# Neutrosophic Approaches to Soliton Solutions for Nonlinear Time-Fractional Coupled Jaulent–Miodek System Using a Modified Laplace Adomian Dec omposition Method

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## Abstract

This paper presents a modified Laplace Adomian decomposition method (MLADM) to solve the nonlinear time-fractional coupled Jaulent–Miodek system. The proposed approach provides convergent series solutions with easily computable components, demonstrating both accuracy and simplicity in its application. By employing the Caputo fractional derivative, this study establishes a robust framework for analyzing nonlinear behavior in fractional differential equations. The effectiveness of the method is validated through comparisons with previous studies, with results illustrated using graphical representations. The solutions proposed herein are significant for modeling complex and dynamic real-world phenomena across various scientific disciplines. All computations and graphical results were carried out using Mathematica, emphasizing the method's reliability, precision, and ease of application to nonlinear fractional systems. The study of fractional nonlinear systems is crucial for modeling complex, dynamic, and uncertain processes, which are core aspects of neutrosophic science. By addressing the intricate behavior of the nonlinear time-fractional coupled Jaulent–Miodek system, this work advances mathematical models that encapsulate uncertainty, indeterminacy, and complex interactions. Such an alignment with the principles of neutrosophic science underscores the relevance of our approach to the objectives of the International Journal of Neutrosophic Science, highlighting its potential to enhance the understanding and practical applications of complex systems.

**Keywords:** Approximate solutions; Jaulent-Miodek system; Adomian Method; Soliton solutions

## 1 Introduction

Non-linearity is a fascinating aspect of nature. A thorough understanding of nonlinear science is crucial for fully comprehending the universe. Nonlinear differential equations can be used to model a wide range of real-world phenomena. Engineers, physicists, and mathematicians have recently focused their attention on nonlinear partial differential equations.[1] The Jaulent-Miodek system, which has an energy-dependent Schrödinger potential, is one example of a nonlinear partial differential equation.[2–5] Fractional differential equations have gained popularity recently because of their demonstrated applications in several seemingly unrelated scientific and engineering domains. Because fractional nonlinear systems of partial differential equations can be used to model a wide range of phenomena in physics, engineering, and biology, they have garnered a lot of research interest. It is now evident that the study of fractional calculus, which deals with fractional derivatives and integrals, is essential for modeling anomalous diffusion, long-range interactions, and memory effects in real-world problems. The study of complex systems with nonlinear interactions between their component parts is known as nonlinear dynamics. Fractional nonlinear PDE systems, which can represent the intricate dynamics of numerous physical and biological systems, have been developed by combining these two fields. In this work, we apply Laplace transform and Adomian method to solve the fractional Jaulent-Miodek system equations, where the required degree of accuracy is attained. The following is the definition of the nonlinear time-fractional coupled Jaulent–Miodek system of equations, for $0 < \alpha \leqslant 1$:

$$\begin{gathered} D_t^{\alpha}u(x,t) + u_{xxx} + \frac{3}{2}vv_{xxx} + \frac{9}{2}v_x v_{xx} - 6uu_x - 6uvv_x - \frac{3}{2}u_x v^2 = 0 \\ D_t^{\alpha}v(x,t) + v_{xxx} - 6u_x v - 6uv_x - \frac{15}{2}v_x v^2 = 0. \end{gathered} \tag{1}$$

Here $D_t^{\alpha} = \frac{\partial^{\alpha}}{\partial t^{\alpha}}$ denoted the Caputo fractional derivative of order $\alpha$, where $0 < \alpha \leqslant 1$ denotes the fractional-order derivative. These equation systems are widely used as models to solve practical problems in many different fields of engineering and the natural sciences. An evolution equation known as the Jaulent–Miodek equation is found in several fields of physics, including fluid dynamics, optics, and enhanced matter physics. Numerous scientists from diverse fields are interested in the JM-type equations. Given the importance and distinctiveness of these nonlinear partial differential equations in the aforementioned fields, a comprehensive understanding of their numerical and analytic-approximate solutions is essential. Research on nonlinear fractional-order coupled Jaulent–Miodek equations is an important field in many scientific and engineering domains, such as plasma and condensed matter physics. Current research is focused on a complete mathematical analysis of these equations.

Numerous approximation techniques, such as the homotopy perturbation method[6] and the reduced transformation,[7,8] have been used to study this model. Comparable numerical methods applied to fractional systems of partial differential equation solutions.[4,9–11] The suggested technique is simple and straightforward. The current method achieves desired accuracy when compared to other analytical techniques. With power law non-linearity, Biswas and Kara examined the classical Jaulent-Miodek equation and created conservation laws for the model.[12] For the classical Jaulent-Miodek system, Nadjafikhah and Hesamiarshad have also completed symmetry analysis and conservation law.[13] Atana and Baleanu have presented a novel non-local model for coupled Jaulent-Miodek equations.[14] They suggested time fractional coupled Jaulent-Miodek equations (1), which they solved by combining the Sumudu transform with a modified homotopy perturbation technique. To find the approximate solution of the governing problem, Gupta and Saha Ray[15] used the operational matrices method based on the Hermite wavelet basis functions.

Several analytical and numerical techniques have been developed to investigate the properties of these systems, including the Laplace transform, the Adomian decomposition method (ADM), and numerical simulations. The convergence analysis of the ADM was covered in.[16] Many earlier studies have produced wave-like solutions, or soliton solutions to differential equations, and the properties of those solutions have been examined, these include.[17–19] The ADM is very helpful for solving linear and nonlinear ordinary and partial differential equations, algebraic equations, functional equations, and integral differential equations. The decomposition method's new proof of convergence analysis is provided by Y. Cherruault and G. Adomian.[20] The order of the Adomian method's convergence is defined by E. Babolian and J. Biazar in.[21] Subsequently, a multitude of researchers attempted to build the precision or broaden the scope of this approach through numerous modifications.[22,23] The work in[24] presents an operational calculus framework for solving differential equations within the neutrosophic environment. It demonstrates how neutrosophic logic and calculus can be applied to deal with uncertainty and imprecision in differential systems. This aligns with our work on fractional nonlinear systems, where modeling and uncertainty quantification play a critical role.

In this work, approximate solutions of fractional Jaulent-Miodek are found using Adomian's decomposition method in conjunction with Laplace transform. The next section provided a detailed explanation of the method; the third section included a written proof of the method's convergence; the last section covered some cases and numerical results

## 2 Modified Laplace Adomian Decomposition Method

This section outlines the Modified Laplace Adomian Decomposition Method (MLADM) process for solving fractional partial differential equations that are nonlinear systems. Have a look at the general nonlinear system below:

$$\begin{aligned} D_t^\alpha u(x,t) =& R_1(u(x,t),v(x,t)) + N_1(u(x,t),v(x,t)) + g_1(x,t), \\ D_t^\alpha v(x,t) =& R_2(u(x,t),v(x,t)) + N_2(u(x,t),v(x,t)) + g_2(x,t), \end{aligned} \tag{2}$$

subject to the following initial conditions:

$$u(x,0) = f_1(x), \quad v(x,0) = f_2(x), \tag{3}$$

where

- $D_t^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$: denoted the Caputo fractional derivative of order $\alpha, 0 < \alpha \leqslant 1$.
- $R_1(u,v), \quad R_2(u,v)$: are the linear functions.
- $N_1(u,v), \quad N_2(u,v)$: are the nonlinear functions.
- $g_1(x,t), \quad g_2(x,t)$: are the source functions.

