



Relations between W_d -fuzzy implication algebras and other logical algebras

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Abstract

In this paper, we continue the study W_d -fuzzy implication algebras which are subalgebras of fuzzy implication algebras. Properties and axiomatic systems for W_d -fuzzy implication algebras are presented, then a few new results on W_d -fuzzy implication algebras have been added. We showed that there are relations between W_d -fuzzy implication algebras and some of other fuzzy logical algebras such as FI-algebras, RFI-algebras, CFI-algebras, HFI-algebras. In particular, the relations between W_d -fuzzy implication algebras and L-algebras are investigated, and we prove that every W_d -fuzzy implication algebras is a proper subclass of L-algebras. Finally, we introduce the notions of GW_d -FI algebras, whose some properties of it are investigated. The relations between distributive GW_d -FI-algebras, Hilbert algebras, BE -algebras and W-eo algebras have been obtained.

Keywords: Fuzzy implication algebra; W_d -Fuzzy implication algebras; L-algebra; GW_d -fuzzy implication algebras

1 Introduction

In the past years, fuzzy algebras and their axiomatization have become important topics in theoretical research and in the applications of fuzzy logic. The implication connective plays a crucial role in fuzzy logic and reasoning [1]. Wu introduced a class of fuzzy implication algebras, FI-algebras for short, in 1990 [2]. Recently, some authors studied fuzzy implications from different perspectives [3]. Naturally, it is meaningful to investigating the common properties of some important fuzzy implications used in fuzzy logic. Various interesting properties of FI-algebras [4,5], regular FI-algebras [1,6], W_d -FI-algebras [7], and other kinds of FI-algebras [8] were reported and some concepts of filter, ideal and fuzzy filter of FI-algebras were proposed [2,3,9]. Relationships among FI-algebra and BCK-algebra [10,11], MV-algebras [12], Rough algebras [13], BL-algebras were partly investigated, and FI-algebras were axiomatized [14,15].

L-algebras, which are related to algebraic logic and quantum structures, were introduced by Rump (2008) [16]. Many examples shown that L-algebras are very useful. Yang and Rump (2012)[17], characterized pseudo-MV-algebras and Bosbachs non-commutative bricks as L-algebras. Wu and Yang proved that orthomodular lattices form a special class of L-algebras in different ways (2020) [18]. It was shown that every lattice-ordered effect algebra has an underlying L-algebra structure in Wu et al. (2019) [19]. Basic algebras

and L-algebras, State L-algebras and derivations of L-algebras, L-algebras in logic, algebra, geometry, and topology, Prime L-algebras and right-angled Artin groups were systematically discussed [20-24].

The paper is organized as follows: preliminary notions and results are introduced in Section 2. Main properties of W_d -FI algebras are included in Section 3. In Section 4, relationships between W_d -FI algebras and several classes of important logical algebras are discussed. In Sect.5, we introduce a new algebraic structures known as generalized W_d -fuzzy implication algebras, and we construct some examples to show the existence of the structures.

2 Preliminary notions of W_d -FI algebras

In this part, we firstly review some relevant concepts and definitions.

Definition 2.1 ([2]) Let X be a set with the algebraic constant $0 \in X$, and $\rightarrow$ be a binary operation on X , where $0 \rightarrow 0 = 1$. A $(2,0)$ -type algebra $(X, \rightarrow, 0)$ is called a fuzzy implication algebra, shortly, FI-algebra, if the following five conditions hold for all $x, y, z \in X$:

$$(I_1)x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z);$$

$$(I_2)(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1;$$

$$(I_3)x \rightarrow x = 1;$$

$$(I_4)x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y;$$

$$(I_5)0 \rightarrow x = 1.$$

Example1. Let $X = \{0, a, b, 1\}$ be a finite set of distinct elements, and $0 < a < b < 1$. We define

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

It is easy to check that $(X, \rightarrow, 0)$ is an FI- algebra.

Definition 2.2 ([2]) Let X be a set with the algebraic constant $0 \in X$, and $\rightarrow$ be a binary operation on X , where $1 = 0 \rightarrow 0$. A $(2,0)$ -type algebra $(X, \rightarrow, 0)$ is called an Heyting fuzzy implication algebra, shortly, HFI-algebra, if the following five conditions hold for all $x, y, z \in X$:

$$(HFI_1)x \rightarrow (y \rightarrow x) = 1;$$

$$(HFI_2)(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1;$$

$$(HFI_3)1 \rightarrow x = 1 \Rightarrow x = 1;$$

$$(HFI_4)x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y;$$

$$(HFI_5)0 \rightarrow x = 1.$$

On an FI-algebra $(X, \rightarrow, 0)$, one can define a binary relation $\leq$ and operators $'$, T , S as follows.

$$x \leq y \iff x \rightarrow y = 1, x, y \in X; \quad (1)$$

$$x' = x \rightarrow 0, x \in X; \quad (2)$$

$$T(x, y) = (x \rightarrow y')', x, y \in X. \quad (3)$$

$$S(x, y) = x' \rightarrow y, x, y \in X. \quad (4)$$

Usually, we also say that X is an FI-algebra for convenience.

Obviously, the relation " $\leq$ " is a partial ordering on X , i.e., the relation is reflexive, antisymmetric and transitive (see [2]). In fuzzy logic, the property (1) is called the ordering property.

The operator " $'$ " defined in the above definition is a negation on X , i.e., the operator is order-inverting and satisfies $0' = 1$, and $1' = 0$. " $\leq$ " and " $'$ " are called the partial ordering and the negation induced by the FI-algebra X , respectively.

Definition 2.3 ([2]) Let X be an FI-algebra.

