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# The Dominator Coloring of Some Graph Classes

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## Abstract

A proper vertex coloring of a graph $G(V,E)$ is an assignment of colors to the vertices of $G$ so that no two adjacent vertices have the same color. A dominator coloring of $G$ is a proper vertex coloring for which every vertex is adjacent to all the vertices of at least one color class. The minimum number of colors required to establish a proper dominator coloring on $G$ is called the dominator coloring number and is denoted by $\chi_d(G)$. In this paper, we determine the dominator coloring number of strong grid graphs $P_m \boxtimes P_n$ when $m,n \geq 3$. We also determine the dominator coloring number of the Queen graph $Q_{2,n}$ for $n \geq 2$.

**Keywords:** Graph coloring; Dominator coloring number; Strong grid; Queen graph

## 1. Introduction

Let $G(V,E)$ be a graph with $|V| = n$ vertices and $|E| = m$ edges. The degree of a vertex $v$ (denoted by $\deg(v)$ ) is the number of all vertices that are adjacent to $v$. Let $Y \subseteq V$ and let $F$ be a subset of $E$ such that $F$ consists of all edges of $G$ which have endpoints in $Y$, then $H = (Y,F)$ is called an induced subgraph of $G$ by $Y$ and is denoted by $G[Y]$. Let $P_m$, $P_n$ be two paths. We define the strong product of $P_m$ and $P_n$ as the graph $P_m \boxtimes P_n$ such that $V(P_m \boxtimes P_n) = V(P_m \times P_n) = \{(i,j): 1 \leq i \leq m,\ 1 \leq j \leq n\}$ and two vertices $(i,i'), (j,j')$ are adjacent if and only if:

- $i$ is adjacent to $j$ and $i' = j'$.
- $i = j$ and $i'$ is adjacent to $j'$.
- $i$ is adjacent to $j$ and $i'$ is adjacent to $j'$.

A strong product $P_m \boxtimes P_n$ is called a strong grid. For more information on strong grids see [3]. The Queen graph $Q_{m,n}$ [1] (also called the rectangular Queen graph) is a graph with $mn$ vertices in which each vertex represents a square in a $m \times n$ chessboard, and each edge corresponds to a legal move by a queen. A subset $S \subseteq V(G)$ is called a dominating set of $G$ if every vertex in $V - S$ is adjacent to a vertex in $S$. The minimum cardinality of all dominating sets in $G$ is called the domination number of $G$ and is denoted by $\gamma\ (G)$. A proper vertex coloring of a graph $G$ is a mapping $C: V(G) \rightarrow X$ where $X$ is a set of colors and no two adjacent vertices have the same color, which means $C$ divides $V(G)$ into color classes $C = (V_1, V_2, \ldots)$ each of which contains all vertices of a certain color. The chromatic number is the minimum number of colors required for a proper coloring of $V(G)$ and is denoted by $\chi(G)$.A dominator coloring of $G$ is a proper vertex coloring for which every vertex dominates all the vertices of at least one color class. The minimum number of colors required to conduct a proper dominator coloring on $G$ is called the dominator coloring number and is denoted by $\chi_d(G)$. The concept of dominator coloring was presented by Gera et al. in [4]. It was also studied by Heditniemi et al. Since then, the problem was widely studied. In [5], Gera et al. studied the dominator coloring of bipartite graphs. Merouane et al. studied the problem on trees in [8]. Kalavivani et al. investigated the dominator coloring of some graph products in [6]. For more information about dominator coloring, see [2], [7] and [9]. In this paper, we determine the dominator coloring number of strong grid graphs $P_m \boxtimes P_n$ when $m,n \geq 3$. We also determine the dominator coloring number of the Queen graph $Q_{2,n}$ for $n \geq 2$.

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**Proposition 1 [4].** For any graph $G$;

$$\chi(G) \leq \chi_d(G)$$

## 2. Main Results

In this section, we determine the dominator coloring number of strong grid graphs $P_m \boxtimes P_n$ when $m, n \geq 3$. We also determine the dominator coloring number of the Queen graph $Q_{2,n}$ for $n \geq 2$.

**Theorem 1.** Let $P_3 \boxtimes P_n$ be a strong grid with $n \geq 3$;

$$\chi_d(P_3 \boxtimes P_n) = \left\lceil \frac{n}{3} \right\rceil + 4.$$

**Proof:** We consider the following cases:

Case 1. $n \equiv 0(mod\ 3)$.

We divide $P_3 \boxtimes P_n$ into $\frac{n}{3}$ blocks each of which consists of nine vertices. We denote these blocks by $B_j: 1 \leq j \leq \frac{n}{3}$ so that:

$$B_j = \{(1,3j-2), (1,3j-1), (1,3j), (2,3j-2),$$

$(2,3j-1), (2,3j), (3,3j-2), (3,3j-1), (3,3j)\}$.

Now we try to apply a proper dominator coloring on an arbitrary block $B_j$, We make the following observations:

**Observation 1**. Earlier, we defined $C(B_j) = \{c((k,l)): 1 \leq k \leq 3; 3j-2 \leq l \leq 3j\}$. It is obvious that $|C(B_{i,j})| \geq 4$, otherwise, at least two of $(1,3j-2), (1,3j-1), (2,3j), (3,3j-2), (3,3j-1)$ are adjacent and have the same color which is a contradiction.

