Operations on Translation of Fermatean Neutrosophic INK-Algebra

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Abstract

This paper investigates the theoretical basis of fermatean neutrosophic sets, which were first introduced by Smarandache, to clarify the relationship between single-valued fermatean neutrosophic sets and their role as specific subsets in the wider context of fermatean neutrosophic sets, particularly in science and engineering. This study investigates fermatean neutrosophic INK-ideals within INK-algebras using the translation concept, which is proposed as an extension of intuitionistic fuzzy sets. First, translation fermatean neutrosophic INK-algebras are presented and their fundamental features are studied. Furthermore, the research investigates properties related to the translation of INK-subalgebras and INK-ideals, as well as the dynamics of their unions, intersections, and multiplications for fermatean neutrosophic INK-ideals. The article adds definitions and theorems to provide a complete grasp of the problems of fermatean neutrosophic INK-algebras.

Keywords: Neutrosophic sets; Single-valued neutrosophic sets; INK-algebras; Translation; Intuitionistic fuzzy sets; Neutrosophic INK-ideals

Zadeh proposed the idea of fuzzy sets in 1965,[9] and it has been useful in handling uncertainties in several real-world situations. Iseki work on BCI-algebras[30] provides foundational insights into mathematical structures, focusing on algebraic systems with a particular type of operation. Imai and Iseki collaborative effort[31] delves into axiom systems of propositional calculi, contributing to the development of logical systems and formal reasoning. Iseki and Tanaka introduction to the theory of BCK algebras[32] offers a comprehensive overview of these algebraic structures, enriching the understanding of mathematical concepts related to logic and algebra. In 1983,[2] Atanassov extended this idea even further with the introduction of intuitionistic fuzzy sets, which are now used in a wide range of fields, including finance, psychological research, sales analysis, marketing of new goods, and negotiating strategies. A 1994 paper by Jun and Meng in Mathematica Japonica examined fuzzy p-ideals inside BCI-algebras.[28] Their objective was to comprehend the characteristics and properties of these fuzzy p-ideals inside BCI-algebras, which are algebraic structures used in mathematical logic and computer science. Smarandache[5] created the notion of neutrosophic sets, which include neutrosophic probability, set, and logic. He later created novel varieties of soft sets, IndetermHyperSoft Set, IndetermSoft Set, TreeSoft Set and including HyperSoft Set.[6] These additions build on classic soft set theory, seeking to better resolve uncertainty in decision-making processes. The focus of Jun and Roh is on BCK/BCI-algebras, a particular subfield of algebra. Within this framework, they present a notion in this study termed MBJ-neutrosophic ideals.[7] Indhria et al.[10] investigated neutrosophic sets with similar algebraic structures and fuzzy p-ideals in INK-Algebra. These discoveries are likely to increase our understanding of fuzzy and neutrosophic concepts in the context of INK-Algebra. Smarandache created the idea of neutrosophic sets, which provide a larger context for intuitionistic fuzzy sets. Lee, Jun, and Doh's work, published in reference,[8] investigates fuzzy multiplications and translations within BCK/BCI-algebras, most likely studying the properties and ramifications of

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these operations on algebraic structures. Davvaz and Agboola[1] introduced Neutrosophic BCI/BCK-Algebras and gave an outline of their algebraic structures.

Senapati researched fuzzy structures in BCK/BCI algebras with fuzzy translations and intuitionistic fuzzy sets, contributing to a better understanding of fuzzy ideas in this context,[20].[21] Smarandache introduced SuperHyperSoft sets, which provide a larger view on fuzzy sets and their possible applications in various domains.[22]

Al-Omeri et al.[27] investigated the Ra operator in ideal topological spaces, adding to our knowledge of its role in these spaces. Indhira and Kaviyarasu[13] examined intuitionistic fuzzy INK-ideals of INK-algebras, providing insight into these algebraic structures. Mohseni Takallo et al.[16] presented MBJ-neutrosophic structures and investigated their applications in BCK/BCI algebras, providing novel insights into algebraic systems. Al-Omeri, Khalil, and Ghareeb[26] investigated the degree of (L, M)-fuzzy semi-precontinuous and (L, M)-fuzzy semi-preirresolute functions, shedding light on their properties within mathematical settings. In 2019, Manokaran et al.[14] addressed MBJ-Neutrosophic B-Subalgebras, most likely researching their features and applications in the context of neutrosophic algebras. Khalid et al.[15] investigated the properties and applications of MBJ-neutrosophic T-ideals on B-algebras in their 2020 paper. Similarly, in 2020, Kaviyarasu et al.[12] examined the direct product of Neutrosophic INK-Algebras, which added to our knowledge of these algebraic structures. Al-Omereti al.[23] investigated neutrosophic fixed point theorems and cone metric spaces, providing insight into fixed point features in the context of neutrosophic sets.

In another paper, Al-Omeri[24] investigates mixed b-fuzzy topological spaces, presumably examining their properties and applications in fuzzy topology. Al-Omeri may also investigate virtually e-I-continuous functions in a different work,[25] possibly looking into their continuity qualities and their applications in mathematical analysis. In their 2021 study, Song et al.[19]use MBJ-Neutrosophic structures to examine commutative ideals of BCI-Algebras, most likely to investigate the properties and applications of these ideals inside BCI-Algebra. The behavior of fuzzy ideals under translations and their theoretical characteristics are studied by Jun.[29]

Bordbar et al.[3] used the framework of neutrosophic sets and falling shadows to investigate the qualities and characteristics of these ideals, providing insights into their behavior in the setting of BCK-algebra. Kaviyarasu and Rajeshwari[17] investigate the features and structures of these direct products, revealing insights and possible applications in the field of neutrosophic algebra. An original method for algebraic structures that may be used in neutrosophic systems is presented by Ranulfo et al. (2023) in their work, Pura Vida Neutrosophic Algebra.[18] Their study is expanded to encompass intuitionistic fuzzy translations, delving into the relationships among fuzzy translations, extensions, and multiplications. Within INK algebras, the researchers study translation INK-ideals using fermatean neutrosophic sets. They define fermatean neutrosophic INK-ideals and study the features of translation INK-subalgebras. Clarifying these points is the main focus of the conversation.

- Create fermatean neutrosophic INK-subalgebras with translations.
- Explain the connection between fermatean neutrosophic translations(FNT) of INK-ideals and INK-subalgebras.
- Examine the relationships between fermatean neutrosophic translations INK-ideals (FNTINK-I)and fermatean neutrosophic translations INK-subalgebras (FNTINK-S).
- Present the criteria for translations fermatean neutrosophic INK-ideals from fermatean neutrosophic INK-algebras.
- Develop the translation feature for fermatean neutrosophic INK-ideals.

Furthermore, the second part summarizes the fundamental notions of INK-algebra(INK-A) and fermatean neutrosophic sets(FNS) required for understanding the discussion. The third part uses FNS to explore translations of INK-subalgebras and INK-ideals(INK-I), and the examination continues in the fourth part.

1 Basic Definitions

This part provides the essential components required for understanding the article.

Definition 1.1. [11] An algebra $(\hbar, \star, 0)$ is called an INK-A if it meets the following requirements for each $s_1, s_2, s_3 \in \hbar$.

