



Ideals and filters on intuitionistic fuzzy lattices

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Abstract

Based on the concept of Atanassov's intuitionistic fuzzy set on a universe X , we introduce the concepts of intuitionistic fuzzy ideals and intuitionistic fuzzy filters on an intuitionistic fuzzy lattice. More specifically, we provide characterizations of these concepts in terms of the intuitionistic fuzzy lattice meet and join operations, in terms of some associated fuzzy sets, as well as, in terms of their crisp level sets. Furthermore, we introduce the concepts of prime intuitionistic fuzzy ideals (resp. filters) as interesting kinds, and investigate their various properties and characterizations.

Keywords: Intuitionistic fuzzy set; Intuitionistic fuzzy order relation; Intuitionistic fuzzy ideal; Intuitionistic fuzzy filter

1 introduction

The concepts of ideal and filter are essential in various branches of classical mathematics and its applications. For example, in topology, they used to express completeness and compactness in metric spaces^{8,31} and, in measure theory, prime filters and ideals can be interpreted as basic components of probability measures. Filters and ideals are also particularly interesting as they are closely related to congruence relations.³⁰ They played a central role in the Stone representation theorem for Boolean lattices²⁵ and in the extensive theory of representation of distributive lattices.^{14,18,24}

After the introduction of fuzzy sets by Zadeh,³² there have been a number of generalizations of this fundamental concept. Atanassov² in 1983, introduced the concept of intuitionistic fuzzy set (IFSs) characterized by a membership function and a non-membership function, which is a generalization of fuzzy set. Intuitionistic fuzzy set theory has been applied in many fields, such as logic programming, decision-making, mathematical modelling, etc... . Further, many authors applied the concept of intuitionistic fuzzy sets to other algebraic structures such as groups, rings, lattices and so on.

Based on the concept of Atanassov's intuitionistic fuzzy set, Burillo and Bustince^{9,10} introduced the concept of an intuitionistic fuzzy relation. In particular, they introduced the intuitionistic fuzzy order relation as a natural generalization of the fuzzy order relation previously introduced by Zadeh.³³ Many authors proposed extended approaches based on intuitionistic fuzzy sets for the concept of ideals and filters. Kim and Jun¹⁹ introduced the concept of intuitionistic fuzzy ideal in a semigroup, Banerjee and Basnet⁵ defined intuitionistic fuzzy ideal on a ring. Recently, Akram and Dudek¹ introduced and studied the concept of intuitionistic fuzzy Lie ideal of Lie algebras. In particular, Thomas and Nair²⁶⁻²⁸ introduced the concept of intuitionistic fuzzy

ideal using the idea of fuzzification the membership function of elements on the carrier of a crisp lattice, and investigated some of its properties. In,²² we have characterized the notion of intuitionistic fuzzy ideal and filter introduced by Thomas and Nair on a crisp lattice in terms of the lattice operations, and in terms of some associated crisp sets. The present study is motivated by the above approaches; our generalization involves the intuitionistic fuzzy ideal and filter on an intuitionistic fuzzy lattice. More specifically, we provide interesting characterizations of these notions in terms of the intuitionistic fuzzy lattice meet and join operations, in terms of their (α, β) -level sets and in terms of some associated fuzzy sets. Moreover, we introduce the notion of prime intuitionistic fuzzy ideal (resp. filter) on an intuitionistic fuzzy lattice, and investigate their various properties and characterizations.

The remainder of this paper is organized as follows. Section 2, summaries preliminary concepts related to intuitionistic fuzzy set theory, mainly, intuitionistic fuzzy relations and the intuitionistic fuzzy lattices. In section 3, we introduce the notion of an intuitionistic fuzzy ideal (resp. filter) on an intuitionistic fuzzy lattice which generalizes the notion of fuzzy ideal (resp. filter) introduced by Mezzomo et al.²⁰ Some related characterizations are then investigated. In Section 4, we characterize the notion of intuitionistic fuzzy ideal (resp. filter) in terms of lattice operations and in terms of their (α, β) -level sets. Section 5 provides some interesting characterizations of prime intuitionistic fuzzy ideals and filters. Finally, we give concluding remarks and a suggestion for further research.

2 Basic concepts

In this section, we collect some basic definitions and properties of intuitionistic fuzzy sets, intuitionistic fuzzy lattices and some related notions and results on which our work in this paper is based. Throughout this paper, X and Y denote nonempty sets.

2.1 Intuitionistic fuzzy sets

A fuzzy set A on X is an object of the form $A = \{ \langle x, \mu_A(x) \rangle \mid x \in X \}$ characterized by a membership function $\mu_A : X \rightarrow [0, 1]$, where $\mu_A(x)$ is interpreted as the degree of membership of the element x in the fuzzy subset A , for each $x \in X$.

In 1983, Atanassov² proposed a generalization of Zadeh's membership degree and introduced the notion of the intuitionistic fuzzy set.

Definition 2.1.² An intuitionistic fuzzy set (IFS, for short) A on X is an object of the form $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}$ characterized by a membership function $\mu_A : X \rightarrow [0, 1]$ and a non-membership function $\nu_A : X \rightarrow [0, 1]$ with the condition:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \text{ for any } x \in X.$$

The class of intuitionistic fuzzy sets on X is denoted by $IFS(X)$.

In the fuzzy set theory, the non-membership degree of an element $x \in X$ is defined as $\nu_A(x) = 1 - \mu_A(x)$ (using the standard strong negation) and thus it is fixed. In intuitionistic fuzzy setting, the non-membership degree is more-or-less independent: the only condition is that $\nu_A(x) \leq 1 - \mu_A(x)$. Certainly, fuzzy sets are intuitionistic fuzzy sets by setting $\nu_A(x) = 1 - \mu_A(x)$.

For any two IFSs A and B on X , several operations are defined (see, e.g., Atanassov,^{3,4} Biswas,⁶ Wang and He¹⁶). Here we will present only those which are related to the present paper.

Let $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}$ and $B = \{ \langle x, \mu_B(x), \nu_B(x) \rangle \mid x \in X \}$ are two intuitionistic fuzzy sets on X . Then

- (i) $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, for all $x \in X$;
- (ii) $A = B$ if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$, for all $x \in X$;
- (iii) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x) \rangle \mid x \in X\}$;
- (iv) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x) \rangle \mid x \in X\}$.

For a given intuitionistic fuzzy set A on X , the following associated fuzzy sets \bar{A} , $[A]$ and $\langle A \rangle$ on X are defined by:

- (v) $\bar{A} = \{\langle x, \nu_A(x), \mu_A(x) \rangle \mid x \in X\}$;
- (vi) $[A] = \{\langle x, \mu_A(x) \rangle \mid x \in X\}$;
- (vii) $\langle A \rangle = \{\langle x, 1 - \nu_A(x) \rangle \mid x \in X\}$.

The support and the kernel of an intuitionistic fuzzy set A on X are defined by:

- (viii) $Supp(A) = \{x \in X \mid \mu_A(x) > 0 \text{ or } (\mu_A(x) = 0 \text{ and } \nu_A(x) < 1)\}$;
- (ix) $Ker(A) = \{x \in X \mid \mu_A(x) = 1 \text{ or } \nu_A(x) = 0\}$.

For further use, we recall the following definition of level sets (often called (α, β) -cuts) of intuitionistic fuzzy set.

Definition 2.2.¹⁷ Let A be an intuitionistic fuzzy set on X . The (α, β) -cut of A is the crisp subset

$$A_{(\alpha, \beta)} = \{x \in X \mid \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq \beta\},$$

where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$.

2.2 Intuitionistic fuzzy relations

Burillo and Bustince^{9,10} introduced the concept of intuitionistic fuzzy relation as a natural generalization of fuzzy relation.

Definition 2.3.^{9,10} An intuitionistic fuzzy binary relation (an intuitionistic fuzzy relation, for short) from X to Y is an intuitionistic fuzzy subset in $X \times Y$, i.e., is an expression R given by

$$R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y) \rangle \mid (x, y) \in X \times Y\},$$

where

$$\mu_R : X \times Y \rightarrow [0, 1], \text{ and } \nu_R : X \times Y \rightarrow [0, 1]$$

satisfy the condition

$$0 \leq \mu_R(x, y) + \nu_R(x, y) \leq 1, \tag{1}$$

for any $(x, y) \in X \times Y$. The value $\mu_R(x, y)$ is called the degree of a membership of (x, y) in R and $\nu_R(x, y)$ is called the degree of a non-membership of (x, y) in R .

Next, we need to recall the following definitions.

Definition 2.4. ^{9,10} Let R and P are two intuitionistic fuzzy relations from X to Y . R is said to be contained in P or we say that P contains R , denoted by $R \subseteq P$, if for all $(x, y) \in X \times Y$ it holds that $\mu_R(x, y) \leq \mu_P(x, y)$ and $\nu_R(x, y) \geq \nu_P(x, y)$.

The transpose (inverse) R^t of R is the intuitionistic fuzzy relation from Y to X defined by

$$R^t = \{ \langle (x, y), \mu_{R^t}(x, y), \nu_{R^t}(x, y) \rangle \mid (x, y) \in X \times Y \},$$

where

$$\begin{cases} \mu_{R^t}(x, y) = \mu_R(y, x) \\ \text{and} \\ \nu_{R^t}(x, y) = \nu_R(y, x), \end{cases}$$

for any $(x, y) \in X \times Y$.

