



The Runge-Kutta Numerical Method of Rank Seven for the Solutions of Some Refined Neutrosophic Differential Problems

Belal Batiha^{1,*}

¹Department of Mathematics, Faculty of Science, Jadara University, Irbid, Jordan

Email: B.bateha@jadara.edu.jo

Abstract

In this paper, we present a numerical approach to the seventh rank refined neutrosophic Runge-Kutta numerical method, where we provide the theoretical basis of this formula to be applicable on refined neutrosophic differential equations. In addition, we provide numerical tables to compare the validity of this new method with other methods, as well as a clear computation of absolute errors in terms of refined neutrosophic numbers.

Keywords: Refined neutrosophic number; Refined neutrosophic Runge-Kutta of rank seven; Numerical error; absolute error

1. Introduction

Neutrosophic logic presented by Smarandache [6] is considered as a new generalization of fuzzy logic that takes into account the idea of indeterminacy and uncertainty in measurements resulting from natural phenomena.

It has been used to study many traditional mathematical concepts, such as algebraic structures, analysis, and even in computer science [10-13].

The Runge-Kutta method is one of the reference methods in numerical analysis that has been dealt with and developed by many researchers around the world. Different ranks of this method have been used in order to find numerical solutions and approximate errors for various problems in applied mathematics [4-9].

The applications of neutrosophic methods in numerical analysis have been studied by many authors see [14-18]. Where, we can see neutrosophic modelled problems with their approximate solutions and absolute errors were presented numerically [19-25].

The main goal of numerical analysis is to study the numerical approximations of solutions for many different problems (algebraic or differential) with a clear computation of errors that emerge from the approximation process [1-3].

This has motivated us to we present a numerical approach to the seventh rank refined neutrosophic Runge-Kutta numerical method, where we provide the theoretical basis of this formula to be applicable on refined neutrosophic differential equations. In addition, we provide numerical tables to compare the validity of this new method with other methods, as well as a clear computation of absolute errors in terms of refined neutrosophic numbers.

2. Main Discussion

The Refined neutrosophic Runge-Kutta method of rank 7:

$$\begin{aligned}
 t_{n+1} + j_{n+1}I_1 + e_{n+1}I_2 \\
 = t_n + j_nI_1 + e_nI_2 + \frac{m + nI_1 + lI_2}{192} (32(s_1 + s_1I_1 + s_1I_2) + 80(s_2 + s_2I_1 + s_2I_2) + 48(s_3 \\
 + s_3I_1 + s_3I_2) + 24(s_4 + s_4I_1 + s_4I_2) + 4(s_5 + s_5I_1 + s_5I_2) + 2(s_6 + s_6I_1 + s_6I_2) + (s_7 \\
 + s_7I_1 + s_7I_2) + (s_8 + s_8I_1 + s_8I_2))
 \end{aligned}$$

Where:

$$s_1 + s_1 I_1 + s_1 I_2 = f(h_n + g_n I_1 + r_n I_2, t_n + j_n I_1 + e_n I_2)$$

$$s_2 + s_2 I_1 + s_2 I_2 = f(h_n + g_n I_1 + r_n I_2 + \frac{m+nI_1+I_2}{2}, t_n + j_n I_1 + e_n I_2 + \frac{1}{2}(m + nI_1 + I_2)(s_1 + s_1 I_1 + s_1 I_2))$$

$$s_3 + s_3 I_1 + s_3 I_2 = f(h_n + g_n I_1 + r_n I_2 + \frac{m+nI_1+I_2}{2}, t_n + j_n I_1 + e_n I_2 + \frac{1}{2}(m + nI_1 + I_2)(s_2 + s_2 I_1 + s_2 I_2))$$

$$s_4 + s_4 I_1 + s_4 I_2 = f(h_n + g_n I_1 + r_n I_2 + m + nI_1 + I_2, t_n + j_n I_1 + e_n I_2 + (m + nI_1 + I_2)(s_3 + s_3 I_1 + s_3 I_2))$$

