



The Applications of Runge-Kutta Numerical Methods to Numerical Solutions of Several Neutrosophic Problems

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Abstract

In This paper, we develop the Runge-Kutta numerical method to be applied on neutrosophic problems of high orders, where we present generalized neutrosophic versions of Runge-Kutta methods of rank five, six and seven to use them in finding numerical solutions for some neutrosophic differential problems. In addition, we apply our generalized methods to some solid problems with many illustrated examples and numerical tables for comparing the results and the absolute errors.

Keywords: Neutrosophic problem; Neutrosophic Runge-Kutta; Numerical stability; Numerical method

1. Introduction

Numerical analysis is a broad branch of applied mathematics concerned with finding numerical solutions and numerical approximations for many mathematical problems and models related to algebraic equations or differential equations alike, through devising some algorithms that are easy to apply and program on the computer [1-3]. Numerical analysis is also concerned with finding good estimates of the error resulting from the approximation process, and with the issue of studying the stability of the resulting numerical solutions

Neutrosophic logic is a revolutionary logic introduced by Smarandache [6] as a new generalization of fuzzy logic that considers the idea of indeterminacy and uncertainty in measurements resulting from natural phenomena. It has been used to study many traditional mathematical concepts, such as algebraic structures, analysis, and even in computer science [10-13]. The Runge-Kutta method is one of the reference methods in numerical analysis that has been dealt with and developed by many researchers around the world. Different ranks of this method have been used to find numerical solutions and approximate errors for various problems in applied mathematics [4-9]. The applications of neutrosophic methods in numerical analysis have been studied by many authors see [14-18]. Where, we can see neutrosophic modelled problems with their approximate solutions and absolute errors were presented numerically [19-25].

This has motivated us to develop the Runge-Kutta numerical method to be applied on neutrosophic problems of high orders, where we present generalized neutrosophic versions of Runge-Kutta methods of rank five, six and seven to use them in finding numerical solutions for some neutrosophic differential problems. In addition, we apply our generalized methods to some solid problems with many illustrated examples and numerical tables for comparing the results and the absolute errors.

2. Main Discussion

The neutrosophic Runge-Kutta method of rank 5:

$$y_{n+1} + z_{n+1}I = y_n + z_nI + \frac{h + dI}{48 + 48I} (8(k_1 + k_1I) + 20(k_2 + k_2I) + 12(k_3 + k_3I) + 6(k_4 + k_4I) + (k_5 + k_5I) + (k_6 + k_6I))$$

Where:

$$\begin{aligned}
 k_1 + k_1 I &= f(x_n + t_n I, y_n + z_n I) \\
 k_2 + k_2 I &= f\left(x_n + t_n I + \frac{h + dI}{2}, y_n + z_n I + \frac{1}{2}(h + dI)(k_1 + k_1 I)\right) \\
 k_3 + k_3 I &= f\left(x_n + t_n I + \frac{h + dI}{2}, y_n + z_n I + \frac{1}{2}(h + dI)(k_2 + k_2 I)\right) \\
 k_4 + k_4 I &= f(x_n + t_n I + h + dI, y_n + z_n I + (h + dI)(k_3 + k_3 I)) \\
 k_5 + k_5 I &= f(x_n + t_n I + h + dI, y_n + z_n I + (h + dI)(k_4 + k_4 I)) \\
 k_6 + k_6 I &= f(x_n + t_n I + h + dI, y_n + z_n I + (h + dI)(k_5 + k_5 I))
 \end{aligned}$$

The neutrosophic Runge-Kutta method of rank 6:

$$\begin{aligned}
 y_{n+1} + z_{n+1} I &= y_n + z_n I + \frac{h + dI}{96 + 96I} (16(k_1 + k_1 I) + 40(k_2 + k_2 I) + 24(k_3 + k_3 I) + 12(k_4 + k_4 I) + 2(k_5 \\
 &+ k_5 I) + (k_6 + k_6 I) + (k_7 + k_7 I))
 \end{aligned}$$