- (i) X is a regular FI-algebra, or an RFI-algebra, if $x'' = x$, for all $x \in X$.
- (ii) X is commutative, or a CFI-algebra, if the binary operation " T " defined by (3) is commutative, or the following condition (I_6) holds for all $x, y \in X$:

$$(I_6)(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x.$$

For the related background of W_d - Fuzzy implication algebra, we refer to Deng and Li(1996). A W_d - Fuzzy implication algebra is an algebra of type(2,0). The notion was first formulated in 1996 by Deng and some properties were obtained[7]. This notion was originated from the motivation based on fuzzy implication algebra introduced by Wu [2].

Definition 2.4 ([7]) A (2,0)-type algebra $(X, \rightarrow, 0)$ is a W_d -Fuzzy implication algebra, shortly, W_d -FI algebra, if the following conditions hold for all $x, y, z \in X$:

$$(W_1)x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z);$$

$$(W_2)(x \rightarrow y) \rightarrow z = (z \rightarrow y) \rightarrow x;$$

$$(W_3)x \rightarrow x = 1;$$

$$(W_4)x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y;$$

$$(W_5)0 \rightarrow x = 1, \text{ where } 1 = 0 \rightarrow 0.$$

Example 2. Consider $X = [0, 1]$, for every $x, y \in X$, defined $x \rightarrow y = 1$, then $(X, \rightarrow, 0)$ is a W_d -FI algebra.

Example 3. Let $X = \{0, a, 1\}$ be a finite set of distinct elements. We define

$\rightarrow$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

Then $(X, \rightarrow, 0)$ is a FI- algebra, but not W_d -FI algebras. In fact, $(1 \rightarrow a) \rightarrow 0 = a \rightarrow 0 = 0$, but $(0 \rightarrow a) \rightarrow 1 = 1 \rightarrow 1 = 1$, so (W_2) does not hold.

In Sect.4, we will prove that any W_d -FI algebra is an FI-algebra, but the inverse is not true. Every W_d -FI algebra is an RFI- algebra, we show that an RFI- algebra is W_d -FI algebra is not correct in general.

3 Main properties of W_d -FI algebra

Given a W_d -FI algebra $(X, \rightarrow, 1)$, a binary relation " $\leq$ " is defined by

$$x \leq y \Leftrightarrow x \rightarrow y = 1,$$

Then " $\leq$ " is a partial order which is called the natural order. Relative to the natural order on X , 1 is the greatest element.

In fact, by Definition 2.4, $(W_3)x \rightarrow x = 1$ we get $x \leq x$, i.e., reflexive hold, using (W_4) , we have antisymmetric are satisfied, i.e., $x \leq y, y \leq x \Rightarrow x = y$. Next, by $x \rightarrow z = x \rightarrow (1 \rightarrow z) = x \rightarrow ((y \rightarrow z) \rightarrow z) = x \rightarrow ((z \rightarrow z) \rightarrow y) = x \rightarrow (1 \rightarrow y) = x \rightarrow y = 1$. Hence, transitive hold. So, $\leq$ is a partial order on X .

Proposition 3.1 If $(X, \rightarrow, 0)$ is a W_d -FI algebra and $x, y, z \in X$, then:

$$(W_6)x \rightarrow 1 = 1, 1 \rightarrow x = x,$$

$$(W_7)x \rightarrow y = 1, y \rightarrow z = 1 \Rightarrow x \rightarrow z = 1,$$

$$(W_8)(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1,$$

$$(W_9)(y \rightarrow z) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1,$$

$$(W_{10})(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = x.$$

Proof. (W_6) Indeed $x \rightarrow 1 = x \rightarrow (0 \rightarrow x) = 0 \rightarrow (x \rightarrow x) = 0 \rightarrow 1 = 1$. Thus, we have verified that $x \rightarrow 1 = 1$.

Besides, $(1 \rightarrow x) \rightarrow x = (x \rightarrow x) \rightarrow 1 = 1 \rightarrow 1 = 1$, $x \rightarrow (1 \rightarrow x) = 1 \rightarrow (x \rightarrow x) = 1 \rightarrow 1 = 1$. That is $1 \rightarrow x = x$.

(W_7) If $x \rightarrow y = 1, y \rightarrow z = 1$ holds, then $x \rightarrow z = 1 \rightarrow (x \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z) = ((x \rightarrow z) \rightarrow y) \rightarrow x = ((y \rightarrow z) \rightarrow x) \rightarrow x = (1 \rightarrow x) \rightarrow x = x \rightarrow x = 1$, Thus, $x \rightarrow z = 1$.

(W_8) Indeed $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = (x \rightarrow y) \rightarrow (((x \rightarrow z) \rightarrow z) \rightarrow y) = (x \rightarrow y) \rightarrow (((z \rightarrow z) \rightarrow x) \rightarrow y) = (x \rightarrow y) \rightarrow ((1 \rightarrow x) \rightarrow y) = (x \rightarrow y) \rightarrow (x \rightarrow y) = 1$.

Hence, for all $x, y, z \in X$, $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$.

(W_9) We have $(y \rightarrow z) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = (x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = (x \rightarrow y) \rightarrow (((x \rightarrow z) \rightarrow z) \rightarrow y) = (x \rightarrow y) \rightarrow (((z \rightarrow z) \rightarrow x) \rightarrow y) = (x \rightarrow y) \rightarrow ((1 \rightarrow x) \rightarrow y) = (x \rightarrow y) \rightarrow (x \rightarrow y) = 1$.

