Soft Weak θ-Continuity and Preservations of Soft Hyperconnectedness and Soft Near Compactness

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Abstract

In this paper, we introduce a new weak form of soft continuity called soft weak θ-continuity in soft topological spaces and investigate the relationships between soft weak θ-continuity and θ-continuity (resp. soft weak continuity and soft δ-continuity). We obtain several characterizations of soft weak θ-continuity. Also, we give sufficient conditions for the equivalence between soft weak θ-continuity and soft θ-continuity (resp. soft δ-continuity). Moreover, we investigate the link between soft weak θ-continuity and weak θ-continuity in classical topology. Furthermore, via soft weak θ-continuity, we obtain preservation theorems of soft hyperconnectedness and soft near compactness. Finally, we obtain soft restriction, soft product, and soft graph theorems of soft weak θ-continuity.

Keywords: Soft θ-continuous functions; Soft weakly-continuous functions; Soft hypreconnected spaces; Soft nearly compact spaces

1 Introduction and Preliminaries

Traditional mathematics theories struggle to solve complex problems that involve uncertain information in various fields like engineering, the environment, economics, medicine, and social science. Theories of probability, fuzzy sets, rough sets, intuitionistic fuzzy sets, and vague sets are ways to handle uncertain situations using math. Molodtsov [1] explained that each of these theories has its challenges. These problems mostly arise from the limitations of the tools used to define the theories. To handle uncertainties and challenges, Molodtsov created the idea of soft sets. A number of researchers have introduced soft sets theory (references [2, 3]). The authors [1, 4] used soft sets in many areas, such as Riemann integration, Perron integration, function smoothness, operations research, game theory, probability, and measurement theory. Authors [5] used soft sets in decision-making problems. In [6], Shabir and Naz presented a theory of soft topological spaces defined over an initial universe with a predetermined set of parameters. Their work focused on theoretical explorations of these spaces. Then several subclasses of soft topological spaces were introduced. Majumdar and Samanta [7] explored how soft sets can be employed to enhance the process of medical diagnosis. Kharal and Ahmed [8] presented and examined the concept of soft mapping, and [9] proposed soft continuity for soft mappings. The literature reviews listed contain a large number of publications that are dedicated to the study of soft continuity and its characterizations.

The authors of [10] proposed soft θ-continuity. The authors in [11] presented soft weak continuity, a weaker type of soft θ-continuity. In [12], the authors defined soft strong θ-continuity and soft almost strong θ-continuity as two strong forms of soft θ-continuity. The purpose of this paper is to introduce and investigate a new class of soft functions called soft weakly θ-continuous functions. This class is contained in the class of soft weakly continuous functions and contains the class of soft θ-continuous functions.

The rest of the paper is organized as follows:

In the rest of this section, we provide some definitions that are required to understand this work.

In Section 2, we give several characterizations of weakly θ-continuous functions and the basic properties of such functions, respectively. Moreover, we investigate the relations between soft weak θ-continuity and θ-continuity (resp. soft weak continuity and soft δ-continuity).

In Section 3, we give soft restriction, soft product, and soft graph theorems of soft weak θ-continuity.

In Section 4, we give some preservation theorems of soft hyperconnectedness and soft near compactness.

To be explicit, we shall refer to concepts and phrases from [13] throughout this paper. Topological space and soft topological space are represented by the abbreviations TS and STS, respectively.

The following definitions will be used in the sequel:

Definition 1.1. A function $g : (Y, \Im) \longrightarrow (Z, \aleph)$ is called

(a) [14] θ-continuous (θ-c) if for every $y \in Y$ and every $B \in \aleph$ such that $g(y) \in B$, we find $A \in \Im$ such that $y \in A$ and $g(Cl_{\Im}(A)) \widetilde{\subseteq} Cl_{\aleph}(B)$.

(b) [15] weakly continuous if for every $y \in Y$ and every $B \in \aleph$ such that $g(y) \in B$, we find $A \in \Im$ such that $y \in A$ and $g(A) \widetilde{\subseteq} Cl_{\aleph}(B)$.

(c) [16] weakly θ-continuous (w-θ-c) if for every $y \in Y$ and every $B \in \aleph$ such that $g(y) \in B$, we find $A \in \Im$ such that $y \in A$ and

$$g(Int_{\Im}(Cl_{\Im}(A))) \widetilde{\subseteq} Cl_{\aleph}(B).$$

Definition 1.2. A soft function $f_{sn} : (Y, \phi, \mathcal{M}) \longrightarrow (Z, \lambda, \mathcal{N})$ is called

(a) [11] soft weakly continuous if for every $a_y \in SP(Y, \mathcal{M})$ and every $G \in \lambda$ such that $f_{sn}(a_y) \widetilde{\in} G$, we find $H \in \phi$ such that $a_y \widetilde{\in} H$ and $f_{sn}(H) \widetilde{\subseteq} Cl_{\lambda}(G)$.

(b) [10] soft θ-continuous (soft θ-c) if for every $a_y \in SP(Y, \mathcal{M})$ and every $G \in \lambda$ such that $f_{sn}(a_y) \widetilde{\in} G$, we find $H \in \phi$ such that $a_y \widetilde{\in} H$ and $f_{sn}(Cl_{\phi}(H)) \widetilde{\subseteq} Cl_{\lambda}(G)$.

(c) [17] soft δ-continuous (soft δ-c) if for every $a_y \in SP(Y, \mathcal{M})$ and every $G \in \lambda$ such that $f_{sn}(a_y) \widetilde{\in} G$, we find $H \in \phi$ such that $a_y \widetilde{\in} H$ and $f_{sn}(Int_{\phi}(Cl_{\phi}(H))) \widetilde{\subseteq} Int_{\lambda}(Cl_{\lambda}(G))$.

(d) [12] soft almost strongly θ-continuous if for every $a_y \in SP(Y, \mathcal{M})$ and every $G \in \lambda$ such that $f_{sn}(a_y) \widetilde{\in} G$, we find $H \in \phi$ such that $a_y \widetilde{\in} H$ and $f_{sn}(Cl_{\phi}(H)) \widetilde{\subseteq} Int_{\lambda}(Cl_{\lambda}(G))$.

(e) [18] soft almost-open if $f_{sn}(Int_{\phi}(Cl_{\phi}(H))) \in \lambda$ for every $H \in \phi$.

Definition 1.3. A STS $(Y, \phi, \mathcal{M})$ is called

(a) [19] soft Hausdorff if for each $a_y, b_z \in SP(Y, \mathcal{M})$ such that $a_y \neq b_z$, we find $U, V \in \phi$ such that $a_y \widetilde{\in} U$, $b_z \widetilde{\in} V$, and $U \widetilde{\cap} V = 0_{\mathcal{M}}$.

(a) [20] soft hyperconnected if any $U \in \phi - \{0_{\mathcal{M}}\}$ is soft dense in $(Y, \phi, \mathcal{M})$.

(b) [21] soft extremally disconnected if for any $U \in \phi$, $Cl_{\phi}\left(U\right) \in \phi$.

Definition 1.4. [22] Let $(Y, \phi, \mathcal{M})$ be a STS and let $H \in SS\left(Y, \mathcal{M}\right)$. Then

(a) H is a soft regular-open set in $(Y, \phi, \mathcal{M})$ if $H = Int_{\phi}\left(Cl_{\phi}\left(H\right)\right)$. The collection of all soft regular open sets in $(Y, \phi, \mathcal{M})$ is denoted by $RO\left(\phi\right)$.

(b) H is a soft regular-closed set in $(Y, \phi, \mathcal{M})$ if $1_{\mathcal{M}} - H \in RO\left(\phi\right)$. The collection of all soft regular closed sets in $(Y, \phi, \mathcal{M})$ is denoted by $RC\left(\phi\right)$.

Definition 1.5. [10, 23] Let $(Y, \phi, \mathcal{M})$ be a STS and let $H \in SS\left(Y, \mathcal{M}\right)$.

(a) The soft θ-closure (resp. δ-closure) of H in $(Y, \phi, \mathcal{M})$ is denoted by $\theta Cl_{\phi}\left(H\right)$ (resp. $\delta Cl_{\phi}\left(H\right)$), where

$a_y \tilde{\in} \theta Cl_{\phi}\left(H\right)$ (resp. $a_y \tilde{\in} \delta Cl_{\phi}\left(H\right)$) iff for each $K \in \phi$ such that $a_y \tilde{\in} K$, $H \tilde{\cap} Cl_{\phi}\left(K\right) \neq 0_{\mathcal{M}}$ (resp. $H \tilde{\cap} Int_{\phi}$ $\left(Cl_{\phi}\left(K\right)\right) \neq 0_{\mathcal{M}}$).

(b) H is soft θ-closed (resp. soft δ-closed) in $(Y, \phi, \mathcal{M})$ if $H = \theta Cl_{\phi}\left(H\right)$ (resp. $H = \delta Cl_{\phi}\left(H\right)$).

(c) H is soft θ-open (resp. soft δ-open) in $(Y, \phi, \mathcal{M})$ if $1_{\mathcal{M}} - H$ is soft θ-closed (resp. soft δ-closed) in $(Y, \phi, \mathcal{M})$.

(d) The collection of all soft θ-open (resp. soft δ-open) sets in $(Y, \phi, \mathcal{M})$ is denoted by ϕ_{θ} (resp. ϕ_{δ}).

Definition 1.6. [24] Let $(Y, \phi, \mathcal{M})$ be a STS and let $K \in SS\left(Y, \mathcal{M}\right)$. Then

(a) K is soft nearly compact relative to $(Y, \phi, \mathcal{M})$ if for any $\Psi \subseteq RO\left(\phi\right)$ such that $K \tilde{\subseteq} \tilde{\cup}_{H \in \Psi} H$, we find a finite sub-collection $\Psi_1 \subseteq \Psi$ such that $K \tilde{\subseteq} \tilde{\cup}_{H \in \Psi_1} H$.

(b) $(Y, \phi, \mathcal{M})$ is soft nearly compact if $1_{\mathcal{M}}$ is soft nearly compact relative to $(Y, \phi, \mathcal{M})$.

Definition 1.7. [25] Let $(Y, \phi, \mathcal{M})$ be a STS and let $K \in SS\left(Y, \mathcal{M}\right)$. Then

(a) K is soft quasi H-closed relative to $(Y, \phi, \mathcal{M})$ if for any $\Psi \subseteq \phi$ such that $K \tilde{\subseteq} \tilde{\cup}_{H \in \Psi} H$, we find a finite sub-collection $\Psi_1 \subseteq \Psi$ such that $K \tilde{\subseteq} \tilde{\cup}_{H \in \Psi_1} Cl_{\phi}\left(H\right)$.

(b) $(Y, \phi, \mathcal{M})$ is soft quasi H-closed if $1_{\mathcal{M}}$ is soft quasi H-closed relative to $(Y, \phi, \mathcal{M})$.

For a soft function $f_{sn} : SP\left(Y, \mathcal{M}\right) \longrightarrow SP\left(Z, \mathcal{N}\right)$, the soft set

$\tilde{\cup} \left\{ \left(a, n\left(a\right)\right)_{\left(y, s(y)\right)} : a \in \mathcal{M} \text{ and } y \in Y \right\}$ is represented by $Graph\left(f_{sn}\right)$ and is called the soft graph of f_{sn}. So, $\left(a, b\right)_{\left(y, z\right)} \tilde{\in} Graph\left(f_{sn}\right)$ iff $f_{sn}(a_y) = b_z$ iff $s(y) = z$ and $n\left(a\right) = b$.

2 Characterizations

Definition 2.1. A soft function $f_{sn} : (Y, \phi, \mathcal{M}) \longrightarrow (Z, \lambda, \mathcal{N})$ is called soft weakly θ-continuous (soft w-θ-c, for short) if $a_y \in SP(Y, \mathcal{M})$ and each $G \in \lambda$ such that $f_{sn}(a_y) \widetilde{\in} G$, we find $H \in \phi$ such that $a_y \widetilde{\in} H$ and $f_{sn}(Int_\phi(Cl_\phi(H))) \widetilde{\subseteq} Cl_\lambda(G)$.