Where:

$$\begin{aligned}
 k_1 + k_1 I &= f(x_n + t_n I, y_n + z_n I) \\
 k_2 + k_2 I &= f\left(x_n + t_n I + \frac{h}{2}, y_n + z_n I + \frac{1}{2}(h + dI)(k_1 + k_1 I)\right) \\
 k_3 + k_3 I &= f\left(x_n + t_n I + \frac{h}{2}, y_n + z_n I + \frac{1}{2}(h + dI)(k_2 + k_2 I)\right) \\
 k_4 + k_4 I &= f(x_n + t_n I + h, y_n + z_n I + (h + dI)(k_3 + k_3 I)) \\
 k_5 + k_5 I &= f(x_n + t_n I + h, y_n + z_n I + (h + dI)(k_4 + k_4 I)) \\
 k_6 + k_6 I &= f(x_n + t_n I + h, y_n + z_n I + (h + dI)(k_5 + k_5 I)) \\
 k_7 + k_7 I &= f(x_n + t_n I + h, y_n + z_n I + (h + dI)(k_6 + k_6 I))
 \end{aligned}$$

The neutrosophic Runge-Kutta method of rank 7:

$$\begin{aligned}
 y_{n+1} + z_{n+1} I &= y_n + z_n I + \frac{h + dI}{192} (32(k_1 + k_1 I) + 80(k_2 + k_2 I) + 48(k_3 + k_3 I) + 24(k_4 + k_4 I) \\
 &+ 4(k_5 + k_5 I) + 2(k_6 + k_6 I) + (k_7 + k_7 I) + (k_8 + k_8 I))
 \end{aligned}$$

Where:

$$\begin{aligned}
 k_1 + k_1 I &= f(x_n + t_n I, y_n + z_n I) \\
 k_2 + k_2 I &= f\left(x_n + t_n I + \frac{h + dI}{2}, y_n + z_n I + \frac{1}{2}(h + dI)(k_1 + k_1 I)\right) \\
 k_3 + k_3 I &= f\left(x_n + t_n I + \frac{h + dI}{2}, y_n + z_n I + \frac{1}{2}(h + dI)(k_2 + k_2 I)\right) \\
 k_4 + k_4 I &= f(x_n + t_n I + h + dI, y_n + z_n I + (h + dI)(k_3 + k_3 I)) \\
 k_5 + k_5 I &= f(x_n + t_n I + h + dI, y_n + z_n I + (h + dI)(k_4 + k_4 I)) \\
 k_6 + k_6 I &= f(x_n + t_n I + h + dI, y_n + z_n I + (h + dI)(k_5 + k_5 I)) \\
 k_7 + k_7 I &= f(x_n + t_n I + h + dI, y_n + z_n I + (h + dI)(k_6 + k_6 I)) \\
 k_8 + k_8 I &= f(x_n + t_n I + h + dI, y_n + z_n I + (h + dI)(k_7 + k_7 I))
 \end{aligned}$$

The stability analysis:

Consider the following initial values:

$$\begin{aligned}
 (y + zI)' &= f(x + tI, y + zI), (y + zI)(x_0 + t_0 I) = y_0 + z_0 I, x + tI \in [a + cI, b + sI] \\
 (1)
 \end{aligned}$$

The general formula:

$$y_{n+1} + z_{n+1}I = y_n + z_nI + (h + dI)\phi(x_n + t_nI, y_n + z_nI, h + dI) \quad (2)$$

The general formula o of phase R :

$$(y + zI)(x_{n+1} + t_{n+1}I) - (y_{n+1} + z_{n+1}I) = O((h + dI)^{p+1}) \quad (3)$$

we get the following:

$$(y + zI)(x_{n+1} + t_{n+1}I) = E(\lambda(h + dI))(y_n + z_nI) \quad (4)$$

we get :

$$(y + zI)' = \lambda(y + zI), (y + zI)(x_0 + t_0I) = (y + zI)_0 \quad (5)$$

Thus:

$$E(\overline{(h + dI)}) = 1 + \overline{(h + dI)} + \frac{1}{2!}\overline{(h + dI)}^2 + \dots + \frac{1}{p!}\overline{(h + dI)}^p + O(\overline{(h + dI)}^{p+1}) \quad (6)$$

