



On Gr -2-nil ideals of graded commutative rings

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Abstract

Let G be a group with identity e and let W be a G -graded ring. In this paper, we introduce and study the concept of Gr -2-nil ideals of W . We obtain many results concerning gr -2-nil ideals. Some characterizations of gr -2-nil ideals and their homogeneous components are given. A proper graded ideal I of W is said to be a gr -2-nil ideal of W if whenever $r_g, s_h, t_i \in h(W)$ with $r_g s_h t_i \in I$, then either $r_g s_h \in Gr(0)$ or $r_g t_i \in I$ or $s_h t_i \in I$.

Keywords: gr -2-Nil ideals; gr -2-n-ideals; gr -(2, n)-ideals

1 Introduction

Throughout this article, we assume that W is a commutative G -graded ring with identity. Graded primary ideals are widely recognized for their pivotal role in commutative ring theory. Consequently, numerous researchers have extensively studied and generalized them. Below, we highlight some generalizations of graded primary ideals. The concept of graded (weakly) prime ideal of a graded ring was introduced in.^{6,12} The concept of graded primary ideal was introduced and studied in.¹¹ In⁷ the authors, introduced a generalization of graded primary ideals called graded 2-absorbing primary ideals. In 2019 the notion of graded n -ideals was first introduced and studied.³ The concept of graded 2-absorbing ideal of a commutative graded rings have been introduced and studied in.² The concept of graded strongly 1-absorbing primary ideals have been introduced and studied in.¹ Recently, the authors, in⁴ introduced and studied the concept of graded $(2, n)$ -ideals. In this paper, we introduce and study the concept of graded 2-nil (gr -2-nil) ideals of graded commutative rings. We obtain many results concerning gr -2-nil ideals. For example, we give a characterization of gr -2-nil ideals and their homogeneous components. Also, we study the relations between gr -2-nil and others that already exist, namely, the graded primary ideals, the graded 2-absorbing primary ideals, the graded n -ideals and graded $(2, n)$ -ideal.

2 Preliminaries

The purpose of this section is to provide the definitions and results that will be needed in the next section.

- Definition 2.1.** 1. Let G be a group with identity e and W be a commutative ring with identity 1_W . Then W is G -graded ring if there exist additive subgroups W_g of W indexed by the elements $g \in G$ such that $W = \bigoplus_{g \in G} W_g$ and $W_g W_h \subseteq W_{gh}$ for all $g, h \in G$. The elements of W_g are called homogeneous of degree g . The set of all homogeneous elements of W is denoted by $h(W)$, i.e. $h(W) = \bigcup_{g \in G} W_g$, (see^{7,8,10}).
2. Let $W = \bigoplus_{g \in G} W_g$ be G -graded ring, an ideal K of W is called a graded ideal if $K = \sum_{h \in G} K \cap W_h = \sum_{h \in G} K_h$, (see^{9,10}).
3. The *graded radical* of a graded ideal K , denoted by $Grad(K)$, is the set of all $r = \sum_{g \in G} r_g \in W$ such that for each $g \in G$ there exists $n_g \in \mathbb{N}$ with $r_g^{n_g} \in K$. Note that, if t is a homogeneous element, then $t \in Grad(K)$ if and only if $t^n \in K$ for some $n \in \mathbb{N}$, (see¹²).
4. A proper graded ideal K of W is said to be a *graded prime* (briefly, *gr-prime*) if whenever $r_g, s_h \in h(W)$ with $r_g s_h \in K$, then either $r_g \in K$ or $s_h \in K$, (see¹²).
5. A proper graded ideal K of W is called a *graded primary* (briefly, *gr-primary*) ideal if whenever $r_g, s_h \in h(W)$ and $r_g s_h \in K$, then either $r_g \in K$ or $s_h \in Grad(K)$, (see¹¹).
6. A proper graded ideal K of W is said to be a *graded 2-absorbing* (briefly, *gr-2-absorbing*) ideal of W if whenever $r_g, s_h, t_i \in h(W)$ with $r_g s_h t_i \in K$, then $r_g s_h \in K$ or $r_g t_i \in K$ or $s_h t_i \in K$, (see²).
7. A proper graded ideal K of W is said to be a *graded 2-absorbing primary* (briefly, *gr-2-absorbing primary*) ideal of W if whenever $r_g, s_h, t_i \in h(W)$ with $r_g s_h t_i \in K$, then $r_g s_h \in K$ or $r_g t_i \in Grad(K)$ or $s_h t_i \in Grad(K)$, (see⁷).
8. A proper graded ideal K of W is said to be a *graded n -ideal* (briefly, *gr- n -ideal*) of W if whenever $r_g, s_h \in h(W)$ with $r_g s_h \in K$ and $r_g \notin Grad(0)$, then $s_h \in K$, (see³).
9. A proper graded ideal K of W is said to be a *graded $(2, n)$ -ideal* (briefly, *gr- $(2, n)$ -ideal*) of W if whenever $r_g s_h t_i \in K$ where $r_g, s_h, t_i \in h(W)$, then either $r_g t_i \in K$ or $r_g s_h \in Grad(0)$ or $s_h t_i \in Grad(0)$, (see⁴).
10. A G -graded ring W is called *graded reduced ring* if whenever $r \in h(W)$ with $r^2 = 0$, then $r = 0$, i.e. $Grad(0) = \{0\}$.
11. For G -graded rings W and W' , a G -graded ring homomorphism $f : W \rightarrow W'$ is a ring homomorphism with $f(W_g) \subseteq W'_g$ for every $g \in G$, (see¹⁰).