Properties of fractional calculus and how to use them to construct the solution to the problem at hand can be found in.[25–27] Several definitions of fractional derivatives have been proposed in the past[28,29] however, in this case, we will employ the improved Caputo definition for the fractional differentiation operator $D^\alpha f(v)$, as follows

**Definition 1.** *Caputo introduces derivatives of fractional orders of a function $f(x)$ as follows*

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \tag{4}$$

$m-1 < \alpha \leqslant m, m \in \mathbf{N}, x > 0$.

**Definition 2.** *The Laplace transform of the Caputo derivative for a function $f(v)$ is defined as:*

$$\mathcal{L}[D^\alpha f(v)] = s^\alpha F(s) - \sum_{k=0}^{m-1} s^{\alpha-1-k} f^{(k)}(0), \quad m-1 < \alpha \leqslant m.$$

*where $m \in N$ and $F(s)$ denotes the Laplace transform of the function $f(v)$.*

Now, we discuss the necessary steps of the Laplace transform and Adomian method for building approximate solution for the system in equation (2).

- Step 1: Taking the Laplace transform of system (2) and according to use of the above definition, with $m = 1$, and using the initial conditions (3) we get:

$$\mathcal{L}\left[u(x,t)\right] = \frac{f_1(x)}{s} + \frac{1}{s^\alpha}\mathcal{L}\left[g_1(x,t) + R_1(u(x,t),v(x,t)) + N_1(u(x,t),v(x,t))\right],$$

$$\mathcal{L}\left[v(x,t)\right] = \frac{f_2(x)}{s} + \frac{1}{s^\alpha}\mathcal{L}\left[g_2(x,t) + R_2(u(x,t),v(x,t)) + N_2(u(x,t),v(x,t))\right].$$

- Step 2: As an infinite series, the solution is constructed

$$u(x,t)=\sum_{k=0}^{\infty}u_k(x,t), \quad v(x,t)=\sum_{k=0}^{\infty}v_k(x,t). \tag{5}$$

The nonlinear terms can be expressed as:

$$N_1\Big(u(x,t),v(x,t)\Big)=\sum_{k=0}^{\infty}A_k(x,t),$$

$$N_2\Big(u(x,t),v(x,t)\Big)=\sum_{k=0}^{\infty}B_k(x,t)$$

called Adomian polynomials, such that $A_k(x,t),\ B_k(x,t)$, $k\geqslant 0$. They are defined as follows:

$$\begin{aligned}A_k(x,t)&=\frac{1}{k!}\frac{d^k}{d\lambda^k}\left[N_1\Big(\sum_{i=0}^{k}\lambda^i u_i(x,t),\sum_{i=0}^{k}\lambda^i v_i(x,t)\Big)\right]\Big|_{\lambda=0},\\ B_k(x,t)&=\frac{1}{k!}\frac{d^k}{d\lambda^k}\left[N_2\Big(\sum_{i=0}^{k}\lambda^i u_i(x,t),\sum_{i=0}^{k}\lambda^i v_i(x,t)\Big)\right]\Big|_{\lambda=0}.\end{aligned} \tag{6}$$

Linear terms can be expressed as:

$$R_1\Big(u(x,t),v(x,t)\Big)=R_1\Big(\sum_{k=0}^{\infty}u_k(x,t),\sum_{k=0}^{\infty}v_k(x,t)\Big),$$

and,

$$R_2\Big(u(x,t),v(x,t)\Big)=R_2\Big(\sum_{k=0}^{\infty}u_k(x,t),\sum_{k=0}^{\infty}v_k(x,t)\Big).$$

After substituting all of these into the resulting equation in step 1, we get:

$$\begin{aligned}\mathcal{L}\left[\sum_{k=0}^{\infty}u_k(x,t)\right]&=\frac{f_1(x)}{s}+\frac{1}{s^\alpha}\mathcal{L}\left[g_1(x,t)+R_1\Big(\sum_{k=0}^{\infty}u_k(x,t),\sum_{k=0}^{\infty}v_k(x,t)\Big)+\sum_{k=0}^{\infty}A_k(x,t)\right],\\ \mathcal{L}\left[\sum_{k=0}^{\infty}v_k(x,t)\right]&=\frac{f_2(x)}{s}+\frac{1}{s^\alpha}\mathcal{L}\left[g_2(x,t)+R_2\Big(\sum_{k=0}^{\infty}u_k(x,t),\sum_{k=0}^{\infty}v_k(x,t)\Big)+\sum_{k=0}^{\infty}B_k(x,t)\right].\end{aligned} \tag{7}$$

- Step 3: By lining up the above Equation left and right sides, we can use the corresponding iteration formulas to find the component functions $\Big\{u_k(x,t),v_k(x,t)\Big\}_{k=0}^{\infty}$. We can then obtain the iterations as follows:

$$\begin{aligned}\mathcal{L}[u_0(x,t)]&=\frac{f_1(x)}{s}+\frac{1}{s^\alpha}\mathcal{L}[g_1(x,t)],\\ \mathcal{L}[u_1(x,t)]&=\frac{1}{s^\alpha}\mathcal{L}[A_0(x,t)+R_1(u_0,v_0)],\\ \mathcal{L}[u_2(x,t)]&=\frac{1}{s^\alpha}\mathcal{L}[A_1(x,t)+R_1(u_1,v_1)],\\ \mathcal{L}[u_{k+1}(x,t)]&=\frac{1}{s^\alpha}\mathcal{L}[A_k(x,t)+R_1(u_k,v_k)], k\geqslant 2,\end{aligned} \tag{8}$$

and

$$\begin{aligned}
\mathcal{L}[v_0(x,t)] &= \frac{f_2(x)}{s} + \frac{1}{s^\alpha}\mathcal{L}[g_2(x,t)],\\
\mathcal{L}[v_1(x,t)] &= \frac{1}{s^\alpha}\mathcal{L}[B_0(x,t) + R_2(u_0,v_0)],\\
\mathcal{L}[v_2(x,t)] &= \frac{1}{s^\alpha}\mathcal{L}[B_1(x,t) + R_2(u_1,v_1)],\\
\mathcal{L}[v_{k+1}(x,t)] &= \frac{1}{s^\alpha}\mathcal{L}[B_k(x,t) + R_2(u_k,v_k)], k \geqslant 2.
\end{aligned} \tag{9}$$

- Step 4: By applying the Laplace transform's inverse to Equations (8) and (9), we arrive at:

$$\begin{aligned}
u_0(x,t) &= \mathcal{L}^{-1}\Big[\frac{f_1(x)}{s} + \frac{1}{s^\alpha}\mathcal{L}[g_1(x,t)]\Big],\\
u_1(x,t) &= \mathcal{L}^{-1}\Big[\frac{1}{s^\alpha}\mathcal{L}[A_0(x,t) + R_1(u_0,v_0)]\Big],\\
u_2(x,t) &= \mathcal{L}^{-1}\Big[\frac{1}{s^\alpha}\mathcal{L}[A_1(x,t) + R_1(u_1,v_1)]\Big],\\
u_{k+1}(x,t) &= \mathcal{L}^{-1}\Big[\frac{1}{s^\alpha}\mathcal{L}[A_k(x,t) + R_1(u_k,v_k)]\Big], k \geqslant 2,
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
v_0(x,t) &= \mathcal{L}^{-1}\Big[\frac{f_2(x)}{s} + \frac{1}{s^\alpha}\mathcal{L}[g_2(x,t)]\Big],\\
v_1(x,t) &= \mathcal{L}^{-1}\Big[\frac{1}{s^\alpha}\mathcal{L}[B_0(x,t) + R_2(u_0,v_0)]\Big],\\
v_2(x,t) &= \mathcal{L}^{-1}\Big[\frac{1}{s^\alpha}\mathcal{L}[B_1(x,t) + R_2(u_1,v_1)]\Big],\\
v_{k+1}(x,t) &= \mathcal{L}^{-1}\Big[\frac{1}{s^\alpha}\mathcal{L}[B_k(x,t) + R_2(u_k,v_k)]\Big], \quad k \geqslant 2.
\end{aligned} \tag{11}$$