RK		stability period
5	$1 + \overline{(h + dI)} + \frac{\overline{(h + dI)}^2}{2!} + \frac{\overline{(h + dI)}^3}{3!} + \frac{\overline{(h + dI)}^4}{4!} + \frac{\overline{(h + dI)}^5}{5!}$	$(-3.100183 + 2.1I, 0)$
6	$1 + \overline{(h + dI)} + \frac{\overline{(h + dI)}^2}{2!} + \frac{\overline{(h + dI)}^3}{3!} + \frac{\overline{(h + dI)}^4}{4!} + \frac{\overline{(h + dI)}^5}{5!} + \frac{\overline{(h + dI)}^6}{6!}$	$(-3.55642 - 1.6I, 0)$
7	$1 + \overline{(h + dI)} + \frac{\overline{(h + dI)}^2}{2!} + \frac{\overline{(h + dI)}^3}{3!} + \frac{\overline{(h + dI)}^4}{4!} + \frac{\overline{(h + dI)}^5}{5!} + \frac{\overline{(h + dI)}^6}{6!} + \frac{\overline{(h + dI)}^7}{7!}$	$(-3.235 + 1.067I, 0)$

Direct applications

Example (1)

Solve the following system of neutrosophic differential equations:

$$(y + zI)'_1 = (y + zI)_2, (y + zI)_1(0) = 1 + I$$

$$(y + zI)'_2 = -1001 - 1001I, (y + zI)_2 - 1000I (y + zI)_1, (y + zI)_2(I) = -1 - I$$

We will take the value of the step length $h = 0.001 + 0.001I$.

Example (2)

Solve the following system of differential equations:

$$(y + zI)'_1 = (600 + 300I)(y + zI)_1^2 ((y + zI)_2 - (y + zI)_1^3), (y + zI)_1(I) = 0.1 + I$$

$$(y + zI)'_2 = (-200 - 100I)((y + zI)_2 - (y + zI)_1^3) + 2(1 - (y + zI)_2), (y + zI)_2(I) = -0.1 - I$$

Table 1: the results of solving the problem in the first example.

ERK method of the fifth rank error values		Truth values		ERK method-Kutta of the fifth rank values	
y_1 error values	y_2 error values	y_1 Values	y_2 Values	y_1 Values	y_2 Values
I	I	1+I	-1-I	1+I	-1-I
3.2355e-14+I	3.2213e-14+I	0.928+0.928I	-0.928-0.928I	0.928+0.928I	-0.928-0.928I
6.31e-14+I	6.31e-14+I	0.93311+I	-(0.93311+I)	0.9447+I	-(0.9447+I)
9.3452e-14+I	9.3452e-14+I	0.9332+I	-(0.9332+I)	0.9332+I	-(0.9332+I)
1.22173e-11+I	1.22173e-11+I	0.96643+I	-(0.96643+I)	0.96643+I	-(0.96643+I)
1.10342e-11+I	1.10342e-11+I	0.9115+I	-(0.9115+I)	0.9115+I	-(0.9115+I)

1.330913e-11+I	1.330913e-11+I	0.658807+I	-(0.658807+I)	0.658807+I	-(0.658807+I)
2.11295e-11+I	2.11295e-11+I	0.9443861+I	0.9443861+I	0.9443861+I	0.9443861+I
2.3387e-11+I	2.3387e-11+I	0.981293+I	-(0.981293+I)	0.981293+I	-(0.981293+I)
2.208876e-11+I	2.208876e-11+I	0.9088+I	-(0.9088+I)	0.9088+I	-(0.9088+I)
3.11045e-11+I	3.11045e-11+I	0.77602+I	-(0.77602+I)	0.77602+I	-(0.77602+I)

Table 2: results of solving the problem in the second example.