$$s_5 + s_5 I_1 + s_5 I_2 = f(h_n + g_n I_1 + r_n I_2 + m + nI_1 + I_2, t_n + j_n I_1 + e_n I_2 + (m + nI_1 + I_2)(s_4 + s_4 I_1 + s_4 I_2))$$

$$s_6 + s_6 I_1 + s_6 I_2 = f(h_n + g_n I_1 + r_n I_2 + m + nI_1 + I_2, t_n + j_n I_1 + e_n I_2 + (m + nI_1 + I_2)(s_5 + s_5 I_1 + s_5 I_2))$$

$$s_7 + s_7 I_1 + s_7 I_2 = f(h_n + g_n I_1 + r_n I_2 + m + nI_1 + I_2, t_n + j_n I_1 + e_n I_2 + (m + nI_1 + I_2)(s_6 + s_6 I_1 + s_6 I_2))$$

$$s_8 + s_8 I_1 + s_8 I_2 = f(h_n + g_n I_1 + r_n I_2 + m + nI_1 + I_2, t_n + j_n I_1 + e_n I_2 + (m + nI_1 + I_2)(s_7 + s_7 I_1 + s_7 I_2))$$

The stability analysis for refined variables:

$$t_{n+1} + j_{n+1} I_1 + e_{n+1} I_2 =$$

$$t_n + j_n I_1 + e_n I_2 + (m + nI_1 + I_2)\phi(h_n + g_n I_1 + r_n I_2, t_n + j_n I_1 + e_n I_2, +nI_1 + I_2) \quad (1)$$

The general is :

$$(t_n + j_n I_1 + e_n I_2)(h_n + g_n I_1 + r_n I_2) - (t_{n+1} + j_{n+1} I_1 + e_{n+1} I_2) = O((m + nI_1 + I_2)^{p+1}) \quad (2)$$

we get the following:

$$(t_n + j_n I_1 + e_n I_2)(h_n + g_n I_1 + r_n I_2) = E(\lambda(m + nI_1 + I_2))(t_n + j_n I_1 + e_n I_2) \quad (3)$$

we get :

$$(t_n + j_n I_1 + e_n I_2)' = \lambda(t_n + j_n I_1 + e_n I_2), (t_n + j_n I_1 + e_n I_2)(h_0 + g_0 I_1 + r_0 I_2) = (t + j I_1 + e I_2)_0 \quad (4)$$

Thus:

$$E(\overline{(m + nI_1 + I_2)}) = 1 + \overline{(m + nI_1 + I_2)} + \frac{1}{2!} \overline{(m + nI_1 + I_2)}^2 + \dots + \frac{1}{p!} \overline{(m + nI_1 + I_2)}^p + O(\overline{(m + nI_1 + I_2)}^{p+1}) \quad (5)$$

RK		stability period
7	$1 + \overline{(m + nI_1 + I_2)} + \frac{\overline{(m + nI_1 + I_2)}^2}{2!} + \frac{\overline{(m + nI_1 + I_2)}^3}{3!} + \frac{\overline{(m + nI_1 + I_2)}^4}{4!} + \frac{\overline{(m + nI_1 + I_2)}^5}{5!} + \frac{\overline{(m + nI_1 + I_2)}^6}{6!} + \frac{\overline{(m + nI_1 + I_2)}^7}{7!}$	$(-2.12265 + 1.1937I_1 + 0.0012I_2)$

Numerical applications:

Example.