Considering the aforementioned in Equations (10) and (11), we can conclude that since the zeroth term affects all other terms in a significant way, it should have the fewest terms possible. In the event that the series converges suitably, we observe that

$$u(x,t) = \sum_{k=0}^{N-1} u_k(x,t), \quad v(x,t) = \sum_{k=0}^{N-1} v_k(x,t)$$

are approximate solutions of system (2), hence:

$$\begin{cases}
u_{approx.}(x,t,\alpha) = \sum_{k=0}^{N-1} u_k(x,t)\\
v_{approx.}(x,t,\alpha) = \sum_{k=0}^{N-1} v_k(x,t)
\end{cases}$$

where $N$ represents the total number of terms discovered. The above mentioned solution series demonstrated convergence in earlier research. We present the convergence proof of the method applied to the equation under consideration in the next section so that readers can gain a deeper understanding of the explanation and presentation of the MLADM method. This allows for the computation of a sufficient number of terms for the series approximate solution. It should be mentioned that a reasonably accurate solution can be obtained with just a few terms. Often, the problem's exact solution is known, particularly when $\alpha$ has an integer value.

The convergence of the Adomian decomposition series and the series of the Adomian polynomials has previously been demonstrated by a number of researchers, including Cherruault and colleagues,[20] among others.[30] The decomposition series' convergence was proven by Cherruault and Adomian[20,30] without the use of the fixed point theorem, which is overly restrictive for a wide range of technical and physical applications.

The Banach-space analogue of the Taylor expansion series is computationally rearranged around the initial solution component function by the Adomian decomposition series, which enables recursion-based solution. The remarkable success of the MLADM is demonstrated by its wide adoption and the numerous modifications made to it to make it more compatible for particular uses, like the different modified recursion schemes. The analyst can create altered recursion schemes for simpler computation in practical systems thanks to the non-unique decomposition.

# 3 Convergence and Uniqueness of Solutions

Ouyang examines the uniqueness and existence of solutions for fractional-order PDEs in.[31] He gave several theorems defining the uniqueness and existence of solutions to fractional PDEs with nonlinearities. This paper examines the J-M system, which is the subject of the study. See[31] for more information on the MLADM method's convergence analysis and the uniqueness of the solutions found. In this paper, we examine the convergence analysis of the MLADM for the J-M system, following in the footsteps of[20] and.[31] The Hilbert Space $\mathcal{H}$, as presented in,[31] is under consideration. The convergence of MLADM is achieved when the following assumptions are met.

**H1** $(\mathcal{P}_i(v) - \mathcal{P}_i(u), v - u) \geqslant k \parallel v - u \parallel^2,\ k > 0, \forall u, v \in \mathcal{H}.$

**H2** There exists constants, $\delta_1(M), \delta_2(M)$ such that for $u, v \in \mathcal{H}$ with $\parallel u \parallel, \parallel v \parallel \leqslant M$, where $M > 0$, we have

$$(\mathcal{P}_i(v) - \mathcal{P}_i(u), y) \leqslant \delta_i(M) \parallel v - u \parallel \parallel y \parallel,\ \forall y \in \mathcal{H}, i = 1, 2.$$

Our claim is that the MLADM applied to the coupled fractional system $J - M$ converges to a particular solution. The convergence of the Adomian decomposition method was examined by multiple authors.[32–34] In this paper, we follow the methodology in[32,33] for studying the convergence analysis of the MLADM for the $J - M$ system. Considering Hilbert Space $\mathcal{H}$, which is defined by $\mathcal{H} = \mathbf{L^2}((a, b) \times [0, T])$ the set of applications:

$$u : (a, b) \times \mathbf{R} \to \mathbf{R}$$

with

$$\int_{(a,b)\times[0,T]} u^2(x, s) ds d\tau < \infty,$$

and the following scalar products:

- $(u, v)_{\mathcal{H}} = \int_{(a,b)\times[0,T]} u(x, s)v(x, s) ds d\tau$
- $\parallel u \parallel_{\mathcal{H}}^2 = \int_{(a,b)\times[0,T]} u^2(x, s) ds d\tau.$

Let us define the following operators $\mathcal{P}_1(u), \mathcal{P}_2(v)$ in the $J - M$ system:

$$\mathcal{P}_1(u) := D_t^\alpha u(t) = \mathcal{G}_1(x, t, u, v) \tag{12}$$

$$\mathcal{P}_2(v) := D_t^{\alpha} v(t) = \mathcal{G}_2(x,t,u,v), \tag{13}$$

where

$$\mathcal{G}_1(x,t,u,v) = -\frac{\partial u}{\partial x^3} - \frac{3}{2}v\frac{\partial v}{\partial x^3} - \frac{9}{2}\frac{\partial v}{\partial x}\frac{\partial v}{\partial x^2} + 6u\frac{\partial u}{\partial x} + 6uv\frac{\partial v}{\partial x} + \frac{3}{2}v^2\frac{\partial u}{\partial x} \tag{14}$$

$$\mathcal{G}_2(x,t,u,v) = -\frac{\partial v}{\partial x^3} + 6v\frac{\partial u}{\partial x} + 6u\frac{\partial v}{\partial x} + \frac{15}{2}v^2\frac{\partial v}{\partial x} \tag{15}$$

MLADM converges if the following hypotheses are satisfied.

**H1:** $(\mathcal{P}_i(u) - \mathcal{P}_i(w), u - w) \geqslant k||u - w||^2, \; k > 0, \; \forall w, u \in \mathcal{H}$.