ERK method of the fifth rank error values		Truth values		ERK method of the fifth rank values	
y ₁ error values	y ₂ error values	y ₁ Values	y ₂ Values	y ₁ Values	y ₂ Values
I	I	0.1+I	-0.1-I	0.1+I	-0.1-I
5.46645e-9+I	1.212e-7+I	0.092237+I	-0.073498-I	0.092234+I	-0.073498-I
1.221376e-8+I	4.2437e-7+I	0.0911369+I	-0.0379445-I	0.0911366+I	-0.0379444-I
1.0987e-8+I	5.7787e-7+I	0.094696+I	-0.046064-I	0.094693+I	-0.049621-I
2.311209e-8+I	6.52014e-7+I	0.092064+I	-0.022398-I	0.092060+I	-0.0333913-I
2.3329e-9+I	6.66783e-8+I	0.095833+I	-0.231998+I	0.095830+I	-0.011690133-I
2.1092e-9+I	6.11038e-8+I	0.094155+I	-0.0100078-I	0.094151+I	-0.0017110-I
2.51375e-9+I	6.1167e-05+I	0.097964+I	-0.01437-I	0.097961+I	-0.01433-I
2.256248e-9+I	6.2334e-05+I	0.03326+I	-0.01125-I	0.03323+I	-0.01121-I
2.3143e-9+I	5.673108e-05+I	0.08873+I	-0.008514-I	0.08870+I	-0.008510-I
2.100123e-9+I	5.4431e-05+I	0.0897664+I	-0.0023638-I	0.0897661+I	-0.0023634-I

Neutrosophic IRK Method of the fifth rank:

Depending on the neutrosophic explicit Runge-Kutta method of fifth rank, the neutrosophic implicit Runge-Kutta formula of fifth rank is in the following form:

$$y_n + z_n I = y_{n+1} + z_{n+1} I + \frac{h + dI}{48 + 48I} (8(k_1 + k_1 I) + 20(k_2 + k_2 I) + 12(k_3 + k_3 I) + 6(k_4 + k_4 I) + (k_5 + k_5 I) + (k_6 + k_6 I))$$

Where:

$$\begin{aligned} k_1 + k_1 I &= f(x_{n+1} + t_{n+1} I, y_{n+1} + z_{n+1} I) \\ k_2 + k_2 I &= f(x_{n+1} + t_{n+1} I - \frac{h + dI}{2}, y_{n+1} + z_{n+1} I - \frac{1}{2}(h + dI)(k_1 + k_1 I)) \\ k_3 + k_3 I &= f(x_{n+1} + t_{n+1} I - \frac{h + dI}{2}, y_{n+1} + z_{n+1} I - \frac{1}{2}(h + dI)(k_2 + k_2 I)) \\ k_4 + k_4 I &= f(x_{n+1} + t_{n+1} I - h - dI, y_{n+1} + z_{n+1} I - (h + dI)(k_3 + k_3 I)) \\ k_5 + k_5 I &= f(x_{n+1} + t_{n+1} I - h - dI, y_{n+1} + z_{n+1} I - (h + dI)(k_4 + k_4 I)) \\ k_6 + k_6 I &= f(x_{n+1} + t_{n+1} I - h - dI, y_{n+1} + z_{n+1} I - (h + dI)(k_5 + k_5 I)). \end{aligned}$$

Neutrosophic IRK Method of the sixth rank:

Depending on the neutrosophic explicit Runge-Kutta method of the sixth rank, the neutrosophic implicit Runge-Kutta formula of the sixth rank is in the following form:

$$y_n + z_n I = y_{n+1} + z_{n+1} I - \frac{h + dI}{96 + 96I} (16(k_1 + k_1 I) + 40(k_2 + k_2 I) + 24(k_3 + k_3 I) + 12(k_4 + k_4 I) + 2(k_5 + k_5 I) + (k_6 + k_6 I) + (k_7 + k_7 I))$$