Solve the following system of neutrosophic differential equations:

$$(t + j I_1 + e I_2)'_1 = (t + j I_1 + e I_2)_2, (t + j I_1 + e I_2)_1(0) = 1 + I_1 + I_2$$

$$(t + j I_1 + e I_2)'_2 = -1001 - 1001 I_1 - 1001 I_2, (t + j I_1 + e I_2)_2 - 1000(I_1 + I_2) (t + j I_1 + e I_2)_1, (t + j I_1 + e I_2)_2(I) = -1 - I_1 - I_2$$

We will take the value of the step length $h = 0.001(1 + I_1 + I_2)$

Example:

Solve the following system of differential equations:

$$(t + jI_1 + eI_2)'_1 = (600 + 300I)(t + jI_1 + eI_2)_1^2 ((t + jI_1 + eI_2)_2 - (t + jI_1 + eI_2)_1^3), (t + jI_1 + eI_2)_1(I) = 0.1 + I$$

$$(t + jI_1 + eI_2)'_2 = (-200 - 100(I_1 + I_2))((t + jI_1 + eI_2)_2 - (t + jI_1 + eI_2)_1^3) + 2(1 - (t + jI_1 + eI_2)_2), (t + jI_1 + eI_2)_2(I_1 + I_2) = -0.1 - (I_1 + I_2)$$

Table 1: the results of solving the problem in the first example

Refined method of Seventh rank error values		Truth values		refined method-Kutta of the Seventh values	
First variable error values	Second variable error values	First variable Values	Second variable Values	First variable Values	Second variable Values
$(I_1 + I_2)$	$(I_1 + I_2)$	$1 + (I_1 + I_2)$	$-1 - (I_1 + I_2)$	$1 + (I_1 + I_2)$	$-1 - (I_1 + I_2)$
$2.16675e-14 + I_1 + I_2$	$3.2213e-14 + I_1 + I_2$	$0.0638 + 0.928 + I_1 + I_2$	$-0.0638 - 0.928(I_1 + I_2)$	$0.0638 + 0.0638((I_1 + I_2))$	$-0.0638 - 0.0638(I_1 + I_2)$
$5.23639e-14 + I_1 + I_2$	$6.31e-14 + I_1 + I_2$	$0.93311 + I_1 + I_2$	$-(0.93311 + I_1 + I_2)$	$0.9447 + I_1 + I_2$	$-(0.90637 + I_1 + I_2)$
$9.114e-14 + I_1 + I_2$	$9.3452e-14 + I_1 + I_2$	$0.90632 + I_1 + I_2$	$-(0.9332 + I_1 + I_2)$	$0.9332 + I_1 + I_2$	$-(0.9332 + I_1 + I_2)$
$1.20063e-11 + I_1 + I_2$	$1.22173e-11 + I_1 + I_2$	$0.9673913 + I_1 + I_2$	$-(0.9739143 + I_1 + I_2)$	$0.9739143 + I_1 + I_2$	$-(0.9739143 + I_1 + I_2)$
$1.10342e-11 + I_1 + I_2$	$1.10342e-11 + I_1 + I_2$	$0.9115 + I_1 + I_2$	$-(0.9115 + I_1 + I_2)$	$0.973915 + I_1 + I_2$	$-(0.0739115 + I_1 + I_2)$
$1.309049113e-11 + I_1 + I_2$	$1.27391913e-11 + I_1 + I_2$	$0.60904807 + I_1 + I_2$	$-(0.6570904107 + I_1 + I_2)$	$0.65739107 + I_1 + I_2$	$-(0.73090407 + I_1 + I_2)$
$2.11295e-11 + I_1 + I_2$	$2.11295e-11 + I_1 + I_2$	$0.9443861 + I_1 + I_2$	$0.9443861 + I_1 + I_2$	$0.9443861 + I_1 + I_2$	$0.9443861 + I_1 + I_2$
$2.3090487e-11 + I_1 + I_2$	$2.090487e-11 + I_1 + I_2$	$0.909041293 + I_1 + I_2$	$-(0.909041293 + I_1 + I_2)$	$0.909041293 + I_1 + I_2$	$-(0.909041293 + I_1 + I_2)$
$2.20904876e-11 + I_1 + I_2$	$2.09048876e-11 + I_1 + I_2$	$0.9088 + I_1 + I_2$	$-(0.9088 + I_1 + I_2)$	$0.9088 + I_1 + I_2$	$-(0.9088 + I_1 + I_2)$
$3.0904145e-11 + I_1 + I_2$	$3.090445e-11 + I_1 + I_2$	$0.709042 + I_1 + I_2$	$-(0.77602 + I_1 + I_2)$	$0.7709042 + I_1 + I_2$	$-(0.7709042 + I_1 + I_2)$