3 Results

Definition 3.1. Let P be a graded ideal of W . Then P is said to be a *gr-2-nil ideal* if whenever, $a, b, c \in h(W)$ with $abc \in P$ then $ab \in Grad(0)$ or $ac \in P$ or $bc \in P$.

From the definitions, we immediately obtain the following.

Theorem 3.2. *gr-prime ideals and gr- n -ideals are gr-2-nil ideals are gr-2-absorbing primary.*

In the following example, the first three parts show that the converse implications do not hold and the fourth one shows that a gr-2-nil ideal is not necessarily a gr- $(2, n)$ -ideal.

Example 3.3. 1. Let p be a prime number and $G = \mathbb{Z}_2$. Then $W = \mathbb{Z}$ is a G -graded ring with $W_0 = \mathbb{Z}$ and $W_1 = \{0\}$. The proper graded ideal $P = p\mathbb{Z}$ of \mathbb{Z} is a gr-2-nil ideal, but not a gr- n -ideal since $1.p \in P$ but neither $1 \in P$ nor $p \in Grad(0)$.

2. Let p be a prime number and $G = \mathbb{Z}_2$. Then $W = \mathbb{Z}_{p^3}$ is a G -graded ring with $W_0 = \mathbb{Z}_{p^3}$ and $W_1 = \{0\}$. Then every proper graded ideal of W is a gr-2-nil ideal. For example $P = p^2\mathbb{Z}_{p^3}$ is a gr-2-nil ideal which is not a gr-prime ideal.

3. Let p and q be a prime numbers and $G = \mathbb{Z}_2$. Then $W = \mathbb{Z}$ is a G -graded ring with $W_0 = \mathbb{Z}$ and $W_1 = \{0\}$. The proper graded ideal $P = pq\mathbb{Z}$ is a gr-2-absorbing ideal and then P is a gr-2-absorbing primary by [13, Lemma 2.2]. However P is not gr-2-nil ideal since $p.q.1 \in P$ but neither $p.q \in Grad(0)$ nor $p.1 \in P$ nor $q.1 \in P$.
4. Let $G = \mathbb{Z}_2$. Then $W = \mathbb{Z}$ is a G -graded ring with $W_0 = \mathbb{Z}$ and $W_1 = \{0\}$. The proper graded ideal $P = 2\mathbb{Z}$ is a gr-2-nil ideal but it is not a gr-(2,n)-ideal as $1.1.2 \in P$ but neither $1.1 \in P$ nor $1.2 \in Grad(0)$.

Theorem 3.4. *Let P be a gr-2-nil ideal of W . Then*

1. $Grad(P)$ is a gr-2-nil ideal of W ;
2. $Grad(P) = P \cup Grad(0)$;
3. $Grad(P) = P$ or $Grad(P) = Grad(0)$.

Proof. (1) Let $abc \in Grad(P)$ for some $a, b, c \in h(W)$ and $ab \notin Grad(0)$. Then $(abc)^n = a^n b^n c^n \in P$ for some $n \in \mathbb{Z}^+$ and $a^n b^n \notin Grad(0)$. It follows $b^n c^n \in P$ or $a^n c^n \in P$, and so $bc \in Grad(P)$ or $ac \in Grad(P)$. Thus $Grad(P)$ is a gr-2-nil ideal of W .