Where:

$$\begin{aligned} k_1 + k_1 I &= f(x_{n+1} + t_{n+1} I, y_{n+1} + z_{n+1} I) \\ k_2 + k_2 I &= f(x_{n+1} + t_{n+1} I - \frac{(h + dI)}{2}, y_{n+1} + z_{n+1} I - \frac{1}{2}(h + dI)(k_1 + k_1 I)) \\ k_3 + k_3 I &= f(x_{n+1} + t_{n+1} I - \frac{(h + dI)}{2}, y_{n+1} + z_{n+1} I - \frac{1}{2}(h + dI)(k_2 + k_2 I)) \\ k_4 + k_4 I &= f(x_{n+1} + t_{n+1} I - (h + dI), y_{n+1} + z_{n+1} I - (h + dI)(k_3 + k_3 I)) \end{aligned}$$

$$k_5 + k_5 I = f(x_{n+1} + t_{n+1} I - (h + dI), y_{n+1} + z_{n+1} I - (h + dI)(k_4 + k_4 I))$$

$$k_6 + k_6 I = f(x_{n+1} + t_{n+1} I - (h + dI), y_{n+1} + z_{n+1} I - (h + dI)(k_5 + k_5 I))$$

$$k_7 + k_7 I = f(x_{n+1} + t_{n+1} I - (h + dI), y_{n+1} + z_{n+1} I - (h + dI)(k_6 + k_6 I))$$

Stability of Higher-order NRK methods:

Consider the general formula of the neutrosophic Runge-Kutta method from the phase:

$$y_{n+1} + z_{n+1} I = y_n + (h + dI)\phi(x_{n+1} + t_{n+1} I, y_{n+1} + z_{n+1} I, h + dI) \quad (7)$$

The neutrosophic implicit Runge-Kutta formula (7) of phase R is said to be of rank P if:

$$(y + zI)(x_{n+1} + t_{n+1} I) - y_{n+1} - z_{n+1} I = O((h + dI)^{p+1})$$

Using formula (7) on the equation of differences we get

$$(y + zI)_{n+1} = E(\lambda h)(y + zI)_n \quad (8)$$

Using the test question:

$$(y + zI)' = \lambda(y + zI), (y + zI)((x + tI)_0) = (y + zI)_0 \quad (9)$$

We get:

$$E(\overline{h + dI}) = 1 + I + \overline{h + dI} + \frac{1}{2!}(\overline{h + dI})^2 + \dots + \frac{1}{(p+1)!}(\overline{h + dI})^{p+1} + O((\overline{h + dI})^{p+1}) \quad (10)$$

Where $\overline{h + dI} = \lambda(h + dI)$ are polynomials of degree R in $\overline{h + dI}$

Therefore, the absolute stability intervals of the implicit Runge-Kutta methods are of higher rank, as in the following table. Considering that these periods were found using the following diagram:

RK	r_2	stability period
5	$1 + \overline{h + dI} + \frac{\overline{h + dI}^2}{2!} + \frac{\overline{h + dI}^3}{3!} + \frac{\overline{h + dI}^4}{4!} + \frac{\overline{h + dI}^5}{5!} + \frac{\overline{h + dI}^6}{6!}$	$(-3.123+I, I)$
6	$1 + \overline{h + dI} + \frac{\overline{h + dI}^2}{2!} + \frac{\overline{h + dI}^3}{3!} + \frac{\overline{h + dI}^4}{4!} + \frac{\overline{h + dI}^5}{5!} + \frac{\overline{h + dI}^6}{6!} + \frac{\overline{h + dI}^7}{7!}$	$(-3.223+I, I)$
7	$1 + \overline{h + dI} + \frac{\overline{h + dI}^2}{2!} + \frac{\overline{h + dI}^3}{3!} + \frac{\overline{h + dI}^4}{4!} + \frac{\overline{h + dI}^5}{5!} + \frac{\overline{h + dI}^6}{6!} + \frac{\overline{h + dI}^7}{7!} + \frac{\overline{h + dI}^8}{8!}$	$(-4.112809+I, I)$

Direct Applications:

Example 3

Solve the system of equations in the first example using the neutrosophic IRK method of the fifth rank.

We will take the step length value $h=0.002+0.001I$.

Example 4

Solve the system of equations in the second example using the neutrosophic IRK method of the fifth rank.

We will take the step length value $h=0.001+ 0.001I$.

Table 3: results of neutrosophic IRK of fifth order for the first example.

