0'.$$

Therefore $\varrho(\xi + p) - \varrho\xi \in$ Ker $\mathfrak{T}$. Thus ker $\mathfrak{T}$ is a left NINA of $Y(I)$.

Hence Ker $\mathfrak{T}$ is an ideal of $Y(I)$ over the NF $X(I)$.

Theorem 3.11: Let $Y(I)$ and $Y'(I)$ be two NNAs over a NF $X(I)$. Let $\mathfrak{T}: Y(I)\ \to Y'(I)$ be a NNAH. If $\mathfrak{S}(I)$ is an ideal of $Y(I)$, then $\mathfrak{T}\big(\mathfrak{S}(I)\big)$ is an ideal of $\mathfrak{T}\big(Y(I)\big)$.

Proof: We know that $\mathfrak{T}\big(Y(I)\big) = \{\ \xi' \in Y'(I): \mathfrak{T}(\ \xi) =\ \xi',\ \xi \in Y(I)\}$ and

$\mathfrak{T}\big(\mathfrak{S}(I)\big) = \big\{\mathfrak{T}(\varrho): \varrho \in \mathfrak{S}(I) \subseteq Y(I), \mathfrak{T}(\varrho) = \varrho', \varrho' \in \mathfrak{T}\big(Y(I)\big)\big\}$. Then $\mathfrak{T}\big(\mathfrak{S}(I)\big) \subseteq \mathfrak{T}\big(Y(I)\big)$. Let $g, d \in X(I)$; $p', n' \in \mathfrak{T}\big(\mathfrak{S}(I)\big)$. Then $p', n' \in \mathfrak{T}\big(Y(I)\big)$ and there exists $p, n \in \mathfrak{S}(I)$ such that $\mathfrak{T}(p) = p'$, $\mathfrak{T}(n) = n'$. Since $\mathfrak{S}(I)$ is an ideal in $Y(I)$, we get $gp + dn \in \mathfrak{S}(I)$ for every $g, d \in X(I); p, n \in \mathfrak{S}(I)$. Now $gp' + dn' = g\mathfrak{T}(p) + d\mathfrak{T}(n) = \mathfrak{T}(gp) + \mathfrak{T}(dn) = \mathfrak{T}(gp + dn) \in \mathfrak{T}\big(\mathfrak{S}(I)\big)$. Therefore for every $g, d \in X(I)$; $p', n' \in \mathfrak{T}\big(\mathfrak{S}(I)\big), gp' + dn' \in \mathfrak{T}\big(\mathfrak{S}(I)\big)$. Thus $\mathfrak{T}\big(\mathfrak{S}(I)\big)$ is a neutrosophic linear subspace of $\mathfrak{T}\big(Y(I)\big)$. Let $p' \in \mathfrak{T}\big(\mathfrak{S}(I)\big), \xi' \in \mathfrak{T}\big(Y(I)\big)$. Then $p' \in \mathfrak{T}\big(Y(I)\big)$ and there exists $p \in \mathfrak{S}(I), \xi \in Y(I)$ such that $\mathfrak{T}(p) = p', \mathfrak{T}(\xi) = \xi'$. Since $\mathfrak{S}(I)$ is an ideal in $Y(I)$, we get $p\xi \in \mathfrak{S}(I)$ for every $p \in \mathfrak{S}(I), \xi \in Y(I)$. Now $p'.\xi' =\ \mathfrak{T}(p)\,.\mathfrak{T}(\xi) = \mathfrak{T}(p.\xi) \in\ \mathfrak{T}\big(\mathfrak{S}(I)\big)$. Thus $\mathfrak{T}\big(\mathfrak{S}(I)\big)$ is a right NINA of $\mathfrak{T}\big(Y(I)\big)$. Let $p' \in \mathfrak{T}\big(\mathfrak{S}(I)\big), \xi', \varrho' \in \mathfrak{T}\big(Y(I)\big)$. Then $p' \in \mathfrak{T}\big(Y(I)\big)$ and there exists $p \in \mathfrak{S}(I), \xi, \varrho \in Y(I)$ such that $\mathfrak{T}(\ \xi) =\ \xi', \mathfrak{T}(\varrho) = \varrho', \mathfrak{T}(p) = p'$. Since $\mathfrak{S}(I)$ is an ideal in $Y(I)$, we get $\varrho(\ \xi + p) - \varrho\xi \in \mathfrak{S}(I)$ for every $p \in \mathfrak{S}(I),\ \ \xi, \varrho \in Y(I)$. Now

