A New Algebraic Approach of Neutrosophic Lie Algebra by AH Isometry

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Abstract

This paper introduces a novel approach to the concept of neutrosophic Lie algebra by leveraging the AH isometry framework. We establish foundational properties of neutrosophic Lie algebra, demonstrating that each neutrosophic algebra inherently fulfills the criteria of a Lie algebra. Moreover, we introduce distinct neutrosophic Lie algebraic structures, providing illustrative examples to support these constructs. By integrating neutrosophic logic, our approach effectively addresses indeterminacy, ambiguity, and imprecision, enhancing the classical algebraic structures with new dimensions of flexibility. The potential applications of neutrosophic Lie algebra are vast, particularly in fields requiring nuanced treatments of uncertainty.

Key words: Neutrosophic rings; Neutrosophic algebra, AH Isometry; Neutrosophic Lie Algebra; Neutrosophic Lie Subalgebra

1. Introduction

Lie algebra is a branch of abstract algebra named after the Norwegian mathematician Sophus Lie, who developed these structures in the 19th century to study continuous symmetries in mathematical systems. Lie algebras provide a powerful framework for analyzing the local properties of Lie groups—groups that describe continuous transformations. Defined over vector spaces, a Lie algebra consists of a set of elements combined with a binary operation called the Lie bracket. This operation is not commutative but adheres to two primary properties: antisymmetric and the Jacobi identity, making Lie algebras distinct from other algebraic structures.

The defining properties of a Lie algebra, particularly the Lie bracket, allow for the study of symmetries and structures in various fields. The antisymmetric of the Lie bracket means that for any elements x, y, and z in a Lie algebra $\mathcal{g}$, the relationship $[x, y] = -[y, x]$, holds. Additionally, the Jacobi identity, which states that $\big[x, [y, z]\big] + \big[y, [z, x]\big] + \big[z, [x, y]\big] = 0$ for any elements x, y, and z in $\mathcal{g}$, imposes a specific structural consistency on the algebra. Together, these properties ensure that Lie algebras can model complex, non-commutative relationships between elements, making them ideal for representing systems with rotational or other continuous symmetries.

Lie algebras are fundamental in many areas of pure and applied mathematics, including geometry, number theory, and mathematical physics. In theoretical physics, for example, they are essential for understanding the symmetries in quantum mechanics, particle physics, and field theory. Lie algebras also play a key role in differential equations and representation theory, providing a means to study linear transformations and invariant properties within vector spaces.

The versatility and broad applicability of Lie algebras have led to numerous extensions and generalizations, including infinite-dimensional Lie algebras and applications in quantum groups, string theory, and cryptography. These extensions build upon the classical structure to accommodate more complex systems, providing powerful tools for researchers across a variety of mathematical disciplines [1-5].

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The vagueness or uncertainty representation of imperfect knowledge becomes a crucial issue in the areas of computer science and artificial intelligence. To deal with the uncertainty, the fuzzy set proposed by Zadeh [34] allows the uncertainty of a set with a membership degree between 0 and 1. Then, [35] introduced an intuitionistic Fuzzy set (IFS) as a generalization of the Fuzzy set.

The IFS represents the uncertainty with respect to both membership and non-membership. However, it can only handle incomplete information but not the indeterminate and inconsistent information, which exists commonly in real situations. Therefore, Smarandache [36] proposed a neutrosophic set. It can independently express truth-membership degree, indeterminacy-membership degree, and false membership degree and deal with incomplete, indeterminate, and inconsistent information.

Neutrosophic field of reals is a generalization of classical field reals adding an indeterminacy element(I), where $I^2 = I,\ I^n = I\ ;\ n \in N$ and I^{-1} is undefined [6, 7]. Neutrosophic has been applied in many areas, such as decision-making [8, 9], artificial intelligence and machine learning [10,11], intelligent disease diagnosis[12, 13], communication services [14], artificial pattern recognition [15], e-learning [16], physics [17, 18], and more.

In [19], the single-valued neutrosophic probability was introduced by F. Smarandache as a function $P_N: X \to [0,1]^3$ where X is a space that contains indeterminant components, i.e., neutrosophic sample space and defined the neutrosophic probability function of event A by $P_N(A) = \big(T(A), I(neutA), F(antiA)\big) = (T, I, F)$ where $0 \leq T, I, F \leq 1$ and $0 \leq T + I + F \leq 3$.