1. $s_1 \star 0 = s_1$
2. $s_1 \star s_1 = 0$
3. $(s_1 \star s_2) \star s_3 = (s_1 \star s_3) \star s_2$
4. $(s_2 \star s_1) \star (s_2 \star s_3) = (s_1 \star s_3)$

where $\star$ represents a binary operation and the 0 is a constant of $\hbar$.

Example 1.2. Consider the INK-A $\hbar = \{0, 1, 2, 3\}$ with the Cayley table below.

$\cdot$	0	1	2	3
0	0	1	2	3
1	1	0	3	3
2	2	3	0	3
3	3	3	3	0

Table 1: INK-algebra

Definition 1.3. [11] If the operation $s_1 \star s_2$ holds for elements s_1 and s_2 in Ş, a non-empty subset Ş of an INK-A $(\hbar, \star, 0)$ is regarded as an INK-S of $\hbar$.

Definition 1.4. [12] In an INK-A $(\hbar, \star, 0)$, an INK-I set is a non-empty subset of $\hbar$ that matches the following conditions:

1. $0 \in$ Į
2. $((s_3 \star s_1) \star (s_3 \star s_2)) \in$ Į and $s_2 \in$ Į imply $s_1 \in$ Į, $\forall\, s_1, s_2, s_3 \in \hbar$.

Definition 1.5. [4] In a nonempty set $\hbar$, a FNS $\wp$ defines the structure of a form as $\wp = \left\{ \left(\hbar, \aleph_{\wp}^{T}(\check{l}), \aleph_{\wp}^{I}(\check{l}), \aleph_{\wp}^{F}(\check{l})\right) | \check{l} \in \hbar \right\}$, where $\aleph_{\wp}^{T} : \hbar \to [0,1]$, $\aleph_{\wp}^{I} : \hbar \to [0,1]$ and $\aleph_{\wp}^{F} : \hbar \to [0,1]$ denotes a membership, indeterminate and a non-membership function respectively.

Definition 1.6. [12] An FNS $\wp = \left\langle \aleph^{T}, \aleph^{I}, \aleph^{F} \right\rangle$ is called a FNINK-S. If it satisfies all the conditions of INK algebra,
1. $\left(\aleph_{\wp}^{T}\right)(s_1 \star s_2) \leq \wedge \left\{ \left(\aleph_{\wp}^{T}\right)(s_1), \left(\aleph_{\wp}^{T}\right)(s_2) \right\}$
2. $\left(\aleph_{\wp}^{I}\right)(s_1 \star s_2) \geq \vee \left\{ \left(\aleph_{\wp}^{I}\right)(s_1), \left(\aleph_{\wp}^{I}\right)(s_2) \right\}$
3. $\left(\aleph_{\wp}^{F}\right)(s_1 \star s_2) \geq \vee \left\{ \left(\aleph_{\wp}^{F}\right)(s_1), \left(\aleph_{\wp}^{F}\right)(s_2) \right\}, \ \forall\, s_1, s_2, \in \hbar.$

Example 1.7. Define $\wp = \left\langle \aleph^{T}, \aleph^{I}, \aleph^{F} \right\rangle$ be an fermatean neutrosophic(FN) subset of $\hbar$ in Table 1.
Then $\wp$ is a FNINK-S of $\hbar$.

A	0	1	2	3
$\aleph^{T}$	0.8	0.6	0.4	0.3
$\aleph^{I}$	0.2	0.5	0.7	0.8
$\aleph^{F}$	0.1	0.3	0.6	0.9

Table 2: FNINK-S

Definition 1.8. [12] Let $\hbar$ represent an INK-algebra. FNS $\wp = \left\{\left(\hbar, \aleph^T_\wp(s_1), \aleph^I_\wp(s_1), \aleph^F_\wp(s_1)\right) | s_1 \in \hbar\right\}$ in $\hbar$ is called FN-I of $\hbar$, if it satisfies the subsequent requirements

1. $\aleph^T_\wp(0) \leq \aleph^T_\wp(s_1), \aleph^I_\wp(0) \geq \aleph^I_\wp(s_1)$ and $\aleph^F_A(0) \geq \aleph^F_\wp(s_1)$
2. $\aleph^T_\wp(s_1) \leq \wedge\left\{\aleph^T_\wp(s_1 \star s_2), \aleph^T_\wp(s_2)\right\}$
3. $\aleph^I_\wp(s_1) \geq \vee\left\{\aleph^I_\wp(s_1 \star s_2), \aleph^I_\wp(s_2)\right\}$
4. $\aleph^F_\wp(s_1) \geq \vee\left\{\aleph^F_\wp(s_1 \star s_2), \aleph^F_\wp(s_2)\right\}. \forall\, s_1, s_2 \in \hbar.$

Definition 1.9. [12] Let $\hbar$ be an INK-A. FNS $\wp = \left\{\left(\hbar, \aleph^T_\wp(s_1), \aleph^I_\wp(s_1), \aleph^F_\wp(s_1)\right) | s_1 \in \hbar\right\}$ in $\hbar$ is referred to as FNINK-I of $\hbar$, if it fulfills the following conditions

1. $\aleph^T_\wp(0) \leq \aleph^T_\wp(s_1), \aleph^I_\wp(0) \geq \aleph^I_\wp(s_1)$ and $\aleph^F_\wp(0) \geq \aleph^F_\wp(s_1)$
2. $\aleph^T_\wp(s_1) \leq \wedge\left\{\aleph^T_\wp((s_3 \star s_1) \star (s_3 \star s_2)), \aleph^T_\wp(s_2)\right\}$
3. $\aleph^I_\wp(s_1) \geq \vee\left\{\aleph^I_\wp((s_3 \star s_1) \star (s_3 \star s_2)), \aleph^I_\wp(s_2)\right\}$
4. $\aleph^F_\wp(s_1) \geq \vee\left\{\aleph^F_\wp((s_3 \star s_1) \star (s_3 \star s_2)), \aleph^F_\wp(s_2)\right\}, \ \forall\, s_1, s_2, s_3 \in \hbar.$

2 FNTINK-S

To simplify, we will use the symbol $\wp = \left(\aleph^T_\wp, \aleph^I_\wp, \aleph^F_\wp\right)$. for FN- subset $\wp = \left(s_1, \aleph^T_\wp, \aleph^I_\wp, \aleph^F_\wp; s_1 \in \hbar\right)$. Throughout this paper, we take $\c{T} = 1 - sup\left\{\aleph^T_\wp(s_l) | s_1 \in \hbar\right\}$ and $\c{I}, \c{F} = inf\left\{\aleph^F_\wp(s_1), \aleph^F_\wp(s_1) | s_1 \in \hbar\right\}$ for any FNS $\wp = \left(\aleph^T_\wp, \aleph^I_\wp, \aleph^F_\wp\right)$ of $\hbar$.

Definition 2.1. A FNS $\wp = \left(\aleph^T_\wp, \aleph^I_\wp, \aleph^F_\wp\right)$ be a FNSof $\hbar$ and $\iota, \rho, \omega \in [0, \tau]$. An object having the form $\left(\wp^{\c{T},\c{I},\c{F}}_{\iota,\rho,\omega}\right)^\tau = \left\langle \left(\aleph^T_\wp\right)^\tau_\iota, \left(\aleph^I_\wp\right)^\tau_\rho, \left(\aleph^F_\wp\right)^\tau_\omega \right\rangle$ is called a FN ι, ρ, ω translations of FNS, if it satisfies $\left(\aleph^T_\wp\right)^\tau_\iota(s_1) = \aleph^T_\wp(s_1) + \iota, \left(\aleph^I_\wp\right)^\tau_\rho(s_1) = \aleph^I_\wp(s_1) - \rho$ and $\left(\aleph^F_\wp\right)^\tau_\omega(s_1) = \aleph^F_\wp(s_1) - \omega$.