$$\varrho'(\ \xi' + p') - \varrho'\xi' =\ \mathfrak{T}(\varrho)(\mathfrak{T}(\ \xi) + \mathfrak{T}(p)) - \mathfrak{T}(\varrho)\mathfrak{T}(\ \xi)$$

$$= \mathfrak{T}(\varrho)\mathfrak{T}(\ \xi + p) - \mathfrak{T}(\varrho\xi) = \mathfrak{T}(\varrho(\ \xi + p)) -\ \mathfrak{T}(\varrho\ \xi)$$

$$= \mathfrak{T}(\varrho(\ \xi + p) - \varrho\ \xi) \in \mathfrak{T}\big(\mathfrak{S}(I)\big).$$

Thus $\mathfrak{T}\big(\mathfrak{S}(I)\big)$ is a left NINA of $\mathfrak{T}\big(Y(I)\big)$. Hence $\mathfrak{T}\big(\mathfrak{S}(I)\big)$ is an ideal of $\mathfrak{T}\big(Y(I)\big)$.

Theorem 3.12: Let $Y(I)$ and $Y'(I)$ be NNAs over a NF $X(I)$, and $\mathfrak{T}: Y(I) \to Y'(I)$ be a NNAH. If $\mathfrak{S}_2(I)$ is an ideal of $Y'(I)$, then $\mathfrak{T}^{-1}(\mathfrak{S}_2(I))$ is an ideal of $Y(I)$.

Proof: Put $\mathfrak{S}(I) = \mathfrak{T}^{-1}(\ \mathfrak{S}_2(I)) = \{\ \xi \in Y(I): \mathfrak{T}(\ \xi) \in\ \mathfrak{S}_2(I)\}$.

(i) Let $\xi, \varrho \in \mathfrak{S}_1(I)$ and $g, d \in X(I)$. Then $\mathfrak{T}(\xi), \mathfrak{T}(\varrho) \in\ \mathfrak{S}_2(I)$. This implies that $g\mathfrak{T}(\xi), d\mathfrak{T}(\varrho) \in\ \mathfrak{S}_2(I)$. Now $\mathfrak{S}_2(I)$ is an ideal, then $g\mathfrak{T}(\xi) + d\mathfrak{T}(\varrho) \in\ \mathfrak{S}_2(I)$. Thus $\mathfrak{T}(g\xi) + \mathfrak{T}(d\varrho) \in\ \mathfrak{S}_2(I), so\ \mathfrak{T}(g\xi + d\varrho) \in$

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$\mathfrak{S}_2(I)$, $thus\ g\xi + d\varrho \in \mathfrak{T}^{-1}(\mathfrak{S}_2(I)) = \mathfrak{S}_1(I)$. Therefore, for every $\xi, \varrho \in \mathfrak{T}^{-1}(\mathfrak{S}_2(I)), g, d \in X(I)$, we get $g\xi + d\varrho \in \mathfrak{T}^{-1}(\mathfrak{S}_2(I))$. Thus $\mathfrak{T}^{-1}(\mathfrak{S}_2(I))$ is a neutrosophic linear subspace of $Y(I)$.

(ii) Let $\xi \in Y(I), p \in \mathfrak{S}_1(I)$. Then $\mathfrak{T}(p) \in \mathfrak{S}_2(I), \mathfrak{T}(\xi) \in Y'(I)$. Now $\mathfrak{S}_2(I)$ is an ideal in $Y'(I)$, then $\mathfrak{T}(p).\mathfrak{T}(\xi) \in \mathfrak{S}_2(I)$. Thus $\mathfrak{T}(p.\xi) \in \mathfrak{S}_2(I)$ and $p.\xi \in \mathfrak{S}_1(I)$. Therefore, for every $\xi \in Y(I), p \in \mathfrak{S}_1(I)$, we get $p.\xi \in \mathfrak{S}_1(I)$. Thus $\mathfrak{T}^{-1}(\mathfrak{S}_2(I))$ is a right NINA of $Y(I)$.