Recently, Abobala et.al, have presented the concept of two-dimensional AH-isometry to study the correspondence between neutrosophic plane $\boldsymbol{R(I)} \times \boldsymbol{R(I)}$ and the classical module $\boldsymbol{R^2} \times \boldsymbol{R^2}$. Also, the one-dimensional AH-isometry between $\boldsymbol{R(I)}$ and $\boldsymbol{R} \times \boldsymbol{R}$. This isometry was useful in defining inner products and norms ordering and neutrosophic geometrical shapes [23,24].

In this research, by incorporating an indeterminacy component, we propose a generalization of classical Lie algebra to handle imprecision, uncertainty, ambiguity, vagueness, and enigmatic aspects.

This leads to the introduction of Neutrosophic Lie Algebra. We explore new Lie neutrosophic algebraic structures, providing examples to illustrate them.

2. Backgrounds

This section provides some definitions of neutrosophic logic and neutrosophic probability. Preliminaries

2.1 Definition [20] Let $X \neq \Phi$ be. A neutrosophic set $NS(A)$ it can be defined by its elements with triples of the form $\{x, (T_{NS(A)}(x), I_{NS(A)}(x), F_{NS(A)}(x)): x \in X\}$, where $T_{NS(A)}$, $I_{NS(A)}(x)\ and\ F_{NS(A)}A(x)$ represent the degree of membership, degree of indeterminacy, and degree of non-membership, respectively, of each element $x\ \in\ X$ to the set $NS(A)$.

2.2 Definition [21] Let F be a field, the neutrosophic field $NS(F)$ is a field that is generated by $\langle F \cup I \rangle$ and we usually denote it by $NS(F) = \langle F \cup I \rangle$.

2.3 Definition [22] The literal neutrosophic number has the form $X(I) = x + \alpha I$ where x, α are real or complex numbers, and I is an algebraic element that satisfies $0 \cdot I = 0$ and $I^2 = I$.

2.4 Definition [19] The single-valued neutrosophic probability of event A is $P_N(A) = \big(T(A), I(neutA), F(antiA)\big)$where T, I, F take its values in [0,1].

2.7 Definition [23]: Let $f: R(I) \to R(I); f = f(X)$ and $X = x + yI \in R(I)$ the f is called a neutrosophic real function with one neutrosophic variable.

a neutrosophic real function $f(X)$ written as follows:

$$f(X) = f(x + yI) = f(x) + I[f(x + y) - f(x)]$$

2.8 Definition[2]**:** Let R be a commutative unitary ring and L is an R-module. Let us define on L the following binary operation:

$$[\ ,] :\ L \times L \longrightarrow L$$

$$(x, y) \longrightarrow [x, y]$$

We call that L is a Lie algebra over R if the following conditions hold:

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1. The mapping [,] bilinear i.e.,

$$[x_1 + x_2, y] = [x_1, y] + [x_2, y] \quad ; \forall\ x_1, x_2, y \in\ L$$

$$[x, y_1 + y_2] = [x, y_1] + [x, y_2] \quad ; \forall\ x, y_1, y_2 \in\ L$$

$$[\alpha x, y] = \alpha[x, y] \quad ; \forall\, \alpha \in R\,, \forall\ x, y \in\ L$$

$$[x, \beta\, y] = \beta[\,x, y] \quad ; \ \forall\ \beta \in R\,, \forall\ x, y \in\ L$$

2. $[x, x] = 0, \forall x \in L$

3. Jacobi identity: $\big[x, [y, z]\big] + \big[y, [z, x]\big] + \big[z, [x, y]\big] = 0 \quad ; \forall\, x, y, z\ \in L.$

2.9 Corollaries:

1. $[x, y] = -[y, x] \quad ; \forall\, x, y\ \in L$

2. Lie algebra L is not commutative, unless it is trivial, i.e.,

$$[x, y] = 0 \quad ; \forall\, x, y\ \in L$$

Lie algebra with a trivial Lie bracket is called a commutative Lie algebra.

3. From Jacobi identity, we find that Lie algebra is not associative.

4. Lie algebra is not a unitary. We note that L is unitary if and only if $L = \{0\}$.

Definition: An algebra is called an associative, if it holds the following

$\big[x, [y, z]\big] = \big[[x, y], z\big] \ , \qquad \forall\ x, y, z \in L.$ [4]

2.10 Theorem: Let L be an associative algebra over a unitary commutative ring R. We define a multiplication operation for a Lie bracket, as follows

$$[x, y] = xy - yx\ ; \quad \forall\, x, y\ \in L$$

L is a Lie algebra.