Table 2: results of solving the problem in the second example

Refined method of Seventh rank error values		Truth values		refined method-Kutta of the Seventh values	
First variable error values	Second variable error values	First variable Values	Second variable Values	First variable Values	Second variable Values
$I_1 + I_2$	$I_1 + I_2$	$0.1 + I_1 + I_2$	$-0.1 - I_1 + I_2$	$0.1 + I_1 + I_2$	$-0.1 - I_1 + I_2$
$5.46645e-9 + I_1 + I_2$	$1.212e-7 + I_1 + I_2$	$0.092237 + I_1 + I_2$	$-0.073498 - I_1 + I_2$	$0.092234 + I_1 + I_2$	$-0.073498 - I_1 + I_2$
$1.221376e-8 + I_1 + I_2$	$4.2437e-7 + I_1 + I_2$	$0.0911369 + I_1 + I_2$	$-0.0379445 - I_1 + I_2$	$0.0911366 + I_1 + I_2$	$-0.0379444 - I_1 + I_2$
$1.0987e-8 + I_1 + I_2$	$5.7787e-7 + I_1 + I_2$	$0.094696 + I_1 + I_2$	$-0.046064 - I_1 + I_2$	$0.094693 + I_1 + I_2$	$-0.049621 - I_1 + I_2$

2.311209e-8+ I_1+I_2	6.52014e-7+ I_1+I_2	0.092064+ I_1+I_2	-0.022398- I_1+I_2	0.092060+ I_1+I_2	-0.0333913- I_1+I_2
2.3329e-9+ I_1+I_2	6.66783e-8+ I_1+I_2	0.095833+ I_1+I_2	-0.231998+ I_1+I_2	0.095830+ I_1+I_2	-0.011690133- I_1+I_2
2.1092e-9+ I_1+I_2	6.11038e-8+ I_1+I_2	0.094155+ I_1+I_2	-0.0100078- I_1+I_2	0.094151+ I_1+I_2	-0.0017110- I_1+I_2
2.51375e-9+ I_1+I_2	6.1167e-05+ I_1+I_2	0.097964+ I_1+I_2	-0.01437- I_1+I_2	0.097961+ I_1+I_2	-0.01433- I_1+I_2
2.256248e-9+ I_1+I_2	6.2334e-05+ I_1+I_2	0.03326+ I_1+I_2	-0.01125- I_1+I_2	0.03323+ I_1+I_2	-0.01121+ I_1+I_2
2.3143e-9+ I_1+I_2	5.673108e-05+ I_1+I_2	0.08873+ I_1+I_2	-0.008514- I_1+I_2	0.08870+ I_1+I_2	-0.008510- I_1+I_2
2.100123e-9+ I_1+I_2	5.4431e-05+ I_1+I_2	0.0897664+ I_1+I_2	-0.0023638- I_1+I_2	0.0897661+ I_1+I_2	-0.0023634- I_1+I_2

Refined Implicit RK Method of the seventh rank:

$$t_{n+1} + j_{n+1}I_1 + e_{n+1}I_2 = t_n + j_nI_1 + e_nI_2 - \frac{m + nl_1 + ll_2}{192} (32(s_1 + s_1I_1 + s_1I_2) + 80(s_2 + s_2I_1 + s_2I_2) + 48(s_3 + s_3I_1 + s_3I_2) + 24(s_4 + s_4I_1 + s_4I_2) + 4(s_5 + s_5I_1 + s_5I_2) + 2(s_6 + s_6I_1 + s_6I_2) + (s_7 + s_7I_1 + s_7I_2) + (s_8 + s_8I_1 + s_8I_2))$$