**Observation 2**. Some vertices of $B_j$ must form an entire color class. Otherwise, the middle vertex $(3i-1,3j)$ does not dominate an entire color class and that is a contradiction.

From Observation 1 we conclude that $C(B_1) \geq 4$, and if $|C(B_1)| = 4$, one of the four colors is unique to $B_1$, From Observation 2 we conclude that $P_3 \boxtimes P_n$ has at least $\frac{n}{3}$ different color classes, each of which corresponds to a block $B_j$, which proves the inequality $\chi_d(P_m \boxtimes P_n) \geq \frac{n}{3} + 3$.

Let $\chi_d(P_m \boxtimes P_n) = \frac{mn}{9} + 3$, this means only three colors can be repeated in two different blocks. Let us color $B_1$ with four colors, we notice that:

- $C((1,1)) = 1; C((1,2)) = 2; C((2,1)) = 3; C((2,2)) = 4$. Otherwise, we have a contradiction.
- $C((1,3)), C((2,3)) \in \{1,2\}$, otherwise we have a contradiction.

Now let us color $B_2$ with colors 1,2,3 and 5 which resembles the unique color class of $B_2$. We notice that:

- Since $C((1,3)), C((2,3)) \in \{1,2\}$ then either $C((1,4)) = 5$ or $C((2,4)) = 5$. Otherwise, a contradiction occurs.
- Let $C((1,4)) = 3, C((2,4)) = 5$. Then $C((1,5)), C((2,5)) \in \{1,2\}$ or else a contradiction occurs. However, this means $C((1,6)), C((2,6)) \in \{3,5\}$, therefore $(1,4)$ does not dominate any color class and that is a contradiction. The same argument applies if $C((1,4)) = 5, C((2,4)) = 3$.

We conclude that it is impossible to apply a proper dominator coloring on $B_1, B_2$ using only five colors. Without loss of generality, which means:

$$\chi_d(P_3 \boxtimes P_n) \geq \frac{n}{3} + 4 \tag{1}$$

Now we establish the upper bound by applying a proper dominator coloring on the strong grid $P_3 \boxtimes P_n$ using $\frac{n}{3} + 4$ colors. We define this dominator coloring as $C = (V_1, V_2, \ldots, V_{\frac{n}{3}+3}, V_{\frac{n}{3}+4})$ with:

$$V_1 = \left\{(1,2j-1), (3,2j-1): 1 \leq j \leq \left\lceil \frac{n}{2} \right\rceil\right\};$$

$$V_2 = \left\{(2,1+6j), (2,3+6j): 0 \leq j \leq \left\lfloor \frac{n}{3} \right\rfloor - 2\right\};$$

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$V_3 = \left\{(1,2j),(3,2j): 1 \le i \le \left\lceil \frac{m}{2} \right\rceil; 1 \le j \le \left\lfloor \frac{n}{2} \right\rfloor\right\};$

$V_4 = \left\{(2,4+6j),(2,6+6j): 0 \le j \le \left\lfloor \frac{n}{3} \right\rfloor - 2\right\};$

For $5 \le i \le \frac{n}{3} + 4$; $V_i = \{(2,3i-13)\}$.

Since $C$ forms a proper dominator coloring of $\frac{n}{3} + 4$ colors on $P_3 \boxtimes P_n$, then:

$$\chi_d(P_3 \boxtimes P_n) \le \frac{n}{3} + 4 \qquad (2)$$

Figure 1 illustrates that $\chi_d(P_3 \boxtimes P_{12}) \le 8$. From (1) and (2) we conclude that $\chi_d(P_3 \boxtimes P_n) = \frac{n}{3} + 4$ if $n \equiv 0 (mod\ 3)$.

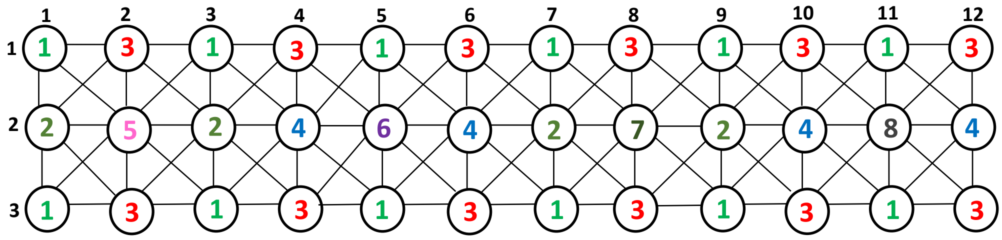

**Figure 1.** $\chi_d(P_3 \boxtimes P_{12}) \le 8$.

Case 2. $n \equiv 1 (mod\ 3)$.

In a similar way to Case 1, we divide $P_3 \boxtimes P_n$ into $\left\lfloor \frac{n}{3} \right\rfloor$ blocks each of which consists of nine vertices, denoted by $B_j: 1 \le j \le \left\lfloor \frac{n}{3} \right\rfloor$ so that:

$$B_j = \{(1,3j-2),(1,3j-1),(1,3j),(2,3j-2),$$

$(2,3j-1),(2,3j),(3,3j-2),(3,3j-1),(3,3j)\}$.