The intersection of two intuitionistic fuzzy relations R and P from X to Y is defined as

$$R \cap P = \{ \langle (x, y), \mu_{R \cap P}(x, y), \nu_{R \cap P}(x, y) \rangle \},$$

where $\mu_{R \cap P}(x, y) = \min(\mu_R(x, y), \mu_P(x, y))$ and $\nu_{R \cap P}(x, y) = \max(\nu_R(x, y), \nu_P(x, y))$ for any $(x, y) \in X \times Y$.

The union of two intuitionistic fuzzy relations R and P from X to Y is defined as

$$R \cup P = \{ \langle (x, y), \mu_{R \cup P}(x, y), \nu_{R \cup P}(x, y) \rangle \},$$

where $\mu_{R \cup P}(x, y) = \max(\mu_R(x, y), \mu_P(x, y))$ and $\nu_{R \cup P}(x, y) = \min(\nu_R(x, y), \nu_P(x, y))$ for any $(x, y) \in X \times Y$.

In general, if A is a set of intuitionistic fuzzy relations from X to Y , then

$$\bigcap_{R \in A} R = \{ \langle (x, y), \mu_{\bigcap_{R \in A} R}(x, y), \nu_{\bigcap_{R \in A} R}(x, y) \rangle \},$$

where $\mu_{\bigcap_{R \in A} R}(x, y) = \inf_{R \in A} \mu_R(x, y)$ and $\nu_{\bigcap_{R \in A} R}(x, y) = \sup_{R \in A} \nu_R(x, y)$ for any $(x, y) \in X \times Y$;

$$\bigcup_{R \in A} R = \{ \langle (x, y), \mu_{\bigcup_{R \in A} R}(x, y), \nu_{\bigcup_{R \in A} R}(x, y) \rangle \},$$

where $\mu_{\bigcup_{R \in A} R}(x, y) = \sup_{R \in A} \mu_R(x, y)$ and $\nu_{\bigcup_{R \in A} R}(x, y) = \inf_{R \in A} \nu_R(x, y)$ for any $(x, y) \in X \times Y$.

Definition 2.5. ^{9,10} Let R be an intuitionistic fuzzy relation from X into itself (intuitionistic fuzzy relation on X , for short). The following properties are crucial in this paper (see, e.g., ^{9-13,15,21,29,34}):

(i) *Reflexivity*: $\mu_R(x, x) = 1$, for any $x \in X$. In this case, notice that $\nu_R(x, x) = 0$, for any $x \in X$.

(ii) *Antisymmetry*: for any $x, y \in X$ if $x \neq y$ then

$$\begin{cases} \mu_R(x, y) \neq \mu_R(y, x) \\ \nu_R(x, y) \neq \nu_R(y, x), \\ \pi_R(x, y) = \pi_R(y, x) \end{cases}$$

where $\pi_R(x, y) = 1 - \mu_R(x, y) - \nu_R(x, y)$.

(iii) *Perfect antisymmetry*: for any $x, y \in X$, if $x \neq y$ and

$$\begin{cases} \mu_R(x, y) > 0 \\ \text{or} \\ \mu_R(x, y) = 0 \text{ and } \nu_R(x, y) < 1 \end{cases},$$

then

$$\begin{cases} \mu_R(y, x) = 0 \\ \text{and} \\ \nu_R(y, x) = 1. \end{cases}$$

(iv) *Transitivity*: for any $x, y \in X$,

$$\begin{cases} \mu_R(x, y) \wedge \mu_R(y, z) \leq \mu_R(x, z) \\ \text{and} \\ \nu_R(x, y) \vee \nu_R(y, z) \geq \nu_R(x, z). \end{cases}$$

Remark 2.6. The definition of perfect antisymmetry given in (iii) is equivalent to the following condition:

(iii)' for any $x, y \in X$, $(\mu_R(x, y) > 0 \text{ and } \mu_R(y, x) > 0)$ or $(\nu_R(x, y) < 1 \text{ and } \nu_R(y, x) < 1)$ implies that $x = y$.

Note that Bustince and Burillo,¹² mentioned that the definition of intuitionistic fuzzy antisymmetry does not recover the fuzzy antisymmetry for the case in which the considered relation R is fuzzy. However, the definition of intuitionistic fuzzy perfect antisymmetry recovers the definition of fuzzy antisymmetry given by Zadeh³³ when the considered relation is fuzzy. This note justifies the following definition of intuitionistic fuzzy order used in this paper.

Definition 2.7.^{9,10} Let X be a nonempty crisp set and $R = \{(x, y), \mu_R(x, y), \nu_R(x, y) \mid x, y \in X\}$ be an intuitionistic fuzzy relation on X . R is called an intuitionistic fuzzy order or a partial intuitionistic fuzzy order if it is reflexive, transitive and perfect antisymmetric.

A nonempty set X with an intuitionistic fuzzy order R defined on it is called an intuitionistic fuzzy ordered set and we denote it by (X, μ_R, ν_R) .

Note that any partially ordered set (X, \leq) , and more generally, any fuzzy ordered set (X, R) can be viewed as intuitionistic fuzzy ordered sets.

Example 2.8. Let $X = \{a, b, c, d, e\}$. Then the intuitionistic fuzzy relation R defined on X by

$$R = \{(x, y), \mu_R(x, y), \nu_R(x, y) \mid x, y \in X\},$$

where μ_R and ν_R are given by the following tables:

$\mu_R(.,.)$	a	b	c	d	e	$\nu_R(.,.)$	a	b	c	d	e
a	1	0	0	0.55	0.40	a	0	1	0.40	0.45	0.25
b	0	1	0	0.35	0.45	b	0.30	0	0.20	0.35	0.10
c	0	0	1	0	0.70	c	1	1	0	0.85	0.15
d	0	0	0	1	0	d	1	1	1	0	1
e	0	0	0	0	1	e	1	1	1	0.90	0

is an intuitionistic fuzzy order on X .

2.3 Intuitionistic fuzzy lattice

For an intuitionistic fuzzy ordered set (X, μ_R, ν_R) and $x \in X$, the intuitionistic fuzzy sets $R_{\geq[x]}$ and $R_{\leq[x]}$ are defined on X as:

$$R_{\geq[x]} = \{(y, \mu_{R_{\geq[x]}}(y), \nu_{R_{\geq[x]}}(y)) \mid y \in X\}, \text{ where } \mu_{R_{\geq[x]}}(y) = \mu_R(x, y) \text{ and } \nu_{R_{\geq[x]}}(y) = \nu_R(x, y),$$

$$R_{\leq[x]} = \{(y, \mu_{R_{\leq[x]}}(y), \nu_{R_{\leq[x]}}(y)) \mid y \in X\}, \text{ where } \mu_{R_{\leq[x]}}(y) = \mu_R(y, x) \text{ and } \nu_{R_{\leq[x]}}(y) = \nu_R(y, x).$$

They are called the dominating class of x and the class dominated by x , respectively.

Remark 2.9. The dominating class of x and the class dominated by x are generalizations of the classical notions $\uparrow x$ and $\downarrow x$ in a usual poset.

Next, we recall definitions of the set of upper bounds, the set of lower bounds, the supremum and the infimum on intuitionistic fuzzy ordered sets.

Definition 2.10. ²⁹ Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set and A be a subset of X .

(i) The set of upper bounds of A with respect to R is the intuitionistic fuzzy subset of X defined by

$$U(R, A)(y) = \bigcap_{x \in A} R_{\geq [x]}, \tag{2}$$

(ii) The set of lower bounds of A with respect to R is the intuitionistic fuzzy subset of X defined by

$$L(R, A)(y) = \bigcap_{x \in A} R_{\leq [x]}, \tag{3}$$

Definition 2.11. ²⁹ Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set and A be a subset of X . An element $x \in X$ is called the least upper bound (or a supremum) of A with respect to R if:

- (i) $x \in Supp(U(R, A))$ and
- (ii) for all other $y \in Supp(U(R, A))$, $\mu_R(x, y) > 0$ or $(\mu_R(x, y) = 0$ and $\nu_R(x, y) < 1)$.

An element $x \in X$ is called the greatest lower bound (or an infimum) of A with respect to R if:

- (i) $x \in Supp(L(R, A))$ and
- (ii) for all $y \in Supp(L(R, A))$, $\mu_R(y, x) > 0$ or $(\mu_R(y, x) = 0$ and $\nu_R(y, x) < 1)$.

Remark 2.12. Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set and A be a subset of X . If a supremum and an infimum of A with respect to R exist, then from the perfect antisymmetry of R they are unique and denoted by $\sqcup_R(A)$ and $\sqcap_R(A)$, respectively.

Definition 2.13. ²⁹ An intuitionistic fuzzy ordered set (X, μ_R, ν_R) is called an intuitionistic fuzzy lattice (IF-lattice, for short) with respect to the intuitionistic fuzzy order R if each pair of elements $\{x, y\}$ of X has a supremum and an infimum. In particular, if R is only a fuzzy order, then under the same conditions (X, R) is called a fuzzy lattice.

Remark 2.14. To avoid any confusion or miss understanding of some equations, we use the notation (R, \sqcap_R, \sqcup_R) to refer the (IF-order, min, max) on the IF-lattice X and (\leq, \wedge, \vee) to refer the (usual order, min, max) on the real interval $[0, 1]$.