Definition 2.2. A FNS $\wp = \left\langle \aleph^{Tr}, \aleph^I, \aleph^F \right\rangle$ is referred to as a FNTINK-S if it satisfies every INK-A criterion,
1. $\left(\aleph^T_\wp\right)^\tau_\iota(s_1 \star s_2) = \wedge\left\{\left(\aleph^T_\wp\right)^\tau_\iota(s_1), \left(\aleph^T_\wp\right)^\tau_\iota(s_2)\right\}$
2. $\left(\aleph^I_\wp\right)^\tau_\rho(s_1 \star s_2) = \vee\left\{\left(\aleph^I_\wp\right)^\tau_\rho(s_1), \left(\aleph^I_\wp\right)^\tau_\rho(s_2)\right\}$
3. $\left(\aleph^F_\wp\right)^\tau_\omega(s_1 \star s_2) = \vee\left\{\left(\aleph^F_\wp\right)^\tau_\omega(s_1), \left(\aleph^F_\wp\right)^\tau_\omega(s_2)\right\}$.

Example 2.3. Let's examine INK-A in Table 2. Assume $\tau = 0.25$ and select $\iota = 0.2$, $\rho = 0.15$, and $\omega = 0.1$. In this case, the FN $\iota, \rho\, \omega$ translation is define by $(\aleph^T_\wp)_\iota, (\aleph^I_\wp)_\rho\ and\ (\aleph^F_\wp)_\omega$ of $\wp$ in $\hbar$.

A	0	1	2	3
$(\aleph^T_\wp)_\iota$	1	0.8	0.6	0.5
$(\aleph^I_\wp)_\rho$	0.05	0.35	0.55	0.65
$(\aleph^F_\wp)_\omega$	0	0.2	0.5	0.8

Table 3: FNTINK-Algebra

Then $\wp$ is a FNTINK-S of $\hbar$.

Definition 2.4. Let $\wp$ be a $\mathbb{N}_s$ of $\hbar$ and $\iota, \rho, \omega \in [0, 1]$. An item with the form $\left(\wp^{\c{T},\c{I},\c{F}}\right)^M_{\iota,\rho,\omega} = \left\langle \left(\aleph^T_\wp\right)^M_\iota, \left(\aleph^I_\wp\right)^M_\rho, \left(\aleph^F_\wp\right)^M_\omega \right\rangle$ is called a FN-M of $\wp$ if, $\left(\aleph^T_\wp\right)^M_\iota(s_1) = (\aleph^T_\wp)^M \cdot \iota, \left(\aleph^I_\wp\right)^M_\rho(s_1) = (\aleph^I_\wp)^M \cdot \rho$ and $\left(\aleph^F_\wp\right)^M_\omega(s_1) = (\aleph^F_\wp)^M \cdot \omega$, for all $s_1 \in \wp$.

A	0	1	2	3
$(\aleph_{\wp}^{T})_{\iota}^{M}$	0.32	0.24	0.16	0.12
$(\aleph_{\wp}^{I})_{\rho}^{M}$	0.06	0.15	0.21	0.24
$(\aleph_{\wp}^{F})_{\omega}^{M}$	0.02	0.06	0.12	0.18

Table 4: FN-M INK-A

Example 2.5. In the INK-algebra shown in Table 2, let $\tau = 0.5$, $\iota = 0.4$, $\rho = 0.3$, and $\omega = 0.2$. In this case, the FN $\iota, \rho, \ \omega$ multiplication is define by $(\aleph_{\wp}^{T})_{\iota}^{M}, (\aleph_{\wp}^{I})_{\rho}^{M}, (\aleph_{\wp}^{F})_{\omega}^{M})$ of $\wp$ in $\hbar$.

Then $\wp$ is a FNTINK-S of $\hbar$.

Theorem 2.6. *Let $\wp$ be a FNS of $\hbar$ such that the FNT $\left(\wp^{T,I,F}\right)_{\iota,\rho,\omega}^{\tau}$ of $\wp$ is a FNINK-S of $\hbar$, for some $\iota, \rho, \omega \in [0, \tau]$. Then $\wp$ is a FNINK-S of $\hbar$.*

Proof. Let $\left(\wp^{T,I,F}\right)_{\iota,\rho,\omega}^{\tau}$ is a FNINK-S of $\hbar$. For some $\iota, \rho, \omega \in [0, \tau]$.

$$\begin{aligned}
\aleph_{\wp}^{T}(s_1 \star s_2) + \iota &= (\aleph_{\wp}^{T})_{\iota}^{\tau}(s_1 \star s_2) \\
&\leq \wedge \left\{(\aleph_{\wp}^{T})_{\iota}^{\tau}(s_1), (\aleph_{\wp}^{T})_{\iota}^{\tau}(s_2)\right\} \\
&= \wedge \left\{\aleph_{\wp}^{T}(s_1) + \iota, \aleph_{\wp}^{T}(s_2) + \iota\right\} \\
&= \wedge \left\{\aleph_{\wp}^{T}(s_1), \aleph_{\wp}^{T}(s_2)\right\} + \iota \\
&= \wedge \left\{\aleph_{\wp}^{T}(s_1), \aleph_{\wp}^{T}(s_2)\right\} \\
\aleph_{\wp}^{I}(s_1 \star s_2) - \rho &= (\aleph_{\wp}^{I})_{\rho}^{\tau}(s_1 \star s_2) \\
&\geq \vee \left\{(\aleph_{\wp}^{I})_{\rho}^{\tau}(s_1), (\aleph_{\wp}^{I})_{\rho}^{\tau}(s_2)\right\} \\
&= \vee \left\{\aleph_{\wp}^{I}(s_1) - \rho, \aleph_{\wp}^{I}(s_2) - \rho\right\} \\
&= \vee \left\{\aleph_{\wp}^{I}(s_1), \aleph_{\wp}^{I}(s_2)\right\} - \rho \\
&= \vee \left\{\aleph_{\wp}^{I}(s_1), \aleph_{\wp}^{I}(s_2)\right\} \\
\aleph_{\wp}^{F}(s_1 \star s_2) - \omega &= (\aleph_{\wp}^{F})_{\omega}^{\tau}(s_1 \star s_2) \\
&\geq \vee \left\{(\aleph_{\wp}^{F})_{\omega}^{\tau}(s_1), (\aleph_{\wp}^{F})_{\omega}^{\tau}(s_2)\right\} \\
&= \vee \left\{\aleph_{\wp}^{F}(s_1) - \omega, \aleph_{\wp}^{F}(s_2) - \omega\right\} \\
&= \vee \left\{\aleph_{\wp}^{F}(s_1), \aleph_{\wp}^{F}(s_2)\right\} - \omega \\
&= \vee \left\{\aleph_{\wp}^{F}(s_1), \aleph_{\wp}^{F}(s_2)\right\}
\end{aligned}$$