Theorem 2.2. Let $\{(Y, \phi_a) : a \in \mathcal{M}\}$ and $\{(Z, \lambda_b) : b \in \mathcal{N}\}$ be two collections of TSs. Let $s : Y \longrightarrow Z$ and $n : \mathcal{M} \longrightarrow \mathcal{N}$ be functions where n is bijective. Then $f_{sn} : (Y, \oplus_{a\in\mathcal{M}}\phi_a, \mathcal{M}) \longrightarrow (Z, \oplus_{b\in\mathcal{N}}\lambda_b, \mathcal{N})$ is soft w-θ-c iff $s : (Y, \phi_a) \longrightarrow \left(Z, \lambda_{n(a)}\right)$ is w-θ-c for all $a \in \mathcal{M}$.

Proof. *Necessity*. Let $f_{sn} : (Y, \oplus_{a\in\mathcal{M}}\phi_a, \mathcal{M}) \longrightarrow (Z, \oplus_{b\in\mathcal{N}}\lambda_b, \mathcal{N})$ be soft w-θ-c. Let $a \in \mathcal{M}$. Let $y \in Y$ and let $V \in \lambda_{n(a)}$ such that $s(y) \in \lambda_{n(a)}$. Then we have $f_{sn}(a_y) = (n(a))_{s(y)} \widetilde{\in} (n(a))_V \in \oplus_{b\in\mathcal{N}}\lambda_b$. Thus, we find $H \in \oplus_{a\in\mathcal{M}}\phi_a$ such that $a_y \widetilde{\in} H$ and $f_{sn}(Int_{\oplus_{a\in\mathcal{M}}\phi_a}(Cl_{\oplus_{a\in\mathcal{M}}\phi_a}(H))) \widetilde{\subseteq} Cl_{\oplus_{b\in\mathcal{N}}\lambda_b}((n(a))_V)$ and hence $(f_{sn}(Int_{\oplus_{a\in\mathcal{M}}\phi_a}(Cl_{\oplus_{a\in\mathcal{M}}\phi_a}(H))))(n(a)) \subseteq (Cl_{\oplus_{b\in\mathcal{N}}\lambda_b}((n(a))_V))(n(a))$. Since n is bijective, then

$$(f_{sn}(Int_{\oplus_{a\in\mathcal{M}}\phi_a}(Cl_{\oplus_{a\in\mathcal{M}}\phi_a}(H))))(n(a)) = s((Int_{\phi_a}(Cl_{\phi_a}(H)))(a)).$$

We have $(Int_{\phi_a}(Cl_{\phi_a}(H)))(a) = Int_{\phi_a}(Cl_{\phi_a}(H(a)))$ and $(Cl_{\oplus_{b\in\mathcal{N}}\lambda_b}((n(a))_V))(n(a)) = Cl_{\lambda_{n(a)}}((n(a))_V((n(a)))) = Cl_{\lambda_{n(a)}}(V)$. Therefore, we have $y \in H(a) \in \phi_a$ and

$$\begin{array}{rcl} s(Int_{\phi_a}(Cl_{\phi_a}(H(a)))) & = & s((Int_{\oplus_{a\in\mathcal{M}}\phi_a}(Cl_{\oplus_{a\in\mathcal{M}}\phi_a}(H)))(a)) \\ & = & (f_{sn}(Int_{\oplus_{a\in\mathcal{M}}\phi_a}(Cl_{\oplus_{a\in\mathcal{M}}\phi_a}(H))))(n(a)) \\ & \widetilde{\subseteq} & (Cl_{\oplus_{b\in\mathcal{N}}\lambda_b}((n(a))_V))(n(a)) \\ & = & Cl_{\lambda_{n(a)}}(V). \end{array}$$

It follows that $s : (Y, \phi_a) \longrightarrow \left(Z, \lambda_{n(a)}\right)$ is w-θ-c.

Sufficiency. Let $s : (Y, \phi_a) \longrightarrow \left(Z, \lambda_{n(a)}\right)$ be w-θ-c for all $a \in \mathcal{M}$. Let $a_y \in SP(Y, \mathcal{M})$ and let $G \in \oplus_{b\in\mathcal{N}}\lambda_b$ such that $f_{sn}(a_y) = (n(a))_{s(y)} \widetilde{\in} G$. Then, we have $s(y) \in G(n(a)) \in \lambda_{n(a)}$. Since $s : (Y, \phi_a) \longrightarrow \left(Z, \lambda_{n(a)}\right)$ is w-θ-c, then we find $U \in \phi_a$ such that $y \in U$ and $s(Int_{\phi_a}(Cl_{\phi_a}(U))) \subseteq Cl_{\lambda_{n(a)}}(G(n(a)))$. Therefore, we have $a_U \in \oplus_{a\in\mathcal{M}}\phi_a$ such that $a_y \widetilde{\in} a_U$ and $f_{sn}(Int_{\oplus_{a\in\mathcal{M}}\phi_a}(Cl_{\oplus_{a\in\mathcal{M}}\phi_a}(a_U))) \widetilde{\subseteq} Cl_{\oplus_{b\in\mathcal{N}}\lambda_b}(G)$. This shows that f_{sn} is soft w-θ-c.

Corollary 2.3. Let $s : (Y, \Im) \longrightarrow (Z, \aleph)$ and $n : \mathcal{M} \longrightarrow \mathcal{N}$ be two functions where n is bijective. Then $s : (Y, \Im) \longrightarrow (Z, \aleph)$ is w-θ-c iff $f_{sn} : (Y, \tau(\Im), \mathcal{M}) \longrightarrow (Z, \tau(\aleph), \mathcal{N})$ is soft w-θ-c.

Proof. For every $a \in \mathcal{M}$ and $b \in \mathcal{N}$, let $\phi_a = \Im$ and $\lambda_b = \aleph$. Then $\tau(\Im) = \oplus_{a\in\mathcal{M}}\lambda_a$ and $\tau(\aleph) = \oplus_{b\in\mathcal{N}}\lambda_b$. By Theorem 2.2, we get the result.

Theorem 2.4. Soft θ-c functions are soft w-θ-c.

Proof. Let $f_{sn} : (Y, \phi, \mathcal{M}) \longrightarrow (Z, \lambda, \mathcal{N})$ be soft θ-c. Let $a_y \in SP(Y, \mathcal{M})$ and let $G \in \lambda$ such that $f_{sn}(a_y) \widetilde{\in} G$. Then, by θ-continuity of f_{sn}, we find $H \in \phi$ such that $a_y \widetilde{\in} H$ and $f_{sn}(Cl_\phi(H)) \widetilde{\subseteq} Cl_\lambda(G)$; hence,

$$f_{sn}(Int_\phi(Cl_\phi(H))) \widetilde{\subseteq} f_{sn}(Cl_\phi(H)) \widetilde{\subseteq} Cl_\lambda(G).$$ Therefore, f_{sn} is soft w-θ-c.

Theorem 2.5. Soft w-θ-c functions are soft weakly continuous.

Proof. Let $f_{sn} : (Y, \phi, \mathcal{M}) \longrightarrow (Z, \lambda, \mathcal{N})$ be soft w-θ-c. Let $a_y \in SP(Y, \mathcal{M})$ and let $G \in \lambda$ such that $f_{sn}(a_y) \widetilde{\in} G$. Then, by soft weak θ-continuity of f_{sn}, we find $H \in \phi$ such that $a_y \widetilde{\in} H$ and $f_{sn}(Int_\phi(Cl_\phi(H))) \widetilde{\subseteq} Cl_\lambda(G)$; hence, $f_{sn}(H) \widetilde{\subseteq} f_{sn}(Int_\phi(Cl_\phi(H))) \widetilde{\subseteq} Cl_\lambda(G)$. Therefore, f_{sn} is soft weakly continuous.

Theorem 2.6. Soft δ-c functions are soft w-θ-c.

Proof. Let $f_{sn} : (Y, \phi, \mathcal{M}) \longrightarrow (Z, \lambda, \mathcal{N})$ be soft δ-c. Let $a_y \in SP(Y, \mathcal{M})$ and let $G \in \lambda$ such that $f_{sn}(a_y) \tilde{\in} G$. Then, by δ-continuity of f_{sn}, we find $H \in \phi$ such that $a_y \tilde{\in} H$ and $f_{sn}(Int_\phi(Cl_\phi(H))) \tilde{\subseteq} Int_\lambda$ $(Cl_\lambda(G)) \tilde{\subseteq} Cl_\lambda(G)$. Therefore, f_{sn} is soft w-θ-c.

The implications of Theorems 2.4, 2.5, and 2.6 are generally irreversible, as demonstrated by the following three examples:

Example 2.7. We utilize Example 3.3 of [16]. Let $Y = \mathbb{R}^2 \cup \{j\}$, where $j \notin \mathbb{R}^2$. Let $\Re$ be the usual topology on $\mathbb{R}^2$. Let

$$\begin{array}{ccc} \oplus = & \left\{H \cap \left(\mathbb{R}^2 - \{(0,0)\}\right) : H \in \Re\right\} & \cup \\ & \left\{\left\{(y,z) : y^2 + z^2 < \frac{1}{m^2}, z > 0\right\} \cup \{(0,0)\} : m \in \mathbb{N}\right\} & \cup \\ & \left\{\left\{(y,z) : y^2 + z^2 < \frac{1}{m^2}, z < 0\right\} \cup \{j\} : m \in \mathbb{N}\right\}. & \end{array}$$

Let $\Im$ be the topology on Y generated by $\oplus$ as a base. Let $Z = \{a, b, c\}$ and $\aleph = \{\varnothing, Z, \{a\}, \{c\}, \{a, c\}\}$. Consider the functions $s : Y \longrightarrow Z$ and $n : \mathbb{Q} \longrightarrow \mathbb{Q}$ defined as follows:

$$s(y,z) = \begin{cases} a & \text{if} \quad z \geqslant 0 \\ b & \text{if} \quad z < 0 \\ c & \text{if} \quad (y,z) = j \end{cases} \quad \text{and } n(a) = a \text{ for all } a \in \mathbb{Q}.$$

It is proved in Example 3.3 of [16] that $s : (Y, \Im) \longrightarrow (Z, \aleph)$ is w-θ-c but not θ-c. Therefore, $f_{sn} : (Y, \tau(\Im), \mathbb{Q}) \longrightarrow (Z, \tau(\aleph), \mathbb{Q})$ is soft w-θ-c but not soft θ-c.

Example 2.8. Let $Y = \{a, b, c, d\}$, $\mathcal{M} = \mathbb{R}$, and

$$\Im = \{\varnothing, Y, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{a, b, c\}, \{b, c, d\}\}.$$

Define $s : Y \longrightarrow Y$ and $n : \mathcal{M} \longrightarrow \mathcal{M}$ as follows: $s(a) = c$, $s(b) = d$, $s(c) = b$, $s(d) = a$, and $n(a) = a$ for all $a \in \mathcal{M}$. Then as proved in Example 3.2 of [16], $s : (Y, \Im) \longrightarrow (Y, \Im)$ is weakly continuous but not w-θ-c. Therefore, $f_{sn} : (Y, \tau(\Im), \mathcal{M}) \longrightarrow (Y, \tau(\aleph), \mathcal{M})$ is soft weakly continuous but not soft w-θ-c.

Example 2.9. Let $Y = \{a, b, c\}$, $\Im = \{\varnothing, Y, \{a\}, \{c\}, \{a, c\}, \{a, b\}\}$, $\aleph = \{\varnothing, Y, \{a\}, \{c\}, \{a, c\}\}$, and $\mathcal{M} = \mathbb{R}$. Consider the identity functions $s : Y \longrightarrow Y$ and $n : \mathcal{M} \longrightarrow \mathcal{M}$. Then $f_{sn} : (Y, \tau(\Im), \mathcal{M}) \longrightarrow (Y, \tau(\aleph), \mathcal{M})$ is soft w-θ-c but not soft δ-c.