(iii) Let $\xi, \varrho \in Y(I), p \in \mathfrak{S}_1(I)$. Now $\mathfrak{T}(p) \in \mathfrak{S}_2(I), \mathfrak{T}(\xi), \mathfrak{T}(\varrho) \in Y'(I)$.

Then $\mathfrak{T}(\varrho)\,\big(\mathfrak{T}(\xi) + \mathfrak{T}(p)\big) - \mathfrak{T}(\varrho).\mathfrak{T}(\xi) \in \mathfrak{S}_2(I)$. Thus $\mathfrak{T}(\varrho)\,.\mathfrak{T}(\xi + p) - \mathfrak{T}(\varrho.\xi) \in \mathfrak{S}_2(I)$. So

$\mathfrak{T}(\varrho(\xi + p) - \varrho.\xi) \in \mathfrak{S}_2(I)$. Therefore $\varrho(\xi + p) - \varrho.\xi \in \mathfrak{S}_1(I)$. Thus $\mathfrak{T}^{-1}(\mathfrak{S}_2(I))$ is a left NINA of $Y(I)$. Hence $\mathfrak{T}^{-1}(\mathfrak{S}_2(I))$ is an ideal of $Y(I)$.

Theorem 3.13: If $\mathfrak{T}: Y(I) \to Y'(I)$ is a NNAH, then $\mathfrak{T}$ is one-one iff Ker $\mathfrak{T} = \{0\}$.

Proof: Suppose that $\mathfrak{T}$ is one-one. Now, we have to prove that Ker $\mathfrak{T} = \{0\}$.

Let $\xi \in$ Ker $\mathfrak{T}$. Then by the definition, we have $\mathfrak{T}(\xi) = 0'$,where $0'$is the additive identity in $Y'(I)$. Now, $\mathfrak{T}(\xi) = 0'$ implies $\mathfrak{T}(\xi) = \mathfrak{T}(0)$, so $\xi = 0$. Hence Ker $\mathfrak{T} = \{0\}$.

Conversely, suppose that Ker $\mathfrak{T} = \{0\}$. Now we have to prove that $\mathfrak{T}$ is one-one.

Let $\mathfrak{T}(\xi) = \mathfrak{T}(\varrho)$, for every $\xi, \varrho \in Y(I)$. Then $\mathfrak{T}(\xi) = \mathfrak{T}(\varrho)$. Thus $\mathfrak{T}(\xi - \varrho) = 0'$. So

$\xi - \varrho \in$ Ker $\mathfrak{T}$. Therefore $\xi - \varrho = 0$. Hence $\xi = \varrho$. Therefore $\mathfrak{T}$ is one-one.

Definition 3.14: Let $\mathfrak{S}(I)$ be a NINA $Y(I)$ over a NF $X(I)$. Then for any element $\xi \in Y(I)$ the set $\xi + \mathfrak{S}(I) = \{\xi + p / p \in \mathfrak{S}(I)\}$ is called a coset of $\mathfrak{S}(I)$ in $Y(I)$, generated by ξ.

Definition 3.15: Let $\mathfrak{S}(I)$ be a NINA $Y(I)$ over a NF $X(I)$. Then the set of all cosets of $\mathfrak{S}(I)$ in $Y(I)$ is denoted by $Y(I)/\mathfrak{S}(I)$ and is defined by $Y(I)/\mathfrak{S}(I) = \{\xi + \mathfrak{S}(I) /\ for\ every\ \xi \in Y(I)\}$.