ERK method of the fifth rank error values		Truth values		ERK method of the fifth rank values	
y_1 error values	y_2 error values	y_1 Values	y_2 Values	y_1 Values	y_2 Values
I	I	I	-I	$1+I$	$-1-I$
$3.110572e-15+I$	$3.11057e-15+I$	$0.996+I$	$-(0.996+I)$	$0.996+I$	$-(0.996+I)$
$6.7886e-15+I$	$6.7882e-15+I$	$0.995+I$	$-(0.995+I)$	$0.995+I$	$-(0.995+I)$
$1.009551e-15+I$	$1.009531e-15+I$	$0.994+I$	$-(0.994+I)$	$0.994+I$	$-(0.994+I)$
$1.32219e-14+I$	$1.32216e-14+I$	$0.992+I$	$-(0.992+I)$	$0.992+I$	$-(0.992+I)$

1.649304e-14+I	1.649301e-14+I	0.991+I	-(0.991+I)	0.991+I	-(0.991+I)
2.08629e-14+I	2.08625e-14+I	0.98831+I	-(0.98831+I)	0.98831+I	-(0.98831+I)
2.3198e-14+I	2.3194e-14+I	0.9842+I	-(0.9842+I)	0.9842+I	-(0.9842+I)
2.6497e-14+I	2.6492e-14+I	0.9833+I	-(0.9833+I)	0.9833+I	-(0.9833+I)
3.06558e-14+I	2.6492e-14+I	0.9821+I	-(0.9821+I)	0.9821+I	-(0.9821+I)
3.32185e-14+I	3.32180e-14+I	0.9811+I	-(0.9811+I)	0.9811+I	-(0.9811+I)

Table 4: results of neutrosophic IRK of fifth order for the second example.

ERK of the fifth rank error values		Truth values		ERK of the fifth rank values	
y_1 error values	y_2 error values	y_1 Values	y_2 Values	y_1 Values	y_2 Values
I	I	0.1+I	-0.1-I	0.1+I	-0.1-I
2.2231e-07+I	8.98476e-06+I	0.09221+I	-0.0797-I	0.09211+I	-0.079715-I
9.113634e-07+I	2.5845e-05+I	0.09864+I	-0.06415-I	0.09851+I	-0.063142-I
1.32565e-06+I	3.589395e-05+I	0.0986675+I	-0.049564-I	0.098101+I	-0.049602-I
1.773654e-06+I	4.968te-05+I	0.092316+I	-0.033798-I	0.09793+I	-0.03284-I
1.4895e-06+I	4.10496e-05+I	0.0989886+I	-0.023369-I	0.098234	-0.029302-I
1.5856375e-07+I	368485e-06+I	0.09847364+I	-0.0211988-I	0.091341+I	-0.0221784-I
1.1124e-07+I	3.104e-06+I	0.097111+I	-0.017769-I	0.0966+I	-0.01422313-I
1.00285e-07+I	2.8406e-06+I	0.097332+I	-0.010015-I	0.09665+I	-0.01100173-I
1.393e-07+I	2.10038e-06+I	0.0969984+I	-0.0056315-I	0.0965543+I	-0.005567-I
1.0440285e-07+I	1.938694e-06+I	0.0964453+I	-0.0011286-I	0.097754+I	-0.00111101-I

Table 5: results of neutrosophic IRK of sixth order for the second example.