Theorem 2.10. If $f_{sn} : (Y, \phi, \mathcal{M}) \longrightarrow (Z, \lambda, \mathcal{N})$ is soft w-θ-c and $(Y, \phi, \mathcal{M})$ is soft extremally disconnected, then f_{sn} is soft θ-c.

Proof. Let $a_y \in SP(Y, \mathcal{M})$ and let $G \in \lambda$ such that $f_{sn}(a_y) \tilde{\in} G$. Since f_{sn} is soft w-θ-c, we find $H \in \phi$ such that $a_y \tilde{\in} H$ and $f_{sn}(Int_\phi(Cl_\phi(H))) \tilde{\subseteq}$

$Cl_\lambda(G)$. Since $(Y, \phi, \mathcal{M})$ is soft extremally disconnected, then $Cl_\phi(H) \in \phi$, and so, $Int_\phi(Cl_\phi(H)) = Cl_\phi(H)$. Therefore, $f_{sn}(Cl_\phi(H)) \tilde{\subseteq} Cl_\lambda(G)$. Hence, f_{sn} is soft θ-c.

Theorem 2.11. If $f_{sn} : (Y, \phi, \mathcal{M}) \longrightarrow (Z, \lambda, \mathcal{N})$ is soft w-θ-c and soft almost-open, then f_{sn} is soft δ-c.

Proof. Let $a_y \in SP(Y, \mathcal{M})$ and let $G \in \lambda$ such that $f_{sn}(a_y) \tilde{\in} G$. Since f_{sn} is soft w-θ-c, then we find $H \in \phi$ such that $a_y \tilde{\in} H$ and $f_{sn}(Int_\phi(Cl_\phi(H))) \tilde{\subseteq}$

$Cl_\lambda(G)$. Since f_{sn} is soft almost-open, then $f_{sn}(Int_\phi(Cl_\phi(H))) \in \lambda$; hence,

$f_{sn}(Int_{\phi}(Cl_{\phi}(H)))\tilde{\subseteq}Int_{\lambda}(Cl_{\lambda}(G))$. Therefore, f_{sn} is soft δ-c.

Theorem 2.12. For a soft function $f_{sn}:(Y,\phi,\mathcal{M})\longrightarrow(Z,\lambda,\mathcal{N})$, the following are equivalent:

(a) f_{sn} is soft w-θ-c.

(b) For each $a_y\in SP(Y,\mathcal{M})$ and each $G\in\lambda$ such that $f_{sn}(a_y)\tilde{\in}G$, we find $K\in RO(\phi)$ such that $a_y\tilde{\in}K$ and $f_{sn}(K)\tilde{\subseteq}Cl_{\lambda}(G)$.

(c) For each $a_y\in SP(Y,\mathcal{M})$ and each $G\in\lambda$ such that $f_{sn}(a_y)\tilde{\in}G$, we find $K\in\phi_{\delta}$ such that $a_y\tilde{\in}K$ and $f_{sn}(K)\tilde{\subseteq}Cl_{\lambda}(G)$.

Proof. (a) $\longrightarrow$ (b): Let $a_y\in SP(Y,\mathcal{M})$ and let $G\in\lambda$ such that $f_{sn}(a_y)\tilde{\in}G$. Then by (a), we find $H\in\phi$ such that $a_y\tilde{\in}H$ and

$f_{sn}(Int_{\phi}(Cl_{\phi}(H)))\tilde{\subseteq}Cl_{\lambda}(G)$. Let $K=Int_{\phi}(Cl_{\phi}(H))$. Then $K\in RO(\phi)$ such that $a_y\tilde{\in}K$ and $f_{sn}(K)\tilde{\subseteq}Cl_{\lambda}(G)$.

(b) $\longrightarrow$ (c): Let $a_y\in SP(Y,\mathcal{M})$ and let $G\in\lambda$ such that $f_{sn}(a_y)\tilde{\in}G$. Then by (b), we find $K\in RO(\phi)\subseteq\phi_{\delta}$ such that $a_y\tilde{\in}K$ and $f_{sn}(K)\tilde{\subseteq}Cl_{\lambda}(G)$.

(c) $\longrightarrow$ (a): Let $a_y\in SP(Y,\mathcal{M})$ and let $G\in\lambda$ such that $f_{sn}(a_y)\tilde{\in}G$. Then by (c), we find $K\in\phi_{\delta}$ such that $a_y\tilde{\in}K$ and $f_{sn}(K)\tilde{\subseteq}Cl_{\lambda}(G)$. Since $a_y\tilde{\in}K\in\phi_{\delta}$, we find $H\in\phi$ such that $a_y\tilde{\in}H\tilde{\subseteq}Int_{\phi}(Cl_{\phi}(H))\tilde{\subseteq}K$; hence, $f_{sn}(Int_{\phi}(Cl_{\phi}(H)))\tilde{\subseteq}Cl_{\lambda}(G)$. It follows that f_{sn} is soft w-θ-c.

Theorem 2.13. For a soft function $f_{sn}:(Y,\phi,\mathcal{M})\longrightarrow(Z,\lambda,\mathcal{N})$, the following are equivalent:

(a) f_{sn} is soft w-θ-c.

(b) $f_{sn}(\delta Cl_{\phi}(A))\tilde{\subseteq}\theta Cl_{\lambda}(f_{sn}(A))$ for each $A\in SS(Y,\mathcal{M})$.

(c) $\delta Cl_{\phi}\left(f_{sn}^{-1}(B)\right)\tilde{\subseteq}f_{sn}^{-1}(\theta Cl_{\lambda}(B))$ for each $B\in SS(Z,\mathcal{N})$.

(d) $f_{sn}^{-1}(\theta Int_{\lambda}(B))\tilde{\subseteq}\delta Int_{\phi}\left(f_{sn}^{-1}(B)\right)$ for each $B\in SS(Z,\mathcal{N})$.

(e) $\delta Cl_{\phi}\left(f_{sn}^{-1}(G)\right)\tilde{\subseteq}f_{sn}^{-1}(Cl_{\lambda}(G))$ for each $G\in\lambda$.

(f) $f_{sn}^{-1}(G)\tilde{\subseteq}\delta Int_{\phi}\left(f_{sn}^{-1}(Cl_{\lambda}(G))\right)$ for each $G\in\lambda$.

(g) $Cl_{\phi}\left(f_{sn}^{-1}(Int_{\lambda}(T))\right)\tilde{\subseteq}f_{sn}^{-1}(T)$ for each $T\in\lambda^{c}$.

Proof. (a) $\longrightarrow$ (b): Let $A\in SS(Y,\mathcal{M})$. Suppose to the contrary that we find $b_z\tilde{\in}f_{sn}(\delta Cl_{\phi}(A))-\theta Cl_{\lambda}(f_{sn}(A))$. Choose $a_y\tilde{\in}\delta Cl_{\phi}(A)$ such $f_{sn}(a_y)=b_z$. Since $b_z\tilde{\notin}\theta Cl_{\lambda}(f_{sn}(A))$, then there is $G\in\lambda$ such that $b_z\tilde{\in}G$ and $Cl_{\lambda}(G)\tilde{\cap}f_{sn}(A)=0_{\mathcal{N}}$. Since $f_{sn}(a_y)\in G\in\lambda$, then by Theorem 2.12 (b), there is $K=Int_{\phi}(Cl_{\phi}(K))\in RO(\phi)$ such that $a_y\tilde{\in}K$ and $f_{sn}(K)\tilde{\subseteq}Cl_{\lambda}(G)$. Since $Cl_{\lambda}(G)\tilde{\cap}f_{sn}(A)=0_{\mathcal{N}}$ and $f_{sn}(K)\tilde{\subseteq}Cl_{\lambda}(G)$, then $f_{sn}(K)\tilde{\cap}f_{sn}(A)=0_{\mathcal{N}}$. Since $f_{sn}(K\tilde{\cap}A)\tilde{\subseteq}f_{sn}(K)\tilde{\cap}f_{sn}(A)=0_{\mathcal{N}}$, then $f_{sn}(K\tilde{\cap}A)=0_{\mathcal{N}}$ and hence $K\tilde{\cap}A=0_{\mathcal{M}}$. Therefore, we have $a_y\tilde{\in}K\in\phi$ and $Int_{\phi}(Cl_{\phi}(K))\tilde{\cap}A=0_{\mathcal{M}}$; hence $a_y\tilde{\notin}\delta Cl_{\phi}(A)$, which is a contradiction.

(b) $\longrightarrow$ (c): Let $B\in SS(Z,\mathcal{N})$. Then by (b), we have

$$f_{sn}(\delta Cl_{\phi}\left(f_{sn}^{-1}(B)\right))\tilde{\subseteq}\theta Cl_{\lambda}\left(f_{sn}(f_{sn}^{-1}(B))\right)\tilde{\subseteq}\theta Cl_{\lambda}(B),$$

and so,

$$\delta Cl_{\phi}\left(f_{sn}^{-1}(B)\right)\tilde{\subseteq}f_{sn}^{-1}\left(f_{sn}(\delta Cl_{\phi}\left(f_{sn}^{-1}(B)\right))\right)\tilde{\subseteq}f_{sn}^{-1}(\theta Cl_{\lambda}(B)).$$

(c) $\longrightarrow$ (d): Let $B \in SS(Z, \mathcal{N})$. Then by (c),

$$\begin{array}{rcl} f_{sn}^{-1}(\theta Int_{\lambda}(B)) & = & f_{sn}^{-1}(1_{\mathcal{N}} - \theta Cl_{\lambda}(B)) \\ & = & 1_{\mathcal{M}} - f_{sn}^{-1}(\theta Cl_{\lambda}(B)) \\ & \tilde{\subseteq} & 1_{\mathcal{M}} - \delta Cl_{\phi}\left(f_{sn}^{-1}(B)\right) \\ & = & \delta Int_{\phi}\left(f_{sn}^{-1}(B)\right). \end{array}$$

(d) $\longrightarrow$ (e): Let $G \in \lambda$. Then $\theta Cl_{\lambda}(G) = Cl_{\lambda}(G)$ and by (d),

$$\begin{array}{rcl} \delta Cl_{\phi}\left(f_{sn}^{-1}(G)\right) & = & 1_{\mathcal{M}} - \delta Int_{\phi}\left(1_{\mathcal{M}} - f_{sn}^{-1}(G)\right) \\ & = & 1_{\mathcal{M}} - \delta Int_{\phi}\left(f_{sn}^{-1}(1_{\mathcal{N}} - G)\right) \\ & \tilde{\subseteq} & 1_{\mathcal{M}} - f_{sn}^{-1}(\theta Int_{\lambda}(1_{\mathcal{N}} - G)) \\ & = & f_{sn}^{-1}(1_{\mathcal{N}} - \theta Int_{\lambda}(1_{\mathcal{N}} - G)) \\ & = & f_{sn}^{-1}(\theta Cl_{\lambda}(G)) \\ & & f_{sn}^{-1}(Cl_{\lambda}(G)). \end{array}$$

(e) $\longrightarrow$ (f): Let $G \in \lambda$. Then $1_{\mathcal{N}} - Cl_{\lambda}(G) \in \lambda$ and by (e),

$\delta Cl_{\phi}\left(f_{sn}^{-1}(1_{\mathcal{N}} - Cl_{\lambda}(G))\right) \tilde{\subseteq} f_{sn}^{-1}(Cl_{\lambda}(1_{\mathcal{N}} - Cl_{\lambda}(G)))$. Now, we have

$$\begin{array}{rcl} \delta Cl_{\phi}\left(f_{sn}^{-1}(1_{\mathcal{N}} - Cl_{\lambda}(G))\right) & = & \delta Cl_{\phi}\left(1_{\mathcal{M}} - f_{sn}^{-1}(Cl_{\lambda}(G))\right) \\ & = & 1_{\mathcal{M}} - \delta Int_{\phi}\left(f_{sn}^{-1}(Cl_{\lambda}(G))\right) \end{array}$$

and

$$\begin{array}{rcl} f_{sn}^{-1}(Cl_{\lambda}(1_{\mathcal{N}} - Cl_{\lambda}(G))) & = & f_{sn}^{-1}(1_{\mathcal{N}} - Int_{\lambda}(Cl_{\lambda}(G))) \\ & = & 1_{\mathcal{M}} - f_{sn}^{-1}(Int_{\lambda}(Cl_{\lambda}(G))) \\ & \tilde{\subseteq} & 1_{\mathcal{M}} - f_{sn}^{-1}(G). \end{array}$$

Therefore, we obtain $f_{sn}^{-1}(G) \tilde{\subseteq} \delta Int_{\phi}\left(f_{sn}^{-1}(Cl_{\lambda}(G))\right)$.