Where:

$$s_1 + s_1I_1 + s_1I_2 = f(h_n + g_nI_1 + r_nI_2, t_n + j_nI_1 + e_nI_2)$$

$$s_2 + s_2I_1 + s_2I_2 = f(h_n + g_nI_1 + r_nI_2 - \frac{m+nl_1+ll_2}{2}, t_n + j_nI_1 + e_nI_2 - \frac{1}{2}(m + nl_1 + ll_2)(s_1 + s_1I_1 + s_1I_2))$$

$$s_3 + s_3I_1 + s_3I_2 = f(h_n + g_nI_1 + r_nI_2 - \frac{m+nl_1+ll_2}{2}, t_n + j_nI_1 + e_nI_2 - \frac{1}{2}(m + nl_1 + ll_2)(s_2 + s_2I_1 + s_2I_2))$$

$$s_4 + s_4I_1 + s_4I_2 = f(h_n + g_nI_1 + r_nI_2 - (m + nl_1 + ll_2), t_n + j_nI_1 + e_nI_2 - (m + nl_1 + ll_2)(s_3 + s_3I_1 + s_3I_2))$$

$$s_5 + s_5I_1 + s_5I_2 = f(h_n + g_nI_1 + r_nI_2 - (m + nl_1 + ll_2), t_n + j_nI_1 + e_nI_2 - (m + nl_1 + ll_2)(s_4 + s_4I_1 + s_4I_2))$$

$$s_6 + s_6I_1 + s_6I_2 = f(h_n + g_nI_1 + r_nI_2 - (m + nl_1 + ll_2), t_n + j_nI_1 + e_nI_2 - (m + nl_1 + ll_2)(s_5 + s_5I_1 + s_5I_2))$$

$$s_7 + s_7I_1 + s_7I_2 = f(h_n + g_nI_1 + r_nI_2 - (m + nl_1 + ll_2), t_n + j_nI_1 + e_nI_2 - (m + nl_1 + ll_2)(s_6 + s_6I_1 + s_6I_2))$$

$$s_8 + s_8I_1 + s_8I_2 = f(h_n + g_nI_1 + r_nI_2 - (m + nl_1 + ll_2), t_n + j_nI_1 + e_nI_2 - (m + nl_1 + ll_2)(s_7 + s_7I_1 + s_7I_2))$$

Stability of Seventh Order refined neutrosophic RK method:

Consider the general formula:

$$t_{n+1} + j_{n+1}I_1 + e_{n+1}I_2 = y_n + (m + nl_1 + ll_2)\phi(h_{n+1} + g_{n+1}I_1 + r_{n+1}I_2, t_{n+1} + j_{n+1}I_1 + e_{n+1}I_2, m + nl_1 + ll_2) \quad (6)$$

The refined neutrosophic IRK formula (6) of phase R is said to be of rank P if:

$$(t + jI_1 + eI_2)(h_{n+1} + g_{n+1}I_1 + r_{n+1}I_2) - (t_{n+1} + j_{n+1}I_1 + e_{n+1}I_2) = O((m + nl_1 + ll_2)^{p+1})$$

$$\text{Hence, } (t + jI_1 + eI_2)_{n+1} = E(\lambda(m + nl_1 + ll_2))(t + jI_1 + eI_2)_n \quad (7)$$

So that:

$$(t + jI_1 + eI_2)' = \lambda(t + jI_1 + eI_2), (t + jI_1 + eI_2)((h_n + g_nI_1 + r_nI_2)_0) = t + jI_1 + eI_2)_0 \quad (8)$$

We get:

$$E(\overline{m + nI_1 + lI_2}) = 1 + I + \overline{m + nI_1 + lI_2} + \frac{1}{2!}(\overline{m + nI_1 + lI_2})^2 + \dots + \frac{1}{(p+1)!}(\overline{m + nI_1 + lI_2})^{p+1} + O((\overline{m + nI_1 + lI_2})^{p+1}) \quad (9)$$