(2) Clearly, $P \cup Grad(0) \subseteq Grad(P)$. Take $a \in Grad(P) \cap h(W)$. If $a \in P$, then obviously $a \in P \cup Grad(0)$. If $a \notin P$, then there exists $n \geq 2$ such that $a^n \in P$ and $a^{n-1} \notin P$. Then $a^n = 1.a.a^{n-1} \in P$, $1.a \notin P$ and $1.a^{n-1} \notin P$, hence $a^n = a.a^{n-1} \in Grad(0)$ since P is a gr-2-nil ideal. Thus $a \in Grad(0) \subseteq P \cup Grad(0)$. This proves the converse inclusion $Grad(P) \subseteq P \cup Grad(0)$.

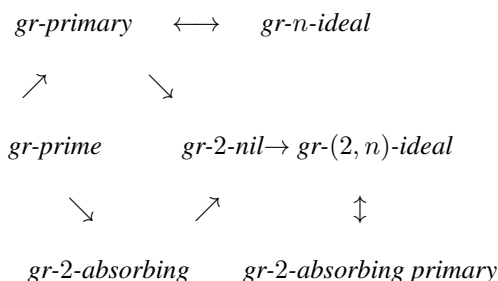
(3) Assume that $P \subsetneq Grad(P)$ and take $b \in Grad(P) \cap h(W) - P$. If $a \in P \cap h(W)$, then $a + b \in Grad(P) - P$. From the proof of part (ii), it follows that $a + b, b \in Grad(0)$, hence $a \in Grad(0)$. This proves that $P \subseteq Grad(0)$, and part (ii) tells us that $Grad(P) = P \cup Grad(0) = Grad(0)$. □

The converse of Theorem 3.4 does not hold in general, as illustrated by the following example.

Example 3.5. Let p and q be prime integers. Let $G = \mathbb{Z}_2$. Then $W = \mathbb{Z}_{p^2q}$ is a G -graded ring with $W_0 = W$ and $W_1 = \{0\}$. The graded zero ideal P satisfies the conditions (i-iii) in Theorem 3.4 but it is not a gr-2-nil ideal since $p.p.q \in P$, $p.p \notin Grad(0)$ and $p.q \notin P$.

If $P \subseteq Grad(0)$, then $Grad(P) = Grad(0)$. Comparing the definitions, we easily see that we have the following implications.

Theorem 3.6. *Let $P \subseteq Grad(0)$ be a graded ideal of W . Then we have the following diagram of implications between properties of P .*



The following example shows that a gr-primary ideal $P \not\subseteq Grad(0)$ is not necessarily a gr-2-nil ideal.

Example 3.7. Let p be a prime number. Let $G = \mathbb{Z}_2$ and $W = \mathbb{Z}$ be a G -graded ring with $W_0 = \mathbb{Z}$ and $W_1 = \{0\}$. The graded ideal $P = p^2\mathbb{Z} \subset \mathbb{Z}$ is a gr-primary but not gr-2-nil, since $p.p.1 \in P$, $p.p \notin Grad(0)$ and $p.1 \notin P$.

Theorem 3.8. Let $P \subseteq \text{Grad}(0)$ be a graded ideal of a G -graded divided ring W . If $\text{Grad}(0)$ is a gr-prime, then P is a gr-2-nil ideal if and only if P is a gr-primary ideal.

Proof. \Rightarrow) Suppose that $ab \in P$ and $a \notin \text{Grad}(0) = \text{Grad}(P)$ for some $a, b \in h(W)$. Since $\text{Grad}(0)$ is a gr-prime, $ab \in \text{Grad}(0)$ and $a \notin \text{Grad}(0)$ implies that $b \in \text{Grad}(0)$. Now, since W is graded divided, there exists an element $c \in h(W)$ such that $b = ac$. Hence $ab = aac \in P$. Observe that $a^2 \notin \text{Grad}(0)$. Since P is a gr-2-nil, it follows that $ac = b \in P$. This proves that if P is a gr-n-ideal, and therefore a gr-primary ideal.

\Leftarrow) See Theorem 3.6. □

The set of all minimal gr-prime ideals is denoted by $\text{Min}_g(W)$.