3. A Neutrosophic Lie Algebra

3.1 Definition: A Neutrosophic Lie algebra over a Neutrosophic field NF is a Neutrosophic vector space $NL = \langle L, [\ ,\ ], I\rangle$ over NF together with a binary operation $[,]^N$ on NL such that the following:

$$[X, Y]^N = [x + yI, z + tI] = [x, z] + ([x + y, z + t] - [x, z])I$$

The Neutrosophic Lie bracket

$$[\ ,\ ]^N : NL\ \times\ NL\ \rightarrow\ NL$$

3.2 Theorem: Neutrosophic Lie algebra is lie algebra

Proof: Satisfying the following properties:

1. $[\ ,\ ]^N$ is bi-linear in the sense that

$$\begin{aligned}
[(a + bI)(x + yI) + (z + tI), n + mI] &= [ax + z + (ay + bx + by + t)I, n + mI] \\
&= [ax + z, n] \quad + ([ax + z + ay + bx + by + t, n + m] - [ax + z, n])I \\
&= a[x, n] + [z, n] + ([(a + b)(x + y) + z + t, n + m] - a[x, n] - [z, n])I \\
&= a[x, n] + [z, n] + (a + b)[x + y, n + m]I + [z + t, n + m]I - a[x, n]I - [z, n]I \\
&= a[x, n] + [z, n] + a[x, n]I + a[x, m]I + a[y, n]I + a[y, m]I + b[x, n]I + b[x, m]I \\
&\quad + b[y, n]I + b[y, m]I + [z, n]I + [z, m]I + [t, n]I + [t, m]I - a[x, n]I - [z, n]I \\
&= a[x, n] + [z, n] + a[x, m]I + a[y, n]I + a[y, m]I + b[x, n]I + b[x, m]I + b[y, n]I \\
&\quad + b[y, m]I + [z, m]I + [t, n]I + [t, m]I \\
&= (a + bI)[x, n] + (aI + bI)[x, m] + (aI + bI)[y, n] + (aI + bI)[y, m] + [z, n] + [z, m]I \\
&\quad + [t, n]I + [t, m]I = L_1
\end{aligned}$$

$$\begin{aligned}(a+bI)[x+yI,n+mI]+[z+tI,n+mI]\\ &=(a+bI)\{[x,n]+([x+y,n+m]-[x,n])I\}+[z,n]+([z+t,n+m]-[z,n])I\\ &=(a+bI)[x,n]+(a+bI)[x+y,n+m]I-(a+bI)[x,n]I+[z,n]+[z+t,n+m]I\\ &-[z,n]I\\ &=(a+bI)[x,n]+(a+bI)[x,n]I+(a+bI)[x,m]I+(a+bI)[y,n]I+(a+bI)[y,m]I\\ &-(a+bI)[x,n]I+[z,n]+[z,n]I+[z,m]I+[t,n]I+[t,m]I-[z,n]I\\ &=(a+bI)[x,n]+(a+bI)[x,m]I+(a+bI)[y,n]I+(a+bI)[y,m]I+[z,n]+[z,m]I\\ &+[t,n]I+[t,m]I=L_2\end{aligned}$$

2. $[,]^N$ is an alternating map in the sense that

$$[X,X]^N=[x+yI,x+yI]=[x,x]+([x+y,x+y]-[x,x])I=0+(0-0)I=0$$

3.$\big[x+yI,[z+tI,n+mI]\big]=[x+yI,[z,n]+([z+t,n+m]-[z,n])I]=\big[x,[z,n]\big]+\big(\big[x+y,[z,n]+[z+t,n+m]-[z,n]\big]-\big[x,[z,n]\big]\big)I=\big[x,[z,n]\big]+\big[x+y,[z+t,n+m]\big]I-\big[x,[z,n]\big]I=L_1$

$$\begin{aligned}\big[n+mI,[x+yI,z+tI]\big]&=[n+mI,[x,z]+([x+y,z+t]-[x,z])I]\\ &=\big[n,[x,z]\big]+\big(\big[n+m,[x,z]+[x+y,z+t]-[x,z]\big]-\big[n,[x,z]\big]\big)I\\ &=\big[n,[x,z]\big]+\big(\big[n+m,[x+y,z+t]\big]-\big[n,[x,z]\big]\big)I\\ &=\big[n,[x,z]\big]+\big[n+m,[x+y,z+t]\big]I-\big[n,[x,z]\big]I=L_2\end{aligned}$$

$$\begin{aligned}\big[z+tI,[n+mI,x+yI]\big]&=[z+tI,[n,x]+([n+m,x+y]-[n,x])I]\\ &=\big[z,[n,x]\big]+\Big(\big[z+t,[n,x]+[n+m,x+y]-[n,x]\big]-\Big[\big[z,[n,x]\big]\Big]\Big)I\\ &=\big[z,[n,x]\big]+\big[z+t,[n+m,x+y]\big]I-\big[z,[n,x]\big]I=L_3\end{aligned}$$