□

Theorem 2.7. *If $\wp$ be a FNINK-S of $\hbar$,then the FN-M of $\wp$ is a FNINK-S a of $\hbar$ for all $\iota, \rho, \omega \in [0, 1]$.*

Proof. Assume that $\wp = \left\langle \aleph_{\wp}^{T}, \aleph_{\wp}^{I}, \aleph_{\wp}^{F} \right\rangle$ be a FNINK-S of $\hbar, \forall \iota, \rho, \omega \in [0, 1]$.

$$\begin{aligned}
(\aleph_{\wp}^{T})_{\iota}^{M}(s_1 \star s_2) &= \iota \cdot \aleph_{\wp}^{T}(s_1 \star s_2) \\
&= \iota \cdot \wedge \left\{\aleph_{\wp}^{T}(s_1), \aleph_{\wp}^{T}(s_2)\right\} \\
&= \wedge \left\{\iota \cdot \aleph_{\wp}^{T}(s_1), \iota \cdot \aleph_{\wp}^{T}(s_2)\right\} \\
&\leq \wedge \left\{(\aleph_{\wp}^{T})_{\iota}^{M}(s_1), (\aleph_{\wp}^{T})_{\iota}^{M}(s_2)\right\} \\
(\aleph_{\wp}^{I})_{\rho}^{M}(s_1 \star s_2) &= \rho \cdot \aleph_{\wp}^{I}(s_1 \star s_2)
\end{aligned}$$

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$$\begin{aligned}
&= \rho \cdot \vee \{\aleph_{\wp}^{I}(s_1), \aleph_{\wp}^{I}(s_2)\} \\
&= \vee \{\rho \cdot \aleph_{\wp}^{I}(s_1), \rho \cdot \aleph_{\wp}^{I}(s_2)\} \\
&\geq \vee \{(\aleph_{\wp}^{I})_{\rho}^{M}(s_1), (\aleph_{\wp}^{I})_{\rho}^{M}(s_2)\} \\
(\aleph_{\wp}^{F})_{\omega}^{M}(s_1 \star s_2) &= \omega \cdot \aleph_{\wp}^{F}(s_1 \star s_2) \\
&= \omega \cdot \vee \{\aleph_{\wp}^{F}(s_1), \aleph_{\wp}^{F}(s_2)\} \\
&= \vee \{\omega \cdot \aleph_{\wp}^{F}(s_1), \omega \cdot \aleph_{\wp}^{F}(s_2)\} \\
&\geq \vee \{(\aleph_{\wp}^{F})_{\omega}^{M}(s_1), (\aleph_{\wp}^{F})_{\omega}^{M}(s_2)\}
\end{aligned}$$

Hence $(\aleph_{\wp}^{T})_{\iota}^{M}$, $(\aleph_{\wp}^{I})_{\rho}^{M}$ and $(\aleph_{\wp}^{F})_{\omega}^{M}$ is a multiplication of FNINK-S of $\hbar$. □

3 Translation of FNINK-ideal

FNTINK-Is in INK-A is defined and its characteristics are examined in this section.

Definition 3.1. A FNS $\wp = \langle \aleph^{T}, \aleph^{I}, \aleph^{F} \rangle$ is considered as a FNINK-I of INK-A if,

1. $\left(\aleph_{\wp}^{T}\right)_{\iota}^{\tau}(0) \leq \left\{\left(\aleph_{\wp}^{T}\right)_{\iota}^{\tau}(s_1)\right\}, \left(\aleph_{\wp}^{I}\right)_{\rho}^{\tau}(0) \geq \left\{\left(\aleph_{\wp}^{I}\right)_{\rho}^{\tau}(s_1)\right\} and$ $\left(\aleph_{\wp}^{F}\right)_{\omega}^{\tau}(0) \geq \left\{\left(\aleph_{\wp}^{F}\right)_{\omega}^{\tau}(s_1)\right\}$
2. $\left(\aleph_{\wp}^{T}\right)_{\iota}^{\tau}(s_1) \leq \wedge \left\{\left(\aleph_{\wp}^{T}\right)_{\iota}^{\tau}((s_3 \star s_1) \star (s_3 \star s_2)), \left(\aleph_{\wp}^{T}\right)_{\iota}^{\tau}(s_2)\right\}$
3. $\left(\aleph_{\wp}^{I}\right)_{\rho}^{\tau}(s_1) \geq \vee \left\{\left(\aleph_{\wp}^{I}\right)_{\rho}^{\tau}((s_3 \star s_1) \star (s_3 \star s_2), \left(\aleph_{\wp}^{I}\right)_{\rho}^{\tau}(s_2)\right\}$
4. $\left(\aleph_{\wp}^{F}\right)_{\omega}^{\tau}(s_1) \geq \vee \left\{\left(\aleph_{\wp}^{F}\right)_{\omega}^{\tau}((s_3 \star s_1) \star (s_3 \star s_2), \left(\aleph_{\wp}^{F}\right)_{\omega}^{\tau}(s_2)\right\}$

Theorem 3.2. *If the FNT $\left(\wp^{T,I,F}\right)_{\iota,\rho,\omega}^{\tau}$ of $\wp$ is a FNINK-I of $\hbar$ for some $\iota, \rho, \omega \in [0, \tau]$, it must be a FN-ideal of $\hbar$.*

Proof. Let $\left(\wp^{T,I,F}\right)_{\iota,\rho,\omega}^{\tau}$ be FNT of $\hbar$.Then we have

$$\begin{aligned}
\left(\aleph_{\wp}^{Tr}\right)_{\iota}^{\tau}(s_1) &\leq \wedge \left\{\left(\aleph_{\wp}^{T}\right)_{\iota}^{\tau}((s_3 \star s_1) \star (s_3 \star s_2)), \left(\aleph_{\wp}^{T}\right)_{\iota}^{\tau}(s_2)\right\} \\
&put\ s_3 = 0 \\
\left(\aleph_{\wp}^{T}\right)_{\iota}^{\tau}(s_1) &\leq \wedge \left\{\left(\aleph_{\wp}^{T}\right)_{\iota}^{\tau}(0 \star s_1) \star (0 \star s_2), \left(\aleph_{\wp}^{T}\right)_{\iota}^{\tau}(s_2)\right\} \\
\left(\aleph_{\wp}^{T}\right)_{\iota}^{\tau}(s_1) &\leq \wedge \left\{\left(\aleph_{\wp}^{T}\right)_{\iota}^{\tau}(s_1 \star s_2), \left(\aleph_{\wp}^{T}\right)_{\iota}^{\tau}(s_2)\right\} \\
\left(\aleph_{\wp}^{I}\right)_{\rho}^{\tau}(s_1) &\geq \vee \left\{\left(\aleph_{\wp}^{I}\right)_{\rho}^{\tau}((s_3 \star s_1) \star (s_3 \star s_2)), \left(\aleph_{\wp}^{I}\right)_{\rho}^{\tau}(s_2)\right\} \\
&put\ s_3 = 0 \\
\left(\aleph_{\wp}^{I}\right)_{\rho}^{\tau}(s_1) &\geq \vee \left\{\left(\aleph_{\wp}^{I}\right)_{\rho}^{\tau}(0 \star s_1) \star (0 \star s_2), \left(\aleph_{\wp}^{I}\right)_{\rho}^{\tau}(s_2)\right\} \\
\left(\aleph_{\wp}^{I}\right)_{\rho}^{\tau}(s_1) &\geq \vee \left\{\left(\aleph_{\wp}^{I}\right)_{\rho}^{\tau}(s_1 \star s_2), \left(\aleph_{\wp}^{I}\right)_{\rho}^{\tau}(s_2)\right\} \\
\left(\aleph_{\wp}^{F}\right)_{\omega}^{\tau}(s_1) &\geq \vee \left\{\left(\aleph_{\wp}^{F}\right)_{\omega}^{\tau}((s_3 \star s_1) \star (s_3 \star s_2)), \left(\aleph_{\wp}^{F}\right)_{\omega}^{\tau}(s_2)\right\} \\
&put\ s_3 = 0 \\
\left(\aleph_{\wp}^{F}\right)_{\omega}^{\tau}(s_1) &\geq \vee \left\{\left(\aleph_{\wp}^{F}\right)_{\omega}^{\tau}(0 \star s_1) \star (0 \star s_2), \left(\aleph_{\wp}^{F}\right)_{\omega}^{\tau}(s_2)\right\} \\
\left(\aleph_{\wp}^{F}\right)_{\omega}^{\tau}(s_1) &\geq \vee \left\{\left(\aleph_{\wp}^{F}\right)_{\omega}^{\tau}(s_1 \star s_2), \left(\aleph_{\wp}^{F}\right)_{\omega}^{\tau}(s_2)\right\}
\end{aligned}$$