Example 2.15. Let R be an intuitionistic fuzzy relation on $X = \{a, b, c, d, e\}$ defined by the following tables:

$\mu_R(.,.)$	a	b	c	d	e	$\nu_R(.,.)$	a	b	c	d	e
a	1	0	0	0.7	0	a	0	1	1	0.2	1
b	0	1	0	0.5	0	b	1	0	1	0.3	1
c	0.2	0.3	1	0.4	0	c	0.8	0.5	0	0.3	1
d	0	0	0	1	0	d	1	1	1	0	1
e	0.2	0.3	0.35	0.6	1	e	0.5	0.4	0.35	0.22	0

It is easy to verify that R is reflexive, perfect antisymmetric, max-min and min-max transitive. Then, (X, μ_R, ν_R) is an intuitionistic fuzzy ordered set. The following table describes the supremum and the infimum of each pair of elements $\{x, y\}$ on X .

$\{x, y\}$	$x \sqcup_R y$	$x \sqcap_R y$
$\{a, b\}$	d	c
$\{a, c\}$	a	c
$\{a, d\}$	d	a
$\{a, e\}$	a	e
$\{b, c\}$	b	c
$\{b, d\}$	d	b
$\{b, e\}$	b	e
$\{c, d\}$	d	c
$\{c, e\}$	c	e
$\{d, e\}$	d	e

So, (X, μ_R, ν_R) is an IF-lattice.

Definition 2.16. ²⁹ Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set.

- (i) An element $\top \in X$ is called the greatest element (the maximum) of X with respect to R or the intuitionistic fuzzy maximum of X if

$$\begin{cases} \mu_R(x, \top) > 0 \\ \text{or} \\ \mu_R(x, \top) = 0 \text{ and } \nu_R(x, \top) < 1, \end{cases}$$

for all $x \in X$.

- (ii) An element $\perp \in X$ is called the least element (the minimum) of X with respect to R or the intuitionistic fuzzy minimum if

$$\begin{cases} \mu_R(\perp, x) > 0 \\ \text{or} \\ \mu_R(\perp, x) = 0 \text{ and } \nu_R(\perp, x) < 1, \end{cases}$$

for all $x \in X$.

In,²⁰ Mezzomo et al. characterized the fuzzy ideals and fuzzy filters in terms of the support and in terms of their level sets.

Proposition 2.17. ²⁰ Let (X, R) be a fuzzy lattice and A be a fuzzy set on X . The following implications hold:

- (i) If A is a fuzzy ideal on (X, R) , then its support $Supp(A)$ is an ideal on $(X, Supp(R))$;
- (ii) If A is a fuzzy filter on (X, R) , then its support $Supp(A)$ is a filter on $(X, Supp(R))$.

Theorem 2.18. ²⁰ Let (X, R) be a fuzzy lattice and A be a fuzzy set on X . The following equivalences hold:

- (i) A is a fuzzy ideal on (X, R) if and only if its level set A_α is an ideal on (X, R_α) , for any $\alpha \in]0, 1]$;
- (ii) A is a fuzzy filter on (X, R) if and only if its level set A_α is a filter on (X, R_α) , for any $\alpha \in]0, 1]$.

3 Intuitionistic fuzzy ideals and filters on an intuitionistic fuzzy lattice

This section introduces the notions of intuitionistic fuzzy ideals (resp. filters) on an intuitionistic fuzzy lattice, and some basic properties are given.

Definition 3.1. Let (X, μ_R, ν_R) be an IF-lattice. An intuitionistic fuzzy subset A on X is called an intuitionistic fuzzy ideal (IF-ideal, for short) on (X, μ_R, ν_R) if for all $x, y \in X$ the following conditions hold:

- (i) $\mu_A(x \sqcup_R y) \geq \mu_A(x) \wedge \mu_A(y)$;
- (ii) $\nu_A(x \sqcup_R y) \leq \nu_A(x) \vee \nu_A(y)$;
- (iii) $\mu_A(x \sqcap_R y) \geq \mu_A(x) \vee \mu_A(y)$;
- (iv) $\nu_A(x \sqcap_R y) \leq \nu_A(x) \wedge \nu_A(y)$.

Definition 3.2. Let (X, μ_R, ν_R) be an IF-lattice. An intuitionistic fuzzy subset A on X is called an intuitionistic fuzzy filter (IF-filter, for short) on (X, μ_R, ν_R) if for all $x, y \in X$ the following conditions hold:

- (i) $\mu_A(x \sqcup_R y) \geq \mu_A(x) \vee \mu_A(y)$;
- (ii) $\nu_A(x \sqcup_R y) \leq \nu_A(x) \wedge \nu_A(y)$;
- (iii) $\mu_A(x \sqcap_R y) \geq \mu_A(x) \wedge \mu_A(y)$;
- (iv) $\nu_A(x \sqcap_R y) \leq \nu_A(x) \vee \nu_A(y)$.

Example 3.3. Consider the IF-lattice (X, μ_R, ν_R) of Example 2.15, A and B are two intuitionistic fuzzy sets on X . Then

- (i) $A = \{ \langle a, 0.5, 0.4 \rangle, \langle b, 0.4, 0.5 \rangle, \langle c, 0.6, 0.3 \rangle, \langle d, 0.3, 0.6 \rangle \}$ is an IF-ideal on (X, μ_R, ν_R) ;
- (ii) $B = \{ \langle a, 0.3, 0.6 \rangle, \langle b, 0.5, 0.4 \rangle, \langle c, 0.2, 0.7 \rangle, \langle d, 0.7, 0.1 \rangle \}$ is an IF-filter on (X, μ_R, ν_R) .

The following propositions are immediate.

Proposition 3.4. Let (X, μ_R, ν_R) be an IF-lattice and A be an IF-ideal on it. Then it holds that

- (i) $\mu_A(x) \geq \mu_A(y) \wedge \mu_R(x, y)$;
- (ii) $\nu_A(x) \leq \nu_A(y) \vee \nu_R(x, y)$.

Proposition 3.5. Let (X, μ_R, ν_R) be an IF-lattice and A be an IF-filter on it. Then it holds that

- (i) $\mu_A(x) \geq \mu_A(y) \wedge \mu_R(y, x)$;
- (ii) $\nu_A(x) \leq \nu_A(y) \vee \nu_R(y, x)$.

The following proposition expresses the relationship between an IF-ideal and an IF-filter on an IF-lattice. Its proof is straightforward.

Proposition 3.6. Let $(X, \mu_R, \nu_R, \sqcap_R, \sqcup_R)$ be an IF-lattice, $(X, \mu_{R^t}, \nu_{R^t}, \sqcup_R, \sqcap_R)$ be its dual order IF-lattice and $A \in IFS(X)$. Then it holds that A is an IF-ideal on $(X, \mu_R, \nu_R, \sqcap_R, \sqcup_R)$ if and only if A is an IF-filter on $(X, \mu_{R^t}, \nu_{R^t}, \sqcup_R, \sqcap_R)$ and vice versa.

Combining Definitions 3.1 and 3.2 easily leads to the following propositions.

Proposition 3.7. Let (X, μ_R, ν_R) be an IF-lattice, A and B are two intuitionistic fuzzy sets on X . Then it holds that

- (i) if A and B are two IF-ideals on (X, μ_R, ν_R) , then $A \cap B$ is an IF-ideal on (X, μ_R, ν_R) ;
- (ii) if A and B are two IF-filters on (X, μ_R, ν_R) , then $A \cap B$ is an IF-filter on (X, μ_R, ν_R) .

Proof. (i) Let $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}$

and $B = \{ \langle x, \mu_B(x), \nu_B(x) \rangle \mid x \in X \}$ be two intuitionistic fuzzy sets. Then $A \cap B = \{ \langle x, \mu_{A \cap B}(x), \nu_{A \cap B}(x) \rangle \mid x \in X \}$, where $\mu_{A \cap B}(x) = \mu_A(x) \wedge \mu_B(x)$ and $\nu_{A \cap B}(x) = \nu_A(x) \vee \nu_B(x)$.

$$\begin{aligned} \mu_{A \cap B}(x \sqcup_R y) &= \mu_A(x \sqcup_R y) \wedge \mu_B(x \sqcup_R y) \\ &\geq (\mu_A(x) \wedge \mu_A(y)) \wedge (\mu_B(x) \wedge \mu_B(y)) \\ &\geq (\mu_A(x) \wedge \mu_B(x)) \wedge (\mu_A(y) \wedge \mu_B(y)) \\ &\geq \mu_{A \cap B}(x) \wedge \mu_{A \cap B}(y). \end{aligned}$$

Similarly, we get that $\nu_{A \cap B}(x \sqcup_R y) \leq \nu_{A \cap B}(x) \vee \nu_{A \cap B}(y)$.

$$\begin{aligned} \mu_{A \cap B}(x \sqcap_R y) &= \mu_A(x \sqcap_R y) \wedge \mu_B(x \sqcap_R y) \\ &\geq (\mu_A(x) \vee \mu_A(y)) \wedge (\mu_B(x) \vee \mu_B(y)) \\ &\geq (\mu_A(x) \wedge \mu_B(x)) \vee (\mu_A(y) \wedge \mu_B(y)) \\ &\geq \mu_{A \cap B}(x) \vee \mu_{A \cap B}(y). \end{aligned}$$

Similarly, we get that $\nu_{A \cap B}(x \sqcap_R y) \leq \nu_{A \cap B}(x) \wedge \nu_{A \cap B}(y)$.

Hence, $A \cap B$ is an IF-ideal on (X, μ_R, ν_R) .