Thus, we get

$$L_1+L_2+L_3=0+0I-0I=0$$

The third identity is termed as the Neutrosophic Jacobi identity.

3.3 Results

1. $[X,Y]^N=-[Y,X]^N$ hold for all $X,Y\in NL$.

As the Neutrosophic Lie bracket $[,]^N$ is bilinear, we have

$$0=[X+Y,X+Y]=[X,X]+[X,Y]+[Y,X]+[Y,Y]=[X,Y]+[Y,X]$$

Hence implies

$[X,Y]=-[Y,X]$ for all $X,Y\in NL$.

2. Any Neutrosophic vector space NL has a Neutrosophic Lie bracket defined by $[X,Y]^N=0$ for all $,Y\in NL$. This is the Neutrosophic Abelian Lie algebra structure on .

3. $[X,0]^N=0=[0,X]^N$ for all $X\in NL$.

Proof

$$[X,0]^N=[x+yI,0+0I]=[x,0]+([x+y,0]-[x,0])I=0+0I$$

In the same way, we can proof that $0=[0,X]^N$.

3.4 Example: Suppose that NV is a finite-dimensional Neutrosophic vector space over NF.

Write $gl(NV)$ for the set of all linear maps from NV to NV. This is becomes a Neutrosophic Lie algebra, known as the Neutrosophic general linear algebra, if we define the Neutrosophic Lie bracket by

$[X,Y]^N$ by $[X,Y]^N=[x+yI,z+tI]=[x,z]+([x+y,z+t]-[x,z])I$

for $,g\in gl(NV)$, where $[f,g]=f\circ g-g\circ f$.

3.5 Example. The usual Neutrosophic vector space NR^3 over NR is a Neutrosophic Lie algebra over NR.

3.6 Remark: Because of Jacobi identity holds. i.e.,

$$\big[x+yI,[z+tI,n+mI]\big]+\big[n+mI,[x+yI,z+tI]\big]$$
$$+\big[z+tI,[n+mI,x+yI]\big]=0$$

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Then we can prove the following relation:

$$\big[[x+yI,z+tI],n+mI\big]+\big[[n+mI,x+yI],z+tI\big]$$
$$+\big[[z+tI,n+mI],x+yI\big]=0$$

3.7 Theorem: Let NL be an associative Neutrosophic Lie algebra over a unitary commutative ring R. We define a multiplication operation for Lie bracket, as follows

$$[x+yI,z+tI]=(x+yI)(z+tI)-(z+tI)(x+yI)\;;\quad \forall\, x+yI,z+tI\;\in NL$$
$$[x+yI,z+tI]=(xz-zx)+(xt-tx+yt-tu+yz-zy)I$$
$$;\quad x+yI,z+tI\;\in NL$$

Thus, NL is a Neutrosophic Lie algebra

Proof:

1. [,] is a bilinear.

$$1)\;[\;x+yI+z+tI,n+mI]=[\;(x+z)+(y+t)I,n+mI]$$
$$=(\,(x+z)+(y+t)I).(\,n+mI)-(\,n+mI).(\,(x+z)+(y+t)I)$$
$$=xn+xmI+zn+zmI+ynI+ymI+tnI+tmI-nx-nz-mxI-mzI-nyI-ntI-myI-mtI$$
$$=(xn-nx)+(zn-nz)+(xm-mx)I+(zm-mz)I+(yn-ny)I+(ym-my)I+(tn-nt)I+(tm-mt)I$$
$$=[x,n]+[z,n]+[x,m]I+[z,m]I+[y,n]I+[y,m]I+[t,n]I+[t,m]I$$
$$=[x,n]+[z,n]+([x,m]+[z,m]+[y,n]+[y,m]+[t,n]+[t,m])I$$
$$[x+yI,n+mI]+[z+tI,n+mI]=(x+yI)(\,n+mI)-(\,n+mI)(x+yI)$$
$$+(z+tI)\,(n+mI)-\;(n+mI)(z+tI)$$
$$=xn+(xm+yn+ym)I-nx-(mx+ny+my)I+$$
$$zn+(zm+tn+tm)I\;-nz-(mz+nt+mt)I$$
$$=xn-nx+(xm+yn+ym-mx-ny-my)I+$$
$$zn\;-nz+(zm+tn+tm-mz-nt-mt)I=$$
$$=[x,n]+[z,n]+([x,m]+[z,m]+[y,n]+[y,m]+[t,n]+[t,m])$$