(f) $\longrightarrow$ (g): Let $T \in \lambda^{c}$. Then $1_{\mathcal{N}} - T \in \lambda$ and by (f),

$f_{sn}^{-1}(1_{\mathcal{N}} - T) \tilde{\subseteq} \delta Int_{\phi}\left(f_{sn}^{-1}(Cl_{\lambda}(1_{\mathcal{N}} - T))\right)$. Now, we have

$$f_{sn}^{-1}(1_{\mathcal{N}} - T) \quad = \quad 1_{\mathcal{M}} - f_{sn}^{-1}(T)$$

and

$$\begin{array}{rcl} \delta Int_{\phi}\left(f_{sn}^{-1}(Cl_{\lambda}(1_{\mathcal{N}} - T))\right) & = & \delta Int_{\phi}\left(f_{sn}^{-1}(1_{\mathcal{N}} - Int_{\lambda}(T))\right) \\ & = & \delta Int_{\phi}\left(1_{\mathcal{M}} - f_{sn}^{-1}(Int_{\lambda}(T))\right) \\ & = & 1_{\mathcal{M}} - \delta Cl_{\phi}\left(f_{sn}^{-1}(Int_{\lambda}(T))\right). \end{array}$$

Therefore, we obtain $1_{\mathcal{M}} - f_{sn}^{-1}(T) \tilde{\subseteq} 1_{\mathcal{M}} - \delta Cl_{\phi}\left(f_{sn}^{-1}(Int_{\lambda}(T))\right)$. Hence,

$\delta Cl_{\phi}\left(f_{sn}^{-1}(Int_{\lambda}(T))\right) \tilde{\subseteq} f_{sn}^{-1}(T)$.

(g) $\longrightarrow$ (f): Let $G \in \lambda$. Then $1_{\mathcal{N}} - G \in \lambda^{c}$ and by (g),

$$Cl_{\phi}\left(f_{sn}^{-1}(Int_{\lambda}(1_{\mathcal{N}} - G))\right) \tilde{\subseteq} f_{sn}^{-1}(1_{\mathcal{N}} - G).$$

Now, we have

$$\begin{array}{rcl} \delta Cl_{\phi}\left(f_{sn}^{-1}(Int_{\lambda}(1_{\mathcal{N}} - G))\right) & = & \delta Cl_{\phi}\left(f_{sn}^{-1}(1_{\mathcal{N}} - Cl_{\lambda}(G))\right) \\ & = & \delta Cl_{\phi}\left(1_{\mathcal{M}} - f_{sn}^{-1}(Cl_{\lambda}(G))\right) \\ & & 1_{\mathcal{M}} - \delta Int_{\phi}\left(f_{sn}^{-1}(Cl_{\lambda}(G))\right) \end{array}$$

and

$$f_{sn}^{-1}(1_{\mathcal{N}} - G) \quad = \quad 1_{\mathcal{M}} - f_{sn}^{-1}(G).$$

Therefore, we obtain $1_{\mathcal{M}} - \delta Int_{\phi}\left(f_{sn}^{-1}(Cl_{\lambda}(G))\right) \widetilde{\subseteq} 1_{\mathcal{M}} - f_{sn}^{-1}(G)$. Hence,

$f_{sn}^{-1}(G) \widetilde{\subseteq} \delta Int_{\phi}\left(f_{sn}^{-1}(Cl_{\lambda}(G))\right)$.

(f) $\longrightarrow$ (a): Let $a_y \in SP(Y, \mathcal{M})$ and let $G \in \phi$ such that $f_{sn}(a_y) \widetilde{\in} G$. Then by (f), $f_{sn}^{-1}(G) \widetilde{\subseteq} \delta Int_{\phi}\left(f_{sn}^{-1}(Cl_{\lambda}(G))\right)$. Since $a_y \widetilde{\in} f_{sn}^{-1}(G) \widetilde{\subseteq} \delta Int_{\phi}\left(f_{sn}^{-1}(Cl_{\lambda}(G))\right)$, then we find $H \in RO(\phi)$ such that $a_y \widetilde{\in} H \widetilde{\subseteq} \delta\ Int_{\phi}\left(f_{sn}^{-1}(Cl_{\lambda}(G))\right) \widetilde{\subseteq} f_{sn}^{-1}(Cl_{\lambda}(G))$; hence, $f_{sn}(Int_{\phi}(Cl_{\phi}(H))) = f_{sn}(H) \widetilde{\subseteq} f_{sn}\left(f_{sn}^{-1}(Cl_{\lambda}\ (G))\right) \widetilde{\subseteq} Cl_{\lambda}(G)$. This shows that f_{sn} is soft w-θ-c.

Theorem 2.14. If $f_{sn} : (Y, \phi, \mathcal{M}) \longrightarrow (Z, \lambda, \mathcal{N})$ is soft w-θ-c. then

(a) $f_{sn}^{-1}(T) \in (\phi_{\delta})^{c}$ for each $T \in (\phi_{\theta})^{c}$.

(b) $f_{sn}^{-1}(G) \in \phi_{\delta}$ for each $G \in \phi_{\theta}$.

Proof. (a) Let $T \in (\phi_{\theta})^{c}$. Then $\theta Cl_{\lambda}(T) = T$ and by Theorem 2.13 (c), $\delta Cl_{\phi}\left(f_{sn}^{-1}(T)\right) \widetilde{\subseteq} f_{sn}^{-1}(\theta Cl_{\lambda}(T)) = f_{sn}^{-1}(T)$. Hence, $f_{sn}^{-1}(T) \in (\phi_{\delta})^{c}$.

(b) Let $G \in \phi_{\theta}$. Then $1_{\mathcal{N}} - G \in (\phi_{\theta})^{c}$ and by (a), $f_{sn}^{-1}(1_{\mathcal{N}} - G) = 1_{\mathcal{M}} - f_{sn}^{-1}(G) \in (\phi_{\delta})^{c}$. Thus, $f_{sn}^{-1}(G) \in \phi_{\delta}$.

The opposite of Theorem 2.14 is not true in general, as demonstrated by the example below.

Example 2.15. Let $Y = \{a, b, c\}$, $\Im = \{\varnothing, Y, \{a\}, \{b\}, \{a, b\}\}$, $\aleph = \{\varnothing, Y, \{a\}, \{a\}, \{a, c\}\}$, and $\mathcal{M} = \mathbb{R}$. Let $s : Y \longrightarrow Y$ and $n : \mathcal{M} \longrightarrow \mathcal{M}$ be the identity functions. Then $f_{sn} : (Y, \tau(\Im), \mathcal{M}) \longrightarrow (Y, \tau(\aleph), \mathcal{M})$ satisfies (a) of Theorem 2.14 but it is not soft w-θ-c.

3 Soft Restriction, Product, and Graph

The following two lemmas will be used in the next main result:

Lemma 3.1. Let $(Y, \phi, \mathcal{M})$ be a STS and let $X \subseteq Y$ such that C_X is soft dense in $(Y, \phi, \mathcal{M})$. Then for any $K \in SS(X, \mathcal{M})$, $Int_{\phi_X}(Cl_{\phi_X}(K)) = Int_{\phi}(Cl_{\phi}(K)) \widetilde{\cap} C_X$.

Proof. Let C_X be soft dense in $(Y, \phi, \mathcal{M})$ and let $K \in SS(X, \mathcal{M})$. To see that $Int_{\phi_X}(Cl_{\phi_X}(K)) \widetilde{\subseteq} Int_{\phi}(Cl_{\phi}(K)) \widetilde{\cap} C_X$, let $a_y \widetilde{\in} Int_{\phi_X}(Cl_{\phi_X}(K))$. Since $Int_{\phi_X}(Cl_{\phi_X}(K)) \in \varphi_X$, then we find $H \in \phi$ such that $Int_{\phi_X}(Cl_{\phi_X}(K)) = H \widetilde{\cap} C_X$. Thus, we have $a_y \widetilde{\in} H \widetilde{\cap} C_X \widetilde{\subseteq} Cl_{\phi_X}(K) = Cl_{\phi}(K) \widetilde{\cap} C_X$.

Claim. $H \widetilde{\subseteq} Cl_{\phi}(K)$.

Proof of Claim. Suppose to the contrary that $H \widetilde{\cap} (1_{\mathcal{M}} - Cl_{\phi}(K)) \neq 0_{\mathcal{M}}$. Since C_X is soft dense in $(Y, \phi, \mathcal{M})$, then $H \widetilde{\cap} (1_{\mathcal{M}} - Cl_{\phi}(K)) \widetilde{\cap} C_X \neq 0_{\mathcal{M}}$. Pick $b_z \widetilde{\in} H \widetilde{\cap} (1_{\mathcal{M}} - Cl_{\phi}(K)) \widetilde{\cap} C_X$. Then, we have $b_z \widetilde{\in} 1_{\mathcal{M}} - Cl_{\phi}(K)$ and $b_y \widetilde{\in} H \widetilde{\cap} C_X = Int_{\phi_X}(Cl_{\phi_X}(K)) \widetilde{\subseteq} Cl_{\phi_X}(K) = Cl_{\phi}(K) \widetilde{\cap} C_X \widetilde{\subseteq} Cl_{\phi}(K)$, which is a contradiction.

Therefore, by the above Claim, we must have $a_y \widetilde{\in} H \widetilde{\subseteq} Cl_{\phi}(K)$, and hence, $a_y \widetilde{\in} Int_{\phi}(Cl_{\phi}(K))$. Therefore, $a_y \widetilde{\in} Int_{\phi}(Cl_{\phi}(K)) \widetilde{\cap} C_X$.

To see that $Int_{\phi}\left(Cl_{\phi}\left(K\right)\right)\tilde{\cap}C_X\tilde{\subseteq}Int_{\phi_X}\left(Cl_{\phi_X}\left(K\right)\right)$, let $a_y\tilde{\in}Int_{\phi}\left(Cl_{\phi}\left(K\right)\right)\tilde{\cap}C_X$. Since $a_y\tilde{\in}Int_{\phi}\left(Cl_{\phi}\left(K\right)\right)\in\phi$, then there is $H\in\phi$ such that $a_y\tilde{\in}H\tilde{\subseteq}Cl_{\varphi}(K)$, and so, $a_y\tilde{\in}H\tilde{\cap}C_X\tilde{\subseteq}Cl_{\phi}(K)\tilde{\cap}C_X = Cl_{\phi_X}\left(K\right)$. Since $H\tilde{\cap}C_X\in\varphi_X$, then $a_y\tilde{\in}Int_{\phi_X}\left(Cl_{\phi_X}\left(K\right)\right)$.

Lemma 3.2. Let $(Y,\phi,\mathcal{M})$ be a STS and let $X\subseteq Y$ such that C_X is soft dense in $(Y,\phi,\mathcal{M})$. Then

(a) If $G\in RO\left(\phi\right)$, then $G\tilde{\cap}C_X\in RO\left(\phi_X\right)$.