Theorem 3.9. If W has at most two minimal gr-prime ideals or $|\text{Min}_g(W)| \leq 2$, then there exists a gr-2-nil ideal. $\text{Grad}(0)$ is immediately gr-2-nil in this case.

Proof. Assume that W has at most two minimal gr-prime ideals. First, let's assume W has only one minimal gr-prime ideal say P , hence $\text{Grad}(0) = P$, so by Theorem 3.6, $\text{Grad}(0)$ is gr-2-absorbing ideal and then $\text{Grad}(0)$ is gr-2-nil ideal by Theorem 3.6. Assume that W has exactly two minimal gr-prime ideals P_1 and P_2 , then $\text{Grad}(0) = P_1 \cap P_2$, so by [4, Lemma 2.15], $\text{Grad}(0)$ is gr-2-absorbing ideal, then $\text{Grad}(0)$ is a gr-2-nil ideal by Theorem 3.6. □

Theorem 3.10. If every gr-2-nil ideal of W is a gr-prime, then either W is a graded reduced ring or $|\text{Min}_g(W)| \geq 3$ or $\text{Grad}(0)$ is a gr-prime ideal which is not a gr-maximal.

Proof. Assume that $\{0\} \subsetneq \text{Grad}(0)$ and that $|\text{Min}_g(W)| \leq 2$. If $|\text{Min}_g(W)| \leq 2$, then $\text{Grad}(0)$ is a gr-2-nil ideal by Theorem 3.9

We have two cases :

Case1: Suppose $|\text{Min}_g(W)| = 2$, in this case we find that $\text{Grad}(0)$ is a gr-2-nil ideal but not gr-prime ideal.

Case2: Suppose $|\text{Min}_g(W)| = 1$, then $\text{Grad}(0)$ is a gr-prime ideal, if $\text{Grad}(0)$ is a gr-maximal ideal, then every proper graded ideal of W is a gr-n-ideal by [4, Theorem 2.11], hence gr-2-nil ideal by Theorem 3.2. So every proper graded ideal of W is a gr-prime by our assumption. Thus W is a graded field, and so $\text{Grad}(0) = \{0\}$ is a contradiction. Therefore, $\text{Grad}(0)$ is not a gr-maximal. □

Theorem 3.11. Let P be a gr-2-nil ideal of W , and $U = \bigoplus_{g \in G} U_g$ be a graded ideal of W . If $a, b \in h(W)$ and $g \in G$ such that $abU_g \subseteq P$, then $ab \in \text{Grad}(0)$ or $aU_g \subseteq P$ or $bU_g \subseteq P$.

Proof. Assume that $abU_g \subseteq P$ but neither $ab \in \text{Grad}(0)$ or $aU_g \subseteq P$ or $bU_g \subseteq P$. Then there exist $u_g, u'_g \in U_g$ such that $au_g \notin P$ and $bu'_g \notin P$. Since $abu_g \in P$, $ab \notin \text{Grad}(0)$ and $au_g \notin P$, we have $bu_g \in P$ as P is gr-2-nil ideal of W . Similarly, by $abu'_g \in P$, $ab \notin \text{Grad}(0)$ and $bu'_g \notin P$, we have $au'_g \in P$. By $u_g + u'_g \in U_g$, we get $ab(u_g + u'_g) \in P$. Then either $b(u_g + u'_g) \in P$ or $a(u_g + u'_g) \in P$ as P is gr-2-nil ideal and $ab \notin \text{Grad}(0)$. If $a(u_g + u'_g) = au_g + au'_g \in P$, then $au_g \in P$, Contradiction. If $b(u_g + u'_g) = bu_g + bu'_g \in P$, then $bu'_g \in P$, Contradiction. This implies that $au_g \in P$ or $bu'_g \in P$. Since $bu_g \in P$ or $au'_g \in P$ which is a contradiction. □

Theorem 3.12. Let P be a gr-2-nil ideal of W . Let $U = \bigoplus_{g \in G} U_g$ and $V = \bigoplus_{h \in G} V_h$ be two graded ideals of W . If $a_i \in h(W)$ and $g, h \in G$ with $a_i U_g V_h \subseteq P$, then either $U_g V_h \subseteq \text{Grad}(0)$ or $a_i V_h \subseteq P$ or $a_i U_g \subseteq P$.