(ii) Follows from Proposition 3.6 and (i). □

Remark 3.8. The union of two IF-ideals (resp. IF-filters) is not necessarily an IF-ideal (resp. IF-filter). Indeed, let us consider the IF-lattice (X, μ_R, ν_R) given in Example 2.15, A and B are two intuitionistic fuzzy sets on (X, μ_R, ν_R) defined by $A = \{ \langle a, 0.5, 0.4 \rangle, \langle b, 0.4, 0.5 \rangle, \langle c, 0.6, 0.3 \rangle, \langle d, 0.3, 0.6 \rangle \}$ and $B = \{ \langle a, 0.6, 0.4 \rangle, \langle b, 0.5, 0.5 \rangle, \langle c, 0.7, 0.3 \rangle, \langle d, 0.4, 0.4 \rangle \}$. We can see that A and B are two IF-ideals on (X, μ_R, ν_R) . We have

$A \cup B = \{ \langle a, 0.6, 0.4 \rangle, \langle b, 0.5, 0.5 \rangle, \langle c, 0.7, 0.3 \rangle, \langle d, 0.4, 0.4 \rangle \}$. We can see that $\nu_{A \cup B}(b) = 0.5 \not\leq \nu_{A \cup B}(d) \vee \nu_R(b, d) = 0.4$, which prove that $A \cup B$ is not an IF-ideal on (X, μ_R, ν_R) .

4 Characterizations of IF-ideals and IF-filters on an IF-lattice

This section provides some interesting characterizations of IF-ideals and IF-filters on an IF-lattice.

4.1 Auxiliary results

In this section, we show some auxiliary results which will be helpful to characterize the IF-ideals (resp. filters) on an IF-lattice. First, we need the following key results.

Theorem 4.1. Let (X, μ_R, ν_R) be an IF-lattice and $A \in IFS(X)$. For all $x, y \in X$, the following equivalences hold:

- (i) $\mu_A(x \sqcap_R y) \geq \mu_A(x) \vee \mu_A(y)$ if and only if $(\mu_R(x, y) > 0$ or $(\mu_R(x, y) = 0$ and $\nu_R(x, y) < 1)) \Rightarrow (\mu_A(x) \geq \mu_A(y))$;
- (ii) $\mu_A(x \sqcup_R y) \geq \mu_A(x) \vee \mu_A(y)$ if and only if $(\mu_R(x, y) > 0$ or $(\mu_R(x, y) = 0$ and $\nu_R(x, y) < 1)) \Rightarrow (\mu_A(x) \leq \mu_A(y))$;
- (iii) $\nu_A(x \sqcap_R y) \leq \nu_A(x) \wedge \nu_A(y)$ if and only if $(\mu_R(x, y) > 0$ or $(\mu_R(x, y) = 0$ and $\nu_R(x, y) < 1)) \Rightarrow (\nu_A(x) \leq \nu_A(y))$;
- (iv) $\nu_A(x \sqcup_R y) \leq \nu_A(x) \wedge \nu_A(y)$ if and only if $(\mu_R(x, y) > 0$ or $(\mu_R(x, y) = 0$ and $\nu_R(x, y) < 1)) \Rightarrow (\nu_A(x) \geq \nu_A(y))$.

Proof. We only prove the first equivalence (i). The other equivalences can be proved analogously. Let $x, y \in X$.

(i) Suppose that $\mu_A(x \sqcap_R y) \geq \mu_A(x) \vee \mu_A(y)$. If $\mu_R(x, y) > 0$ or $(\mu_R(x, y) = 0$ and $\nu_R(x, y) < 1)$ then $x \sqcap_R y = x$. Hence, $\mu_A(x) = \mu_A(x \sqcap_R y) \geq \mu_A(x) \vee \mu_A(y)$. Thus, $\mu_A(x) \geq \mu_A(y)$.

Conversely, suppose that $(\mu_R(x, y) > 0$ or $(\mu_R(x, y) = 0$ and $\nu_R(x, y) < 1) \Rightarrow \mu_A(x) \geq \mu_A(y)$. Since $(\mu_R(x \sqcap_R y, x) > 0$ or $(\mu_R(x \sqcap_R y, x) = 0$ and $\nu_R(x \sqcap_R y, x) < 1))$ and $(\mu_R(x \sqcap_R y, y) > 0$ or $(\mu_R(x \sqcap_R y, y) = 0$ and $\nu_R(x \sqcap_R y, y) < 1))$, it follows that $\mu_A(x \sqcap_R y) \geq \mu_A(x)$ and $\mu_A(x \sqcap_R y) \geq \mu_A(y)$. Thus, $\mu_A(x \sqcap_R y) \geq \mu_A(x) \vee \mu_A(y)$.

□

As corollaries, we obtain the following interesting properties of IF-ideals and IF-filters on an IF-lattice.

Corollary 4.2. Let (X, μ_R, ν_R) be an IF-lattice and I be an IF-ideal on it. Then for any $x, y \in X$ it holds that

- (i) if $\mu_R(x, y) > 0$ or $(\mu_R(x, y) = 0$ and $\nu_R(x, y) < 1)$, then $\mu_I(x) \geq \mu_I(y)$, (i.e., the map $\mu_I : X \rightarrow [0, 1]$ is antitone);
- (ii) if $\mu_R(x, y) > 0$ or $(\mu_R(x, y) = 0$ and $\nu_R(x, y) < 1)$, then $\nu_I(x) \leq \nu_I(y)$, (i.e., the map $\nu_I : X \rightarrow [0, 1]$ is monotone).

Corollary 4.3. Let (X, μ_R, ν_R) be an IF-lattice and F be an IF-filter on it. Then for any $x, y \in X$ it holds that

- (i) if $\mu_R(x, y) > 0$ or $(\mu_R(x, y) = 0$ and $\nu_R(x, y) < 1)$, then $\mu_F(x) \leq \mu_F(y)$, (i.e., the map $\mu_F : X \rightarrow [0, 1]$ is monotone);
- (ii) if $\mu_R(x, y) > 0$ or $(\mu_R(x, y) = 0$ and $\nu_R(x, y) < 1)$, then $\nu_F(x) \geq \nu_F(y)$, (i.e., the map $\nu_F : X \rightarrow [0, 1]$ is antitone).

Corollary 4.4. Let (X, μ_R, ν_R) an IF-lattice with the smallest element \perp or the greatest element \top and I be an IF-ideal on it. Then it holds that

- (i) $\mu_I(\perp) = \max \mu_I(X)$ and $\mu_I(\top) = \min \mu_I(X)$, where $\mu_I(X) = \{\mu_I(x) \mid x \in X\}$;
- (ii) $\nu_I(\perp) = \min \nu_I(X)$ and $\nu_I(\top) = \max \nu_I(X)$, where $\nu_I(X) = \{\nu_I(x) \mid x \in X\}$.

Corollary 4.5. Let (X, μ_R, ν_R) an IF-lattice with the smallest element \perp or the greatest element \top and F be an IF-filter on it. Then it holds that

- (i) $\mu_F(\perp) = \min \mu_F(X)$ and $\mu_F(\top) = \max \mu_F(X)$, where $\mu_F(X) = \{\mu_F(x) \mid x \in X\}$;
- (ii) $\nu_F(\perp) = \max \nu_F(X)$ and $\nu_F(\top) = \min \nu_F(X)$, where $\nu_F(X) = \{\nu_F(x) \mid x \in X\}$.

For a given IF-lattice (X, μ_R, ν_R) , the following proposition shows that the crisp structures $(X, Supp(R))$, $(X, Ker(R))$ and $(X, R_{(\alpha, \beta)})$, for any $((\alpha, \beta) \in]0, 1[\times]0, 1[$ satisfying $\alpha + \beta \leq 1$) are crisp lattices.

Proposition 4.6. Let R be an intuitionistic fuzzy relation on X . The following implications hold:

- (i) if (X, μ_R, ν_R) is an IF-lattice, then $(X, Supp(R))$ is a crisp lattice;
- (ii) if (X, μ_R, ν_R) is an IF-lattice, then $(X, Ker(R))$ is a crisp lattice;
- (iii) if (X, μ_R, ν_R) is an IF-lattice, then $(X, R_{(\alpha, \beta)})$ is a crisp lattice.

Proof. We only prove (i), as (ii) and (iii) can be proved analogously. Let (X, μ_R, ν_R) is an IF-lattice, then the fact that R is reflexive implies that $(x, x) \in Supp(R)$.

Let $x, y \in X$ such that $x \neq y$,

$$\begin{cases} \mu_R(x, y) > 0 \text{ or } (\mu_R(x, y) = 0 \wedge \nu_R(x, y) < 1) \\ \text{and} \\ \mu_R(y, x) > 0 \text{ or } (\mu_R(y, x) = 0 \wedge \nu_R(y, x) < 1). \end{cases}$$

Four cases to consider

- (a) If $\mu_R(x, y) > 0$ and $\mu_R(y, x) > 0$, then the fact that R is perfect antisymmetry implies that $\mu_R(y, x) = 0$. This is a contradiction with the hypothesis $\mu_R(y, x) > 0$.
- (b) Suppose that $\mu_R(x, y) > 0$ and $(\mu_R(y, x) = 0 \wedge \nu_R(y, x) < 1)$, the fact that R is perfect antisymmetry and $\mu_R(x, y) > 0$ implies that $\nu_R(y, x) = 1$, and this is a contradiction with the hypothesis $\nu_R(y, x) < 1$.
- (c) Suppose that $(\mu_R(x, y) = 0 \wedge \nu_R(x, y) < 1)$ and $\mu_R(y, x) > 0$. The proof of this case is analogous to that of (b).
- (d) Suppose that $(\mu_R(x, y) = 0 \wedge \nu_R(x, y) < 1)$ and $(\mu_R(y, x) = 0 \wedge \nu_R(y, x) < 1)$, then the fact that R is perfect antisymmetry implies that $\nu_R(y, x) = 1$. This is a contradiction with the hypothesis $\nu_R(y, x) < 1$. Thus, $x = y$.