(b) If $H\in RO\left(\phi_X\right)$, then there is $G\in RO\left(\phi\right)$ such that $H=G\tilde{\cap}C_X$.

Proof. (a) Since $G\in RO\left(\phi\right)\subseteq\phi$, then $G\tilde{\cap}C_X\in\phi_X$. Since $G\tilde{\cap}C_X\tilde{\subseteq}Cl_{\phi_X}\left(G\tilde{\cap}C_X\right)$ and $G\tilde{\cap}C_X\in\phi_X$, then

$$G\tilde{\cap}C_X\tilde{\subseteq}Int_{\phi_X}\left(Cl_{\phi_X}\left(G\tilde{\cap}C_X\right)\right).$$

By Lemma 3.1,

$$Int_{\phi_X}\left(Cl_{\phi_X}\left(G\tilde{\cap}C_X\right)\right)=Int_{\phi}\left(Cl_{\phi}\left(G\tilde{\cap}C_X\right)\right)\tilde{\cap}C_X.$$

Thus, we have

$$\begin{aligned}
G\tilde{\cap}C_X &\tilde{\subseteq} Int_{\phi_X}\left(Cl_{\phi_X}\left(G\tilde{\cap}C_X\right)\right)\\
&= Int_{\phi}\left(Cl_{\phi}\left(G\tilde{\cap}C_X\right)\right)\tilde{\cap}C_X\\
&\tilde{\subseteq} Int_{\phi}\left(Cl_{\phi}\left(G\right)\right)\tilde{\cap}C_X\\
&= G\tilde{\cap}C_X.
\end{aligned}$$

Hence, $Int_{\phi_X}\left(Cl_{\phi_X}\left(G\tilde{\cap}C_X\right)\right)=G\tilde{\cap}C_X$. It follows that $G\tilde{\cap}C_X\in RO\left(\phi_X\right)$.

(b) Since $H\in RO\left(\phi_X\right)$, then $H=Int_{\phi_X}\left(Cl_{\phi_X}\left(H\right)\right)$ and by Lemma 3.1, $Int_{\phi_X}\left(Cl_{\phi_X}\left(H\right)\right)=Int_{\phi}\left(Cl_{\phi}\left(H\right)\right)\tilde{\cap}C_X$. Let $G=Int_{\phi}\left(Cl_{\phi}\left(H\right)\right)$. Then $G\in RO\left(\phi\right)$ and $H=G\tilde{\cap}C_X$.

Theorem 3.3. If $f_{sn}:(Y,\phi,\mathcal{M})\longrightarrow(Z,\lambda,\mathcal{N})$ is soft w-θ-c and $X\subseteq Y$ such that C_X is soft dense in $(Y,\phi,\mathcal{M})$, then $\left(f_{sn}\right)_{|C_X}:(X,\phi_X,\mathcal{M})\longrightarrow(Z,\lambda,\mathcal{N})$ is soft w-θ-c.

Proof. Let $a_x\in SP\left(X,\mathcal{M}\right)\subseteq SP\left(Y,\mathcal{M}\right)$ and let $G\in\lambda$ such that $f_{sn}\left(a_x\right)\tilde{\in}G$. Then, by Theorem 2.8 (b), we find $K\in RO\left(\phi\right)$ such that $a_x\tilde{\in}K$ and $f_{sn}(K)\tilde{\subseteq}Cl_{\lambda}\left(G\right)$. By Lemma 3.2 (a), $K\tilde{\cap}C_X\in RO\left(\phi_X\right)$. Moreover, $\left(f_{sn}\right)_{|C_X}\left(K\tilde{\cap}C_X\right)=f_{sn}\left(K\tilde{\cap}C_X\right)\tilde{\subseteq}f_{sn}(K)\tilde{\subseteq}Cl_{\lambda}\left(G\right)$. Thus, by Theorem 2.12 (b), $\left(f_{sn}\right)_{|C_X}$ is soft w-θ-c.

The following lemma will be used in the next main result:

Lemma 3.4. Let $(Y,\phi,\mathcal{M})$ be a STS and let $X\subseteq Y$ such that $C_X\in\phi-\{0_{\mathcal{M}}\}$. If $G\in RO\left(\phi\right)$, then $G\tilde{\cap}C_X\in RO\left(\phi_X\right)$.

Proof. Let $C_X\in\phi-\{0_{\mathcal{M}}\}$ and let $G\in RO\left(\phi\right)$. Since $C_X\in\phi$, then

$$\begin{aligned}
Int_{\phi_X}\left(Cl_{\phi_X}\left(G\tilde{\cap}C_X\right)\right) &= Int_{\phi}\left(Cl_{\phi_X}\left(G\tilde{\cap}C_X\right)\right)\tilde{\cap}C_X\\
&= Int_{\phi}\left(Cl_{\phi}\left(G\tilde{\cap}C_X\right)\tilde{\cap}C_X\right)\tilde{\cap}C_X\\
&\tilde{\subseteq} Int_{\phi}\left(Cl_{\phi}\left(G\right)\right)\tilde{\cap}C_X\\
&= G\tilde{\cap}C_X.
\end{aligned}$$

Since $G\in RO\left(\phi\right)\subseteq\phi$, then $G\tilde{\cap}C_X\in\phi_X$. Since $G\tilde{\cap}C_X\tilde{\subseteq}Cl_{\phi_X}\left(G\tilde{\cap}C_X\right)$ and $G\tilde{\cap}C_X\in\phi_X$, then $G\tilde{\cap}C_X\tilde{\subseteq}Int_{\phi_X}\left(Cl_{\phi_X}\left(G\tilde{\cap}C_X\right)\right)$. This shows that $Int_{\phi_X}\left(Cl_{\phi_X}\left(G\tilde{\cap}C_X\right)\right)=G\tilde{\cap}C_X$. Hence, $G\tilde{\cap}C_X\in RO\left(\phi_X\right)$.

Theorem 3.5. If $f_{sn}:(Y,\phi,\mathcal{M})\longrightarrow(Z,\lambda,\mathcal{N})$ is soft w-θ-c and $X\subseteq Y$ such that $C_X\in\phi-\{0_{\mathcal{M}}\}$, then $(f_{sn})_{|C_X}:(Y,\phi_X,\mathcal{M})\longrightarrow(Z,\lambda,\mathcal{N})$ is soft w-θ-c.

Proof. Let $a_x\in SP(X,\mathcal{M})\subseteq SP(Y,\mathcal{M})$ and let $G\in\lambda$ such that $f_{sn}(a_y)\tilde{\in}G$. Then, by Theorem 2.12 (b), we find $K\in RO(\phi)$ such that $a_x\tilde{\in}K$ and $f_{sn}(K)\tilde{\subseteq}Cl_\lambda(G)$. By Lemma 3.4, $K\tilde{\cap}C_X\in RO(\phi_X)$. Moreover, $(f_{sn})_{|C_X}(K\tilde{\cap}C_X)=f_{sn}(K\tilde{\cap}C_X)\tilde{\subseteq}f_{sn}(K)\tilde{\subseteq}Cl_\lambda(G)$. Thus, by Theorem 2.12 (b), $(f_{sn})_{|C_X}$ is soft w-θ-c.

Theorem 3.6. Let $f_{s_1n_1}:(Y,\phi,\mathcal{M})\longrightarrow(Z,\lambda,\mathcal{N})$ and $f_{s_2n_2}:(W,\mu,\mathcal{L})\longrightarrow(T,\sigma,\mathcal{F})$ be two soft functions. Let $s^*:Y\times W\longrightarrow Z\times T$ and $n^*:\mathcal{M}\times\mathcal{L}\longrightarrow\mathcal{N}\times\mathcal{F}$ be the functions defined by $s^*(y,w)=(s_1(y),s_2(w))$ and $n^*(a,b)=(n_1(a),n_2(b))$. Then $f_{s^*n^*}:(Y\times W,pr(\phi\times\mu),\mathcal{M}\times\mathcal{L})\longrightarrow(Z\times T,pr(\lambda\times\sigma),\mathcal{N}\times\mathcal{F})$ is soft w-θ-c iff $f_{s_1n_1}$ and $f_{s_2n_2}$ are soft w-θ-c.

Proof. *Necessity.* Let $f_{s^*n^*}$ be soft w-θ-c. Let $a_y\in SP(Y,\mathcal{M})$ and $b_w\in SP(W,\mathcal{L})$. Let $T\in\lambda$ and $R\in\sigma$ such that $f_{s_1n_1}(a_y)\tilde{\in}T$ and $f_{s_2n_2}(b_w)\tilde{\in}R$. Then we have $f_{s^*n^*}\left((a,b)_{(y,w)}\right)\tilde{\in}T\times R\in pr(\lambda\times\sigma)$. Since $f_{s^*v^*}$ is soft w-θ-c, then we find $O\in pr(\phi\times\mu)$ such that $(a,b)_{(y,w)}\tilde{\in}O$ and

$$f_{s^*n^*}(Int_{pr(\phi\times\mu)}\left(Cl_{pr(\phi\times\mu)}(O)\right))\tilde{\subseteq}Cl_{pr(\lambda\times\sigma)}(T\times R)=Cl_\lambda(T)\times Cl_\sigma(R).$$

Choose $U\in\phi$ and $V\in\mu$ such that $(a,b)_{(y,w)}\tilde{\in}U\times V\tilde{\subseteq}O$. Thus,

$$\begin{aligned}
f_{s_1n_1}(Int_\phi(Cl_\phi(U)))\times f_{s_2n_2}(Int_\mu(Cl_\mu(V))) &=\\
f_{s^*n^*}(Int_\phi(Cl_\phi(U))\times Int_\mu(Cl_\mu(V))) &=\\
f_{s^*n^*}(Int_{pr(\phi\times\mu)}\left(Cl_{pr(\phi\times\mu)}(U\times V)\right)) &\tilde{\subseteq}\\
f_{s^*n^*}(Int_{pr(\phi\times\mu)}\left(Cl_{pr(\phi\times\mu)}(O)\right)) &\tilde{\subseteq}\\
Cl_{pr(\lambda\times\sigma)}(T\times R) &=\\
Cl_\lambda(T)\times Cl_\sigma(R). &
\end{aligned}$$

Therefore, we have $a_y\tilde{\in}U\in\phi$, $b_w\tilde{\in}V\in\sigma$, $f_{s_1n_1}(Int_\phi(Cl_\phi(U)))\tilde{\subseteq}Cl_\lambda(T)$, and $f_{s_2n_2}(Int_\mu(Cl_\mu(V)))$ $\tilde{\subseteq}Cl_\sigma(R)$. This shows that $f_{s_1n_1}$ and $f_{s_2n_2}$ are soft w-θ-c.