□

Theorem 3.3. *If FNT $\left(\wp^{T,I,F}\right)^{\tau}_{\iota,\rho,\omega}$ of $\wp$ is a FNINK-I of $\hbar$, for $\iota,\rho,\omega \in [0,1]$. If $s_1 \leq s_2$, then $(\aleph^T_\wp)^\tau_\iota(s_1) \leq (\aleph^T_\wp)^\tau_\iota(s_2)$, $(\aleph^I_\wp)^\tau_\rho(s_1) \geq (\aleph^I_\wp)^\tau_\rho(s_2)$ and $(\aleph^F_\wp)^\tau_\omega(s_1) \geq (\aleph^F_\wp)^\tau_\omega(s_2)$.*

Proof. Let $s_1, s_2, s_3 \in \hbar$ we have,

$$\begin{aligned}
(\aleph^T_\wp)^\tau_\iota(s_1) &\leq \wedge\left\{(\aleph^T_\wp)^\tau_\iota((s_3 \star s_1)\star(s_3\star s_2)), (\aleph^T_\wp)^\tau_\iota(s_2)\right\}\\
&= \wedge\left\{(\aleph^T_\wp)^\tau_\iota(s_1 \star s_2), (\aleph^T_\wp)^\tau_\iota(s_2)\right\}\\
&= \wedge\left\{(\aleph^T_\wp)^\tau_\iota(0), (\aleph^T_\wp)^\tau_\iota(s_2)\right\}\\
&= (\aleph^T_\wp)^\tau_\iota(s_2)\\
(\aleph^I_\wp)^\tau_\rho(s_1) &\geq \vee\left\{(\aleph^I_{\text{Ł}})^\tau_\rho((s_3 \star s_1)\star(s_3\star s_2)), (\aleph^I_A)^\tau_\rho(s_2)\right\}\\
&= \vee\left\{(\aleph^I_\wp)^\tau_\rho(s_1 \star s_2), (\aleph^I_\wp)^\tau_\rho(s_2)\right\}\\
&= \vee\left\{(\aleph^I_\wp)^\tau_\rho(0), (\aleph^I_\wp)^\tau_\rho(s_2)\right\}\\
&= (\aleph^I_\wp)^\tau_\rho(s_2)\\
(\aleph^F_\wp)^\tau_\omega(s_1) &\geq \vee\left\{(\aleph^F_\wp)^\tau_\omega((s_3 \star s_1)\star(s_3\star s_2)), (\aleph^F_\wp)^\tau_\omega(s_2)\right\}\\
&= \vee\left\{(\aleph^F_\wp)^\tau_\omega(s_1 \star s_2), (\aleph^F_\wp)^\tau_\omega(s_2)\right\}\\
&= \vee\left\{(\aleph^F_\wp)^\tau_\omega(0), (\aleph^F_\wp)^\tau_\omega(s_2)\right\}\\
&= (\aleph^F_\wp)^\tau_\omega(s_2)
\end{aligned}$$

□

Theorem 3.4. *If $\wp$ is a FNINK-I of $\hbar$, then FNT $\left(\wp^{T,I,F}\right)^{\tau}_{\iota,\rho,\omega}$ of A is a FNINK-S of $\hbar$.*

Proof.

$$\begin{aligned}
(\aleph^T_\wp)^\tau_\iota(s_1\star s_2) &= \aleph^T_\wp(s_1\star s_2)+\iota\\
&= \wedge\left\{\aleph^T_\wp((s_3\star(s_1\star s_2)\star(s_3\star s_2)), \aleph^T_\wp(s_2)\right\}+\iota \quad (Def.4.1\ in\ (2))\\
&= \wedge\left\{\aleph^T_\wp((s_1\star s_2)\star(s_3\star s_2)), \aleph^T_\wp(s_2)\right\}+\iota\\
&= \wedge\left\{\aleph^T_\wp(0), \aleph^T_\wp(s_2)\right\}+\iota\\
&\leq \wedge\left\{\aleph^T_\wp(s_1), \aleph^T_\wp(s_2)\right\}+\iota\\
(\aleph^T_\wp)^\tau_\iota(s_1\star s_2) &\leq \wedge\left\{(\aleph^T_\wp)^\tau_\iota(s_1), (\aleph^T_\wp)^\tau_\iota(s_2)\right\}+\iota\\
(\aleph^I_\wp)^\tau_\rho(s_1\star s_2) &= \aleph^I_\wp(s_1\star s_2)-\rho\\
&= \vee\left\{\aleph^I_\wp((s_3\star(s_1\star s_2)\star(s_3\star s_2)), \aleph^I_\wp(s_2)\right\}-\rho \quad (Def.4.1\ in\ (3))\\
&= \vee\left\{\aleph^I_\wp((s_1\star s_2)\star(s_3\star s_2)), \aleph^I_\wp(s_2)\right\}-\rho\\
&= \vee\left\{\aleph^I_\wp(0), \aleph^I_\wp(s_2)\right\}-\rho\\
&\geq \vee\left\{\aleph^I_\wp(s_1), \aleph^I_\wp(s_2)\right\}-\rho\\
&\geq \vee\left\{(\aleph^I_\wp)^\tau_\rho(s_1), (\aleph^I_\wp)^\tau_\rho(s_2)\right\}-\rho\\
(\aleph^F_\wp)^\tau_\omega(s_1\star s_2) &= \aleph^F_\wp(s_1\star s_2)-\omega\\
&= \vee\left\{\aleph^F_\wp((s_3\star(s_1\star s_2)\star(s_3\star s_2)), \aleph^F_\wp(s_2)\right\}-\omega\\
&= \vee\left\{\aleph^F_\wp((s_1\star s_2)\star(s_3\star s_2)), \aleph^F_\wp(s_2)\right\}-\omega \quad (Def.4.1\ in\ (4))\\
&= \vee\left\{\aleph^F_\wp(0), \aleph^F_\wp(s_2)\right\}-\omega\\
&\geq \vee\left\{\aleph^F_\wp(s_1), \aleph^F_\wp(s_2)\right\}-\omega\\
&\geq \vee\left\{(\aleph^F_\wp)^\tau_\omega(s_1), (\aleph^F_\wp)^\tau_\omega(s_2)\right\}-\omega.
\end{aligned}$$