Theorem 3.16: Let $\mathfrak{S}(I)$ be NINA $Y(I)$ over a NF $X(I)$. Then the set $Y(I)/\mathfrak{S}(I)$ is a NNA over a NF $X(I)$ with respect to the operations defined by

$$\big(\xi + \mathfrak{S}(I)\big) + \big(\varrho + \mathfrak{S}(I)\big) = (\xi + \varrho) + \mathfrak{S}(I),$$

$$g\big(\xi + \mathfrak{S}(I)\big) = g\xi + \mathfrak{S}(I),$$

$$\big(\xi + \mathfrak{S}(I)\big)\,\big(\varrho + \mathfrak{S}(I)\big) = (\xi\varrho) + \mathfrak{S}(I)$$

for every $\xi, \varrho \in Y(I), g \in X(I)$.

Proof: First two operations are well defined. Suppose that $\xi + \mathfrak{S}(I) = \xi' + \mathfrak{S}(I), \varrho + \mathfrak{S}(I) = \varrho' + \mathfrak{S}(I)$; where $\xi, \xi', \varrho, \varrho' \in Y(I)$. Then $\xi - \xi', \varrho - \varrho' \in \mathfrak{S}(I)$. Put $\xi - \xi' = b_1,\ \varrho - \varrho' = b_2;\ b_1, b_2 \in \mathfrak{S}(I)$. So $\xi = \xi' + b_1, \varrho = \varrho' + b_2$. This implies that $\xi\varrho = (\xi' + b_1)(\varrho' + b_2) = \xi'(\varrho' + b_2) + b_1(\varrho' + b_2)$. Now $\xi\varrho - \xi'\varrho' = \xi'(\varrho' + b_2) + b_1(\varrho' + b_2) - \xi'\varrho' = \xi'(\varrho' + b_2) - \xi'\varrho' + b_1(\varrho' + b_2) \in \mathfrak{S}(I)$. Therefore $\xi\varrho - \xi'\varrho' \in \mathfrak{S}(I)$. Which implies $\xi\varrho + \mathfrak{S}(I) = \xi'\varrho' + \mathfrak{S}(I)$, and so

$\big(\xi + \mathfrak{S}(I)\big)\big(\varrho + \mathfrak{S}(I)\big) = \big(\xi' + \mathfrak{S}(I)\big)\big(\varrho' + \mathfrak{S}(I)\big)$. Therefore, product of cosets is well defined.

A direct verification demonstration that $Y(I)/\mathfrak{S}(I)$ is a NLS over a NF $X(I)$. Let $\xi + \mathfrak{S}(I), \varrho + \mathfrak{S}(I), z + \mathfrak{S}(I) \in Y(I)/\mathfrak{S}(I)$, where $\xi, \varrho, z \in Y(I), g \in X(I)$. Then

$$\Big(\big(\xi + \mathfrak{S}(I)\big)\big(\varrho + \mathfrak{S}(I)\big)\Big)\,\big(z + \mathfrak{S}(I)\big) = \big((\xi\varrho) + \mathfrak{S}(I)\big)\big(z + \mathfrak{S}(I)\big) = (\xi\varrho)z + \mathfrak{S}(I) = \xi(\varrho z) + \mathfrak{S}(I)$$

$$= \big(\xi + \mathfrak{S}(I)\big)\big(\varrho z + \mathfrak{S}(I)\big) = \big(\xi + \mathfrak{S}(I)\big)\,\Big(\big(\varrho + \mathfrak{S}(I)\big)\big(z + \mathfrak{S}(I)\big)\Big).$$

Now $\big[\big(\xi + \mathfrak{S}(I)\big) + \big(\varrho + \mathfrak{S}(I)\big)\big]\big(z + \mathfrak{S}(I)\big) = [(\xi + \varrho) + \mathfrak{S}(I)]\big(z + \mathfrak{S}(I)\big)$

$$= (\xi + \varrho)z + \mathfrak{S}(I)$$

$$= (\xi z + \varrho z) + \mathfrak{S}(I)$$

$$= \big(\xi z + \mathfrak{S}(I)\big) + \big(\varrho z + \mathfrak{S}(I)\big)$$

$$= \big[\big(\xi + \mathfrak{S}(I)\big)\big(z + \mathfrak{S}(I)\big)\big] + \big[\big(\varrho + \mathfrak{S}(I)\big)\big(z + \mathfrak{S}(I)\big)\big].$$