(W_{10}) Let $x, y, z \in X$. Applying (W_1) and (W_2) , we have

$$(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = (y \rightarrow (x \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = (x \rightarrow y) \rightarrow ((y \rightarrow (x \rightarrow z)) \rightarrow (x \rightarrow z)) = (x \rightarrow y) \rightarrow (((x \rightarrow z) \rightarrow z) \rightarrow y) = (x \rightarrow y) \rightarrow (((z \rightarrow z) \rightarrow x) \rightarrow y) = (x \rightarrow y) \rightarrow ((1 \rightarrow x) \rightarrow y) = (x \rightarrow y) \rightarrow (x \rightarrow y) = 1 \rightarrow x = x.$$

Proposition 3.2 Let X be an W_d -FI algebra, for all x, y, z , we have

$$(W_{11})y \leq x \rightarrow y; x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y).$$

$$(W_{12})x \leq y \Rightarrow z \rightarrow x \leq z \rightarrow y, y \rightarrow z \leq x \rightarrow z.$$

$$(W_{13})x'' = x; (x'' \rightarrow x)' = 0.$$

$$(W_{14})x''' = x';$$

$$(W_{15})x' \rightarrow y' = y \rightarrow x;$$

$$(W_{16})x' \rightarrow y = y' \rightarrow x;$$

$$(W_{17})(x \rightarrow y)' = x;$$

$$(W_{18})(y \rightarrow x) \rightarrow y' \leq x';$$

$$(W_{19})(x' \rightarrow y') \rightarrow x = x' \rightarrow ((x' \rightarrow y')' = ((x \rightarrow y) \rightarrow x')' = y;$$

$$(W_{20})(Commutativity)T(x, y) = T(y, x), S(x, y) = S(y, x);$$

$$(W_{21})(Associativity)T(T(x, y), z) = T(x, T(y, z)), S(S(x, y), z) = S(x, S(y, z));$$

$$(W_{22})(Monotonicity)x \leq y \Rightarrow T(x, z) \leq T(y, z), S(x, Z) \leq S(y, z);$$

$$(W_{23})(Identity)T(x, 1) = x, S(x, 0) = x;$$

$$(W_{24})(Duality)S(x, y) = T(x', y'), T(x, y) = (S(x'), y')';$$

$$(W_{25})S(x, x') = 1, T(x, x') = 0;$$

$$(W_{26})x \rightarrow (y \rightarrow z) = T(x, y) \rightarrow z;$$

$$(W_{27})T((z \rightarrow x), (z \rightarrow y)) = z \rightarrow T(x, y);$$

$$(W_{28})x' = x \rightarrow 0 = 0 \Rightarrow x = 1.$$

Proof. (W_{11}) Applying (W_1), we have $y \rightarrow (x \rightarrow y) = x \rightarrow (y \rightarrow y) = x \rightarrow 1 = 1$. So $y \leq x \rightarrow y$;

By $(x \rightarrow y) \rightarrow ((z \rightarrow x) \rightarrow (z \rightarrow y)) = (z \rightarrow x) \rightarrow ((x \rightarrow y) \rightarrow (z \rightarrow y)) = (z \rightarrow x) \rightarrow (((z \rightarrow y) \rightarrow y) \rightarrow x) = (z \rightarrow x) \rightarrow (((y \rightarrow y) \rightarrow z) \rightarrow x) = (z \rightarrow x) \rightarrow ((1 \rightarrow z) \rightarrow x) = (z \rightarrow x) \rightarrow (z \rightarrow x) = 1$, we have that for all $x, y, z \in X$, $(x \rightarrow y) \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$.

(W_{12}) Assume that $x \leq y$. Then $(z \rightarrow x) \rightarrow (z \rightarrow y) = z \rightarrow ((z \rightarrow x) \rightarrow y) = z \rightarrow ((y \rightarrow x) \rightarrow z) = (y \rightarrow x) \rightarrow (z \rightarrow z) = (y \rightarrow x) \rightarrow 1 = (1 \rightarrow x) \rightarrow y = x \rightarrow y = 1$, which implies that $z \rightarrow x \leq z \rightarrow y$.

Similarly, we have $(y \rightarrow z) \rightarrow (x \rightarrow z) = ((x \rightarrow z) \rightarrow z) \rightarrow y = ((z \rightarrow z) \rightarrow x) \rightarrow y = (1 \rightarrow x) \rightarrow y = x \rightarrow y = 1$.

$$(W_{13})x'' = (x \rightarrow 0) \rightarrow 0 = (0 \rightarrow 0) \rightarrow x = 1 \rightarrow x = x.$$

$$(x'' \rightarrow x)' = (x \rightarrow x)' = 1' = 1 \rightarrow 0 = 0.$$

$$(W_{14})x''' = ((x \rightarrow 0) \rightarrow 0) \rightarrow 0 = (0 \rightarrow 0) \rightarrow (x \rightarrow 0) = 1 \rightarrow (x \rightarrow 0) = x'.$$

$$(W_{15})x' \rightarrow y' = (x \rightarrow 0) \rightarrow (y \rightarrow 0) = y \rightarrow ((x \rightarrow 0) \rightarrow 0) = y \rightarrow ((0 \rightarrow 0) \rightarrow x) = y \rightarrow (1 \rightarrow x) = y \rightarrow x.$$

$$(W_{16})x' \rightarrow y = (x \rightarrow 0) \rightarrow y = (y \rightarrow 0) \rightarrow x = y' \rightarrow x.$$

$$(W_{17})(x \rightarrow y)' = (x \rightarrow y) \rightarrow 0 = (0 \rightarrow y) \rightarrow x = 1 \rightarrow x = x.$$

$$(W_{18})((y' \rightarrow x) \rightarrow y') \rightarrow x' = (x' \rightarrow y') \rightarrow (y \rightarrow x) = (y \rightarrow x) \rightarrow (y \rightarrow x) = 1. \text{ So, } (y \rightarrow x) \rightarrow y' \leq x'.$$

$$(W_{19})(x' \rightarrow y') \rightarrow x = (y \rightarrow x) \rightarrow x = (x \rightarrow x) \rightarrow y = 1 \rightarrow y = y, x \rightarrow (x' \rightarrow y')' = x' \rightarrow (y \rightarrow x)' = (y \rightarrow x) \rightarrow x = (x \rightarrow x) \rightarrow y = 1 \rightarrow y = y. \text{ Therefore, } (x' \rightarrow y') \rightarrow x = x' \rightarrow (x' \rightarrow y')' = y.$$