**H2:** There exists constants $\delta_i(M) > 0 (i = 1, 2)$ , such that for $v, u \in \mathcal{H}$ with $||v||, ||u|| \leqslant M$, where $M > 0, i = 1, 2$ we have

$$(\mathcal{P}_i(u) - \mathcal{P}_i(w), y) \leqslant \delta_i(M)||u - w||||y||, \; \forall y \in \mathcal{H}, \; i = 1, 2.$$

**Theorem 5.1** (Sufficient condition for convergence[16]): Given two Lipschitzian functions $f_1, f_2$ in $\mathcal{H}$, the coupled $J - M$ system was subjected to the MLADM:

$$\mathcal{P}_1(u) = \mathcal{G}_1(x,t,u,v) + f_1(x,t)$$

and,

$$\mathcal{P}_2(v) = \mathcal{G}_2(x,t,u,v) + f_2(x,t)$$

approaches a specific solution without the need for starting and stopping conditions when

$$\mathcal{P}_1(u) = D_t^{\alpha} u(t), \;\; \mathcal{P}_2(v) = D_t^{\alpha} v(t).$$

**Proof:** First, we will demonstrate the validity of hypothesis **H1**. We have $f_1(x,t) = f_2(x,t) = 0$ in the $J - M$ system. Equation (14) gives us the following:

$$\mathcal{P}_1(u) - \mathcal{P}_1(w) = -\frac{\partial}{\partial x^3}(u - w) + 3\frac{\partial}{\partial x}(u^2 - w^2) + 6v\frac{\partial v}{\partial x}(u - w) + \frac{3}{2}v^2(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial x})$$

and,

$$\begin{aligned}(\mathcal{P}_1(u) - \mathcal{P}_1(w), u - w) = &\Big(-\frac{\partial}{\partial x^3}(u - w), u - w\Big) + 3\Big(\frac{\partial}{\partial x}(u^2 - w^2), u - w\Big) \\ &+ 6\Big(v\frac{\partial v}{\partial x}(u - w), u - w\Big) + \frac{3}{2}\Big(v^2(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial x}), u - w\Big).\end{aligned}$$

Since $\frac{\partial}{\partial x}$ and $\frac{\partial^3}{\partial x^3}$ are differential operators in $\mathcal{H}$ and bounded in the $\mathbf{L^2}$ norm of $\mathcal{H}$, there is a real constant $\delta_1$ such that

$$(-\frac{\partial}{\partial x^3}(u - w), u - w) \geqslant \delta_1||u - w||^2 \tag{16}$$

From Schwartz inequality, we have

$$3\Big(\frac{\partial}{\partial x}(u^2 - w^2), u - w\Big) \leqslant \delta_2||u - w||^2||u - w||, \tag{17}$$

for some constant $\delta_2$. Now we use the mean value theorem to obtain

$$\begin{aligned}3\Big(\frac{\partial}{\partial x}(u^2 - w^2), u - w\Big) &\leqslant 3\;\delta_2\;||u - w||^2||u - w|| \\ &= 6\;\delta_2\;\eta\;||u - w||^2 \leqslant 6\;\delta_2\;M\;||u - w||^2\end{aligned}$$

where $w < \eta < u$ and $||w||, ||u|| \leqslant M$. Therefore

$$6\Big(v\frac{\partial v}{\partial x}(u - w), u - w\Big) \leqslant 6\;M^2\;||u - w||^2 \tag{18}$$

where $||\frac{\partial v}{\partial x}|| \leqslant M$, and $||v|| \leqslant M$. Also,

$$\frac{3}{2}\Big(v^2(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial x}), u-w\Big) \leqslant \frac{3}{2}\ M^2\ \gamma_2\ ||u-w||^2. \tag{19}$$

Substituting Equations (16)-(19) into equation (16), we obtain

$$(\mathcal{P}_1(u) - \mathcal{P}_1(w), u-w) \geqslant \Big(\delta_1 - 6\delta_2 M - 6M^2 - \frac{3}{2}M^2\delta_2\Big)||u-w||^2 = k||u-w||^2, \tag{20}$$

where $k = \delta_1 - 6\delta_2 M - 6M^2 - \frac{3}{2}M^2\delta_2$, and so hypothesis $\mathbf{H_1}$ is valid. Similarly, the $v(x,t)$ equation yields the following result.

$$(\mathcal{P}_2(v) - \mathcal{P}_2(\phi), v-\phi) \geqslant (\delta_1 - 6M - 6M\delta_2 - \frac{15}{2}M^2\delta_2)||v-\phi||^2 = c||v-\phi||^2.$$

where $c = \delta_1 - 6M - 6M\delta_2 - \frac{15}{2}M^2\delta_2$. To verify the second convergence hypothesis $\mathbf{H2}$ for operator $\mathcal{P}_1(u)$, we have

$$\begin{aligned} \Big(\mathcal{P}_1(u) - \mathcal{P}_1(w), y\Big) &= \Big(-\frac{\partial}{\partial x^3}(u-w), y\Big) + 3\Big(\frac{\partial}{\partial x}(u^2-w^2), y\Big) + 6v\frac{\partial v}{\partial x}\Big((u-w), y\Big) \\ &+ \frac{3}{2}v^2\Big(\frac{\partial}{\partial x}(u-w), y\Big) \\ &\leqslant -\parallel u-w \parallel\parallel y \parallel + 6M \parallel u-w \parallel\parallel y \parallel \\ &+ 6M^2 \parallel u-w \parallel\parallel y \parallel + \frac{3}{2}M^2 \parallel u-w \parallel\parallel y \parallel \\ &= \gamma_1(M) \parallel u-w \parallel\parallel y \parallel, \end{aligned}$$

where $\gamma_1(M) = \frac{15}{2}M^2 + 6M - 1$. For the $v(x,t)$ equation, we may do the same calculations to get $\gamma_2(M) = \frac{15}{2}M^2 + 12M - 1$. So hypothesis $\mathbf{H_2}$ also holds.

This shows that the solution obtained by MLADM converges to the exact solution when it used to solve the system under consideration $(J-M)$. Therefore, in the next section, we will present the solution of the $J-M$ system using MLADM that has been shown to converges to the exact solution.

# 4 Results and Discussions

The performance and effectiveness of the MLADM for solving nonlinear partial differential equations with time derivatives are illustrated by the example we will examine in this section. Consider the nonlinear system of time-fractional equations as given in Equation (1), subject to the initial conditions (3).

It is known that the system under consideration does not have an exact solution when $\alpha \neq 1$. But, when $\alpha = 1$ it is the only case for which we know the exact solution is given by

$$\begin{aligned} u(x,t) &= \frac{1}{8}\lambda^2\Big(1 - 4sech^2\Big[\frac{1}{2}\lambda\Big(x + \frac{1}{2}\lambda^2 t\Big)\Big]\Big) \\ v(x,t) &= \lambda sech\Big[\frac{1}{2}\lambda\Big(x + \frac{1}{2}\lambda^2 t\Big)\Big]. \end{aligned} \tag{21}$$

where the arbitrary constant $\lambda$ is used. Subject to the initial condition, we solve (1) using the modified approach presented in this paper.

$$\begin{aligned} u(x,0) &= \frac{1}{2}\lambda^2\Big(1 - 4sech^2\Big[\frac{1}{2}\lambda x\Big]\Big) \\ v(x,0) &= \lambda sech\Big[\frac{1}{2}\lambda x\Big]. \end{aligned} \tag{22}$$