$\Big(g\big(\xi + \mathfrak{S}(I)\big)\Big)\,\big(\varrho + \mathfrak{S}(I)\big) = \big(g\xi + \mathfrak{S}(I)\big)\big(\varrho + \mathfrak{S}(I)\big) = (g\xi)\varrho + \mathfrak{S}(I) = g(\xi\varrho) + \mathfrak{S}(I)$

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$$= g\big(\xi\varrho + \mathfrak{S}(I)\big) = g\left(\big(\xi + \mathfrak{S}(I)\big)\big(\varrho + \mathfrak{S}(I)\big)\right).$$

Hence $Y(I)/\mathfrak{S}(I)$ is a NNA over the NF $X(I)$.

Definition 3.17: Let $\mathfrak{S}(I)$ be a NINA $Y(I)$ over a NF $X(I)$. Then the set

$Y(I)/\mathfrak{S}(I) = \{\xi + \mathfrak{S}(I)/\xi \in Y(I)\}$ with respect to the usual induced operations of addition, scalar multiplication and product of cosets defined by

$$\big(\xi + \mathfrak{S}(I)\big) + \big(\varrho + \mathfrak{S}(I)\big) = (\xi + \varrho) + \mathfrak{S}(I),$$

$$g\big(\xi + \mathfrak{S}(I)\big) = g\xi + \mathfrak{S}(I),$$

$$\big(\xi + \mathfrak{S}(I)\big)\big(\varrho + \mathfrak{S}(I)\big) = (\xi\varrho) + \mathfrak{S}(I)$$

for every $\xi + \mathfrak{S}(I), \varrho + \mathfrak{S}(I) \in Y(I)/\mathfrak{S}(I),\ g \in X(I)$ is a NNA. This NNA is called a neutrosophic quotient near algebra.

Theorem 3.18: If $\mathfrak{S}(I)$ is a NINA $Y(I)$ over a NF $X(I)$, then the neutrosophic quotient near algebra $Y(I)/\mathfrak{S}(I)$ is a homomorphic image of $Y(I)$.

Proof: Let $\mathfrak{T}: Y(I) \to Y(I)/\mathfrak{S}(I)$ be a mapping defined by $\mathfrak{T}(\xi) = \xi + \mathfrak{S}(I)$ for all $\xi \in Y(I)$.

Let $\xi, \varrho \in Y(I), g \in X(I)$. Suppose that $\xi = \varrho$. Then $\xi + \mathfrak{S}(I) = \varrho + \mathfrak{S}(I)$ implies that $\mathfrak{T}(\xi) = \mathfrak{T}(\varrho)$. Therefore $\mathfrak{T}$ is well defined. Now,

$\mathfrak{T}(\xi + \varrho) = (\xi + \varrho) + \mathfrak{S}(I) = \big(\xi + \mathfrak{S}(I)\big) + \big(\varrho + \mathfrak{S}(I)\big) = \mathfrak{T}(\xi) + \mathfrak{T}(\varrho),$

$\mathfrak{T}(\xi\varrho) = (\xi\varrho) + \mathfrak{S}(I) = \big(\xi + \mathfrak{S}(I)\big)\big(\varrho + \mathfrak{S}(I)\big) = \mathfrak{T}(\xi)\mathfrak{T}(\varrho),$

$\mathfrak{T}(g\xi) = g\xi + \mathfrak{S}(I) = g\big(\xi + \mathfrak{S}(I)\big) = g\,\mathfrak{T}(\xi).$

Therefore $\mathfrak{T}$ is a NNAH. Let $\xi + \mathfrak{S}(I) \in Y(I)/\mathfrak{S}(I)$. Then $\xi \in Y(I)$. For this $\xi \in Y(I)$ we have $\mathfrak{T}(\xi) = \xi + \mathfrak{S}(I)$. That is for each $\xi + \mathfrak{S}(I) \in Y(I)/\mathfrak{S}(I)$ there exists $\xi \in Y(I)$ such that $\mathfrak{T}(\xi) = \xi + \mathfrak{S}(I)$. Therefore $\mathfrak{T}$ is an onto mapping. Thus $\mathfrak{T}: Y(I) \to Y(I)/\mathfrak{S}(I)$ is an onto homomorphism. Hence $Y(I)/\mathfrak{S}(I)$ is a homomorphic image of $Y(I)$.