Sufficiency. Let $f_{s_1n_1}$ and $f_{s_2n_2}$ be soft w-θ-c. Let $(a,b)_{(y,w)}\in SP(Y\times W,\mathcal{M}\times\mathcal{L})$ and let $G\in pr(\lambda\times\sigma)$ such that $f_{s^*n^*}\left((a,b)_{(y,w)}\right)\tilde{\in}G$. Choose $T\in\lambda$ and $R\in\sigma$ such that $f_{s^*n^*}\left((a,b)_{(y,z)}\right)=(n_1(a),n_2(b))_{(s_1(y),s_2(z))}\tilde{\in}T\times R\tilde{\subseteq}G$. Then, we have $(n_1(a))_{s_1(y)}=f_{s_1n_1}(a_y)\tilde{\in}T\in\lambda$ and $(n_2(b))_{s_2(w)}=f_{s_2n_2}(b_w)\tilde{\in}R\in\sigma$. Since $f_{s_1n_1}$ and $f_{s_2n_2}$ are soft w-θ-c, then there exist $U\in\phi$ and $V\in\mu$ such that $a_y\tilde{\in}U$, $b_w\tilde{\in}V$, $f_{s_1n_1}(Int_\phi(Cl_\phi(U)))\tilde{\subseteq}Cl_\lambda(T)$, and $f_{s_2n_2}(Int_\mu(Cl_\mu(V)))\tilde{\subseteq}Cl_\sigma(R)$. Therefore, we have $(a,b)_{(y,w)}\tilde{\in}U\times V\in pr(\phi\times\mu)$ and

$$\begin{aligned}
f_{s^*n^*}(Int_{pr(\phi\times\mu)}\left(Cl_{pr(\phi\times\mu)}(U\times V)\right)) &=\\
f_{s^*n^*}(Int_\phi(Cl_\phi(U))\times Int_\mu(Cl_\mu(V))) &=\\
f_{s_1n_1}(Int_\phi(Cl_\phi(U)))\times f_{s_2n_2}(Int_\mu(Cl_\mu(V))) &\tilde{\subseteq}\\
Cl_\lambda(T)\times Cl_\sigma(R) &=\\
Cl_{pr(\lambda\times\sigma)}(T\times R) &\tilde{\subseteq}\\
Cl_{pr(\lambda\times\sigma)}(G). &
\end{aligned}$$

It follows that $f_{s^*n^*}$ is soft w-θ-c.

Let X and Y be two non-empty sets. The projection functions $h:X\times Y\longrightarrow X$ and $g:X\times Y\longrightarrow Y$ defined by $h(x,y)=x$ and $g(x,y)=y$ for all $(x,y)\in X\times Y$ will be denoted by π_X and π_Y, respectively.

Theorem 3.7. Let $(Y,\phi,\mathcal{M})$, $(Z,\lambda,\mathcal{N})$, and $(W,\mu,\mathcal{L})$ be three STSs. If $f_{sn}:(Y,\phi,\mathcal{M})\longrightarrow(Z\times W,pr(\lambda\times\mu),\mathcal{N}\times\mathcal{L})$ is soft w-θ-c iff $f_{(\pi_Z\circ s)(\pi_{\mathcal{N}}\circ n)}:(Y,\phi,\mathcal{M})\longrightarrow(Z,\lambda,\mathcal{N})$ and $f_{(\pi_W\circ s)(\pi_{\mathcal{L}}\circ n)}:(Y,\phi,\mathcal{M})\longrightarrow(W,\mu,\mathcal{L})$ are soft w-θ-c.

Proof. *Necessity.* Let f_{sn} be soft w-θ-c. Let $a_y \in SP(Y,\mathcal{M})$, $T \in \lambda$, and $R \in \mu$ such that $f_{(\pi_Z \circ s)(\pi_{\mathcal{N}} \circ n)}(a_y) \tilde{\in} T$ and $f_{(\pi_W \circ s)(\pi_{\mathcal{L}} \circ n)}(a_y) \tilde{\in} R$. Then $f_{sn}(a_y) \in T \times R \in pr(\lambda \times \mu)$. So, there is $H \in \phi$ such that $a_y \tilde{\in} H$ and $f_{sn}(Int_\phi(Cl_\phi(H))) \tilde{\subseteq} Cl_{pr(\lambda\times\mu)}(T \times R) = Cl_\lambda(T) \times Cl_\mu(R)$. Thus,

$f_{(\pi_Z \circ s)(\pi_{\mathcal{N}} \circ n)}(Int_\phi(Cl_\phi(H))) \tilde{\subseteq} Cl_\lambda(T)$ and $f_{(\pi_W \circ s)(\pi_{\mathcal{L}} \circ n)}(Int_\phi(Cl_\phi(H))) \tilde{\subseteq} Cl_\mu(R)$. It follows that $f_{(\pi_Z \circ s)(\pi_{\mathcal{N}} \circ n)}$ and $f_{(\pi_W \circ s)(\pi_{\mathcal{L}} \circ n)}$ are soft w-θ-c.

Sufficiency. Let $f_{(\pi_Z \circ s)(\pi_{\mathcal{N}} \circ n)}$ and $f_{(\pi_W \circ s)(\pi_{\mathcal{L}} \circ n)}$ be soft w-θ-c. Let $a_y \in SP(Y,\mathcal{M})$ and let $G \in pr(\lambda\times\sigma)$ such that $f_{sn}(a_y) \tilde{\in} G$. Choose $T \in \lambda$ and $R \in \mu$ such that $f_{sn}(a_y) \tilde{\in} T \times R \tilde{\subseteq} G$. Then, we have $f_{(\pi_Z \circ s)(\pi_{\mathcal{N}} \circ n)}(a_y) \tilde{\in} T$ and $f_{(\pi_W \circ s)(\pi_{\mathcal{L}} \circ n)}(a_y) \tilde{\in} R$. Since $f_{(\pi_Z \circ s)(\pi_{\mathcal{N}} \circ n)}$ and $f_{(\pi_W \circ s)(\pi_{\mathcal{L}} \circ n)}$ are soft w-θ-c, there exist $U, V \in \phi$ such that $a_y \tilde{\in} U \tilde{\cap} V$, $f_{(\pi_Z \circ s)(\pi_{\mathcal{N}} \circ n)}(Int_\phi(Cl_\phi(U))) \tilde{\subseteq} Cl_\lambda(T)$, and $f_{(\pi_W \circ s)(\pi_{\mathcal{L}} \circ n)}(Int_\phi(Cl_\phi(V))) \tilde{\subseteq} Cl_\mu(R)$. Thus, we have $a_y \tilde{\in} U \tilde{\cap} V \in \phi$, and $f_{sn}(Int_\phi(Cl_\phi(U \tilde{\cap} V)) \tilde{\subseteq} Cl_\lambda(T) \times Cl_\mu(R) = Cl_{pr(\lambda\times\mu)}(T \times R) \tilde{\subseteq} Cl_{pr(\lambda\times\mu)}(G)$. It follows that f_{sn} is soft w-θ-c.

For every function $p : X \longrightarrow Y$, the function $h : X \longrightarrow X \times Y$ defined by $h(x) = (x, h(x))$ is represented by $p^{\#}$.

Theorem 3.8. Let $f_{sn} : (Y,\phi,\mathcal{M}) \longrightarrow (Z,\lambda,\mathcal{N})$ be a soft function. Then $f_{s^{\#}n^{\#}} : (Y,\phi,\mathcal{M}) \longrightarrow (Y \times Z, pr(\phi \times \lambda), \mathcal{M} \times \mathcal{N})$ is soft w-θ-c iff f_{sn} is soft w-θ-c.

Proof. *Necessity.* Let $f_{s^{\#}n^{\#}}$ be soft w-θ-c. Then, by Theorem 3.7, $f_{sn} = f_{(\pi_Z \circ s^{\#})(\pi_{\mathcal{N}} \circ n^{\#})} : (Y,\phi,\mathcal{M}) \longrightarrow (Z,\lambda,\mathcal{N})$ is soft w-θ-c.

Sufficiency. Let f_{sn} be soft w-θ-c. Let $a_y \in SP(Y,\mathcal{M})$ and let $G \in pr(\lambda\times\sigma)$ such that $f_{s^{\#}n^{\#}}(a_y) \tilde{\in} G$. Choose $T \in \lambda$ and $R \in \mu$ such that $f_{s^{\#}n^{\#}}(a_y) = (a, n(a))_{(y,s(y))} \tilde{\in} T \times R \tilde{\subseteq} G$. Since $(n(a))_{s(y)} \tilde{\in} R \in \lambda$, then we find $H \in RO(\phi)$ such that $a_y \tilde{\in} H$ and $f_{sn}(H) \tilde{\subseteq} Cl_\lambda(R)$. Let $K = H \tilde{\cap} Int_\phi(Cl_\phi(T))$. Then $a_y \tilde{\in} K \in RO(\phi)$ and $f_{s^{\#}n^{\#}}(K) \tilde{\subseteq} Int_\phi(Cl_\phi(T)) \times Cl_\mu(R) \tilde{\subseteq} Cl_\phi(T) \times Cl_\mu(R) = Cl_{pr(\phi\times\lambda)}(T \times R) \tilde{\subseteq} Cl_{pr(\phi\times\lambda)}(G)$. It follows that $f_{s^{\#}n^{\#}}$ is soft w-θ-c.

4 Soft Hyperconnectedness and Soft Nearly Compactness

Theorem 4.1. If $(Z,\lambda,\mathcal{N})$ is soft hyperconnected, then every soft function $f_{sn} : (Y,\phi,\mathcal{M}) \longrightarrow (Z,\lambda,\mathcal{N})$ is soft almost strongly θ-continuous.

Proof. Let $a_y \in SP(Y,\mathcal{M})$ and let $G \in \lambda$ such that $f_{sn}(a_y) \tilde{\in} G$. Since $(Z,\lambda,\mathcal{N})$ is soft hyperconnected, then $Cl_\lambda(G) = 1_{\mathcal{N}}$ and $Int_\lambda(Cl_\lambda(G)) = 1_{\mathcal{N}}$. Take $H = 1_{\mathcal{M}}$. Then $a_y \tilde{\in} H \in \phi$ and $f_{sn}(Cl_\phi(H)) \tilde{\subseteq} Int_\lambda(Cl_\lambda(G))$. Therefore, f_{sn} is soft almost strongly θ-continuous.

Corollary 4.2. If $(Z,\lambda,\mathcal{N})$ is soft hyperconnected, then every soft function $f_{sn} : (Y,\phi,\mathcal{M}) \longrightarrow (Z,\lambda,\mathcal{N})$ is soft θ-continuous.

Proof. We derive the proof from from Theorem 4.1 and Corollary 3.11 of [12].

Corollary 4.3. If $(Z,\lambda,\mathcal{N})$ is soft hyperconnected, then every soft function $f_{sn} : (Y,\phi,\mathcal{M}) \longrightarrow (Z,\lambda,\mathcal{N})$ is soft δ-continuous.

Proof. We derive the proof from from Theorem 4.1 and Theorem 3.8 of [12].

Corollary 4.4. If $(Z,\lambda,\mathcal{N})$ is soft hyperconnected, then every soft function $f_{sn} : (Y,\phi,\mathcal{M}) \longrightarrow (Z,\lambda,\mathcal{N})$ is soft w-θ-c.

Proof. We derive the proof from from Theorems 2.4 and 4.1, and Theorem Corollary 4.2.

Theorem 4.5. If $f_{sn} : (Y, \phi, \mathcal{M}) \longrightarrow (Z, \lambda, \mathcal{N})$ is soft w-θ-c surjection and $(Y, \phi, \mathcal{M})$ is soft hyperconnected, then $(Z, \lambda, \mathcal{N})$ is soft hyperconnected.

Proof. Let $G \in \lambda - \{0_{\mathcal{N}}\}$. Since f_{sn} is surjective, then we find $a_y \in SP(Y, \mathcal{M})$ such that $f_{sn}(a_y) \in G$. Since f_{sn} is soft w-θ-c, then we find $H \in \phi$ such that $a_y \tilde{\in} H$ and $f_{sn}(Int_{\phi}(Cl_{\phi}(H))) \tilde{\subseteq} Cl_{\lambda}(G)$. Since $(Y, \phi, \mathcal{M})$ is soft hyperconnected, then $Cl_{\phi}(H) = 1_{\mathcal{M}}$ and $Int_{\phi}(Cl_{\phi}(H)) = 1_{\mathcal{M}}$. Since f_{sn} is surjective, then $f_{sn}(1_{\mathcal{M}}) = 1_{\mathcal{N}}$. Therefore, we have $1_{\mathcal{N}} \tilde{\subseteq} Cl_{\lambda}(G)$ and hence $Cl_{\lambda}(G) = 1_{\mathcal{N}}$. This shows that $(Z, \lambda, \mathcal{N})$ is soft hyperconnected.