Proof. Let $a_i \in h(W)$ and $g, h \in G$ with $a_i U_g V_h \subseteq P$ and $U_g V_h \not\subseteq \text{Grad}(0)$. We show that $a_i V_h \subseteq P$ or $a_i U_g \subseteq P$. Suppose that $a_i V_h \not\subseteq P$ and $a_i U_g \not\subseteq P$. Then there are $u_g \in U_g$ and $v_h \in V_h$ with $a_i u_g \notin P$ and $a_i v_h \notin P$, but $a_i u_g v_h \in P$ so we get $u_g v_h \in \text{Grad}(0)$ as P is gr-2-nil ideal of W . Now, since $U_g V_h \not\subseteq \text{Grad}(0)$, there exist $u'_g \in U_g$ and $v'_h \in V_h$ such that $u'_g v'_h \notin \text{Grad}(0)$. Since $a_i u'_g v'_h \in P$ and $u'_g v'_h \notin \text{Grad}(0)$, $a_i u'_g \in P$ or $a_i v'_h \in P$ as P is gr-2-nil ideal of W . We consider three cases :

Case 1: Assume that $a_i u'_g \in P$ but $a_i v'_h \notin P$. Since $a_i u_g v'_h \in P$, $a_i u_g \notin P$ and $a_i v'_h \notin P$, we have $u_g v'_h \in Grad(0)$. Since $a_i u'_g \in P$ but $a_i u_g \notin P$, we get $a_i(u_g + u'_g) \notin P$. By $a_i(u_g + u'_g)v'_h \in P$ and $a_i v'_h \notin P$, we get $(u_g + u'_g)v'_h = u_g v'_h + u'_g v'_h \in Grad(0)$ as P is gr-2-nil ideal of W . Hence $u'_g v'_h \in Grad(0)$, since $u_g v'_h \in Grad(0)$. Which is contradiction.

Case 2: Assume that $a_i v'_h \in P$ but $a_i u'_g \notin P$, similar to Case 1.

Case 3: Assume that $a_i u'_g \in P$ and $a_i v'_h \in P$. By $a_i v'_h \in P$ and $r v_h \notin P$, we have $a_i(v_h + v'_h) \notin P$. Since $a_i u_g(v_h + v'_h) \in P$, $a_i u_g \notin P$ and $a_i(v_h + v'_h) \notin P$, we have $u_g(v_h + v'_h) \in Grad(0)$ as P is gr-2-nil ideal of W . Since $u_g(v_h + v'_h) = u_g v'_h + u_g v_h \in Grad(0)$ and $u_g v_h \in Grad(0)$, we get $u_g v'_h \in Grad(0)$. By $a_i u'_g \in P$ and $a_i u_g \notin P$, we get $a_i(u_g + u'_g) \notin P$. Since $a_i(u_g + u'_g)v_h \in P$, $a_i v_h \notin P$ and $a_i(u_g + u'_g) \notin P$, we have $(u_g + u'_g)v_h = u_g v_h + u'_g v_h \in Grad(0)$ as P is gr-2-nil ideal of W . Thus $u'_g v_h \in Grad(0)$ since $u_g v_h \in Grad(0)$. As $a_i(u_g + u'_g)(v_h + v'_h) \in P$, $a_i(u_g + u'_g) \notin P$, and $a_i(v_h + v'_h) \notin P$, we have $(u_g + u'_g)(v_h + v'_h) = u_g v_h + u'_g v_h + u_g v'_h + u'_g v'_h \in Grad(0)$. So $u'_g v'_h \in Grad(0)$, a contradiction. \square

Theorem 3.13. Let P be a proper graded ideal of W . Let $U = \bigoplus_{g \in G} U_g$, $V = \bigoplus_{g \in G} V_g$ and $L = \bigoplus_{g \in G} L_g$ be graded ideals of W . Then the following statements are equivalent:

1. P is gr-2-nil ideal of W .
2. For every $g, h, \lambda \in G$, with $V_h U_g L_\lambda \subseteq P$, either $U_g L_\lambda \subseteq Grad(0)$ or $V_h L_\lambda \subseteq P$ or $V_h U_g \subseteq P$.