An approximation of a solution was obtained by using the Mathematica program and the solutions discussed in the previous sections to address the problem in this example. For the sake of this study, we limited our technique to the first four terms in the solution series. We illustrated the solution in multiple Figures for various values of $\alpha$ to demonstrate the degree of its correctness and dependability. The approximate solutions for $u(x,t)$ and $v(x,t)$ for $n = 4$ denoted by $u_A(x,t,\alpha), v_A(x,t,\alpha)$ are plotted in the Figures $3, 4$. It is evident that Figures 3 and 4 which represent the approximate solutions are remarkably similar to Figures 1 and 2, that represent the exact solutions. It is also clear that the numerical results are approaching the corresponding exact solutions with the initial conditions (22) of Equation (1) much more closely as more terms are computed for the decomposition series. Figure 5 illustrates the approximate solution $u$ when $\alpha$ is equal to 1, and Figure 6 illustrates the approximate solution $u$ when $\alpha = 0.75$. The wave started to split into two waves that were the same height as the original, as seen in Figure 6, as time increased. The $n-$term approximations for $u$ and $v$ are typically accurate for very small values of $t$. The modified Adomian decomposition method's effectiveness is supported by our numerical results. It is possible to lower overall errors by including more terms in the decomposition series. Because the decomposition method does not require the discretization of variables (time and space), it uses less computer memory and time. This makes it resistant to computation round off errors. With the current $t$ and $x$ parameters, the decomposition method yields minimal absolute errors and is highly accurate for coupled nonlinear equations. For varying values of $t, \alpha$, we plot a 2-dimension graph for the obtained solution in Figures 7 through 14. The symmetrical nature of the system solutions, as well as their soliton-like form, were confirmed by the results. Furthermore, as the value of the constant $\alpha$ moves away from 1, the solution starts to deviate from the correct behavior, as the graphics 7-4 confirm. Plotting the computed solution at $\alpha = 1$, or very close to 1, alongside the precise solution, we can see that they complement each other, as seen in Figure 7 when dealing with the $u(x,t)$, and in Figure 12 for $v(x,t)$. We plotted the approximate solutions for $u$ and $v$ for various values of $\alpha$ in Figures $8, 9, 10, 11, 13$, and $14$. As $\alpha$ decreases, we can see a difference in the graphs, particularly near zero for the values of $x$.

This paper uses MLADM, an effective technique for solving fractional differential equations, to solve the coupled nonlinear fractional $J-M$ system. Figures are used to illustrate the solutions that are obtained. These examples are a little bit artificial in that the initial conditions are taken straight from the exact solution, and the exact solution, for the special case $\alpha = 1$, is known beforehand. Mathematica is a symbolic calculus software that is used to calculate all of the results. The behavior of the approximate solution's values at various $\alpha$ is found to be identical to that of the values obtained with the exact solution, for which $\alpha = 1$. It is evident from the theory of fractional calculus that the approximate solution of the problem continuously tends to the exact solution when the fractional derivative $\alpha$ tends to the positive integer $\alpha = 1$. Upon closer inspection of Figures 7-11, we can see that our method does indeed possess this feature, see.[35] In Figures $8, 13, 14$ the more we deviate from zero or for higher the values of $|x|$, the more the approximate solution resembles the exact solution ($\alpha$ near 1). As a result, we observe that the lines that weave at the edges match, which is regarded as one of the method's benefits since it indicates precision.

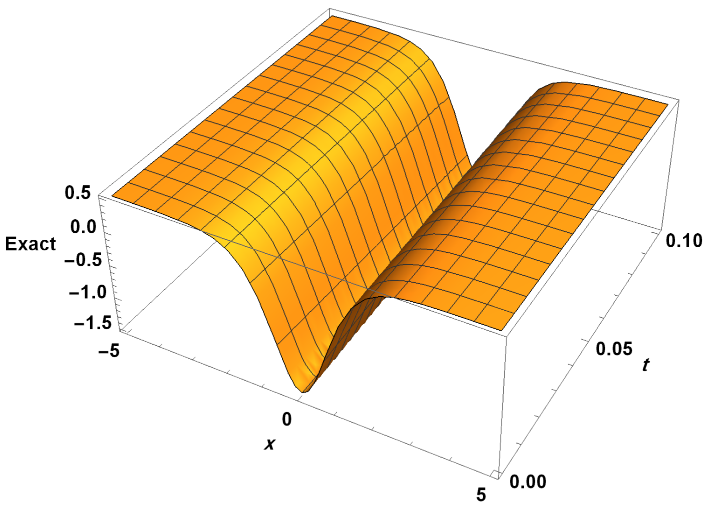


Figure 1: The exact solution of $u(x,t)$ when $-5 < x < 5, ; \; 0 < t < 0.1$

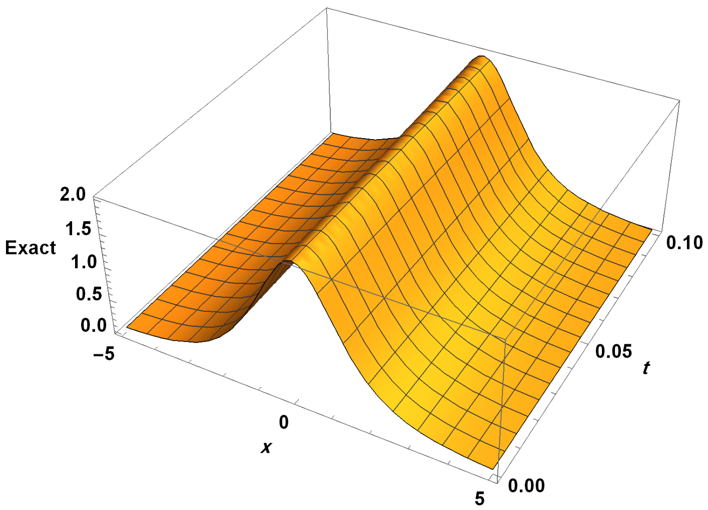


Figure 2: The exact solution of $v(x,t)$ when $-5 < x < 5, ;\ 0 < t < 0.1$.

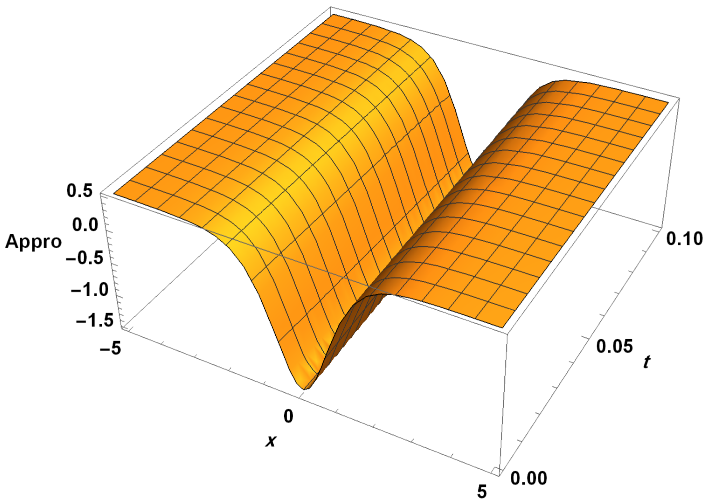


Figure 3: The approximate of $u(x,t)$ when $n = 4$ for $-5 < x < 5$ and $0 < t < 0.1$.

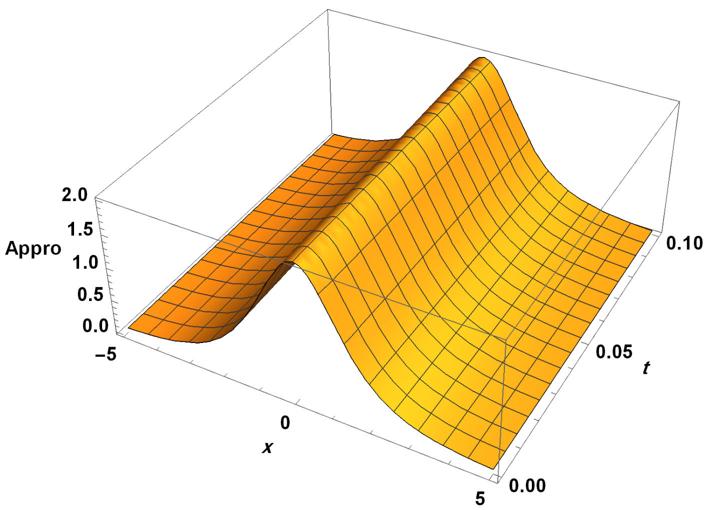


Figure 4: The approximate of $v(x,t)$ when $n = 4$ for $-5 < x < 5$ and $0 < t < 0.1$.