□

Theorem 3.5. *Every FNT $(\aleph^T_\wp)^\tau_{\iota,\rho,\omega}$ of A is a FNINK-I of $\hbar$, if $\wp$ is a FNTINK-I of $\hbar$,for all $\iota,\rho,\omega \in [0,\tau]$.*

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Proof. Assume that $\wp$ is a FNINK-I of $\hbar$ and let $\iota, \rho, \omega \in [0, \tau]$.

$$
\begin{aligned}
(\aleph_{\wp}^{T})_{\iota}^{\tau}(0) &= (\aleph_{\wp}^{T})(s_1) + \iota \\
&\leq \wedge \left\{ (\aleph_{\wp}^{T})((s_3 \star s_1) \star (s_3 \star s_2)), (\aleph_{\wp}^{T})(s_3) \right\} + \iota \\
&= \wedge \left\{ (\aleph_{\wp}^{T})((s_3 \star s_1) \star (s_3 \star s_2)) + \iota, (\aleph_{\wp}^{T})(s_2) + \iota \right\} \\
(\aleph_{\wp}^{T})_{\iota}^{\tau}(0) &= \wedge \left\{ (\aleph_{\wp}^{T})_{\iota}^{\tau}((s_3 \star s_1) \star (s_3 \star s_2)), (\aleph_{\wp}^{T})_{\iota}^{\tau}(s_2) \right\}. \\
(\aleph_{\wp}^{I})_{\iota}^{\tau}(0) &= (\aleph_{\wp}^{I})(s_1) - \rho \\
&\geq \vee \left\{ (\aleph_{\wp}^{I})((s_3 \star s_1) \star (s_3 \star s_2)), (\aleph_{\wp}^{I})(s_2) \right\} - \rho \\
&= \vee \left\{ (\aleph_{\wp}^{I})((s_3 \star s_1) \star (s_3 \star s_2)) + \rho, (\aleph_{\wp}^{I})(s_2) - \rho \right\} \\
(\aleph_{\wp}^{I})_{\rho}^{\tau}(0) &= \vee \left\{ (\aleph_{\wp}^{I})_{\rho}^{\tau}((s_3 \star s_1) \star (s_3 \star s_2)), (\aleph_{\wp}^{I})_{\rho}^{\tau}(s_2) \right\}. \\
(\aleph_{\wp}^{F})_{\omega}^{\tau}(0) &= (\aleph_{\wp}^{F})(s_1) - \omega \\
&\geq \vee \left\{ (\aleph_{\wp}^{F})((s_3 \star s_1) \star (s_3 \star s_2)), (\aleph_{\wp}^{F})(s_2) \right\} - \omega \\
&= \vee \left\{ (\aleph_{\wp}^{F})((s_3 \star s_1) \star (s_1 \star s_2)) + \omega, (\aleph_{\wp}^{F})(s_2) - \omega \right\} \\
(\aleph_{\wp}^{F})_{\aleph}^{\tau}(0) &= \vee \left\{ (\aleph_{\wp}^{F})_{\omega}^{\tau}((s_3 \star s_1) \star (s_3 \star s_2)), (\aleph_{\wp}^{F})_{\omega}^{Ts}(s_2) \right\}.
\end{aligned}
$$

□

Theorem 3.6. *Let $\wp$ be a FN- subset of $\hbar$ such that FN ι, ρ, ω-multiplication of $\wp$ is a FNINK-I $\hbar$ for some $\iota, \rho, \omega \in [0, 1]$, then $\wp$ is a FNINK-I of $\hbar$.*

Proof. Assume that $(\aleph_{\wp}^{T})_{\iota,\rho,\omega}^{M}$ is a FNINK-I of $\hbar$,for some $\iota, \rho, \omega \in [0, 1]$.
Let $\iota, \rho, \omega \in [0, 1]$. Then

$$
\begin{aligned}
\iota \cdot (\aleph_{\wp}^{T})(0) &= (\aleph_{A}^{T})_{\iota}^{M}(0) \\
&\leq (\aleph_{\wp}^{T})_{\iota}^{M}(0) \\
&= \iota \cdot (\aleph_{\wp}^{T})(s_1) \\
(\aleph_{\wp}^{T})(0) &\leq (\aleph_{\wp}^{T})(s_1) \\
\iota \cdot (\aleph_{\wp}^{T})(s_1) &= (\aleph_{\wp}^{T})_{\iota}^{M}(s_1) \\
&\leq \wedge \left\{ \left(\aleph_{\wp}^{T}\right)_{\iota}^{M} ((s_3 \star s_1) \star (s_3 \star s_2)), \left(\aleph_{\wp}^{T}\right)_{\iota}^{M} (s_2) \right\} \\
&\leq \wedge \left\{ \iota \cdot \left(\aleph_{\wp}^{T}\right) ((s_3 \star s_1) \star (s_3 \star s_2)), \iota \cdot \left(\aleph_{\wp}^{T}\right) (s_2) \right\} \\
\iota \cdot (\aleph_{\wp}^{T})(s_1) &= \iota \cdot \wedge \left\{ \left(\aleph_{\wp}^{T}\right) ((s_3 \star s_1) \star (s_3 \star s_2)), \left(\aleph_{\wp}^{T}\right) (s_2) \right\} \\
(\aleph_{\wp}^{T})(s_1) &= \wedge \left\{ \left(\aleph_{\wp}^{T}\right) ((s_3 \star s_1) \star (s_3 \star s_2)), \left(\aleph_{\wp}^{T}\right) (s_2) \right\} \\
\rho \cdot (\aleph_{\wp}^{I})(0) &= (\aleph_{\wp}^{I})_{\rho}^{M}(0) \\
&\geq (\aleph_{\wp}^{I})_{\rho}^{M}(0) \\
&= \rho \cdot (\aleph_{\wp}^{I})(s_1) \\
(\aleph_{\wp}^{I})(0) &\geq (\aleph_{\wp}^{I})(s_1) \\
\rho \cdot (\aleph_{\wp}^{I})(s_1) &= (\aleph_{\wp}^{I})_{\rho}^{M}(s_1) \\
&\geq \vee \left\{ \left(\aleph_{\wp}^{I}\right)_{\rho}^{M} ((s_3 \star s_1) \star (s_3 \star s_2)), \left(\aleph_{\wp}^{I}\right)_{\rho}^{M} (s_2) \right\} \\
&\geq \vee \left\{ \rho \cdot \left(\aleph_{\wp}^{I}\right) ((s_3 \star s_1) \star (s_3 \star s_2)), \rho \cdot \left(\aleph_{\wp}^{I}\right) (s_2) \right\} \\
\rho \cdot (\aleph_{\wp}^{I})(s_1) &= \rho \cdot \vee \left\{ \left(\aleph_{\wp}^{I}\right) ((s_3 \star s_1) \star (s_3 \star s_2)), \left(\aleph_{\wp}^{I}\right) (s_2) \right\} \\
(\aleph_{\wp}^{I})(s_1) &= \vee \left\{ \left(\aleph_{\wp}^{I}\right) ((s_3 \star s_1) \star (s_3 \star s_2)), \left(\aleph_{\wp}^{I}\right) (s_2) \right\} \\
\omega \cdot (\aleph_{\wp}^{F})(0) &= (\aleph_{\wp}^{F})_{\omega}^{M}(0)
\end{aligned}
$$