ERK of the sixth rank error values		Truth values		ERK of the sixth rank values	
y_1 error values	y_2 error values	y_1 Values	y_1 error values	y_2 error values	y_1 Values
I	I	0.1+I	I	I	0.1+I
2.2231e-07+I	8.98476e-06+I	0.09221+0.0001I	2.2231e-07+I	8.98476e-06+0.0001I	0.09221+0.00002I
9.113634e-07+I	2.5845e-05+I	0.09864+0.0002I	9.113634e-07+I	2.5845e-05+0.00001I	0.09864+0.00002I
1.32565e-06+I	3.589395e-05+I	0.0986675+0.0003I	1.32565e-06+I	3.589395e-05+0.00002I	0.0986675+0.00004I
1.773654e-06+I	4.968te-05+I	0.092316+0.0004I	1.773654e-06+I	4.968te-05+0.000002I	0.092316+0.00004I
1.4895e-06+I	4.10496e-05+I	0.0989886+0.0005I	1.4895e-06+I	4.10496e-05+0.00003I	0.0989886+0.00004I
1.5856375e-07+I	368485e-06+I	0.09847364+0.0006I	1.5856375e-07+I	368485e-06+0.00004I	0.09847364+0.0000112I
1.1124e-07+I	3.104e-06+I	0.097111+0.0007I	1.1124e-07+I	3.104e-06+0.000031I	0.097111+0.0000112I
1.00285e-07+I	2.8406e-06+I	0.097332+0.0008I	1.00285e-07+I	2.8406e-06+0.000032I	0.097332+0.0000112I
1.393e-07+I	2.10038e-06+I	0.0969984+0.0009I	1.393e-07+I	2.10038e-06+0.0000112I	0.0969984+0.0000112I
1.0440285e-07+I	1.938694e-06+I	0.0964453+0.001I	1.0440285e-07+I	1.938694e-06+0.000004472I	0.0964453+0.0000112I

3. Conclusion

In This paper, we developed the Runge-Kutta numerical method to be applied on neutrosophic problems of high orders, where we presented generalized neutrosophic versions of Runge-Kutta methods of rank five, six and seven to use them in finding numerical solutions for some neutrosophic differential problems. In addition, we applied our generalized methods to some solid problems with many illustrated examples and numerical tables for comparing the results and the absolute errors.

References

- [1] J.C. Butcher and M.T. Diamantakis, "DESIRE: Diagonally extended singly implicit Runge-Kutta effective order methods," *Numerical Algorithms*, vol. 17, pp. 121-145, 1998.
- [2] J.C. Butcher, "Numerical methods for differential equations and applications," *The Arabian Journal for Science and Engineering*, vol. 22, no. 2Command, pp. 17-29, 1997.
- [3] J.R. Cash, "Block Runge-Kutta methods for numerical integration of initial value problems in ordinary differential equations. Part II: The stiff case," *Mathematics of Computation*, vol. 40, no. 161, pp. 193-206, 1983.
- [4] J.R. Cash, "A class of implicit Runge-Kutta methods for the numerical integration of stiff ordinary differential equations," *Journal of the Association of Computing Machinery*, vol. 22, no. 4, pp. 504-511, 1975.
- [5] J.R. Cash, "Runge-Kutta methods for the solution of stiff two-point boundary value problems," *Applied Numerical Mathematics*, vol. 22, pp. 165-177, 1996.
- [6] M. Roche, C. Lubich, and E. Hairer, "Error of Rosenbrock methods for stiff ordinary differential equations," *BIT Numerical Mathematics*, vol. 29, pp. 77-90, 1989.
- [7] D.A. Voss and M.J. Casper, "Efficient split linear multistep methods for stiff ordinary differential equations," *SIAM Journal on Scientific and Statistical Computing*, vol. 19, no. 5, pp. 990-999, 1989.
- [8] D.A. Voss, "Factored two-step Runge-Kutta methods," *Applied Mathematics and Computation*, vol. 31, pp. 361-368, 1989.
- [9] D.A. Voss, "Fifth-order exponentially fitted formula," *SIAM Journal on Numerical Analysis*, vol. 25, no. 3, p. 15, 1988.
- [10] A.A. Abubaker, M. Abualhomos, K. Matarneh, and A. Al-Husban, "A numerical approach for the algebra of two-fold," *Neutrosophic Sets and Systems*, vol. 75, pp. 181-195, 2025.
- [11] G. Alvarez, C. Gómez, and A. Romero, "Assessment of the educational live action in uncertainty environment under single-valued neutrosophic sets," *International Journal of Neutrosophic Science*, vol., no., pp. 167-175, 2024. DOI: 10.54216/IJNS.230115.
- [12] A.A. Salama, O.M. Mobarez, M.H. Elfar, and R. Alhabib, "Neutrosophic model for measuring and evaluating the role of digital transformation in improving sustainable performance using the balanced scorecard in Egyptian universities," *Neutrosophic Systems with Applications*, vol. 21, pp. 1-24, 2024. DOI: 10.61356/j.nswa.2024.21370.
- [13] S. M., and A.N. Mera, "Fuzzy logic used to solve ODEs of second order under neutrosophic initial conditions," *International Journal of Neutrosophic Science*, vol., no., pp. 51-58, 2024. DOI: 10.54216/IJNS.230104.
- [14] S. Topal, F. Tas, S. Broumi, and O. Ayhan, "Applications of neutrosophic logic of smart agriculture via Internet of Things," *International Journal of Neutrosophic Science*, vol., no., pp. 105-115, 2020. DOI: 10.54216/IJNS.120205.
- [15] A. Shihadeh, K.A.M. Matarneh, R. Hatamleh, M.O. Al-Qadri, and A. Al-Husban, "On the two-fold fuzzy n -refined neutrosophic rings for $n \geq 3n \geq 3$," *Neutrosophic Sets and Systems*, vol. 68, pp. 8-25, 2024.
- [16] A. Shihadeh, K.A.M. Matarneh, R. Hatamleh, R.B.Y. Hijazeen, M.O. Al-Qadri, and A. Al-Husban, "An example of two-fold fuzzy algebras based on neutrosophic real numbers," *Neutrosophic Sets and Systems*, vol. 67, pp. 169-178, 2024.
- [17] A.S. Heilat, R.C. Karoun, A. Al-Husban, A. Abbes, M. Al Horani, G. Grassi, and A. Ouannas, "The new fractional discrete neural network model under electromagnetic radiation: Chaos, control, and synchronization," *Alexandria Engineering Journal*, vol. 76, pp. 391-409, 2023.
- [18] A. Al-Husban, R.C. Karoun, A.S. Heilat, M. Al Horani, A.A. Khennaoui, G. Grassi, and A. Ouannas, "Chaos in a two-dimensional fractional discrete Hopfield neural network and its control," *Alexandria Engineering Journal*, vol. 75, pp. 627-638, 2023.
- [19] B. Batiha, "New solution of the Sine-Gordon equation by the Daftardar-Gejji and Jafari Method," *Symmetry*, vol. 14, no. 1, art. No. 57, 2022.
- [20] B. Batiha, F. Ghanim, O. Alayed, R. Hatamleh, A.S. Heilat, H. Zureigat, and O. Bazighifan, "Solving multispecies Lotka-Volterra equations by the Daftardar-Gejji and Jafari Method," *International Journal of Mathematics and Mathematical Sciences*, vol. 2022, art. No. 1839796, 2022.