Let $x, y, z \in X$ such that $(x, y) \in Supp(R)$ and $(y, z) \in Supp(R)$. Then

$$\begin{cases} \mu_R(x, y) > 0 \text{ or } (\mu_R(x, y) = 0 \wedge \nu_R(x, y) < 1) \\ \text{and} \\ \mu_R(y, z) > 0 \text{ or } (\mu_R(y, z) = 0 \wedge \nu_R(y, z) < 1), \end{cases}$$

We consider four cases:

- (e) If $\mu_R(x, y) > 0$ and $\mu_R(y, z) > 0$, then the fact that R is transitive (i.e., $\mu_R(x, y) \wedge \mu_R(y, z) \leq \mu_R(x, z)$) implies that $\mu_R(x, z) > 0$. Hence, $(x, z) \in Supp(R)$.
- (f) Suppose that $\mu_R(x, y) > 0$ and $(\mu_R(y, z) = 0 \wedge \nu_R(y, z) < 1)$, then the fact that R is transitive implies that $\mu_R(x, z) > 0$ or $(\mu_R(x, z) = 0 \wedge \nu_R(x, z) \leq \nu_R(x, y) \vee \nu_R(y, z) < 1)$. Hence, $(x, z) \in Supp(R)$.
- (g) Suppose that $(\mu_R(x, y) = 0 \wedge \nu_R(x, y) < 1)$ and $\mu_R(y, z) > 0$. The proof of this case is analogous to that of (b).
- (h) Suppose that $(\mu_R(x, y) = 0 \wedge \nu_R(x, y) < 1)$ and $(\mu_R(y, z) = 0 \wedge \nu_R(y, z) < 1)$, then the fact that R is transitive (i.e., $\mu_R(x, y) \wedge \mu_R(y, z) \leq \mu_R(x, z)$ and $\nu_R(x, y) \vee \nu_R(y, z) \geq \nu_R(x, z)$) implies that $\mu_R(x, z) = 0$ and $\nu_R(x, z) < 1$. Hence, $(x, z) \in Supp(R)$.

Thus, $(X, Supp(R))$ is a crisp poset. Moreover, $x \sqcup_R y$ and $x \sqcap_R y$ coincide with $x \sqcup y$ and $x \sqcap y$ on $(X, Supp(R))$. We conclude that $(X, Supp(R))$ is a crisp lattice. □

In the same direction, the following proposition shows that the associated fuzzy structures $(X, [R])$ and $(X, \langle R \rangle)$ of an IF-lattice (X, μ_R, ν_R) are fuzzy lattices.

Proposition 4.7. *Let R be an intuitionistic fuzzy relation on X . The following implications hold:*

- (i) *if (X, μ_R, ν_R) is an IF-lattice, then $(X, [R])$ is a fuzzy lattice;*
- (ii) *if (X, μ_R, ν_R) is an IF-lattice, then $(X, \langle R \rangle)$ is a fuzzy lattice.*

Proof. The proof is similar to that of Proposition 4.6. □

4.2 Characterizations of IF-ideals and IF-filters in terms of the IF-lattice meet and join operations

We start with the following result.

Theorem 4.8. *Let (X, μ_R, ν_R) be an IF-lattice and $I \in IFS(X)$. I is an IF-ideal on (X, μ_R, ν_R) if and only if the following two conditions are satisfied:*

- (i) $\mu_I(x \sqcup_R y) = \mu_I(x) \wedge \mu_I(y)$, for any $x, y \in X$;
- (ii) $\nu_I(x \sqcup_R y) = \nu_I(x) \vee \nu_I(y)$, for any $x, y \in X$.

Proof. Suppose that I is an IF-ideal on (X, μ_R, ν_R) . Then for any $x, y \in X$ it holds that $\mu_I(x \sqcup_R y) \geq \mu_I(x) \wedge \mu_I(y)$ and $\nu_I(x \sqcup_R y) \leq \nu_I(x) \vee \nu_I(y)$. Moreover, since $(\mu_R(x, x \sqcup_R y) > 0$ or $(\mu_R(x, x \sqcup_R y) = 0$ and $\nu_R(x, x \sqcup_R y) < 1)$) and $(\mu_R(y, x \sqcup_R y) > 0$ or $(\mu_R(y, x \sqcup_R y) = 0$ and $\nu_R(y, x \sqcup_R y) < 1)$), it follows from Corollary 4.2 that $\mu_I(x) \geq \mu_I(x \sqcup_R y)$ and $\mu_I(y) \geq \mu_I(x \sqcup_R y)$. Hence, $\mu_I(x) \wedge \mu_I(y) \geq \mu_I(x \sqcup_R y)$ and lastly

$$\mu_I(x \sqcup_R y) = \mu_I(x) \wedge \mu_I(y).$$

In similar way we obtain from Corollary 4.2 that $\nu_I(x) \leq \nu_I(x \sqcup_R y)$ and $\nu_I(y) \leq \nu_I(x \sqcup_R y)$. Hence, $\nu_I(x) \vee \nu_I(y) \leq \nu_I(x \sqcup_R y)$. Thus,

$$\nu_I(x \sqcup_R y) = \nu_I(x) \vee \nu_I(y).$$

Conversely, suppose that $\mu_I(x \sqcup_R y) = \mu_I(x) \wedge \mu_I(y)$ and $\nu_I(x \sqcup_R y) = \nu_I(x) \vee \nu_I(y)$, for any $x, y \in X$. Then it is obvious that

$$\mu_I(x \sqcup_R y) \geq \mu_I(x) \wedge \mu_I(y) \text{ and } \nu_I(x \sqcup_R y) \leq \nu_I(x) \vee \nu_I(y),$$

for any $x, y \in X$. Next, we show that

$\mu_I(x \sqcap_R y) \geq \mu_I(x) \vee \mu_I(y)$ and $\nu_I(x \sqcap_R y) \leq \nu_I(x) \wedge \nu_I(y)$, for any $x, y \in X$. Let $x, y \in X$, since $x \sqcup_R (x \sqcap_R y) = x$ and $y \sqcup_R (x \sqcap_R y) = y$ then it holds that $\mu_I(x \sqcup_R (x \sqcap_R y)) = \mu_I(x)$, $\mu_I(y \sqcup_R (x \sqcap_R y)) = \mu_I(y)$, $\nu_I(x \sqcup_R (x \sqcap_R y)) = \nu_I(x)$ and $\nu_I(y \sqcup_R (x \sqcap_R y)) = \nu_I(y)$. From the hypothesis (i) and (ii) it follows that $\mu_I(x) \wedge \mu_I(x \sqcap_R y) = \mu_I(x)$ and $\mu_I(y) \wedge \mu_I(x \sqcap_R y) = \mu_I(y)$. Hence, $\mu_I(x \sqcap_R y) \geq \mu_I(x)$ and $\mu_I(x \sqcap_R y) \geq \mu_I(y)$. Thus,

$$\mu_I(x \sqcap_R y) \geq \mu_I(x) \vee \mu_I(y),$$

for any $x, y \in X$. In the same way, we obtain that

$$\nu_I(x \sqcap_R y) \leq \nu_I(x) \wedge \nu_I(y),$$

for any $x, y \in X$. Therefore, I is an IF-ideal on (X, μ_R, ν_R) . □

In the same way, the following theorem provides a basic characterization of IF-filters on a lattice.

Theorem 4.9. *Let X be an IF-lattice and $F \in IFS(X)$. F is an IF-filter on (X, μ_R, ν_R) if and only if the following conditions are satisfied:*

- (i) $\mu_F(x \sqcap_R y) = \mu_F(x) \wedge \mu_F(y)$, for any $x, y \in X$;
- (ii) $\nu_F(x \sqcap_R y) = \nu_F(x) \vee \nu_F(y)$, for any $x, y \in X$.

Proof. The proof directly applies Proposition 3.6 and Theorem 4.8. □

The above Theorems 4.8 and 4.9 lead to the following interesting corollaries.

Corollary 4.10. *Let (X, \sqcap, \sqcup) be a crisp lattice and $I, F \in IFS(X)$. Then it holds that*

(i) I is an IF-ideal on (X, \sqcap, \sqcup) if and only if the following two conditions are satisfied:

$$(a) \mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y), \text{ for any } x, y \in X;$$

$$(b) \nu_I(x \sqcup y) = \nu_I(x) \vee \nu_I(y), \text{ for any } x, y \in X.$$

(ii) F is an IF-filter on (X, \sqcap, \sqcup) if and only if the following conditions are satisfied:

$$(a) \mu_F(x \sqcap y) = \mu_F(x) \wedge \mu_F(y), \text{ for any } x, y \in X;$$

$$(b) \nu_F(x \sqcap y) = \nu_F(x) \vee \nu_F(y), \text{ for any } x, y \in X.$$

Corollary 4.11. Let (X, R) be a fuzzy lattice and $I, F \in FS(X)$. Then it holds that

(i) I is a fuzzy ideal on (X, R) if and only if $\mu_I(x \sqcup_R y) = \mu_I(x) \wedge \mu_I(y)$, for any $x, y \in X$;

(ii) F is a fuzzy filter on (X, R) if and only if $\mu_F(x \sqcap_R y) = \mu_F(x) \wedge \mu_F(y)$, for any $x, y \in X$.