$$
\begin{aligned}
& \geq (\aleph_{\wp}^{F})_{\iota}^{M}(0) \\
& = \omega \cdot (\aleph_{\wp}^{F})(s_1) \\
(\aleph_{\wp}^{F})(0) & \geq (\aleph_{\wp}^{F})(s_1) \\
\omega \cdot (\aleph_{\wp}^{F})(s_1) & = (\aleph_{\wp}^{F})_{\omega}^{M}(s_1) \\
& \geq \vee \left\{ \left(\aleph_{\wp}^{F}\right)_{\omega}^{M}((s_3 \star s_1) \star (s_3 \star s_2)), \left(\aleph_{\wp}^{F}\right)_{\omega}^{M}(s_2) \right\} \\
& \geq \vee \left\{ \omega \cdot \left(\aleph_{\wp}^{F}\right)((s_3 \star s_1) \star (s_3 \star s_2)), \omega \cdot \left(\aleph_{\wp}^{F}\right)(s_2) \right\} \\
\omega \cdot (\aleph_{\wp}^{F})(s_1) & = \omega \cdot \vee \left\{ \left(\aleph_{\wp}^{F}\right)((s_3 \star s_1) \star (s_3 \star s_2)), \left(\aleph_{\wp}^{F}\right)(s_2) \right\} \\
(\aleph_{\wp}^{F})(s_1) & = \vee \left\{ \left(\aleph_{\wp}^{F}\right)((s_3 \star s_1) \star (s_3 \star s_2)), \left(\aleph_{\wp}^{F}\right)(s_2) \right\}
\end{aligned}
$$

Hence $(\aleph_{\wp}^{T})_{\iota}^{M}, (\aleph_{\wp}^{T})_{\rho}^{M}$ and $(\aleph_{\wp}^{F})_{\omega}^{M}$ represents the multiple of the FNINK-I of $\hbar$. □

Theorem 3.7. *If the FNT $\left(\wp^{T,T,F}\right)_{\iota,\rho,\omega}^{M}$ of $\wp$ is a FNINK-I of $\hbar$, $(\iota, \rho, \omega) \in [0,1]$, then $\wp$ is a FNINK-S of $\hbar$.*

Proof. Let us assume that $\left(\wp^{T,I,F}\right)_{\iota,\rho,\omega}^{M}$ of $\wp$ is a FNINK-ideal of $\hbar$. Then

$$
\begin{aligned}
\iota \cdot (\aleph_{\wp}^{T})(s_1 \star s_2) & = (\aleph_{\wp}^{T})_{\iota}^{M}(s_1 \star s_2) \\
& \leq \wedge \left\{ \left(\aleph_{\wp}^{T}\right)_{\iota}^{M}((s_3 \star (s_1 \star s_2)) \star (s_3 \star s_2)), \left(\aleph_{\wp}^{T}\right)_{\iota}^{M}(s_2) \right\} \\
& \leq \wedge \left\{ \left(\aleph_{\wp}^{T}\right)_{\iota}^{M}((s_1 \star s_2) \star s_2)), \left(\aleph_{\wp}^{T}\right)_{\iota}^{M}(s_2) \right\} \quad (Def\ 4.1.\ (2)) \\
& \leq \wedge \left\{ \left(\aleph_{\wp}^{T}\right)_{\iota}^{M}(0), \left(\aleph_{\wp}^{T}\right)_{\iota}^{M}(s_2) \right\} \\
& = \wedge \left\{ \left(\aleph_{\wp}^{T}\right)_{\iota}^{M}(s_1), \left(\aleph_{\wp}^{T}\right)_{\iota}^{M}(s_2) \right\} \\
& = \wedge \left\{ \left(\iota \cdot \aleph_{\wp}^{T}\right)(s_1), \iota \cdot \left(\aleph_{\wp}^{T}\right)(s_2) \right\} \\
\iota \cdot (\aleph_{\wp}^{T})(s_1 \star s_2) & = \iota \cdot \wedge \left\{ \left((\aleph_{\wp}^{T}\right)(s_1), \left(\aleph_{\wp}^{T}\right)(s_2) \right\} \\
(\aleph_{\wp}^{T})(s_1 \star s_2) & = \wedge \left\{ \left((\aleph_{\wp}^{T}\right)(s_1), \left(\aleph_{\wp}^{T}\right)(s_2) \right\} \\
\rho \cdot (\aleph_{\wp}^{T})(s_1 \star s_2) & = (\aleph_{\wp}^{I})_{\rho}^{M}(s_1 \star s_2) \\
& \geq \vee \left\{ \left(\aleph_{\wp}^{I}\right)_{\rho}^{M}((s_3 \star (s_1 \star s_2)) \star (s_3 \star s_2)), \left(\aleph_{\wp}^{I}\right)_{\rho}^{M}(s_2) \right\} \\
& \geq \vee \left\{ \left(\aleph_{\wp}^{I}\right)_{\rho}^{M}((s_1 \star s_2) \star s_2)), \left(\aleph_{\wp}^{I}\right)_{\rho}^{M}(s_2) \right\} \quad (Def\ 4.1.\ (3)) \\
& \geq \vee \left\{ \left(\aleph_{\wp}^{I}\right)_{\rho}^{M}(0), \left(\aleph_{\wp}^{I}\right)_{\rho}^{M}(s_2) \right\} \\
& = \vee \left\{ \left(\aleph_{\wp}^{I}\right)_{\rho}^{M}(s_1), \left(\aleph_{\wp}^{I}\right)_{\rho}^{M}(s_2) \right\} \\
& = \vee \left\{ \left(\rho \cdot \aleph_{\wp}^{I}\right)(s_1), \rho \cdot \left(\aleph_{\wp}^{I}\right)(s_2) \right\} \\
\rho \cdot (\aleph_{\wp}^{I})(s_1 \star s_2) & = \rho \cdot \vee \left\{ \left((\aleph_{\wp}^{I}\right)(s_1), \left(\aleph_{\wp}^{I}\right)(s_2) \right\} \\
(\aleph_{\wp}^{I})(s_1 \star s_2) & = \vee \left\{ \left((\aleph_{\wp}^{I}\right)(s_1), \left(\aleph_{\wp}^{I}\right)(s_2) \right\} \\
\omega \cdot (\aleph_{\wp}^{F})(s_1 \star s_2) & = (\aleph_{\wp}^{F})_{\omega}^{M}(s_1 \star s_2) \\
& \geq \vee \left\{ \left(\aleph_{\wp}^{F}\right)_{\omega}^{M}((s_3 \star (s_1 \star s_2)) \star (s_3 \star s_2)), \left(\aleph_{\wp}^{F}\right)_{\omega}^{M}(s_2) \right\} \\
& \geq \vee \left\{ \left(\aleph_{\wp}^{F}\right)_{\omega}^{M}((s_1 \star s_2) \star s_2)), \left(\aleph_{\wp}^{F}\right)_{\omega}^{M}(s_2) \right\} \quad (Def\ 4.1.\ (4)) \\
& \geq \vee \left\{ \left(\aleph_{\wp}^{F}\right)_{\omega}^{M}(0), \left(\aleph_{A}^{F}\right)_{\omega}^{M}(s_2) \right\} \\
& = \vee \left\{ \left(\aleph_{\wp}^{F}\right)_{\omega}^{M}(s_1), \left(\aleph_{\wp}^{F}\right)_{\omega}^{M}(s_2) \right\} \\
& = \vee \left\{ \left(\omega \cdot \aleph_{\wp}^{F}\right)(s_1), \omega \cdot \left(\aleph_{\wp}^{F}\right)(s_2) \right\} \\
\omega \cdot (\aleph_{\wp}^{F})(s_1 \star s_2) & = \omega \cdot \vee \left\{ \left((\aleph_{\wp}^{F}\right)(s_1), \left(\aleph_{\wp}^{F}\right)(s_2) \right\} \\
(\aleph_{\wp}^{F})(s_1 \star s_2) & = \vee \left\{ \left((\aleph_{\wp}^{F}\right)(s_1), \left(\aleph_{\wp}^{F}\right)(s_2) \right\}
\end{aligned}
$$