Where $\overline{m + nI_1 + lI_2} = \lambda(\overline{m + nI_1 + lI_2})$ are polynomials of degree R in $\overline{m + nI_1 + lI_2}$

RK	r_2	stability period
7	$1 + \overline{m + nI_1 + lI_2} + \frac{\overline{m + nI_1 + lI_2}^2}{2!} + \frac{\overline{m + nI_1 + lI_2}^3}{3!} + \frac{\overline{m + nI_1 + lI_2}^4}{4!} + \frac{\overline{m + nI_1 + lI_2}^5}{5!} + \frac{\overline{m + nI_1 + lI_2}^6}{6!} + \frac{\overline{m + nI_1 + lI_2}^7}{7!} + \frac{\overline{m + nI_1 + lI_2}^8}{8!}$	(-4.112809+ I_1 , 0.2213 I_1 + 0.55715 I_2)

Example:

Solve the system of equations in the first example using the neutrosophic IRK method of the seventh rank.

We will take the step length value $h=0.002+0.001I_1 + 0.001I_2$.

Example:

Solve the system of equations in the second example using the neutrosophic IRK method of the seventh rank.

We will take the step length value $h=0.001+ 0.001I_1 + 0.001I_2$.

Table 3: results of neutrosophic IRK of seventh order for the first example

Refined method of Seventh rank error values		Truth values		refined method-Kutta of the Seventh values	
First variable error values	Second variable error values	First variable Values	First variable error values	Second variable error values	First variable Values
+0.00151 I_1 + 0.001661 I_2	+0.00151 I_1 + 0.001661 I_2	+ I_1 + I_2	- I_1 + I_2	1+0.00152 I_1 + 0.001663 I_2	-1-0.00152 I_1 + 0.001663 I_2
3.108872e-15+ I_1 + I_2	3.108877e-15+ I_1 + I_2	0.90887+ I_1 + I_2	-(0.90887+ I_1 + I_2)	0.90887+ I_1 + I_2	-(0.908870887+ I_1 + I_2)
6.7233486e-15+ I_1 + I_2	6.7233482e-15+ I_1 + I_2	0.923345+ I_1 + I_2	-(0.923345+ I_1 + I_2)	0.923345+ I_1 + I_2	-(0.923345+ I_1 + I_2)
1.02334551e-15+ I_1 + I_2	1.02334531e-15+ I_1 + I_2	0.9348344+ I_1 + I_2	-(0.923344+ I_1 + I_2)	0.923344+ I_1 + I_2	-(0.923344+ I_1 + I_2)
1.32219e-14+ I_1 + I_2	1.32216e-14+ I_1 + I_2	0.93482+ I_1 + I_2	-(0.93482+ I_1 + I_2)	0.93482+ I_1 + I_2	-(0.93482+ I_1 + I_2)
1.649304e-14+ I_1 + I_2	1.649301e-14+ I_1 + I_2	0.991+ I_1 + I_2	-(0.991+ I_1 + I_2)	0.991+ I_1 + I_2	-(0.991+ I_1 + I_2)
2.0808879e-14+ I_1 + I_2	2.0808875e-14+ I_1 + I_2	0.9808871+ I_1 + I_2	-(0.9808871+ I_1 + I_2)	0.9808871+ I_1 + I_2	-(0.98088731+ I_1 + I_2)