Thus, by a comparison, we get

$$[\;x+yI+z+tI,n+mI]=[\,x+yI,n+mI]+[z+tI,n+mI]$$

$$2)\;[\,x+yI,z+tI+\;n+mI]=(x+yI).(z+tI+\;n+mI)-(z+tI+\;n+mI)(x+yI)$$
$$=(x+yI)(z+tI)+(x+yI)(\,n+mI)-(z+tI)(x+yI)-(\,n+mI)(x+yI)$$
$$=\big((x+yI)(z+tI)-(z+tI)(x+yI)\big)+\big((\,n+mI)(x+yI)-(x+yI)(\,n+mI)\big)$$
$$=[x+yI,z+tI]+[x+yI,n+mI]\quad;\forall\;\;x+yI,z+tI+\;n+mI\in\;NL$$

$$3)\;[\;\alpha(x+yI),z+tI]=\big(\alpha(x+yI)\big).(z+tI)-(z+tI)\big(\alpha(x+yI)\big)$$
$$=\alpha\big((x+yI)(z+tI)\big)-\alpha\big((z+tI)(x+yI)\big)$$
$$=\alpha\big((x+yI)(z+tI)-(z+tI)(x+yI)\big)=\alpha[\,x+yI,z+tI]\quad;\forall\;\alpha\in R\,,\forall\;\;(x+yI),(z+tI)\in\;NL$$

$$4)\;[\;x+yI,\beta\;(z+tI)]=(x+yI).\big(\beta(z+tI)\big)-(x+yI).\big(\beta(z+tI)\big)$$
$$=\beta\big((x+yI)(z+tI)\big)-\beta\big((x+yI)(z+tI)\big)$$
$$=\beta\big((x+yI)(z+tI)-(z+tI)(x+yI)\big)=\beta[\;x+yI,z+tI]\quad;\;\forall\;\;\beta\in R\,,\forall\,x+yI,z+tI\in\;NL$$

1. Let us prove the first property

$$[x+yI,x+yI]=(x+yI)(x+yI)-(x+yI)(x+yI)=0\qquad\forall\,x+yI\in\;NL$$

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2. Jacobi identity

$$\big[x+yI,[z+tI,n+mI]\big]+\big[z+tI,[n+mI,x+yI]\big]+\big[n+mI,[x+yI,z+tI]\big]=$$

$$=[x+yI,(z+tI)(n+mI)-(n+mI)(z+tI)]+[z+tI,(n+mI)(x+yI)-(x+yI)(n+mI)] \\ +[n+mI,(x+yI)(z+tI)-(z+tI)(x+yI)]=$$

$$(x+yI).\big((z+tI)(n+mI)-(n+mI)(z+tI)\big)-\big((z+tI)(n+mI)-(n+mI)y\big)(x+yI) \\ +(z+tI)\big((n+mI)(x+yI)-(x+yI)(n+mI)\big) \\ -\big((n+mI)(x+yI)-(x+yI)(n+mI)\big)(z+tI)$$

$$+(n+mI)\big((x+yI)(z+tI)-(z+tI)(x+yI)\big)-\big((x+yI)(z+tI)-(z+tI)(x+yI)\big)(n+mI)$$

$$=(x+yI)(z+tI)(n+mI)-(x+yI)(n+mI)(z+tI)-(z+tI)(n+mI)(x+yI) \\ +(n+mI)(z+tI)(x+yI)+(z+tI)(n+mI)(x+yI)-(z+tI)(x+yI)(n+mI) \\ -(n+mI)(x+yI)(z+tI)+(x+yI)(n+mI)(z+tI)+(n+mI)(x+yI)(z+tI) \\ -(n+mI)(z+tI)(x+yI)-(x+yI)(z+tI)(n+mI)+(z+tI)(x+yI)(n+mI) \\ =0 \quad ;\ \forall\, x+yI, z+tI\, , n+mI \in NL$$

Thus, NL is a Neutrosophic Lie algebra

3.8 Example: Let $L = M_n(R)$ algebra of a Neutrosophic square matrices, with a size of n, over a commutative unitary ring. A is an associative algebra, thus if we define the multiplication [,] as the following:

$$[X,Y]=X.Y-Y.X \quad ,\forall\, X=A+BI\, ,Y=N+MI \in NL=M_n(R)$$

Then, we get a Neutrosophic Lie algebra.