Proof. (1) \Rightarrow (2) Assume that P is gr-2-nil ideal of W . Let $g, h, \lambda \in G$, with $V_h U_g L_\lambda \subseteq P$ and $U_g L_\lambda \not\subseteq Grad(0)$. Then for all $v_h \in V_h$ either $v_h L_\lambda \subseteq P$ or $v_h U_g \subseteq P$ by Theorem 3.12. If $v_h U_g \subseteq P$ for all $v_h \in V_h$, then $V_h U_g \subseteq P$, we are done. Similarly, If $v_h L_\lambda \subseteq P$ for all $v_h \in V_h$ then $V_h L_\lambda \subseteq P$, we are done. Suppose that $v_{h_1}, v_{h_2} \in V_h$ are such that $v_{h_1} U_g \not\subseteq P$ and $v_{h_2} L_\lambda \not\subseteq P$. Since $v_{h_1} U_g L_\lambda \subseteq P$, $U_g L_\lambda \not\subseteq Grad(0)$ and $v_{h_1} U_g \not\subseteq P$, by Theorem 3.12., we get $v_{h_1} L_\lambda \subseteq P$. Similarly we have $v_{h_2} U_g \subseteq P$ by $v_h + v'_h \in V_h$, we get $(v_h + v'_h) U_g L_\lambda \subseteq P$. Then either $(v_h + v'_h) U_g \subseteq P$ or $(v_h + v'_h) L_\lambda \subseteq P$ by Theorem 3.12. By $(v_h + v'_h) U_g \subseteq P$ it follows that $v_h U_g \subseteq P$, which is a contradiction. Similarly, by $(v_h + v'_h) L_\lambda \subseteq P$, we get a contradiction. Therefore, either $V_h L_\lambda \subseteq P$ or $V_h U_g \subseteq P$.

(2) \Rightarrow (1) Let $a_g, b_h, c_\lambda \in h(W)$ with $a_g b_h c_\lambda \in P$. Let $U = (a_g)$, $V = (b_h)$ and $L = (c_\lambda)$ be graded ideals of W generated by a_g, b_h, c_λ , respectively. So, $V_h U_g L_\lambda \subseteq P$, and hence either $U_g L_\lambda \subseteq Grad(0)$ or $V_h L_\lambda \subseteq P$ or $V_h U_g \subseteq P$. It follows that $a_g c_\lambda \in Grad(0)$ or $b_h c_\lambda \in P$ or $a_g b_h \in P$. Thus P is gr-2-nil ideal of W . \square

Theorem 3.14. Let W_1 and W_2 be two G -graded rings and $f: W_1 \rightarrow W_2$ be a graded ring homomorphism. Then:

1. If f is a graded monomorphism and P_2 is a gr-2-nil ideal of W_2 , then $f^{-1}(P_2)$ is agr-2-nil ideal of W_1 .
2. If f is a graded epimorphism and P_1 is a gr-2-nil ideal of W_1 containing $Ker(f)$, then $f(P_1)$ is a gr-2-nil ideal of W_2 .

Proof. (1) Take $a, b, c \in h(W_1)$ and assume that $abc \in f^{-1}(P_2)$. Then $f(abc) = f(a) f(b) f(c) \in P_2$, which implies $f(a) f(b) = f(ab) \in Grad(0_{W_2})$ or $f(b) f(c) = f(bc) \in P_2$ or $f(a) f(c) = f(ac) \in P_2$ as P_2 is a gr-2-nil. $f^{-1}(Grad(0_{W_2})) \subseteq Grad(0_{W_1})$ since f is graded ring monomorphism, and it follows that $ab \in Grad(0_{W_1})$ or $bc \in f^{-1}(P_2)$ or $ac \in f^{-1}(P_2)$. Thus $f^{-1}(P_2)$ is a gr-2-nil ideal of W_1 .

(2) Let $abc \in f(P_1)$ for some $a, b, c \in h(W_2)$. Since f is onto, $a = f(x)$, $b = f(y)$ and $c = f(z)$ for some $x, y, z \in h(W_1)$. Then $f(xyz) \in f(P_1)$. Then $xyz \in P_1$ as $ker(f) \subseteq P_1$, and it follows that $xy \in Grad(0_{W_1})$ or $yz \in P_1$ or $xz \in P_1$, hence $ab \in Grad(0_{W_2})$ or $bc \in f(P_1)$ or $ac \in f(P_1)$, as required. \square

In view of Theorem 3.14, we conclude the following result

Corollary 3.15. Let P, U be graded ideals of W such that $U \subseteq P \subset W$. Then

1. If P is a gr -2-nil ideal of W , then P/U is a gr -2-nil ideal of W/U .
2. If A is a graded subring of W and P is a gr -2-nil ideal of W , then $P \cap A$ is a gr -2-nil ideal of A .
3. If P/U is a gr -2-nil ideal of W/U and $U \subseteq Grad(0)$, then P is a gr -2-nil ideal of W .