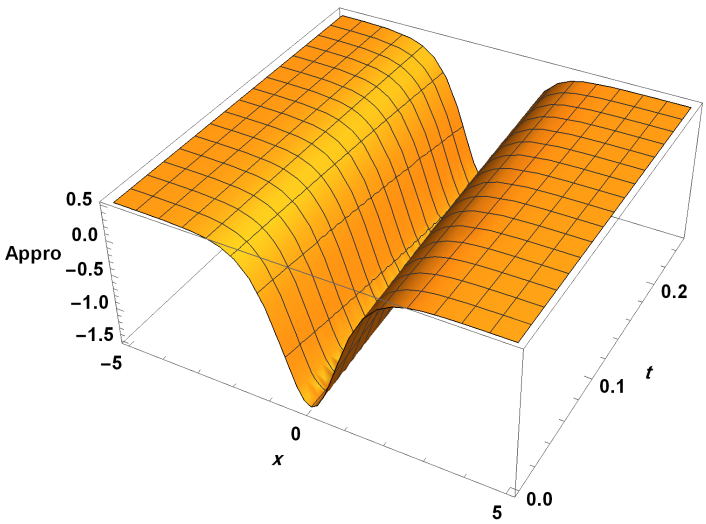


Figure 5: The approximate of $u(x,t)$ when $n = 4, \alpha = 1$ for $-5 < x < 5$ and $0 < t < 0.25$.

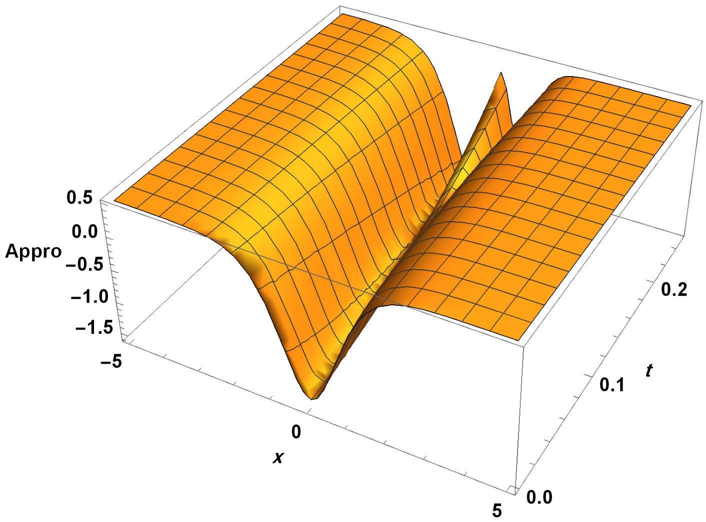


Figure 6: The approximate of $u(x,t)$ when $n = 4, \alpha = 0.75$ for $-5 < x < 5$ and $0 < t < 0.25$.

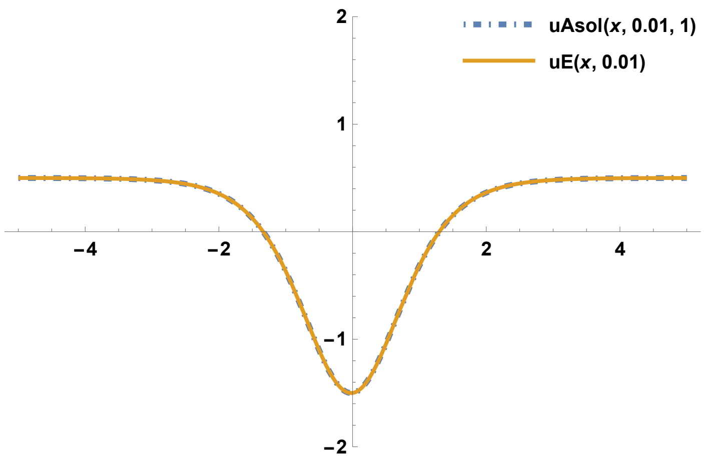


Figure 7: The approximate solution of $u(x,t)$ together with the exact solution when $t = 0.01, \alpha = 1$

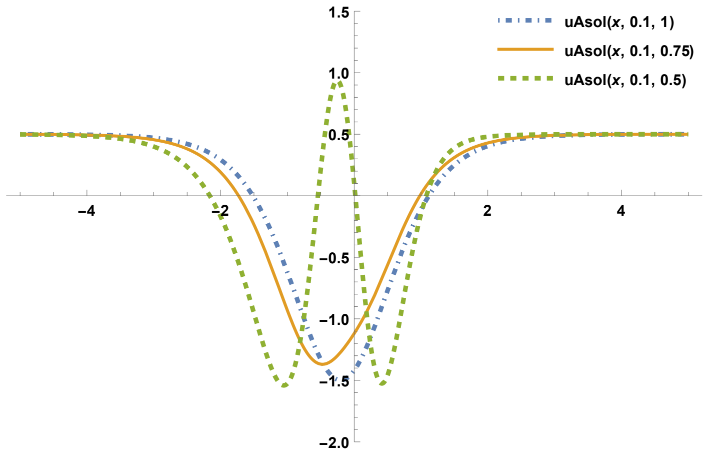


Figure 8: Three plots for the approximate solution of $u(x,t)$ when $t = 0.1, \alpha = 1.0,\ 0.75,\ 0.5$

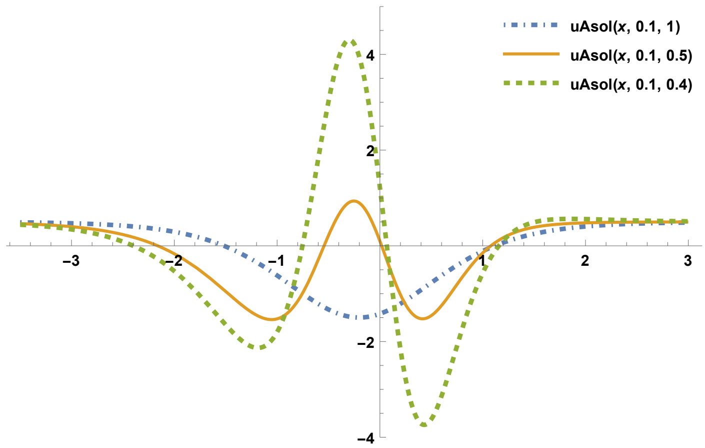


Figure 9: Three plots for the approximate solution of $u(x,t)$ when $t = 0.1, \alpha = 1.0,\ 0.5,\ 0.4$