Corollary 4.6. If $f_{sn} : (Y, \phi, \mathcal{M}) \longrightarrow (Z, \lambda, \mathcal{N})$ is soft θ-c surjection and $(Y, \phi, \mathcal{M})$ is soft hyperconnected, then $(Z, \lambda, \mathcal{N})$ is soft hyperconnected.

Proof. We derive the proof from from Theorems 2.4 and 4.5.

Corollary 4.7. If $f_{sn} : (Y, \phi, \mathcal{M}) \longrightarrow (Z, \lambda, \mathcal{N})$ is soft δ-c surjection and $(Y, \phi, \mathcal{M})$ is soft hyperconnected, then $(Z, \lambda, \mathcal{N})$ is soft hyperconnected.

Proof. We derive the proof from from Theorems 2.6 and 4.5.

Theorem 4.8. If $f_{sn} : (Y, \phi, \mathcal{M}) \longrightarrow (Z, \lambda, \mathcal{N})$ is soft w-θ-c and K is soft nearly compact relative to $(Y, \phi, \mathcal{M})$, then $f_{sn}(K)$ is soft quasi H-closed relative to $(Z, \lambda, \mathcal{N})$.

Proof. Let $\psi \subseteq \lambda$ such that $f_{sn}(K) \tilde{\subseteq} \tilde{\cup}_{V \in \psi} V$. For each $a_y \tilde{\in} K$, we find $V(a_y) \in \psi$ such that $f_{sn}(a_y) \tilde{\in} V(a_y)$. Since f_{sn} is soft w-θ-c, by Theorem 2.12, we find $U(a_y) \in RO(\phi)$ such that $a_y \tilde{\in} U(a_y)$ and $f_{sn}(U(a_y)) \tilde{\subseteq} Cl_{\lambda}$ $(V(a_y))$. Since K is soft nearly compact relative to $(Y, \phi, \mathcal{M})$ and $K \tilde{\subseteq} \tilde{\cup}_{a_y \tilde{\in} K} U(a_y)$, then we find a finite subset $P \subseteq SP(Y, \mathcal{M})$ such that $a_y \tilde{\in} K$ for every $a_y \in P$ and $K \tilde{\subseteq} \tilde{\cup}_{a_y \in P} U(a_y)$. Thus,

$$\begin{aligned} f_{sn}(K) &\tilde{\subseteq} f_{sn}\left(\tilde{\cup}_{a_y \in P} U(a_y)\right) \\ &= \tilde{\cup}_{a_y \in P} f_{sn}(U(a_y)) \\ &\tilde{\subseteq} \tilde{\cup}_{a_y \in P} Cl_{\lambda}(V(a_y)). \end{aligned}$$

It follows that $f_{sn}(K)$ is soft quasi H-closed relative to $(Z, \lambda, \mathcal{N})$.

Corollary 4.9. If $f_{sn} : (Y, \phi, \mathcal{M}) \longrightarrow (Z, \lambda, \mathcal{N})$ is a soft w-θ-c surjection and $(Y, \phi, \mathcal{M})$ is soft nearly compact, then $(Z, \lambda, \mathcal{N})$ is soft quasi H-closed.

Corollary 4.10. If $f_{sn} : (Y, \phi, \mathcal{M}) \longrightarrow (Z, \lambda, \mathcal{N})$ is a soft θ-c surjection and $(Y, \phi, \mathcal{M})$ is soft nearly compact, then $(Z, \lambda, \mathcal{N})$ is soft quasi H-closed.

Proof. We derive the proof from from Theorem 2.4 and Corollary 4.9.

Corollary 4.11. If $f_{sn} : (Y, \phi, \mathcal{M}) \longrightarrow (Z, \lambda, \mathcal{N})$ is a soft δ-c surjection and $(Y, \phi, \mathcal{M})$ is soft nearly compact, then $(Z, \lambda, \mathcal{N})$ is soft quasi H-closed.

Proof. We derive the proof from from Theorem 2.6 and Corollary 4.9.

Lemma 4.12. If $(Y, \phi, \mathcal{M})$ is soft nearly compact and $T \in (\phi_{\delta})^{c} - \{1_{\mathcal{M}}\}$, then T is soft nearly compact relative to $(Y, \phi, \mathcal{M})$.

Proof. Let $\psi \subseteq RO\left(\phi\right)$ such that $T\tilde{\subseteq}\tilde{\cup}_{G\in\psi}G$. Since $T\in\left(\phi_\delta\right)^c$, then $1_{\mathcal{M}}-T\in\phi_\delta$, and so, we find $\Phi\subseteq RO\left(\phi\right)$ such that $1_{\mathcal{M}}-T=\tilde{\cup}_{H\in\Phi}H$. Since $\left(Y,\phi,\mathcal{M}\right)$ is soft nearly compact and $1_{\mathcal{M}}=\left(\tilde{\cup}_{G\in\psi}G\right)\tilde{\cup}\left(\tilde{\cup}_{H\in\Phi}H\right)$, then there exist finite sub-collections $\psi_1\subseteq\psi$ and $\Phi_1\subseteq\Phi$ such that $1_{\mathcal{M}}=\left(\tilde{\cup}_{G\in\psi_1}G\right)\tilde{\cup}\left(\tilde{\cup}_{H\in\Phi_1}H\right)$, and thus,

$$\begin{array}{rcc} T & = & \left(\left(\tilde{\cup}_{G\in\psi_1}G\right)\tilde{\cup}\left(\tilde{\cup}_{H\in\Phi_1}H\right)\right)\tilde{\cap}T \\ & = & \left(\left(\tilde{\cup}_{G\in\psi_1}G\right)\tilde{\cap}T\right)\tilde{\cup}\left(\left(\tilde{\cup}_{H\in\Phi_1}H\right)\tilde{\cap}T\right) \\ & = & \left(\left(\tilde{\cup}_{G\in\psi_1}G\right)\tilde{\cap}T\right)\tilde{\cup}0_{\mathcal{M}} \\ & = & \left(\tilde{\cup}_{G\in\psi_1}G\right)\tilde{\cap}T \end{array}$$

Therefore, $T\tilde{\subseteq}\tilde{\cup}_{G\in\psi_1}G$ and hence, T is soft nearly compact relative to $\left(Y,\phi,\mathcal{M}\right)$.

Theorem 4.13. If $f_{sn}:\left(Y,\phi,\mathcal{M}\right)\longrightarrow\left(Z,\lambda,\mathcal{N}\right)$ is soft w-θ-c such that $\left(Y,\phi,\mathcal{M}\right)$ is soft nearly compact and $\left(Z,\lambda,\mathcal{N}\right)$ is soft Hausdorff, then $f_{sn}\left(T\right)\in\left(\lambda_\delta\right)^c$ for each $T\in\left(\phi_\delta\right)^c$.

Proof. Let $T\in\left(\phi_\delta\right)^c$. Since $\left(Y,\phi,\mathcal{M}\right)$ is soft nearly compact, then by Lemma 4.12, T is soft nearly compact relative to $\left(Y,\phi,\mathcal{M}\right)$. So, by Theorem 4.8, $f_{sn}\left(T\right)$ is soft quasi H-closed relative to $\left(Z,\lambda,\mathcal{N}\right)$. We are going to show that $1_{\mathcal{N}}-f_{sn}\left(T\right)\tilde{\subseteq}1_{\mathcal{N}}-\delta Cl_\lambda\left(f_{sn}\left(T\right)\right)$. Let $b_y\tilde{\in}1_{\mathcal{N}}-f_{sn}\left(T\right)$. Since $\left(Z,\lambda,\mathcal{N}\right)$ is soft Hausdorff, then for each $d_z\tilde{\in}f_{sn}\left(T\right)$, there exist $G\left(d_z\right),H\left(d_z\right)\in\lambda$ such that $f_{sn}\left(b_y\right)\tilde{\in}G\left(d_z\right)$, $d_z\tilde{\in}H\left(d_z\right)$, and $G\left(d_z\right)\tilde{\cap}H\left(d_z\right)=0_{\mathcal{N}}$; hence $Int_\lambda\left(Cl_\lambda\left(G\left(d_z\right)\right)\right)\tilde{\cap}Cl_\lambda\left(H\left(d_z\right)\right)=0_{\mathcal{N}}$. Since $f_{sn}\left(T\right)$ is soft quasi H-closed relative to $\left(Z,\lambda,\mathcal{N}\right)$ and $f_{sn}\left(T\right)\tilde{\subseteq}\tilde{\cup}_{d_z\tilde{\in}f_{sn}\left(T\right)}H\left(d_z\right)$, then we find a finite subset $P\subseteq SP\left(Z,\mathcal{N}\right)$ such that $d_z\tilde{\in}f_{sn}\left(T\right)$ for every $d_z\in P$ and $f_{sn}\left(T\right)\tilde{\subseteq}\tilde{\cup}_{d_z\in P}Cl_\lambda\left(H\left(d_z\right)\right)$. Let $K=\tilde{\cap}_{d_z\in P}G\left(d_z\right)$. Then, we have $b_y\tilde{\in}K\in\phi$ and

$$\begin{array}{rcc} Int_\lambda\left(Cl_\lambda\left(K\right)\right)\tilde{\cap}f_{sn}\left(T\right) & \tilde{\subseteq} & Int_\lambda\left(Cl_\lambda\left(K\right)\right)\tilde{\cap}\left(\tilde{\cup}_{d_z\in P}Cl_\lambda\left(H\left(d_z\right)\right)\right) \\ & = & \tilde{\cup}_{a_y\in P}\left(Int_\lambda\left(Cl_\lambda\left(K\right)\right)\tilde{\cap}Cl_\lambda\left(H\left(d_z\right)\right)\right) \\ & \tilde{\subseteq} & \tilde{\cup}_{a_y\in P}\left(Int_\lambda\left(Cl_\lambda\left(G\left(d_z\right)\right)\right)\tilde{\cap}Cl_\lambda\left(H\left(d_z\right)\right)\right) \\ & = & 0_{\mathcal{N}}. \end{array}$$

Therefore, $b_y\tilde{\in}1_{\mathcal{N}}-\delta Cl_\lambda\left(f_{sn}\left(T\right)\right)$.

Lemma 4.14. Let $f_{sn}:\left(Y,\phi,\mathcal{M}\right)\longrightarrow\left(Z,\lambda,\mathcal{N}\right)$ be a soft function. Then $Graph(f_{sn})\in\left(\left(pr\left(\phi\times\lambda\right)\right)_\delta\right)^c$ iff for each $\left(b,d\right)_{\left(y,z\right)}\tilde{\in}1_{\mathcal{M}\times\mathcal{N}}-Graph(f_{sn})$, there exist $U\in RO\left(\phi\right)$ and $V\in RO\left(\lambda\right)$ such that $b_y\tilde{\in}U$, $d_z\tilde{\in}V$, and $f_{sn}\left(U\right)\tilde{\cap}V=0_{\mathcal{N}}$.

Proof. *Necessity.* Suppose that $Graph(f_{sn})\in\left(\left(pr\left(\phi\times\lambda\right)\right)_\delta\right)^c$. Let $\left(b,d\right)_{\left(y,z\right)}\tilde{\in}1_{\mathcal{M}\times\mathcal{N}}-Graph(f_{sn})$. Since $Graph(f_{sn})\in\left(\left(pr\left(\phi\times\lambda\right)\right)_\delta\right)^c$, then $\left(b,d\right)_{\left(y,z\right)}\tilde{\in}1_{\mathcal{M}\times\mathcal{N}}-\delta Cl_{pr\left(\phi\times\lambda\right)}\left(Graph(f_{sn})\right)$. So, we find $G\in pr\left(\phi\times\lambda\right)$ such that $\left(b,d\right)_{\left(y,z\right)}\tilde{\in}G$ and $Int_{pr\left(\phi\times\lambda\right)}\left(Cl_{pr\left(\phi\times\lambda\right)}\left(G\right)\right)\tilde{\cap}Graph(f_{sn})=0_{\mathcal{M}\times\mathcal{N}}$. Choose $H\in\phi$ and $K\in\lambda$ such that $\left(b,d\right)_{\left(y,z\right)}\tilde{\in}H\times K\tilde{\subseteq}G$. Let $U=Int_\phi\left(Cl_\phi\left(H\right)\right)$ and $V=Int_\lambda\left(Cl_\lambda\left(K\right)\right)$. Then $U\in RO\left(\phi\right)$ and $V\in RO\left(\lambda\right)$ such that $b_y\tilde{\in}U$, $d_z\tilde{\in}V$, and $f_{sn}\left(U\right)\tilde{\cap}V\tilde{\subseteq}f_{sn}\left(Int_\phi\left(Cl_\phi\left(H\right)\right)\right)\tilde{\cap}Int_\lambda\left(Cl_\lambda\left(K\right)\right)=0_{\mathcal{N}}$.