3.9 Example: Let M be an R- Neutrosophic module, where R is a unitary commutative ring and let $NL = End(M)$ be a set of Neutrosophic endomorphizms over M.

3.10 Example: Let R be a Neutrosophic commutative unitary ring, and let us define over $NL = R^3$

4. Neutrosophic Lie Sub algebra

4.1 Definition: Let L be a Lie algebra over the ring R, and $\emptyset \neq M \subseteq L$. We call that M is a Lie subalgebra from L, if it holds the following

1. M is an R submodule.

2. $[x\, ,y\,] \in M, \forall x,y \in M.$

4.2 Definition: Let NL be a Neutrosophic Lie algebra over the ring R, and let $\emptyset \neq NM \subseteq NL$. We call that NM is a Neutrosophic Lie subalgebra from NL, if it holds that

1. NM is an R- Neutrosophic submodule.

2. $[X,Y\,] \in NM, \forall\, X = x_0 + xI\, , Y = y_0 + yI \in NM.$

4.3 Examples:

1. The set of square matrices with trace is equal to 0.

2. $so(n,R)$ the set of square matrices with $A^t = -A$.

4.4 Definition: Let L be an algebraic Lie over the ring R, and let $\emptyset \neq J \subseteq L$. We call that J is an ideal from L, if it holds the following

1. J is an R-submodule.

2. for any $x \in J\ \ , a \in L$, we have $[a,x\,] \in J$ and $[x,a\,] \in J$.

4.5 Remark:

In Lie algebra the ideal will be from left and right, because if J is an ideal from the left, then $[a,x\,] \in J$, but $[x,a\,] = -[a,x\,] \in J$. Thus, $[x,a\,] \in J$, and J is an ideal from the left and right.

4.6 Definition: Let NL be a Neutrosophic Lie algebra over the ring R, and let $\emptyset \neq NJ \subseteq NL$. We call that NJ is an ideal from NL, if it holds the following:

1. NJ is an R- Neutrosophic submodule in NL.

2. for any $X = x_0 + x_1 I \in NJ$ and $a = a_0 + a_1 I \in Nl$, we have $[X, a\,] \in NJ$ or $[a, X\,] \in NJ$.

4.7 Definition: Let L, be a Lie algebra over a ring R. We define the center of L as the following

$$Z(L) = \{x \in L\ ;\ [x,a] = [a,x] \quad \forall\, a \in L\}$$

$$Z(L) = \{x \in L\ ;\ [x,a] = 0 \quad \forall\, a \in L\}$$

Note that $Z(L)$ is an ideal in L.

4.8 Definition: Let NL be a Neutrosophic Lie algebra over the ring R. We define the center of this algebra as

$$Z(NL) = \{x_0 + x_1 I \in NL\ ;\ [x_0 + x_1 I, a_0 + a_1 I] = 0 \quad \forall\, a_0 + a_1 I \in L\}$$

Note:

That $Z(L)$ is an ideal in NL.

4.9 Example: Let we have the following Neutrosophic Lie algebra

$$NL = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \quad ;\ a\,, b, d \in R \right\}$$

NL is a Neutrosophic Lie algebra over the ring R, where

$$[A,B] = A.B - B.A$$

Note that

$$NS = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \quad ;\ a\,, b \in R \right\}$$

is a Neutrosophic ideal in NL, beacause

$$\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} xa & xb \\ 0 & 0 \end{bmatrix} \in NS$$

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} ax & ay + bz \\ 0 & 0 \end{bmatrix} \begin{bmatrix} ax & ay + bz \\ 0 & 0 \end{bmatrix} \in NS$$

5. Conclusion

Neutrosophic Algebra represents a significant advancement in algebraic systems, addressing the challenges posed by uncertainty and imprecision. By integrating Neutrosophy with traditional algebraic concepts, it opens new research avenues and provides innovative solutions for complex problems across multiple disciplines. This introduction integrates the fundamental concepts of Lie algebras, Neutrosophy, and Neutrosophic Algebra, providing a comprehensive overview suitable for academic contexts.

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