4.3 Characterizations of IF-ideals and IF-filters in terms of some associated fuzzy sets

In this subsection, we characterize IF-ideals and IF-filters on an IF-lattice in terms of some of their associated fuzzy sets.

Theorem 4.12. Let (X, μ_R, ν_R) be an IF-lattice and $A \in IFS(X)$. The following equivalences hold:

(i) A is an IF-ideal on (X, μ_R, ν_R) if and only if $[A]$ is a fuzzy ideal on $(X, [R])$ and $\langle A \rangle$ is a fuzzy ideal on $(X, \langle R \rangle)$;

(ii) A is an IF-filter on (X, μ_R, ν_R) if and only if $[A]$ is a fuzzy filter on $(X, [R])$ and $\langle A \rangle$ is a fuzzy filter on $(X, \langle R \rangle)$.

Proof. (i) Let $(X, [R]), (X, \langle R \rangle)$ be the associated fuzzy lattices of the IF-lattice (X, μ_R, ν_R) and $A \in IFS(X)$ be an IF-ideal of (X, μ_R, ν_R) . We show that $[A]$ is a fuzzy ideal on $(X, [R])$ and $\langle A \rangle$ is a fuzzy ideal on $(X, \langle R \rangle)$. We easily verify that

$$\begin{aligned} \mu_{[A]}(x \sqcup_{[R]} y) &= \mu_A(x \sqcup_{[R]} y) \\ &= \mu_A(x \sqcup_R y) \\ &= \mu_A(x) \wedge \mu_A(y) \\ &= \mu_{[A]}(x) \wedge \mu_{[A]}(y). \end{aligned}$$

Also,

$$\begin{aligned} \mu_{\langle A \rangle}(x \sqcup_{\langle R \rangle} y) &= 1 - \nu_A(x \sqcup_{\langle R \rangle} y) \\ &= 1 - \nu_A(x \sqcup_R y) \\ &= 1 - (\nu_A(x) \vee \nu_A(y)) \\ &= (1 - \nu_A(x)) \wedge (1 - \nu_A(y)) \\ &= \mu_{\langle A \rangle}(x) \wedge \mu_{\langle A \rangle}(y). \end{aligned}$$

Corollary 4.11 then guarantees that $[A]$ is a fuzzy ideal on $(X, [R])$ and $\langle A \rangle$ is a fuzzy ideal on $(X, \langle R \rangle)$.

Conversely, suppose that $[A]$ is a fuzzy ideal on $(X, [R])$ and $\langle A \rangle$ is a fuzzy ideal on $(X, \langle R \rangle)$. By using the same steps as above, we get that A is an IF-ideal on (X, μ_R, ν_R) .

(ii) Follows from Proposition 3.6 and (i). □

Proposition 4.13. Let (X, μ_R, ν_R) be an IF-lattice and $A \in IFS(X)$. The following equalities hold:

- (i) $Supp(A) = Supp(\langle A \rangle)$;
- (ii) $Ker(A) = Ker(\langle A \rangle)$.

Proof. (i) Let $A \in IFS(X)$. We show that $Supp(A) = Supp([A]) \cup Supp(\langle A \rangle)$

$$\begin{aligned}
 Supp(A) &= \{x \in X \mid \mu_A(x) > 0 \text{ or } (\mu_A(x) = 0 \text{ and } \nu_A(x) < 1)\} \\
 &= \{x \in X \mid \mu_A(x) > 0\} \cup \{x \in X \mid \mu_A(x) = 0 \text{ and } 1 - \nu_A(x) > 0\} \\
 &= \{x \in X \mid \mu_{[A]}(x) > 0\} \cup \{\{x \in X \mid \mu_A(x) = 0\} \cap \{x \in X \mid \mu_{\langle A \rangle}(x) > 0\}\} \\
 &= Supp([A]) \cup ((Supp([A])^c \cap Supp(\langle A \rangle)) \\
 &= Supp([A]) \cup Supp(\langle A \rangle).
 \end{aligned}$$

Since $Supp([A]) \subseteq Supp(\langle A \rangle)$, it holds that $Supp(A) = Supp(\langle A \rangle)$.

(ii) Now, we show that $Ker(A) = Ker([A]) \cup Ker(\langle A \rangle)$.

$$\begin{aligned}
 Ker(A) &= \{x \in X \mid \mu_A(x) = 1 \text{ or } \nu_A(x) = 0\} \\
 &= \{x \in X \mid \mu_{[A]}(x) = 1 \text{ or } \mu_{\langle A \rangle}(x) = 1 - \nu_A(x) = 1\} \\
 &= Ker([A]) \cup Ker(\langle A \rangle).
 \end{aligned}$$

Since $Ker([A]) \subseteq Ker(\langle A \rangle)$, it holds that $Ker(A) = Ker(\langle A \rangle)$.

□

Combining Theorem 4.12 and Propositions 2.17 and 4.13 leads to the following result.

Corollary 4.14. *Let (X, μ_R, ν_R) be an IF-lattice and $A \in IFS(X)$. The following implications hold:*

- (i) *if A is an IF-ideal on (X, μ_R, ν_R) , then its support $Supp(A)$ is an ideal on $(X, Supp(R))$ and its kernel $Ker(A)$ is an ideal on $(X, Ker(R))$;*
- (ii) *if A is an IF-filter on (X, μ_R, ν_R) , then its support $Supp(A)$ is a filter on $(X, Supp(R))$ and its kernel $Ker(A)$ is a filter on $(X, Ker(R))$.*

4.4 Characterizations of IF-ideals and filters in terms of their level sets

The present section provides an interesting characterization of IF-ideals and IF-filters in terms of their level sets. We start with the following key result.

Proposition 4.15. *Let (X, μ_R, ν_R) be an IF-lattice and $A \in IFS(X)$. Then it holds that*

$$A_{(\alpha, \beta)} = [A]_\alpha \cap \langle A \rangle_{1-\beta} \text{ and } R_{(\alpha, \beta)} = [R]_\alpha \cap \langle R \rangle_{1-\beta},$$

for any $\alpha, \beta \in [0, 1]$ satisfying $\alpha + \beta \leq 1$.

Proof. We only prove the first equality, as the second equality can be proved similarly. For any $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$, it holds that

$$\begin{aligned}
 [A]_\alpha \cap \langle A \rangle_{1-\beta} &= \{x \in X \mid \mu_{[A]}(x) \geq \alpha\} \cap \{x \in X \mid \mu_{\langle A \rangle}(x) \geq 1 - \beta\} \\
 &= \{x \in X \mid \mu_A(x) \geq \alpha\} \cap \{x \in X \mid 1 - \nu_A(x) \geq 1 - \beta\} \\
 &= \{x \in X \mid \mu_A(x) \geq \alpha\} \cap \{x \in X \mid \nu_A(x) \leq \beta\} \\
 &= \{x \in X \mid \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq \beta\} \\
 &= A_{(\alpha, \beta)}.
 \end{aligned}$$

□

In the following theorem, we show a characterization of an IF-ideal (resp. IF-filter) in terms of their level sets.

Theorem 4.16. *Let (X, μ_R, ν_R) be an IF-lattice and $A \in IFS(X)$. The following equivalences hold:*

- (i) *A is an IF-ideal on (X, μ_R, ν_R) if and only if $A_{(\alpha, \beta)}$ is a crisp ideal on $(X, R_{(\alpha, \beta)})$, for any $(\alpha, \beta) \in]0, 1] \times]0, 1[$ with $\alpha + \beta \leq 1$;*
- (ii) *A is an IF-filter on (X, μ_R, ν_R) if and only if $A_{(\alpha, \beta)}$ is a crisp filter on $(X, R_{(\alpha, \beta)})$ $(\alpha, \beta) \in]0, 1] \times]0, 1[$ with $\alpha + \beta \leq 1$.*

Proof. Let $(\alpha, \beta) \in]0, 1] \times]0, 1[$ satisfying $\alpha + \beta \leq 1$, $A \in IFS(L)$ and $A_{\alpha, \beta}$ its level set.

- (i) Suppose that A is an IF-ideal on (X, μ_R, ν_R) and we show that $A_{(\alpha, \beta)}$ is an ideal on $(X, R_{(\alpha, \beta)})$. Proposition 4.6 guarantees that $(X, R_{(\alpha, \beta)})$ is a lattice. Since A is an IF-ideal on (X, μ_R, ν_R) , it follows from Theorem 4.12 that $[A]$ is a fuzzy ideal on $(X, [R])$ and $\langle A \rangle$ is a fuzzy ideal on $(X, \langle R \rangle)$. Furthermore, from Proposition 2.18, it follows that their level sets $[A]_\alpha$ (resp. $\langle A \rangle_\beta$) are ideals on $(X, [R]_\alpha)$ (resp. $(X, \langle R \rangle_\beta)$). Hence, Proposition 4.15 guarantees that $A_{(\alpha, \beta)}$ are crisp ideals on $(X, R_{(\alpha, \beta)})$.

Conversely, suppose that all level sets of A are ideals on $(X, R_{(\alpha, \beta)})$. By using Theorems 2.18 and 4.12 that A is an IF-ideal on (X, μ_R, ν_R) .

- (ii) Follows from Proposition 3.6 and (i).

□

5 Prime intuitionistic fuzzy ideals and filters

In this section, we introduce the concept of prime IF-ideal and prime IF-filter and investigate their various properties and characterizations.

5.1 Definitions and basic properties

Definition 5.1. Let (X, μ_R, ν_R) be an IF-lattice. An IF-ideal (resp. IF-filter) A on X is called proper IF-ideal (resp. IF-filter) if $A \neq X$.