$2.32334e-14+I$	$2.312334e-14+I_1+I_2$	$0.98571534+I_1+I_2$	$-(0.982334+I_1+I_2)$	$0.92334+I_1+I_2$	$-(0.92334+I_1+I_2)$
$2.6497e-14+I_1+I_2$	$2.6492e-14+I_1+I_2$	$0.9833+I_1+I_2$	$-(0.93483+I_1+I_2)$	$0.93483+I_1+I_2$	$-(0.93483+I_1+I_2)$
$3.0887558e-14+I_1+I_2$	$2.0571572e-14+I_1+I_2$	$0.088721+I_1+I_2$	$-(0.0834821+I_1+I_2)$	$0.0834821+I_1+I_2$	$-(0.0834821+I_1+I_2)$
$3.32185e-14+I_1+I_2$	$3.35715180e-14+I_1+I_2$	$0.01611+I_1+I_2$	$-(0.01611+I_1+I_2)$	$0.01621+I_1+I_2$	$-(0.100161+I_1+I_2)$

Table 4: results of neutrosophic IRK of seventh order for the second example

Refined method of Seventh rank error values		Truth values		refined method-Kutta of the Seventh values	
First variable error values	Second variable error values	First variable Values	First variable error values	Second variable error values	First variable Values
$+0.00151I_1+0.001661I_2$	$+0.00151I_1+0.001661I_2$	$+I_1+I_2$	$-I_1+I_2$	$1+0.00152I_1+0.001663I_2$	$-1-0.00152I_1+0.001663I_2$
$2.2231e-07+I_1+I_2$	$8.95715476e-06+I_1+I_2$	$0.0562421+I_1+I_2$	$-0.0797-I_1+I_2$	$0.09211I_1+I_2$	$-0.079715-I_1+I_2$
$9.15624634e-07+I_1+I_2$	$2.556245e-05+I_1+I_2$	$0.09864+I_1+I_2$	$-0.06415-I_1+I_2$	$0.09851+I_1+I_2$	$-0.063142-I_1+I_2$
$1.3562465e-06+I_1+I_2$	$3.585624395e-05+I_1+I_2$	$0.0986675+I_1+I_2$	$-0.049564-I_1+I_2$	$0.098101+I_1+I_2$	$-0.049602-I_1+I_2$
$1.773654e-06+I_1+I_2$	$4.95715e-05+I_1+I_2$	$0.092316+I_1+I_2$	$-0.033798-I_1+I_2$	$0.09793+I_1+I_2$	$-0.03284-I_1+I_2$
$1.4895e-06+I_1+I_2$	$4.10496e-05+I_1+I_2$	$0.0989886+I_1+I_2$	$-0.016901369-I_1+I_2$	0.016901234	$-0.029302-I_1+I_2$
$1.5856375e-07+I_1+I_2$	$3.68485e-06+I_1+I_2$	$0.09847364+I_1+I_2$	$-0.0211988-I_1+I_2$	$0.091657150141+I_1+I_2$	$-0.0221784-I_1+I_2$
$1.11690124e-07+I_1+I_2$	$3.156244e-06+I_1+I_2$	$0.09562411+I_1+I_2$	$-0.017769-I_1+I_2$	$0.0169016+I_1+I_2$	$-0.01422313-I_1+I_2$
$1.01690185e-07+I_1+I_2$	$2.856246e-06+I_1+I_2$	$0.097332+I_1+I_2$	$-0.010015-I_1+I_2$	$0.01690165+I_1+I_2$	$-0.01100173I_1+I_2$
$1.169013e-07+I_1+I_2$	$2.10038e-06+I_1+I_2$	$0.0969984+I_1+I_2$	$-0.0056315I_1+I_2$	$0.0965543+I_1+I_2$	$-0.005567-I_1+I_2$
$1.0169010285e-07+I_1+I_2$	$1.93562494e-06+I_1+I_2$	$0.05715453+I_1+I_2$	$-0.0057152486-I_1+I_2$	$0.057159014+I_1+I_2$	$-0.005624101-I_1+I_2$

3. Conclusion

In This paper, we presented a numerical approach to the seventh rank refined neutrosophic Runge-Kutta numerical method, where we provided the theoretical basis of this formula to be applicable on refined neutrosophic differential equations. In addition, we provided numerical tables to compare the validity of this new method with other methods, as well as a clear computation of absolute errors in terms of refined neutrosophic numbers.

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