Similarly, we have $((x \rightarrow y) \rightarrow x')' = ((x' \rightarrow y) \rightarrow x)' = (((x \rightarrow 0) \rightarrow y) \rightarrow x)' = (((y \rightarrow 0) \rightarrow x) \rightarrow x)' = ((x \rightarrow x) \rightarrow (y \rightarrow 0))' = (y \rightarrow 0) \rightarrow 0 = y$.

$(W_{20}) T(x, y) = (x \rightarrow y')' = (y \rightarrow x')' = T(y, x)$. Similarly, we have $S(x, y) = S(y, x)$.

$(W_{21}) T(T(x, y), z) = T((x \rightarrow y')', z) = ((x \rightarrow y')' \rightarrow z')' = (((x \rightarrow y') \rightarrow 0) \rightarrow z')' = (((x \rightarrow y') \rightarrow 0) \rightarrow (z \rightarrow 0))' = (z \rightarrow (x \rightarrow y'))' = (x \rightarrow (z \rightarrow y'))' = (x \rightarrow (z \rightarrow (y \rightarrow 0)))' = (x \rightarrow (y \rightarrow (z \rightarrow 0)))' = T(x, T(y, z))$.

Similarly, we have $S(S(x, y), z) = S(x, S(y, z))$.

(W_{22}) Due to $x \leq y \Leftrightarrow x \rightarrow y = 1$, then for all $x, y, z \in X$, it is $(x \rightarrow z')' \rightarrow (y \rightarrow z')' = ((x \rightarrow z') \rightarrow 0) \rightarrow ((y \rightarrow z') \rightarrow 0) = (y \rightarrow z') \rightarrow (x \rightarrow z') = x \rightarrow y = 1$. Hence, $T(x, z) \leq T(y, z)$. $S(x, z) \leq S(y, z)$. Similarly, we have $S(x, z) \leq S(y, z)$.

$(W_{23}) T(x, 1) = (x \rightarrow 1')' = (x \rightarrow 1') \rightarrow 0 = (0 \rightarrow 1') \rightarrow x = 1 \rightarrow x = x$.

$S(x, 0) = x' \rightarrow 0 = (x \rightarrow 0) \rightarrow 0 = (0 \rightarrow 0) \rightarrow x = 1 \rightarrow x = x$.

$(W_{24}) (T(x', y'))' = (x' \rightarrow y'')'' = x' \rightarrow y = S(x, y)$.

$(S(x', y'))' = (x'' \rightarrow y')' = (x \rightarrow y')' = T(x, y)$.

$(W_{25}) S(x, x') = x' \rightarrow x' = 1; T(x, x') = (x \rightarrow x'')' = (x \rightarrow x')' = 1' = 1 \rightarrow 0 = 0$.

$(W_{26}) T(x, y) \rightarrow z = (x \rightarrow z')' \rightarrow z = ((x \rightarrow y') \rightarrow 0) \rightarrow z = ((x \rightarrow (y \rightarrow 0)) \rightarrow 0) \rightarrow z = (z \rightarrow 0) \rightarrow (x \rightarrow (y \rightarrow 0)) = x \rightarrow ((z \rightarrow 0) \rightarrow (y \rightarrow 0)) = x \rightarrow (y \rightarrow z)$.

$(W_{27}) T((z \rightarrow x), (z \rightarrow y)) = ((z \rightarrow x), (z \rightarrow y'))' = ((z \rightarrow x) \rightarrow ((z \rightarrow y) \rightarrow 0)) \rightarrow 0 = z \rightarrow x$, and $z \rightarrow T(x, y) = z \rightarrow (x \rightarrow y')' = z \rightarrow ((x \rightarrow (y \rightarrow 0)) \rightarrow 0) = z \rightarrow x$.

(W_{28}) By assume that $x' = x \rightarrow 0 = 0 \Rightarrow 1 = 0 \rightarrow 0 = (x \rightarrow 0) \rightarrow 0 = (0 \rightarrow 0) \rightarrow x = 1 \rightarrow x = x$, we get $x = 1$.

Proposition 3.3 Let X be a W_d -FI algebra and $x, y, z \in X$, then $(x \rightarrow y) \rightarrow z = (z \rightarrow y) \rightarrow x$ implies $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

Proof. By (W_6) , we have $x \rightarrow (y \rightarrow z) = (1 \rightarrow x) \rightarrow (y \rightarrow z) = ((y \rightarrow z) \rightarrow x) \rightarrow 1 = ((x \rightarrow z) \rightarrow y) \rightarrow 1 = (1 \rightarrow y) \rightarrow (x \rightarrow z) = y \rightarrow (x \rightarrow z)$. Hence, $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

Theorem 3.4 A $(2,0)$ -type algebra $(X, \rightarrow, 0)$ is a W_d -fuzzy implication algebra if and only if it satisfies that

$(W1') (x \rightarrow y) \rightarrow z = (z \rightarrow y) \rightarrow x$;

$(W2') 1 \rightarrow x = x$;

$(W3') x \rightarrow x = 1$;

$(W4')$ If $x \rightarrow y = y \rightarrow x = 1$, then $x = y$;

$(W5') 0 \rightarrow x = 1$, where $1 = 0 \rightarrow 0$.