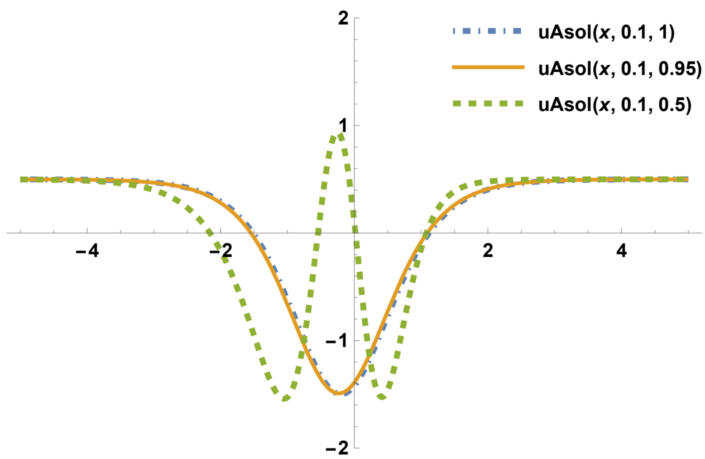


Figure 10: Three plots for the approximate solution of $u(x,t)$ when $t = 0.1, \alpha = 1.0,\ 0.95,\ 0.5$

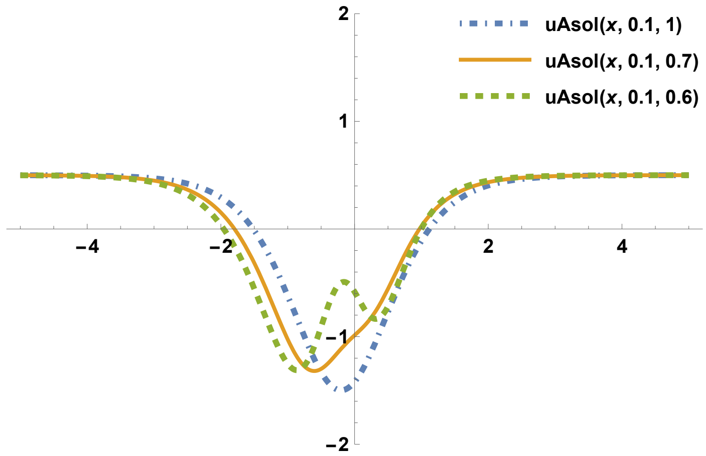


Figure 11: Three plots for the approximate solution of $u(x, t)$ when $t = 0.1, \alpha = 1.0,\ 0.7,\ 0.6$

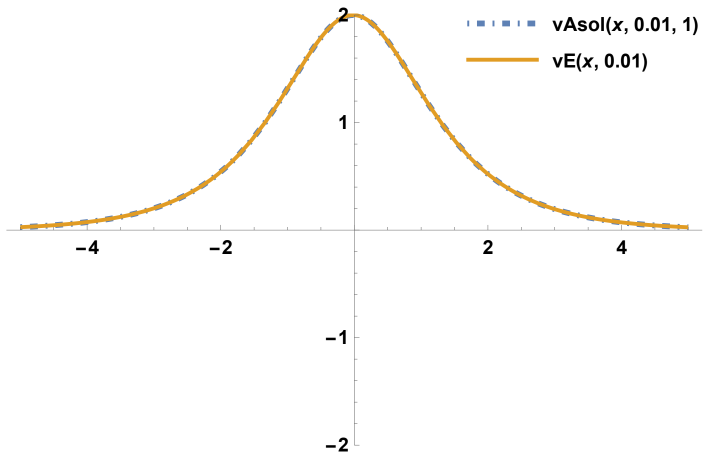


Figure 12: The approximate solution of $v(x, t)$ when $\alpha = 1$ together with the exact solution of $v(x, t)$.

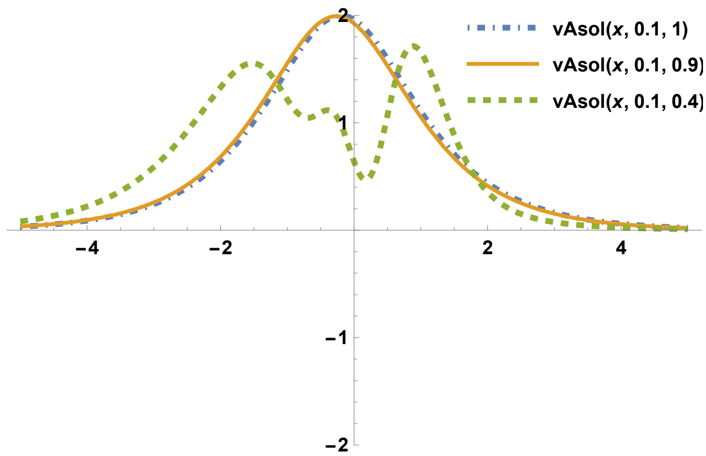


Figure 13: Three plots for the approximate solution of $v(x, t)$ when $t = 0.1, \alpha = 1.0,\ 0.9,\ 0.4$

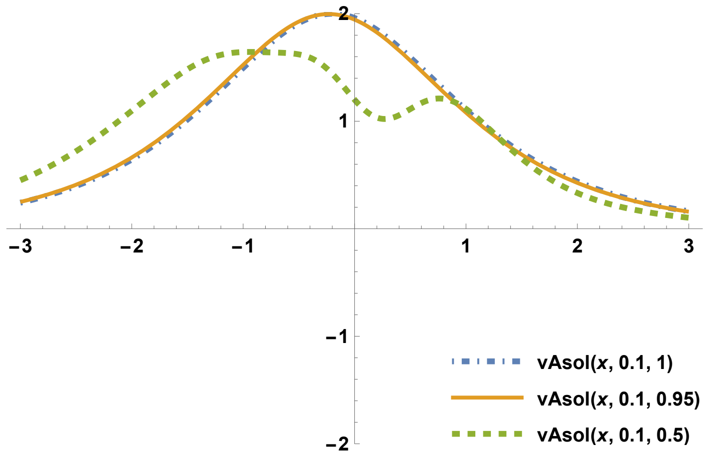


Figure 14: Three plots for the approximate solution of $v(x,t)$ when $t = 0.1, \alpha = 1.0,\ 0.95,\ 0.5$

## Conclusion

The primary objective of this paper was to develop an efficient algorithm for solving nonlinear fractional partial differential equations using the Modified Laplace Adomian Decomposition Method (MLADM), specifically applied to the time-fractional coupled Jaulent-Miodek system. The results demonstrate that MLADM is an effective, precise, and computationally efficient approach, capable of capturing the complex dynamics inherent in fractional systems. By providing accurate approximate and convergent solutions, this method offers a robust framework for modeling complex and uncertain systems—a key focus of neutrosophic science. The study emphasizes the method's potential for advancing the analysis of fractional systems within applied sciences and engineering, contributing to the understanding and application of systems characterized by uncertainty, indeterminacy, and nonlinearity. Future work will extend these techniques to more complex systems, further exploring their application in engineering, physics, and related fields. Ultimately, we draw the conclusion that a large range of nonlinear fractional partial differential equations have very good analytical and numerical solutions that can be found using the Laplace Adomian decomposition method. We plan to solve $3 \times 3$ fractional non-linear partial differential equations in the future, with applications in engineering and physics.

## Authors contributions

This article was written in collaboration with all of the contributors. All writers read and approved the final manuscript.

## Conflicts of interest

There are no competing interests declared by the authors.

## References

[1] Mustafa Inc, Xiao-Jun Yang, and Devendra Kumar. *New Numerical and Analytical Methods for Nonlinear Partial Differential Equations with Applications in Quantum Physics*. Frontiers Media SA, 2023.

[2] Muhammad S Iqbal, Aly R Seadawy, Muhammad Z Baber, and Muhammad Qasim. Application of modified exponential rational function method to Jaulent- Miodek system leading to exact classical solutions. *Chaos, Solitons & Fractals*, 164:112600, 2022.