Definition 5.2. Let (X, μ_R, ν_R) be an IF-lattice. A proper IF-ideal I on X is called a prime IF-ideal if for any $x, y \in X$,

$$\mu_I(x \sqcap_R y) \leq \mu_I(x) \vee \mu_I(y)$$

and

$$\nu_I(x \sqcap_R y) \geq \nu_I(x) \wedge \nu_I(y).$$

Definition 5.3. Let (X, μ_R, ν_R) be an IF-lattice. A proper IF-filter F on X is called a prime IF-filter if for any $x, y \in X$

$$\mu_F(x \sqcup_R y) \leq \mu_F(x) \vee \mu_F(y)$$

and

$$\nu_F(x \sqcup_R y) \geq \nu_F(x) \wedge \nu_F(y).$$

Combining Theorem 4.8 and Definition 3.1 we get the following main characterization of prime IF-ideals on an IF-lattice.

Proposition 5.4. *Let (X, μ_R, ν_R) be an IF-lattice and $I \in IFS(X)$. Then I is a prime IF-ideal on (X, μ_R, ν_R) if and only if the following conditions hold:*

- (i) $\mu_I(x \sqcup_R y) = \mu_I(x) \wedge \mu_I(y)$, for any $x, y \in X$;
- (ii) $\mu_I(x \sqcap_R y) = \mu_I(x) \vee \mu_I(y)$, for any $x, y \in X$;
- (iii) $\nu_I(x \sqcup_R y) = \nu_I(x) \vee \nu_I(y)$, for any $x, y \in X$;
- (iv) $\nu_I(x \sqcap_R y) = \nu_I(x) \wedge \nu_I(y)$, for any $x, y \in X$.

We also get from Theorem 4.9 and Definition 3.2 the following main characterization of prime IF-filters on an IF-lattice.

Proposition 5.5. *Let (X, μ_R, ν_R) be an IF-lattice and $F \in IFS(X)$. Then F is a prime IF-filter on (X, μ_R, ν_R) if and only if the following conditions hold:*

- (i) $\mu_F(x \sqcup_R y) = \mu_F(x) \vee \mu_F(y)$, for any $x, y \in X$;
- (ii) $\mu_F(x \sqcap_R y) = \mu_F(x) \wedge \mu_F(y)$, for any $x, y \in X$;
- (iii) $\nu_F(x \sqcup_R y) = \nu_F(x) \wedge \nu_F(y)$, for any $x, y \in X$;
- (iv) $\nu_F(x \sqcap_R y) = \nu_F(x) \vee \nu_F(y)$, for any $x, y \in X$.

Now we proceed with some basic properties of prime IF-ideals and IF-filters on an IF-lattice including the intersection, the union and the complement.

Proposition 5.6. *Let (X, μ_R, ν_R) be an IF-lattice and $(A_i)_{i \in I}$ be a family of intuitionistic fuzzy sets on X . Then*

- (i) *if $(A_i)_{i \in I}$ is a family of prime IF-ideals on (X, μ_R, ν_R) , then $\bigcap_{i \in I} A_i$ is a prime IF-ideal on it;*
- (ii) *if $(A_i)_{i \in I}$ is a family of prime IF-filters on (X, μ_R, ν_R) , then $\bigcap_{i \in I} A_i$ is a prime IF-filter on it.*

Proof. (i) Suppose that for any $i \in I$, A_i is a prime IF-ideal on (X, μ_R, ν_R) . From Proposition 3.7, it follows that $\bigcap_{i \in I} A_i$ is an IF-ideal on (X, μ_R, ν_R) . It remains to show that $\bigcap_{i \in I} A_i$ is prime. Let $x, y \in L$ such that $x \sqcap_R y \in \bigcap_{i \in I} A_i$. Then we have $x \sqcap_R y \in A_i$, for any $i \in I$. Since for any $i \in I$, A_i is a prime IF-ideal, it follows that

$$\mu_{A_i}(x \sqcap_R y) \leq \mu_{A_i}(x) \vee \mu_{A_i}(y) \text{ and } \nu_{A_i}(x \sqcap_R y) \geq \nu_{A_i}(x) \wedge \nu_I(y),$$

This implies that

$$\mu_{\bigcap_{i \in I} A_i}(x \sqcap_R y) \leq \mu_{A_i}(x \sqcap_R y) \leq \mu_{A_i}(x) \vee \mu_{A_i}(y)$$

and

$$\nu_{\bigcap_{i \in I} A_i}(x \sqcap_R y) \geq \nu_{A_i}(x \sqcap_R y) \geq \nu_{A_i}(x) \wedge \nu_I(y),$$

for any $i \in I$. Hence,

$$\mu_{\bigcap_{i \in I} A_i}(x \sqcap_R y) \leq \bigwedge_{i \in I} (\mu_{A_i}(x) \vee \mu_{A_i}(y)) \text{ and } \nu_{\bigcap_{i \in I} A_i}(x \sqcap_R y) \geq \bigvee_{i \in I} (\nu_{A_i}(x) \wedge \nu_I(y)).$$

Thus, $\mu_{\bigcap_{i \in I} A_i}(x \sqcap_R y) \leq \mu_{\bigcap_{i \in I} A_i}(x) \vee \mu_{\bigcap_{i \in I} A_i}(y)$ and $\nu_{\bigcap_{i \in I} A_i}(x \sqcap_R y) \geq \nu_{\bigcap_{i \in I} A_i}(x) \wedge \nu_{\bigcap_{i \in I} A_i}(y)$. Therefore, $\bigcap_{i \in I} A_i$ is a prime IF-ideal on (X, μ_R, ν_R) .

- (ii) Follows from Proposition 3.6 and (i).

□

Remark 5.7. The union of two prime IF-ideals (resp. IF-filters) is not necessarily a prime IF-ideal (resp. IF-filter). Indeed, from Remark 3.8 the union of two IF-ideals (resp. IF-filters) does not necessarily be an IF-ideal (resp. IF-filter).

Proposition 5.8. Let (X, μ_R, ν_R) be an IF-lattice and $A \in IFS(X)$. The following equivalences hold:

- (i) A is a prime IF-ideal on (X, μ_R, ν_R) if and only if \overline{A} is also a prime IF-filter on it;
- (ii) A is a prime IF-filter on (X, μ_R, ν_R) if and only if \overline{A} is also a prime IF-ideal on it.

Proof. (i) Suppose that A is a prime IF-ideal. Then for any $x, y \in X$ from Proposition 5.4 it follows that

$$\begin{aligned}\mu_{\overline{A}}(x \sqcup_R y) &= \nu_A(x \sqcup_R y) \\ &= \nu_A(x) \vee \nu_A(y) \\ &= \mu_{\overline{A}}(x) \vee \mu_{\overline{A}}(y)\end{aligned}$$

and

$$\begin{aligned}\mu_{\overline{A}}(x \sqcap_R y) &= \nu_A(x \sqcap_R y) \\ &= \nu_A(x) \wedge \nu_A(y) \\ &= \mu_{\overline{A}}(x) \wedge \mu_{\overline{A}}(y).\end{aligned}$$

In a similar way, we can prove that $\nu_{\overline{A}}(x \sqcup_R y) = \nu_{\overline{A}}(x) \wedge \nu_{\overline{A}}(y)$ and $\nu_{\overline{A}}(x \sqcap_R y) = \nu_{\overline{A}}(x) \vee \nu_{\overline{A}}(y)$. Applying Proposition 5.5 it guarantees that \overline{A} is a prime IF-filter on X .

The converse follows directly from Proposition 3.6, the fact that $A = \overline{\overline{A}}$ and the first implication.

- (ii) Follows from Proposition 3.6 and (i).

□

Remark 5.9. Notice that if A is only an IF-ideal (resp. IF-filter), then its complement \overline{A} is not necessarily an IF-ideal (resp. IF-filter).

The following propositions show that the support and the kernel of a prime IF-ideal (resp. IF-filter) are crisp ideals (resp. filters).

Proposition 5.10. Let (X, μ_R, ν_R) be an IF-lattice and $A \in IFS(X)$. The following implications hold:

- (i) if A is a prime IF-ideal on (X, μ_R, ν_R) , then its support $Supp(A)$ is a prime ideal on $(X, Supp(R))$ and $Ker(A)$ is a prime ideal on $(X, Ker(R))$;
- (ii) if A is a prime IF-filter on (X, μ_R, ν_R) , then its support $Supp(A)$ is a prime filter on $(X, Supp(R))$ and $Ker(A)$ is a prime filter on $(X, Ker(R))$.

Proof. (i) Suppose that A is a prime IF-ideal on a lattice (X, μ_R, ν_R) . From Corollary 4.14 it holds that $Supp(A)$ is an ideal on $(X, Supp(R))$. Next, we prove that $Supp(A)$ is prime. Let $x, y \in X$ such that $x \sqcap_R y \in Supp(A)$. Then it holds that $\mu_A(x \sqcap_R y) > 0$ or $(\mu_A(x \sqcap_R y) = 0$ and $\nu_A(x \sqcap_R y) < 1)$. We consider the following cases:

- (a) If $\mu_A(x \sqcap_R y) > 0$, then the fact that A is prime IF-ideal on (X, μ_R, ν_R) implies that

$$\mu_A(x) \vee \mu_A(y) = \mu_A(x \sqcap_R y) > 0.$$

This implies that either $\mu_A(x) > 0$ or $\mu_A(y) > 0$. Hence, either $x \in Supp(A)$ or $y \in Supp(A)$.