Proof. Immediate from Proposition 3.3 and Definition 2.3.

By $(W1')$ of Theorem 3.4, for distinct $x, y \in X$, we obtain

$(W_{29}) ((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$.

Theorem 3.5 Let $(X, \rightarrow, 0)$ be a W_d -fuzzy implication algebras with negation, we define the partial addition

$$x + y := y' \rightarrow x. \quad (5)$$

Then $(X, +, 0)$ is an Abelian monoid.

Proof. It is easy to check that 0 is partial addition unit, i.e $0 + x = x + 0 = x$.

Next, we will show that $x + (y + z) = (x + y) + z, x + y = y + x$.

In fact, by (5), we have $(x + y) + z = z' \rightarrow (x + y) = (z \rightarrow 0) \rightarrow ((y \rightarrow 0) \rightarrow x) = (z \rightarrow 0) \rightarrow (x \rightarrow 0) \rightarrow y = (x \rightarrow 0) \rightarrow (z \rightarrow 0) \rightarrow y = (x \rightarrow 0) \rightarrow (y + z) = ((y + z) \rightarrow 0) \rightarrow x = x + (y + z)$.

Similarly, it can be proved that $x + y = y + x$, for any $x, y \in X$. Therefore, $(X, +, 0)$ is an Abelian monoid.

4 Relation between W_d -FI algebras and other logical algebras

4.1 Relation with FI-algebras

In this subsection, we establish the connections between: **Proposition 4.1** Every W_d -FI algebra is an FI - algebra, but the inverse is not true.

Proof. From the Definition 2.3 and (3) of Proposition 3.1, it is easy to see that any W_d -FI algebra must be an FI - algebra. By example 3, W_d -FI algebra may be an proper subalgebra of FI-algebra, but FI algebra must not be W_d -FI algebra.

Example 4. Let $X = \{0, a, b, 1\}$ be a finite set of distinct elements. We define

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Then, $(X, \rightarrow, 1)$ is an FI-algebra which is not W_d -FI algebra, because $(a \rightarrow b) \rightarrow b = b \rightarrow b = 1 \neq (b \rightarrow b) \rightarrow a = 1 \rightarrow a = a$.

4.2 Relation with RFI-algebras

Proposition 4.2 Every W_d -FI algebra is an RFI- algebra.

Proof. By Proposition 4.1 and Definition 2.3, we have $\forall x \in X, x'' = (x \rightarrow 0) \rightarrow 0 = (0 \rightarrow 0) \rightarrow x = 1 \rightarrow x = x$. Therefore $(X, \rightarrow, 0)$ is an RFI- algebra.

In the following example, we show that converse of Proposition 4.3 is not correct in general.

Example 5. Let $X = [0, 1]$, we define $x \rightarrow y = \min(1, 1 - x + y)$. It is easy to verify that $(X, \rightarrow, 0)$ is an RFI-algebra. But the converse of above Proposition is not true.

In fact, we put $x = 0.1, y = 0.2, z = 0.3$, because $(0.1 \rightarrow 0.2) \rightarrow 0.3 = \min(1, 1 - 0.1 + 0.2) \rightarrow 0.3 = 1 \rightarrow 0.3 = 0.3$, on the other hand, $(0.3 \rightarrow 0.2) \rightarrow 0.1 = \min(1, 1 - 0.3 + 0.2) \rightarrow 0.1 = 0.9 \rightarrow 0.1 = \min(1, 1 - 0.9 + 0.1) = 0.2$, hence, it is not satisfies condition (W_2) .

We have see that W_d -FI algebra classes are a subclasses of RFI-algebras.

4.3 Relation with CFI-algebras

Proposition 4.3 W_d -FI algebra must not be CFI- algebra.

Proof. It is easy to prove that if $x \neq y$, then the condition (I_6) does not hold, i.e., suppose

$$(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x,$$

then clearly

$$(y \rightarrow y) \rightarrow x = (x \rightarrow x) \rightarrow y.$$

Thus, we have $x = y$, which contradicts to assertion.

4.4 Relation with HFI-algebras

Proposition 4.4 ([7]) The relation between W_d -FI algebra and HFI- algebra is as following :

(1) If $(X, \rightarrow, 0)$ is a W_d -FI algebra such that, for all $x, y, z \in X$,

$$x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z) \quad (6)$$

holds, then $(X, \rightarrow, 0)$ is a HFI-algebra.

(2) If $(X, \rightarrow, 0)$ is an HFI-algebra such that, for all $x, y, z \in X$,

$$(x \rightarrow y) \rightarrow z = (z \rightarrow y) \rightarrow x$$

holds, then $(X, \rightarrow, 0)$ is a W_d -FI algebra.

Proof. (1) Assume that $(X, \rightarrow, 0)$ is a W_d -FI algebra, it is easy to verify that conditions (HFI_4) , (HFI_5) of HFI-algebra hold. By $(W_1)x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$, and put $z = x$, we have $x \rightarrow (y \rightarrow x) = y \rightarrow 1 = 1$. So (HFI_1) holds for all $x, y \in X$.