Theorem 3.19: Let $Y(I), Y'(I)$ be two NNAs over a NF $X(I)$. Let $\mathfrak{T}: Y(I) \to Y'(I)$ be a NNAH with Kernal $\mathfrak{S}(I)$. Then $\mathfrak{T}\big(Y(I)\big)$ is isomorphic to $Y(I)/\mathfrak{S}(I)$.

Proof: Suppose that $\mathfrak{S}(I)$ is a NINA of $Y(I)$. Then $Y(I)/\mathfrak{S}(I) = \{\xi + \mathfrak{S}(I)/\xi \in \mathfrak{S}(I)\}$ is a neutrosophic quotient near algebra. Define a mapping $\Psi: Y(I)/\mathfrak{S}(I) \to \mathfrak{T}\big(Y(I)\big)$ by $\Psi\big(\xi + \mathfrak{S}(I)\big) = \mathfrak{T}(\xi)$ for each $\xi + \mathfrak{S}(I) \in Y(I)/\mathfrak{S}(I)$. Let $\xi + \mathfrak{S}(I), \varrho + \mathfrak{S}(I) \in Y(I)/\mathfrak{S}(I)$. Now $\xi + \mathfrak{S}(I) = \varrho + \mathfrak{S}(I)$. Then $\big(\xi + \mathfrak{S}(I)\big) - \big(\varrho + \mathfrak{S}(I)\big) = 0 + \mathfrak{S}(I)$. Thus $(\xi - \varrho) + \mathfrak{S}(I) = \mathfrak{S}(I)$. Hence $\xi - \varrho \in \mathfrak{S}(I)$. Thus $\mathfrak{T}(\xi - \varrho) = 0'$. Which gives $\mathfrak{T}(\xi) - \mathfrak{T}(\varrho) = 0'$. Thus $\mathfrak{T}(\xi) = \mathfrak{T}(\varrho)$. Hence $\Psi\big(\xi + \mathfrak{S}(I)\big) = \Psi\big(\varrho + \mathfrak{S}(I)\big)$. Therefore Ψ is well defined and one-one. Let $\mathfrak{T}(\xi) \in \mathfrak{T}\big(Y(I)\big)$. Then $\xi \in Y(I)$. For this $\xi \in Y(I)$ we have $\xi + \mathfrak{S}(I) \in Y(I)/\mathfrak{S}(I)$ and $\Psi\big(\xi + \mathfrak{S}(I)\big) = \mathfrak{T}(\xi)$. That is for each $\Psi(\xi) \in \Psi\big(Y(I)\big)$ there exists $\xi + \mathfrak{S}(I) \in Y(I)/\mathfrak{S}(I)$ such that $\Psi\big(\xi + \mathfrak{S}(I)\big) = \mathfrak{T}(\xi)$. Thus Ψ is onto. Let $\xi + \mathfrak{S}(I), \varrho + \mathfrak{S}(I) \in Y(I)/\mathfrak{S}(I)$ and $g \in X(I)$, where $\xi, \varrho \in Y(I)$. Then

$$\Psi\left(\big(\xi + \mathfrak{S}(I)\big) + \big(\varrho + \mathfrak{S}(I)\big)\right) = \Psi\big((\xi + \varrho) + \mathfrak{S}(I)\big) = \mathfrak{T}(\xi + \varrho) = \mathfrak{T}(\xi) + \mathfrak{T}(\varrho)$$

$$= \Psi\big(\xi + \mathfrak{S}(I)\big) + \Psi\big(\varrho + \mathfrak{S}(I)\big),$$

$$\Psi\left(\big(\xi + \mathfrak{S}(I)\big)\big(\varrho + \mathfrak{S}(I)\big)\right) = \Psi\big((\xi\varrho) + \mathfrak{S}(I)\big) = \mathfrak{T}(\xi\varrho) = \mathfrak{T}(\xi)\mathfrak{T}(\varrho)$$

$$= \Psi\big(\xi + \mathfrak{S}(I)\big)\Psi\big(\varrho + \mathfrak{S}(I)\big),$$