[3] Saleh Alshammari, M Mossa Al-Sawalha, and Rasool Shah. Approximate analytical methods for a fractional-order nonlinear system of Jaulent–Miodek equation with energy-dependent Schrödinger potential. *Fractal and Fractional*, 7(2):140, 2023.

[4] Jinbing Chen and Xianguo Geng. Decomposition to the modified Jaulent–Miodek hierarchy. *Chaos, Solitons & Fractals*, 30(4):797–803, 2006.

[5] Kamel Al-Khaled, Mahmood Shareef Ajeel, Amer Darweesh, and Hala Al-Khalid. A reliable algorithm for solving Blasius boundary value problem. *Results in Nonlinear Analysis*, 7(4):1–8, 2024-doi.org/10.31838/rna/2024.07.04.001.

[6] M. M. Rashidi, G Domairry, and S Dinarvand. The homotopy analysis method for explicit analytical solutions of Jaulent–Miodek equations. *Numerical Methods for Partial Differential Equations: An International Journal*, 25(2):430–439, 2009.

[7] Yildiray Keskin and Galip Oturanc. Reduced differential transform method for partial differential equations. *International Journal of Nonlinear Sciences and Numerical Simulation*, 10(6):741–750, 2009.

[8] A. Saravanan and N. Magesh. A comparison between the reduced differential transform method and the Adomian decomposition method for the Newell–Whitehead–Segel equation. *Journal of the Egyptian Mathematical Society*, 21(3):259–265, 2013.

[9] Kamel Al-Khaled, Mahmood Shareef Ajeel, Issam Abu-Irwaq, and Hala K. Al-Khalid. An efficient approximate method for solving Bratu's boundary value problem. *International Journal of Electrical and Computer Engineering*, 14(5):5738–5743, 2024.

[10] D. D. Ganji, M Jannatabadi, and E Mohseni. Application of He's variational iteration method to nonlinear Jaulent–Miodek equations and comparing it with ADM. *Journal of Computational and Applied Mathematics*, 207(1):35–45, 2007.

[11] Melih Cinar, Ismail Onder, Aydin Secer, Mustafa Bayram, Tukur Abdulkadir Sulaiman, and Abdullahi Yusuf. Solving the fractional Jaulent–Miodek system via a modified Laplace decomposition method. *Waves in Random and Complex Media*, pages 1–14, 2022- doi.org/10.1080/17455030.2022.2057613.

[12] Anjan Biswas and AH Kara. 1-soliton solution and conservation laws for the Jaulent–Miodek equation with power law nonlinearity. *Applied mathematics and computation*, 217(2):944–948, 2010.

[13] Mehdi Nadjafikhah, Mostafa Hesamiarshad, et al. Analysis of the symmetries and conservation laws of the nonlinear Jaulent-Miodek equation. *Abstract and Applied Analysis*, 2014, Article ID 476025, 7 pages, doi.org/10.1155/2014/476025.

[14] Abdon Atangana, Dumitru Baleanu, et al. Nonlinear fractional Jaulent-Miodek and Whitham-Broer-Kaup equations within Sumudu transform. In *Abstract and Applied Analysis*, volume 2013. article ID 160681. Hindawi, 2013.

[15] A. K. Gupta and S. Saha Ray. An investigation with Hermite wavelets for accurate solution of fractional Jaulent–Miodek equation associated with energy-dependent Schrödinger potential. *Applied Mathematics and Computation*, 270:458–471, 2015.

[16] Yves Cherruault. Convergence of Adomian's method. *Kybernetes*, 18(2):31–38, 1989.

[17] Marwan Alquran, Kamel Al-Khaled, Mohammed Ali, and Ameina Ta'any. The combined laplace transform-differential transform method for solving linear non-homogeneous pdes. *J. Math. Comput. Sci.*, 2(3):690–701, 2012.

[18] Kamel Al-Khaled. Theory and computation in singular boundary value problems. *Chaos, Solitons & Fractals*, 33(2):678–684, 2007.

[19] Ma'mon Abu Hammad, Albandari W. Alrowaily, Rasool Shah, Sherif M. E. Ismaeel, and Samir A. El-Tantawy. Analytical analysis of fractional nonlinear Jaulent–Miodek system with energy-dependent Schrödinger potential. *Front. Phys.*, 11, 2023-doi.org/10.3389/fphy.2023.1148306.

[20] Y. Cherruault and G. Adomian. Decomposition methods: a new proof of convergence. *Mathematical and Computer Modelling*, 18(12):103–106, 1993.

[21] Esmail Babolian and Jafar Biazar. On the order of convergence of Adomian method. *Applied Mathematics and Computation*, 130(2-3):383–387, 2002.

[22] M Almazmumy, F. A. Hendi, H. O. Bakodah, and H Alzumi. Recent modifications of Adomian decomposition method for initial value problem in ordinary differential equations. *American Journal of Computational Mathematics*, 2(03):228–234, 2012.

[23] Saleh Alshammari, M. Mossa Al-Sawalha, and Rasool Shah. Approximate analytical methods for a fractional-order nonlinear system of Jaulent–Miodek equation with energy-dependent Schrödinger potential. *Fractal Fract.*, 7(140), 2023 - doi.org/10.3390/ fractalfract7020140.

[24] Smarandache F. and A. K. Dey. Neutrosophic operational calculus and its applications to differential equations. *Fractal Fract.*, 35:170–180, 2020.

[25] Francesco Mainardi. *Fractional Calculus: some basic problems in continuum and statistical mechanics*. Springer, 1997.

[26] Dumitru Baleanu and Devendra Kumar. *Fractional Calculus and its applications in physics*. Frontiers Media SA, 2019.

[27] Dumitru Baleanu, Kai Diethelm, Enrico Scalas, and Juan Trujillo. *Fractional calculus: models and numerical methods*. World Scientific, 2012.

[28] Constantin Milici, Gheorghe Drăgănescu, and J. Tenreiro Machado. *Introduction to fractional differential equations*. Springer, 2018.

[29] Keith Oldham and Jerome Spanier. *The fractional calculus theory and applications of differentiation and integration to arbitrary order*. Elsevier, 1974.

[30] G. Adomian. A review of the decomposition method in applied mathematics. *Journal of mathematical analysis and applications*, 135(2):501–544, 1988.

[31] Z. Ouyang. Existence and uniqueness of the solutions for a class of nonlinear fractional order partial differential equations with delay. *Comput Math Appl.*, 211(4):860–870, 2011.

[32] Kamel Al-Khaled, Shaher Momani, and Ahmed Alawneh. Approximate wave solutions for generalized Benjamin–Bona–Mahony–Burgers equations. *Applied Mathematics and Computation*, 171(1):281–292, 2005.

[33] Melih Cinar, Ismail Onder, Aydin Secer, Mustafa Bayram, Tukur Abdulkadir Sulaiman, and Abdullahi Yusuf. Solving the fractional Jaulent–Miodek system via a modified Laplace decomposition method. *Waves in Random and Complex Media*, pages 1–14, 2022.

[34] P. Parfeny Saworotnow. A generalized Hilbert space. *Duke Math.*, 35(1):191–197, 1968.

[35] Kamel Al-Khaled. Numerical solution of time-fractional partial differential equations using Sumudu decomposition method. *Romanian Journal of Physics*, 60(1):1–12, 2015.