Next, by $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$, we have $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1$, which implies (HFI_2) holds for all $x, y, z \in X$. By $1 \rightarrow x = x$ and $1 \rightarrow x = 1 \Rightarrow x = 1$, so (HFI_3) holds for all $x \in X$. Hence, $(X, \rightarrow, 0)$ is a HFI-algebra.

(2) It is easy to prove that if an HFI-algebra $(X, \rightarrow, 0)$ satisfies the condition $(x \rightarrow y) \rightarrow z = (z \rightarrow y) \rightarrow x$, then $(X, \rightarrow, 0)$ is a W_d -FI algebra.

Corollary 4.5 Every self distributive W_d -FI algebra is an HFI-algebra.

Proof. By Proposition 4.4 (1), the proof is clear.

4.5 Relation with BCK-algebras

In 1966, Imai and Iseki [28] introduced two classes of abstract algebras, BCK-algebras. It is well known that the class of MV-algebras is a proper subclass of the class of BCK-algebras. Therefore, both BCK-algebras and MV-algebras are important for the study of fuzzy logic.

In this subsection, we investigate the relation between W_d -FI-algebras and BCK-algebras.

Definition 4.1 [28] An algebraic structure $(A, \rightarrow, 1)$ of type $(2, 0)$ is called a BCK-algebra, if for any $x, y, z \in A$, the following conditions hold:

$$(BCK_1)(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1,$$

$$(BCK_2) 1 \rightarrow x = x,$$

$$(BCK_3) x \rightarrow 1 = 1,$$

$$(L_3) \text{ if } x \rightarrow y = y \rightarrow x = 1, \text{ then } x = y.$$

Theorem 4.6 Every W_d -FI-algebras is a BCK -algebras.

Proof. By Definition 2.4 and $(W_6), (W_8)$ in Proposition 3.1, the proof is clear.

By the following example we show that every BCK -algebra is not W_d -FI-algebra, in general.

Example 6. Let $X = \{a, b, c, 1\}$ be a finite set of distinct elements. We define

$\rightarrow$	a	b	c	1
a	1	a	a	1
b	1	1	a	1
c	1	a	1	1
1	a	b	c	1

Then $(X, \rightarrow, 1)$ is a BCK -algebra, and it is not a W_d -FI-algebra, since W_2 is not satisfied

$$(b \rightarrow c) \rightarrow a = a \rightarrow a = 1 \neq a = a \rightarrow b = (a \rightarrow c) \rightarrow b.$$

4.6 Relation with L-algebras

L-algebras, which are related to algebraic logic and quantum structures, were introduced by Rump [16]. It turns out that the concept of L-algebra is fundamental in the sense that various algebraic structures, even with several operations like Heyting algebras, (one-side) hoops, (pseudo) MV-algebras or l-group cones, are definable as L-algebra[17-24].

First, we recall some definitions and properties about L-algebras.

Definition 4.2([16]) An L-algebras is an algebra $(L, \rightarrow, 1)$ of type (2,0) satisfying for all $x, y, z \in L$,

$$(L_1) x \rightarrow x = x \rightarrow 1 = 1, 1 \rightarrow x = x;$$

$$(L_2) (x \rightarrow y) \rightarrow (x \rightarrow z) = (y \rightarrow x) \rightarrow (y \rightarrow z);$$

$$(L_3) x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y.$$

From the above definition of Rump ([16]), we know that Hilbert algebras are equivalent to implicative BCK -algebras, and they are special L-algebras. L-subalgebra $\{y \rightarrow x | y, x \in L\}$ of an L-algebra is an MV-algebra. Next, we give characterizations of L-algebras and discuss relations between W_d -FI algebra and L-algebras.

Theorem 4.7 ([16]) Let $(X, \rightarrow, 1)$ be an L-algebra. The following hold for all $x, y, z \in X$:

$$(1) x \leq y \Rightarrow z \rightarrow x \leq z \rightarrow y;$$

$$(2) \leq \text{ is a partial order on } X;$$

$$(3) x = y \text{ if and only if } x \rightarrow z = y \rightarrow z, \text{ for all } x, y, z \in X;$$

$$(4) x, y \leq z \text{ such that } z \rightarrow x = z \rightarrow y, \text{ then } x = y;$$

$$(5) ((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y;$$

$$(6) ((x \rightarrow y) \rightarrow y) \rightarrow x \rightarrow ((x \rightarrow y) \rightarrow y) \rightarrow z = x \rightarrow z.$$

(7) If X satisfies condition $(x \rightarrow (y \rightarrow z)) \rightarrow (y \rightarrow (x \rightarrow z)) = 1$, then

$$x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z).$$

Theorem 4.8 Any W_d -FI algebra is an L-algebra, but the inverse is not true.

Proof. Let $(X, \rightarrow, 0)$ is a W_d -FI algebra, any $x, y, z \in X$, by equations (W_1) , (W_2) and (W_6) , respectively, it is easy to see that

$$(x \rightarrow y) \rightarrow (x \rightarrow z) = x \rightarrow ((x \rightarrow y) \rightarrow z) = x \rightarrow ((z \rightarrow y) \rightarrow x) = (z \rightarrow y) \rightarrow (x \rightarrow x) = (z \rightarrow y) \rightarrow 1 = 1,$$

$$(y \rightarrow x) \rightarrow (y \rightarrow z) = y \rightarrow ((y \rightarrow x) \rightarrow z) = y \rightarrow ((z \rightarrow x) \rightarrow y) = (z \rightarrow x) \rightarrow (y \rightarrow y) = (z \rightarrow x) \rightarrow 1 = 1,$$

Hence, $(x \rightarrow y) \rightarrow (x \rightarrow z) = (y \rightarrow x) \rightarrow (y \rightarrow z)$. Thus (L_2) holds in X .