$\Psi\left(g\big(\xi + \mathfrak{S}(I)\big)\right) = \Psi\big(g\xi + \mathfrak{S}(I)\big) = \mathfrak{T}(g\xi) = g\mathfrak{T}(\xi) = g\Psi\big(\xi + \mathfrak{S}(I)\big)$. Thus Ψ is a homomorphism.

Hence $Y(I)/\mathfrak{S}(I)$ is isomorphic to $\mathfrak{T}\big(Y(I)\big)$.

4. Conclusion

In this paper, we have presented NINA, NNAH and Kernal of a NNA. There have been introduced several outcomes and examples pertaining to the NINA. Neutrosophic Gamma near algebra, neutrosophic ideal of a gamma near algebra and neutrosophic fuzzy near algebra nearby modules can be included in the scope of neutrosophic theory.

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References

[1] A. A. Agboola and B. Davvaz, "Introduction to neutrosophic nearrings," *Annals of Fuzzy Mathematics and Informatics*, vol. 12, no. 3, pp. 397–410, 2016.

[2] A. A. Agboola and S. A. Akinleye, "Neutrosophic vector spaces," *Neutrosophic Sets and Systems*, vol. 4, 2014.

[3] A. A. Agboola, A. D. Akinola, and O. Y. Oyebola, "Neutrosophic Rings I," *International Journal of Mathematical Combinatorics*, vol. 4, pp. 1–14, 2011.

[4] A. A. Agboola, A. O. Akwu, and Y. T. Oyebo, "Neutrosophic groups and Neutrosophic Subgroups," *International Journal of Mathematical Combinatorics*, vol. 3, pp. 1–9, 2012.

[5] L. Bhaskar, P. Narasimha Swamy, and G. Nanaji Rao, "Vague Γ- Near Algebra over Vague Field," *Southeast Asian Bulletin of Mathematics*, vol. 48, pp. 15–22, 2024.

[6] H. Brown, "Near algebra," *Illinois Journal of Mathematics*, vol. 12, pp. 215–227, 1968.

[7] Harika, K. Rajani, P. Narasimha Swamy, T. Nagaiah, and L. Bhaskar, "Neutrosophic near algebra over neutrosophic field," *International Journal of Neutrosophic Science*, vol. 22, no. 2, pp. 29–34, 2023.

[8] H. Bhurgula, N. S. Pasham, R. Bandaru, and A. S. Alali, "Hybrid near algebra," *Axioms*, vol. 12, p. 877, 2023.

[9] P. Narasimha Swamy, B. Harika, K. Vijay Kumar, B. Jyothi, and K. Rajani, "A note on Hybrid Coset of a Nearring," *Communications on Applied Nonlinear Analysis*, vol. 31, no. 2, 2024.

[10] T. Nagaiah, L. Bhaskar, P. Narasimha Swamy, and S. Broumi, "A Study on Neutrosophic Algebra," *Neutrosophic Sets and Systems*, vol. 50, 2022.

[11] G. Pilz, Near-ring. Amsterdam: North-Holland Publishers, 1983.

[12] F. Smarandache, A Unifying Field in Logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability. Rehoboth: *American Research Press*, 2003.

[13] W. B. Vasantha Kanda Swamy and F. Smarandache, Fuzzy Algebra. Rehoboth: *American Research Press*, 2003.

[14] W. B. Vasantha Kanda Swamy and F. Smarandache, Basic Neutrosophic Algebraic Structures and Their Applications to Fuzzy and Neutrosophic Models. *Church Rock: Hexis*, 2004.

[15] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, pp. 338–353, 1965.

DOI: https://doi.org/10.54216/IJNS.250414