Proof. (1) Consider the natural graded epimorphism $\pi : W \rightarrow W/U$, the result is clear by Theorem 3.14(i).

(2) Consider the graded monomorphism $i : A \rightarrow W$, since P is a gr -2-nil ideal of W , $i^{-1}(P) = P \cap A$ is a gr -2-nil ideal of A , by Theorem 3.14(2).

(3) Suppose that $a, b, c \in h(W)$ with $abc \in P$ and $ab \notin Grad(0)$, hence $(a+U)(b+U)(c+U) = abc + U \in P/U$ and $ab \notin Grad(0_{W/U})$ which implies that $(b+U)(c+U) = bc+U \in P/U$ or $(a+U)(c+U) = ac+U \in P/U$. Thus $bc \in P$ or $ac \in P$, we are done. □

Let P be a graded ideal of W . Then $G-Z_P(W) = \{r \in h(W) : rs \in P \text{ for some } s \in h(W) - P\}$. In particular, $G-Z_0(W) = G-Z(W)$ is the set of graded zero divisors in $h(W)$.

Theorem 3.16. Let $S \subset h(W)$ be multiplicatively closed subset of W . Then:

1. If P is a gr -2-nil ideal of W and $P \cap S = \emptyset$, then $S^{-1}P$ is a gr -2-nil ideal of $S^{-1}W$.
2. If $S^{-1}P$ is a gr -2-nil ideal of $S^{-1}W$ and $S \cap G-Z(W) = S \cap G-Z_P(W) = \emptyset$, then P is a gr -2-nil ideal of W .

Proof. (1) Suppose that $\frac{a}{s_1} \frac{b}{s_2} \frac{c}{s_3} \in S^{-1}P$. For some $\frac{a}{s_1}, \frac{b}{s_2}, \frac{c}{s_3} \in h(S^{-1}W)$. Hence there exists $u \in h(S)$ such that $uabc \in P$, since P is a gr -2-nil ideal of W , then $ab \in Grad(0)$, or $uac \in P$ or $ubc \in P$, this implies $\frac{a}{s_1} \frac{b}{s_2} \in S^{-1}(Grad(0)) \subseteq Grad(0_{S^{-1}W})$ or $\frac{uac}{us_1s_3} \in S^{-1}P$ or $\frac{ubc}{us_2s_3} \in S^{-1}P$, and it follows that $S^{-1}P$ is a gr -2-nil ideal.

(2) If $abc \in P$ for some $a, b, c \in h(W)$, then $\frac{a}{1} \frac{b}{1} \frac{c}{1} \in S^{-1}P$ and it follows that $\frac{a}{1} \frac{b}{1} \in Grad(0_{S^{-1}W})$ or $\frac{b}{1} \frac{c}{1} \in S^{-1}P$ or $\frac{a}{1} \frac{c}{1} \in S^{-1}P$ as $S^{-1}P$ is gr -2-nil ideal of W . If $\frac{a}{1} \frac{b}{1} \in Grad(0_{S^{-1}W})$, then there exists $u \in h(S)$ and $n \in \mathbb{Z}^+$ with $ua^n b^n = 0$. Since $S \cap G-Z(W) = \emptyset$, we have that $a^n b^n = 0$, and $ab \in Grad(0)$. If $\frac{b}{1} \frac{c}{1} \in S^{-1}P$, then $abc \in P$ for some $v \in h(S)$. Our assumption $S \cap G-Z_P(W) = \emptyset$ implies that $bc \in P$. In the third case, we conclude in a similar way that $ac \in P$. Thus P is a gr -2-nil ideal of W . □

4 Conclusions

In this article, we introduced the concept of Gr -2-nil ideals over a commutative graded ring. Several properties, examples and characterizations of Gr -2-nil ideals, have been investigated. Moreover, we explored the behaviour of Gr -2-nil ideals under graded ring homomorphisms and localizations. Also, we study the relations between gr -2-nil and others that already exist, namely, the graded primary ideals, the graded 2-absorbing primary ideals, the graded n -ideals and graded $(2, n)$ -ideal.

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