Sufficiency. We will show that $1_{\mathcal{M}\times\mathcal{N}}-Graph(f_{sn})\tilde{\subseteq}1_{\mathcal{M}\times\mathcal{N}}-\delta Cl_{pr\left(\phi\times\lambda\right)}\left(Graph(f_{sn})\right)$. Let $\left(b,d\right)_{\left(y,z\right)}\tilde{\in}1_{\mathcal{M}\times\mathcal{N}}-Graph(f_{sn})$. Then by assumption, there exist $U\in RO\left(\phi\right)$ and $V\in RO\left(\lambda\right)$ such that $b_y\tilde{\in}U$, $d_z\tilde{\in}V$, and $f_{sn}\left(U\right)\tilde{\cap}V=0_{\mathcal{N}}$. Then, we have $\left(b,d\right)_{\left(y,z\right)}\tilde{\in}U\times V\in pr\left(\phi\times\lambda\right)$ and

$$\begin{array}{cc} Int_{pr\left(\phi\times\lambda\right)}\left(Cl_{pr\left(\phi\times\lambda\right)}\left(U\times V\right)\right)\tilde{\cap}Graph(f_{sn}) & = \\ \left(Int_\phi\left(Cl_\phi\left(U\right)\right)\times Int_\phi\left(Cl_\phi\left(V\right)\right)\right)\tilde{\cap}Graph(f_{sn}) & = \\ \left(U\times V\right)\tilde{\cap}Graph(f_{sn}) & = \\ 0_{\mathcal{M}\times\mathcal{N}}. & \end{array}$$

Therefore, $\left(b,d\right)_{\left(y,z\right)}\tilde{\in}1_{\mathcal{M}\times\mathcal{N}}-\delta Cl_{pr\left(\phi\times\lambda\right)}\left(Graph(f_{sn})\right)$.

Theorem 4.15. If $f_{sn}:\left(Y,\phi,\mathcal{M}\right)\longrightarrow\left(Z,\lambda,\mathcal{N}\right)$ is soft w-θ-c and $\left(Z,\lambda,\mathcal{N}\right)$ is soft Hausdorff, then $Graph(f_{sn})\in\left(\left(pr\left(\phi\times\lambda\right)\right)_\delta\right)^c$.

Proof. We will show that $1_{\mathcal{M}\times\mathcal{N}}-Graph(f_{sn})\tilde{\subseteq}1_{\mathcal{M}\times\mathcal{N}}-\delta Cl_{pr(\phi\times\lambda)}\left(Graph(f_{sn})\right)$. Let $(b,d)_{(y,z)}\tilde{\in}1_{\mathcal{M}\times\mathcal{N}}-Graph(f_{sn})$. Then $f_{sn}\left(b_y\right)\neq d_z$. Since $(Z,\lambda,\mathcal{N})$ is soft Hausdorff, then there exist $G,H\in\lambda$ such that $f_{sn}\left(b_y\right)\tilde{\in}G$, $d_z\tilde{\in}H$, and $G\tilde{\cap}H=0_{\mathcal{N}}$; hence $Cl_\lambda\left(G\right)\tilde{\cap}Int_\lambda\left(Cl_\lambda\left(H\right)\right)=0_{\mathcal{N}}$. Since f_{sn} is soft w-θ-c, then by Theorem 2.12, we find $K\in RO\left(\phi\right)$ such that $b_y\tilde{\in}K$ and $f_{sn}(K)\tilde{\subseteq}Cl_\lambda\left(G\right)$. Thus, we have $b_y\tilde{\in}K\in RO\left(\phi\right)$, $d_z\tilde{\in}Int_\lambda\left(Cl_\lambda\left(H\right)\right)\in RO\left(\lambda\right)$, and $f_{sn}(Int_\phi\left(Cl_\phi\left(K\right)\right))\tilde{\cap}Int_\lambda\left(Cl_\lambda\left(H\right)\right)=0_{\mathcal{N}}$. Therefore, by Lemma 4.14, $Graph(f_{sn})\in\left(\left(pr\left(\phi\times\lambda\right)\right)_\delta\right)^c$.

Theorem 4.16. Let $f_{sn}:(Y,\phi,\mathcal{M})\longrightarrow(Z,\lambda,\mathcal{N})$ be a soft function with $Graph(f_{sn})\in\left(\left(pr\left(\phi\times\lambda\right)\right)_\delta\right)^c$. Then

(a) If T is soft nearly compact relative to $(Z,\lambda,\mathcal{N})$, then $f_{sn}^{-1}\left(T\right)\in\left(\phi_\delta\right)^c$.

(b) If R is soft nearly compact relative to $(Y,\phi,\mathcal{M})$, then $f_{sn}\left(R\right)\in\left(\lambda_\delta\right)^c$.

Proof. (a) We will show that $1_{\mathcal{M}}-f_{sn}^{-1}\left(T\right)\tilde{\subseteq}1_{\mathcal{M}}-\delta Cl_\phi\left(f_{sn}^{-1}\left(T\right)\right)$. Let $a_y\tilde{\in}1_{\mathcal{M}}-f_{sn}^{-1}\left(T\right)$. Then for each $d_z\tilde{\in}T$, $(a,d)_{(y,z)}\tilde{\in}1_{\mathcal{M}\times\mathcal{N}}-Graph(f_{sn})$ and by Lemma 4.14, there exist $U\left(d_z\right)\in RO\left(\phi\right)$ and $V\left(d_z\right)\in RO\left(\lambda\right)$ such that $a_y\tilde{\in}U\left(d_z\right)$, $d_z\tilde{\in}V\left(d_z\right)$, and $f_{sn}\left(U\left(d_z\right)\right)\tilde{\cap}V\left(d_z\right)=0_{\mathcal{N}}$. Since T is soft nearly compact relative to $(Z,\lambda,\mathcal{N})$ and $T\tilde{\subseteq}\tilde{\cup}_{d_z\tilde{\in}T}V\left(d_z\right)$, then we find a finite subset $P\subseteq SP\left(Z,\mathcal{N}\right)$ such that $d_z\tilde{\in}f_{sn}\left(T\right)$ for every $d_z\in P$ and $f_{sn}\left(T\right)\tilde{\subseteq}\tilde{\cup}_{d_z\in P}V\left(d_z\right)$. Let $K=\tilde{\cap}_{d_z\in P}U\left(d_z\right)$. Then, we have, $a_y\tilde{\in}K\in RO\left(\phi\right)$ and $K\tilde{\cap}f_{sn}^{-1}\left(T\right)=0_{\mathcal{M}}$. It follows that $a_y\tilde{\in}1_{\mathcal{M}}-\delta Cl_\phi\left(f_{sn}^{-1}\left(T\right)\right)$.

(b) We will show that $1_{\mathcal{N}}-f_{sn}\left(R\right)\tilde{\subseteq}1_{\mathcal{N}}-\delta Cl_\phi\left(f_{sn}\left(R\right)\right)$. Let $b_z\tilde{\in}1_{\mathcal{N}}-f_{sn}\left(R\right)$. Then for each $a_y\tilde{\in}R$, $(a,b)_{(y,z)}\tilde{\in}1_{\mathcal{M}\times\mathcal{N}}-Graph(f_{sn})$ and by Lemma 4.14, there exist $U\left(a_y\right)\in RO\left(\phi\right)$ and $V\left(a_y\right)\in RO\left(\lambda\right)$ such that $a_y\tilde{\in}U\left(a_y\right)$, $b_z\tilde{\in}V\left(a_y\right)$, and $f_{sn}\left(U\left(a_y\right)\right)\tilde{\cap}V\left(a_y\right)=0_{\mathcal{N}}$. Since R is soft nearly compact relative to $(Y,\phi,\mathcal{M})$ and $R\tilde{\subseteq}\tilde{\cup}_{a_y\tilde{\in}R}U\left(a_y\right)$, then we find a finite subset $P\subseteq SP\left(Y,\mathcal{M}\right)$ such that $a_y\tilde{\in}R$ for every $a_y\in P$ and $R\tilde{\subseteq}\tilde{\cup}_{a_y\in P}U\left(a_y\right)$. Let $H=\tilde{\cap}_{a_y\in P}V\left(a_y\right)$. Then, we have, $b_z\tilde{\in}H\in RO\left(\lambda\right)$ and $H\tilde{\cap}f_{sn}\left(R\right)=0_{\mathcal{N}}$. It follows that $b_z\tilde{\in}1_{\mathcal{N}}-\delta Cl_\lambda\left(f_{sn}\left(R\right)\right)$.

Theorem 4.17. Let $f_{sn}:(Y,\phi,\mathcal{M})\longrightarrow(Z,\lambda,\mathcal{N})$ be a soft function with $(Z,\lambda,\mathcal{N})$ is soft nearly compact and $Graph(f_{sn})\in\left(\left(pr\left(\phi\times\lambda\right)\right)_\delta\right)^c$. Then f_{sn} is soft δ-c.

Proof. Let $T\in RC\left(\lambda\right)$. Since $(Z,\lambda,\mathcal{N})$ is soft nearly compact, then by Lemma 4.12, T is soft nearly compact relative to $(Z,\lambda,\mathcal{N})$. Then by Theorem 4.16, $f_{sn}^{-1}\left(T\right)\in\left(\phi_\delta\right)^c$. Hence, f_{sn} is soft δ-c.

Corollary 4.18. Let $(Z,\lambda,\mathcal{N})$ be soft nearly compact Hausdorff. For a soft function $f_{sn}:(Y,\phi,\mathcal{M})\longrightarrow(Z,\lambda,\mathcal{N})$, the following are equivalent:

(a) f_{sn} is soft w-θ-c.

(b) $Graph(f_{sn})\in\left(\left(pr\left(\phi\times\lambda\right)\right)_\delta\right)^c$.

(c) f_{sn} is soft δ-c.

Proof. (a) $\longrightarrow$ (b): Follows from Theorem 4.15.

(b) $\longrightarrow$ (c): Follows from Theorem 4.17.

(b) $\longrightarrow$ (c): Follows from Theorem 2.6.

5 Conclusion

This paper has successfully introduced and explored the concept of soft weak θ-continuity in soft topological spaces. This paper illuminates the relationships between this notion and other forms of continuity, such as θ-continuity, soft weak continuity, and soft δ-continuity. A deeper understanding of soft weak θ-continuity of communication came through giving characterizations of it and studying its connections to classical topology. Furthermore, this research provided preservation theorems for soft hyperconnectedness and near compactness, as well as soft restriction, product, and graph theorems for soft weak θ-continuity.

These results not only advance soft topology but also provide a platform for future studies in this field. This research highlights the promise of the soft-weak θ-continuity as a tool for exploring the features of soft topological spaces. We encourage further investigation of this concept and its applications.

Future research might look into the following topics: (1) defining soft weakly δ-continuous functions; (2) defining soft strongly δ-continuous functions; and (3) finding a use for these new concepts in a decision-making problem.

6 Author's contributions

This article was written in collaboration by all of the contributors. The final manuscript was read and approved by all writers.

7 Conflicts of interest

There are no competing interests declared by the authors.

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