- [21] B. Batiha, "A variational iteration method for solving the nonlinear Klein-Gordon equation," *Australian Journal of Basic and Applied Sciences*, vol. 3, no. 4, pp. 3876-3890, 2009.
- [22] O. Ala'yed, B. Batiha, R. Abdelrahim, and A. Jawarneh, "On the numerical solution of the nonlinear Bratu type equation via quintic B-spline method," *Journal of Interdisciplinary Mathematics*, vol. 22, no. 4, pp. 405–413, 2019.
- [23] A. Shihadeh, W. Mahmoud, M. Bataineh, H. Al-Tarawneh, A. Alahmade, and A. Al-Husban, "On the geometry of weak fuzzy complex numbers and applications to the classification of some A-curves," *International Journal of Neutrosophic Science*, vol. 24, no. 3, pp. 369-375, 2024. DOI: 10.54216/IJNS.230428.
- [24] A.F. Salamah and R.M. Dallah, "A study of neutrosophic Bernoulli and Riccati equations using the one-dimensional geometric AH-Isometry," *Journal of Neutrosophic and Fuzzy Systems*, vol., no., pp. 30-40, 2023. DOI: 10.54216/JNFS.050104.
- [25] T. Hamadneh, A. Abbes, I.A. Falahah, Y.A. Al-Khassawneh, A.S. Heilat, A. Al-Husban, and A. Ouannas, "Complexity and chaos analysis for two-dimensional discrete-time predator-prey Leslie-Gower model with fractional orders," *Axioms*, vol. 12, no. 6, p. 561, 2023.