From definition 2.3, we see that (L_1) and (L_3) hold in X . Therefore, $(X, \rightarrow, 0)$ be a L-algebra.

Example 7. Let $X = \{0, a, b, c, d, 1\}$ be a set and operation $\rightarrow$ be defined as follows:

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

Then $(X, \rightarrow, 1)$ is an L-algebra which is not a W_d -FI algebra, since $(b \rightarrow c) \rightarrow d = 1 \neq (d \rightarrow c) \rightarrow b = d \rightarrow b = b$. Therefore, the class of W_d -FI algebras is a proper subclass of L-algebras.

Theorem 4.9 Let $(X, \rightarrow, 1)$ is an L-algebra satisfying the condition

$$(x \rightarrow y) \rightarrow z = (z \rightarrow y) \rightarrow x$$

for $x, y, z \in X$, then $(X, \rightarrow, 1)$ is a W_d -FI algebra.

Proof. By Definition 2.4 and Theorem 4.3 we have $x \rightarrow (y \rightarrow z) = (1 \rightarrow x) \rightarrow (y \rightarrow z) = ((y \rightarrow z) \rightarrow x) \rightarrow 1 = ((x \rightarrow z) \rightarrow y) \rightarrow 1 = (1 \rightarrow y) \rightarrow (x \rightarrow z) = y \rightarrow (x \rightarrow z)$. So, $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$. Thus, L-algebra $(X, \rightarrow, 1)$ is a W_d -FI algebra.

Definition 4.3 ([21]) Let $(X, \rightarrow, 1)$ be an L-algebra. The following hold for all $x, y \in X$:

(1) If L-algebra $(X, \rightarrow, 1)$ satisfies

$$x \rightarrow (y \rightarrow x) = 1,$$

then it is called a KL-algebra.

(2) If L-algebra $(X, \rightarrow, 1)$ satisfies

$$(x \rightarrow (y \rightarrow z)) \rightarrow (y \rightarrow (x \rightarrow z)) = 1,$$

then it is called a CL-algebra.

Proposition 4.10 ([21]) Any CL-algebra is a KL-algebra.

Theorem 4.11 If $(X, \rightarrow, 0)$ is a W_d -FI algebra, then $(X, \rightarrow, 0)$ is a CL-algebra and it is also a KL-algebra.

Proof. By Theorem 4.7, any W_d -FI algebra $(X, \rightarrow, 0)$ is an L-algebra, using (W_1) and (W_3) , we get $(x \rightarrow (y \rightarrow z)) \rightarrow (y \rightarrow (x \rightarrow z)) = (y \rightarrow (x \rightarrow z)) \rightarrow (y \rightarrow (x \rightarrow z)) = 1$. Hence we get $(X, \rightarrow, 0)$ is a CL-algebra. By Proposition 4.8, we get $(X, \rightarrow, 0)$ is a KL-algebra.

5 Generalized W_d -FI algebras

In this Sect., the concept of GW_d -FI-algebra is introduced, and some properties of it are investigated. We discuss on a distributive GW_d -FI-algebra. The relationship between distributive GW_d -FI-algebra and Hilbert algebra is considered, and we prove that every distributive GW_d -FI-algebra is a Hilbert algebra, but the converse may not be true, with an example being given to illustrate it. Moreover, we prove that every GW_d -FI-algebra is a BE-algebras.

5.1 Notes of Generalized W_d -FI algebras

The condition (W_2) of Definition 2.4 is a strong condition, in this Sect., we abandon the condition W_2 , redefine generalized W_d -fuzzy implication algebras which is a generalization of W_d -FI-algebra. In order to show the existence of the structures, we presented some examples. We then explore some properties of these structures. We start with the following definition.

Definition 5.1 A (2,0)-type algebra $(X, \rightarrow, 0)$ is said to be a generalized W_d -Fuzzy implication algebra, shortly, GW_d -FI algebra, if the following conditions hold for all $x, y, z \in X$:

$$(GW_1) x \rightarrow x = 1;$$

$$(GW_2) 1 \rightarrow x = x;$$

$$(GW_3) x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z);$$

$$(GW_4) \text{ if } x \rightarrow y = y \rightarrow x = 1, \text{ then } x = y;$$

$$(GW_5) 0 \rightarrow x = 1, \text{ where } 1 = 0 \rightarrow 0.$$

Let us give two examples.

Example 8. Let $X = \{0, a, b, c, 1\}$ be a set and operation $\rightarrow$ be defined as follows:

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	1	1	b	c	1
b	0	a	1	1	1
c	1	1	b	1	1
1	0	a	b	c	1

Then, $(X, \rightarrow, 0)$ is a GW_d -FI-algebra.

Example 9. Let $X = \{0, a, b, c, d, 1\}$ be a set and operation $\rightarrow$ be defined as follows:

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	c	d	1
b	0	a	1	c	d	1
c	0	a	b	1	d	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

Then, $(X, \rightarrow, 0)$ is a GW_d -FI-algebra.

Let us state and prove some properties.