- (b) If $(\mu_A(x \sqcap_R y) = 0$ and $\nu_A(x \sqcap_R y) < 1)$, then the fact that A is prime IF-ideal on (X, μ_R, ν_R) implies that

$$\mu_A(x) \vee \mu_A(y) = \mu_A(x \sqcap_R y) = 0 \text{ and } \nu_A(x) \wedge \nu_A(y) = \nu_A(x \sqcap_R y) < 1.$$

These imply that $(\mu_A(x) = 0 \wedge \mu_A(y) = 0)$ and $(\nu_A(x) < 1 \vee \nu_A(y) < 1)$. Hence, $(\mu_A(x) = 0$ and $\nu_A(x) < 1)$ or $(\mu_A(y) = 0$ and $\nu_A(y) < 1)$. Thus, either $x \in \text{Supp}(A)$ or $y \in \text{Supp}(A)$.

Therefore, $\text{Supp}(A)$ is a prime ideal on $(X, \text{Supp}(R))$.

Also, we prove that $\text{Ker}(A)$ is prime. Let $x, y \in X$ such that $x \sqcap_R y \in \text{Ker}(A)$. Then it holds that $\mu_A(x \sqcap_R y) = 1$ or $\nu_A(x \sqcap_R y) = 0$. We consider the following cases:

- (c) If $\mu_A(x \sqcap_R y) = 1$, then the fact that A is prime IF-ideal on (X, μ_R, ν_R) implies that

$$\mu_A(x) \vee \mu_A(y) = \mu_A(x \sqcap_R y) = 1.$$

This implies that either $\mu_A(x) = 1$ or $\mu_A(y) = 1$. Hence, either $x \in \text{Ker}(A)$ or $y \in \text{Ker}(A)$.

- (d) If $\nu_A(x \sqcap_R y) = 0$, then the fact that A is prime IF-ideal on (X, μ_R, ν_R) implies that

$$\nu_A(x) \wedge \nu_A(y) = \nu_A(x \sqcap_R y) = 0.$$

These imply that $\nu_A(x) = 0$ or $\nu_A(y) = 0$. Thus, either $x \in \text{Ker}(A)$ or $y \in \text{Ker}(A)$.

Therefore, $\text{Ker}(A)$ is a prime ideal on $(X, \text{Ker}(R))$.

- (ii) Follows from Proposition 3.6 and (i).

□

5.2 Characterizations of prime IF-ideals and IF-filters in terms of some associated (crisp) fuzzy sets

In this subsection, we provide some interesting characterizations of prime IF-ideals (resp. IF-filters) on an IF-lattice.

The following proposition shows a characterization of a prime IF-ideal (resp. IF-filter) in terms of some associated fuzzy sets.

Proposition 5.11. *Let (X, μ_R, ν_R) be an IF-lattice and $A \in \text{IFS}(X)$. The following equivalences hold:*

- (i) A is a prime IF-ideal on (X, μ_R, ν_R) if and only if $[A]$ is a prime fuzzy ideal on $(X, [R])$ and $\langle A \rangle$ is a prime fuzzy ideal on $(X, \langle R \rangle)$;
- (ii) A is a prime IF-filter on (X, μ_R, ν_R) if and only if $[A]$ is a prime fuzzy filter on $(X, [R])$ and $\langle A \rangle$ is a prime fuzzy filter on $(X, \langle R \rangle)$.

Proof. (i) Suppose that A is a prime IF-ideal on (X, μ_R, ν_R) . Since A is an IF-ideal on (X, μ_R, ν_R) , it follows from Proposition 4.12 that $[A]$ is a fuzzy ideal on $(X, [R])$ and $\langle A \rangle$ is a fuzzy ideal on $(X, \langle R \rangle)$. Next, we show that $[A]$ and $\langle A \rangle$ are prime. It trivially to verify that

$$\begin{aligned} \mu_{[A]}(x \sqcap_R y) &= \mu_A(x \sqcap_{[R]} y) \\ &= \mu_A(x \sqcap_R y) \\ &= \mu_A(x) \vee \mu_A(y) \\ &= \mu_{[A]}(x) \vee \mu_{[A]}(y) \end{aligned}$$

and

$$\begin{aligned}
 \nu_{[A]}(x \sqcap_R y) &= 1 - \mu_A(x \sqcap_{[R]} y) \\
 &= 1 - \mu_A(x \sqcap_R y) \\
 &= 1 - (\mu_A(x) \vee \mu_A(y)) \\
 &= (1 - \mu_A(x)) \wedge (1 - \mu_A(y)) \\
 &= \nu_{[A]}(x) \wedge \nu_{[A]}(y).
 \end{aligned}$$

Hence, we can conclude that $[A]$ is a prime fuzzy ideal on $(X, [R])$. Similarly, we obtain that $\langle A \rangle$ is prime fuzzy ideal on $(X, \langle R \rangle)$.

Conversely, suppose that $[A]$ is a prime fuzzy ideal on $(X, [R])$ and $\langle A \rangle$ is a prime fuzzy ideal on $(X, \langle R \rangle)$. Using the same method, we easily show that A is a prime IF-ideal on (X, μ_R, ν_R) .

(ii) Follows from Proposition 3.6 and (i). □

In the same line, the following theorem provides a characterization of a prime IF-ideal (resp. IF-filter) in terms of their level sets.

Theorem 5.12. *Let (X, μ_R, ν_R) be an IF-lattice, $A \in IFS(X)$ and $(\alpha, \beta) \in]0, 1] \times [0, 1[$ with $\alpha + \beta \leq 1$. The following equivalences hold:*

- (i) *A is a prime IF-ideal on (X, μ_R, ν_R) if and only if their level sets are prime ideals on $(X, R_{(\alpha, \beta)})$;*
- (ii) *A is a prime IF-filter on (X, μ_R, ν_R) if and only if their level sets are prime filters on $(X, R_{(\alpha, \beta)})$.*

Proof. (i) From Theorem 4.16, A is an IF-ideal on (X, μ_R, ν_R) if and only if $A_{(\alpha, \beta)}$ is an ideal on $(X, R_{(\alpha, \beta)})$, for any $(\alpha, \beta) \in]0, 1] \times [0, 1[$ satisfying $\alpha + \beta \leq 1$. It remains to show the primality. Let A be a prime IF-ideal on (X, μ_R, ν_R) . Let $x, y \in X$ such that $x \sqcap_R y \in A_{\alpha, \beta}$, then from Proposition 5.4 it follows that

$$(\mu_A(x \sqcap_R y) = \mu_A(x) \vee \mu_A(y) \geq \alpha \text{ and } \nu_A(x \sqcap_R y) = \nu_A(x) \wedge \nu_A(y) \leq \beta).$$

These imply that either $(\mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq \beta)$ or $(\mu_A(y) \geq \alpha \text{ and } \nu_A(y) \leq \beta)$. Hence, either $x \in A_{\alpha, \beta}$ or $y \in A_{\alpha, \beta}$. Thus, $A_{(\alpha, \beta)}$ is a prime ideal on $(X, R_{(\alpha, \beta)})$, for any $(\alpha, \beta) \in]0, 1] \times [0, 1[$ satisfying $\alpha + \beta \leq 1$.

Conversely, let $(\alpha, \beta) \in]0, 1] \times [0, 1[$ satisfying $\alpha + \beta \leq 1$. Suppose that $A_{(\alpha, \beta)}$ is a prime ideal on $(X, R_{(\alpha, \beta)})$ and A is not a prime IF-ideal on (X, μ_R, ν_R) . Then it holds that there exist $x, y \in X$ such that $\mu_A(x \sqcap_R y) > \mu_A(x) \vee \mu_A(y)$ and $\nu_A(x \sqcap_R y) < \nu_A(x) \wedge \nu_A(y)$. These imply that $(\mu_A(x \sqcap_R y) > \mu_A(x) \text{ and } \mu_A(x \sqcap_R y) > \mu_A(y))$ and $(\nu_A(x \sqcap_R y) < \nu_A(x) \text{ and } \nu_A(x \sqcap_R y) < \nu_A(y))$. Putting $\mu_A(x \sqcap_R y) = \alpha$ and $\nu_A(x \sqcap_R y) = \beta$, we get $(\mu_A(x) < \alpha \text{ and } \nu_A(x) > \beta)$ and $(\mu_A(y) < \alpha \text{ and } \nu_A(y) > \beta)$. Hence, $x \sqcap_R y \in A_{\alpha, \beta}$ and $x, y \notin A_{\alpha, \beta}$. It is a contradiction with the fact that $A_{(\alpha, \beta)}$ is a prime ideal on $(X, R_{(\alpha, \beta)})$. Thus, A is a prime IF-ideal on (X, μ_R, ν_R) .

(ii) Follows from Proposition 3.6 and (i). □

6 Conclusion

In this paper, we have introduced the notion of IF-ideal (resp. IF-filter) on an IF-lattice based on the intuitionistic fuzzy order relation as a generalization of the notion of fuzzy ideal (resp. fuzzy filter) on a fuzzy lattice defined by Mezzomo, et al., and have investigated their most interesting properties. We have provided a characterization of the IF-ideal (resp. IF-filter) in terms of the IF-lattice meet and join operations, and in terms of their level sets. As an interesting kind, we have introduced the notion of prime IF-ideal (resp. prime IF-filter) and have provided their different characterizations and properties. In future work, we plan to characterize the principal IF-ideal (resp. IF-filter) on an IF-lattice.

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