□

Theorem 3.8. *A FNINK-I of $\hbar$ which contains the intersection and union of two FNT is also an FNINK-I $\hbar$.*

Proof. Let $\left(\wp^{T,I,F}\right)_{\iota_1,\rho_1,\omega_1}$ and $\left(\wp^{T,I,F}\right)_{\iota_2,\rho_2,\omega_2}$ be two FNT of a FNINK-I of $\hbar$ where $(\iota_1,\rho_1,\omega_1,\omega_2,\iota_2,\rho_2)\in[0,1]$. Assume that $\iota_1\leq\iota_2,\rho_1\leq\rho_2$ and $\omega_1\geq\omega_2$. Then $\left(A^{T,Id,Fs}\right)_{\iota_1,\rho_1,\omega_1}$ and $\left(\wp^{T,I,F}\right)_{\iota_2,\rho_2,\omega_2}$ are FNINK-I of $\hbar$.

$$\begin{aligned}
((\aleph_\wp^T)_{\iota_1}\cap(\aleph_\wp^T)_{\iota_2})(s_1) &= \wedge\{(\aleph_\wp^T)_{\iota_1}(s_1),(\aleph_\wp^T)_{\iota_2}(s_1)\}\\
&= \wedge\{(\aleph_\wp(s_1)+\iota_1,\aleph_\wp(s_1)+\iota_2\}\\
&= \aleph_\wp(s_1)+\iota_1\\
((\aleph_\wp^T)_{\iota_1}\cap(\aleph_\wp^T)_{\iota_2})(s_1) &= \aleph_\wp^T(s_1).\\
((\aleph_\wp^T)_{\rho_1}\cap(\aleph_\wp^I)_{\rho_2})(s_1) &= \vee\{(\aleph_\wp^T)_{\rho_1}(s_1),(\aleph_\wp^I)_{\rho_2}(s_1)\}\\
&= \vee\{(\aleph_\wp(s_1)+\rho_1,\aleph_\wp(s_1)+\rho_2\}\\
&= \aleph_\wp(s_1)+\rho_1\\
((\aleph_\wp^T)_{\rho_1}\cap(\aleph_\wp^I)_{\rho_2})(s_1) &= \aleph_\wp^T(s_1).\\
((\aleph_\wp^F)_{\omega_1}\cap(\aleph_\wp^T)_{\omega_2})(s_1) &= \vee\{(\aleph_\wp^F)_{\omega_1}(s_1),(\aleph_\wp^F)_{\omega_2}(s_1)\}\\
&= \vee\{(\aleph_\wp(s_1)+\omega_1,\aleph_\wp(s_1)+\omega_2\}\\
&= \aleph_\wp(s_1)+\omega_1\\
((\aleph_\wp^F)_{\omega_1}\cap(\aleph_\wp^F)_{\omega_2})(s_1) &= \aleph_\wp^F(s_1).
\end{aligned}$$

and

$$\begin{aligned}
((\aleph_\wp^T)_{\iota_1}\cup(\aleph_\wp^T)_{\iota_2})(s_1) &= \wedge\{(\aleph_\wp^T)_{\iota_1}(s_1),(\aleph_\wp^T)_{\iota_2}(s_1)\}\\
&= \wedge\{(\aleph_\wp(s_1)+\iota_1,\aleph_\wp(s_1)+\iota_2\}\\
&= \aleph_\wp(s_1)+\iota_1\\
((\aleph_\wp^T)_{\iota_1}\cup(\aleph_\wp^T)_{\iota_2})(s_1) &= \aleph_\wp^T(s_1).\\
((\aleph_\wp^T)_{\rho_1}\cup(\aleph_\wp^I)_{\rho_2})(s_1) &= \vee\{(\aleph_\wp^T)_{\rho_1}(s_1),(\aleph_\wp^I)_{\rho_2}(s_1)\}\\
&= \vee\{(\aleph_\wp(s_1)+\rho_1,\aleph_\wp(s_1)+\rho_2\}\\
&= \aleph_\wp(s_1)+\rho_1\\
((\aleph_\wp^T)_{\rho_1}\cup(\aleph_\wp^I)_{\rho_2})(s_1) &= \aleph_\wp^T(s_1).
\end{aligned}$$

$$\begin{aligned}
((\aleph_\wp^F)_{\omega_1}\cup(\aleph_\wp^T)_{\omega_2})(s_1) &= \vee\{(\aleph_\wp^F)_{\omega_1}(s_1),(\aleph_\wp^F)_{\omega_2}(s_1)\}\\
&= \vee\{(\aleph_\wp(s_1)+\omega_1,\aleph_\wp(s_1)+\omega_2\}\\
&= \aleph_\wp(s_1)+\omega_1\\
((\aleph_\wp^F)_{\omega_1}\cup(\aleph_\wp^F)_{\omega_2})(s_1) &= \aleph_\wp^F(s_1).
\end{aligned}$$

Hence $\left(\wp^{T,I,F}\right)_{\iota_1,\rho_1,\omega_1}$ and $\left(\wp^{T,I,F}\right)_{\iota_2,\rho_2,\omega_2}$ are FNINK-ideal of $\hbar$. □

4 Conclusion

This paper illuminates the practical consequences of FNINK-ideals by introducing and investigating their translation within the context of INK-algebras. It clarifies the relations between the FN-Ms included in these INK-ideals and fermatean neutrosophic translations, providing a foundation for further studies in INK-algebra theory. In the future, we may investigate the following topics related to the fermatean neutrosophic structure of INK-algebras: translating fermatean neutrosophic soft a-ideals in INK algebras, analyzing the results of translating fermatean neutrosophic soft d-ideals in INK algebras, and mapping out the translation dynamics of fermatean neutrophilic soft INK-ideals in INK-algebras. We think that exploring these areas of research will make a big difference in this field's development. **Funding:** Not applicable.
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