Proposition 5.1 Let $(X, \rightarrow, 0)$ be a generalized GW_d -FI algebra and $x, y \in X$. Then

$$(5-1) x \rightarrow 1 = 1;$$

$$(5-2) x \rightarrow (y \rightarrow x) = 1.$$

Proof. For any $x \in X$, we have $x \rightarrow 1 = x \rightarrow (0 \rightarrow x) = 0 \rightarrow (x \rightarrow x) = 0 \rightarrow 1 = 1$. Hence, $x \rightarrow 1 = 1$.

$$x \rightarrow (y \rightarrow x) = y \rightarrow (x \rightarrow x) = y \rightarrow 1 = 1. \text{ So, } x \rightarrow (y \rightarrow x) = 1.$$

Definition 5.2 ([25,26]) A BE-algebra is a non-empty set X with a constant 1 and a binary operation $\rightarrow$ satisfying the following axioms:

$$(BE_1) x \rightarrow x = 1,$$

$$(BE_2) x \rightarrow 1 = 1,$$

$$(BE_3) 1 \rightarrow x = x,$$

$$(BE_4) x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z), \text{ for all } x, y, z \in X.$$

Proposition 5.2 Every GW_d -FI algebra $(X, \rightarrow, 0)$ is a BE-algebra, but the converse may not be true.

Proof. By Definition 5.1, Definition 5.2 and Proposition 5.1, the proof is completed.

Example 10. Let $X = \{0, a, b, 1\}$ be a set and operation $\rightarrow$ be defined as follows:

$\rightarrow$	0	a	b	1
0	1	a	b	1
a	0	1	b	1
b	0	a	1	1
1	0	a	b	1

It is easy to see that $(X, \rightarrow, 0)$ is a BE-algebra, but it is not GW_d -FI algebra, since $0 \rightarrow a = a \neq 1$.

Definition 5.3 A GW_d -FI algebra $(X, \rightarrow, 1)$ is said to be self-distributive if $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$, for all $x, y, z \in X$.

Definition 5.4 ([27]) A Hilbert algebra is an algebra $(A, \rightarrow, 1)$ of type (2,0) such that the following axioms are fulfilled for every $x, y, z \in A$:

$$(H_1) x \rightarrow (y \rightarrow x) = 1;$$

$$(H_2) (x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1;$$

$$(H_3) \text{ If } x \rightarrow y = y \rightarrow x = 1, \text{ then } x = y.$$

From Definition 5.1, 5.3, and 5.4, we have the following results.

Proposition 5.3 Every self-distributive GW_d -FI algebra $(X, \rightarrow, 1)$ is a Hilbert algebra. In this case, $(X, \leq)$ is a poset by defining an order relation $\leq$ such that $x \leq y$ iff $x \rightarrow y = 1$ (called the natural order on X), with respect to this order, 1 is the greatest element of X .

Proof. (1) By using (5-2) of Proposition 5.1, we get that (H_1) holds. Using the distributive law again, we obtain that (H_2) holds. (GW_4) coincides with (H_3) . The proof is completed.

(2) By (GW_1) , $x \rightarrow x = 1$, we get $x \leq x$. By (GW_4) , we obtain $x \leq y, y \leq x \Rightarrow x = y$. Now, let $x, y, z \in X$, by using self-distributive law again, from $x \leq y, y \leq z$, we get

$$x \rightarrow z = 1 \rightarrow (x \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z) = x \rightarrow (y \rightarrow z) = x \rightarrow 1 = 1.$$

Thus, $x \leq z$. Hence, $(X, \leq)$ is a partial order set.

Remark. In Proposition 5.3, the condition self-distributive is necessary. The following example is given to illustrate it.

Example 11. If we consider Example 8, then we have

$$(b \rightarrow (c \rightarrow a)) \rightarrow ((b \rightarrow c) \rightarrow (b \rightarrow a)) = (b \rightarrow 1) \rightarrow (1 \rightarrow a) = 1 \rightarrow a = a \neq 1.$$

Hence, in Example 8, self-distributive law is not satisfied. So $(X, \rightarrow, 1)$ is a GW_d -FI algebra, but it is not a Hilbert algebra.

5.2 Generalized W_d -FI algebras and W-eo algebra

Definition 5.5 ([30]) Let L be a non-empty set, $\rightarrow: L \times L \rightarrow L$ a binary operation and $\top$ a fixed element of L . The triple $(L; \rightarrow; \top)$ is a weak extended-order algebra, shortly w-eo algebra, if for all $a, b, c \in L$, the following conditions are satisfied:

$$(O_1)a \rightarrow \top = \top \text{ (upper bound condition);}$$

$$(O_2)a \rightarrow a = \top \text{ (reflexivity condition);}$$

$$(O_3)a \rightarrow b = b \rightarrow a = \top \Rightarrow a = b \text{ (antisymmetry condition);}$$

$$(O_4)a \rightarrow b = \top \text{ and } b \rightarrow c = \top \Rightarrow a \rightarrow c = \top \text{ (weak transitivity condition).}$$

Proposition 5.4 Every self-distributive GW_d -FI algebra $(X, \rightarrow, 0)$ is a W-eo algebra.

Proof. By using (5-1) of Proposition 5.1, we get (O_1) hold. By (GW_1) and (GW_4) , we get (O_2) , and (O_3) hold, respectively. We take $1 = \top$, by $a \rightarrow b = 1$ and $b \rightarrow c = 1$, we have $a \rightarrow c = 1 \rightarrow (a \rightarrow c) = (a \rightarrow b) \rightarrow (a \rightarrow c) = a \rightarrow (b \rightarrow c) = a \rightarrow 1 = 1$, so $a \rightarrow c = 1$. Hence, (O_4) holds, i.e., $(X, \rightarrow, 0)$ is a W-eo algebra.

6 Conclusion

The aim of this paper is to study the relations between W_d -FI-algebras and other logical algebras such as FI-algebras, RFI-algebras, CFI-algebras, BCK-algebras, Hilbert-algebras and L-algebras, etc. The concept of GW_d -FI-algebra is introduced, and some properties of it are investigated. For the future research, we will investigate new structures on GW_d -algebras.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Data availability statement

The research data are not shared.

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