The three remaining vertices form a sub-block denoted by $SB = \{(1,n),(2,n),(3,n)\}$. We notice that the same argument of Case 1 applies to the blocks $B_j: 1 \le j \le \left\lfloor \frac{n}{3} \right\rfloor$, which means $C(P_3 \boxtimes P_n[V - \{(1,n),(2,n),(3,n)\}]) = \frac{n-1}{3} + 4$. However, we also notice that $SB$ must contain at least one entire color class. Otherwise, none of $(1,n),(2,n),(3,n)$ dominates an entire color class and that is a contradiction. Thus:

$$\chi_d(P_3 \boxtimes P_n) \ge \frac{n-1}{3} + 5 = \left\lceil \frac{n}{3} \right\rceil + 4 \qquad (3)$$

We conduct a proper dominator coloring on $P_3 \boxtimes P_n$ using $\left\lceil \frac{n}{3} \right\rceil + 4$ colors. We define this dominator coloring as $C = (V_1, V_2, \dots, V_{\left\lceil \frac{n}{3} \right\rceil + 3}, V_{\left\lceil \frac{n}{3} \right\rceil + 4})$ with:

$V_1 = \left\{(1,2j-1),(3,2j-1): 1 \le j \le \left\lceil \frac{n}{2} \right\rceil\right\};$

$V_2 = \left\{(2,1+6j),(2,3+6j): 0 \le j \le \left\lfloor \frac{n}{3} \right\rfloor - 2\right\};$

$V_3 = \left\{(1,2j),(3,2j): 1 \le i \le \left\lceil \frac{m}{2} \right\rceil; 1 \le j \le \left\lfloor \frac{n}{2} \right\rfloor\right\};$

$V_4 = \left\{(2,4+6j),(2,6+6j): 0 \le j \le \left\lfloor \frac{n}{3} \right\rfloor - 2\right\};$

For $5 \le i \le \left\lceil \frac{n}{3} \right\rceil + 3$; $V_i = \{(2,3i-13)\}$.

$V_{\left\lceil \frac{n}{3} \right\rceil + 4} = \{(2,n)\}$.

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Therefore:

$$\chi_d(P_3 \boxtimes P_n) \leq \left\lceil \frac{n}{3} \right\rceil + 4 \tag{4}$$

From (3) and (4) we obtain $\chi_d(P_3 \boxtimes P_n) = \left\lceil \frac{n}{3} \right\rceil + 4$ if $n \equiv 1(mod\ 3)$.

Case 3. $n \equiv 2(mod\ 3)$.

As we implied in Case 2, $P_3 \boxtimes P_n$ can be divided into $\left\lfloor \frac{n}{3} \right\rfloor$ blocks each of which consists of nine vertices, denoted by $B_j : 1 \leq j \leq \left\lfloor \frac{n}{3} \right\rfloor$ so that:

$$B_j = \{(1,3j-2),(1,3j-1),(1,3j),(2,3j-2),$$

$(2,3j-1),(2,3j),(3,3j-2),(3,3j-1),(3,3j)\}$.

While the six remaining vertices form a sub-block denoted by $SB = \{(1,n-1),(2,n-1),(3,n-1),(1,n),(2,n),(3,n)\}$. By applying the same argument of Case 2 we conclude that $C(P_3 \boxtimes P_n[V - \{(1,n-1),(2,n-1),(3,n-1),(1,n),(2,n),(3,n)]) = \frac{n-2}{3} + 4$. Furthermore, $SB$ must contain at least one entire color class. Otherwise, none of its vertices dominates an entire color class and that is a contradiction. This means:

$$\chi_d(P_3 \boxtimes P_n) \geq \frac{n-2}{3} + 5 = \left\lceil \frac{n}{3} \right\rceil + 4 \tag{5}$$

We notice that the same proper dominator coloring from Case 2 is applicable in the case of $n \equiv 2(mod\ 3)$, thus:

$$\chi_d(P_3 \boxtimes P_n) \leq \left\lceil \frac{n}{3} \right\rceil + 4 \tag{6}$$

From (5) and (6) we prove that $\chi_d(P_3 \boxtimes P_n) = \left\lceil \frac{n}{3} \right\rceil + 4$ if $n \equiv 2(mod\ 3)$.

From all the previous cases we conclude the requested.■

**Theorem 2.** Let $P_m \boxtimes P_n$ be a strong grid with $m,n \geq 3$;

$$\chi_d(P_m \boxtimes P_n) = \begin{cases} \frac{mn}{9} + 4 \ \ if \ \ m,n \equiv 0(mod\ 3); \\ \frac{m}{3}\left\lfloor \frac{n}{3} \right\rfloor \ + \frac{m}{3} + 4 \ \ if\ m \equiv 0(mod\ 3)\ and\ n \equiv 1,2(mod\ 3); \\ \left\lfloor \frac{m}{3} \right\rfloor \frac{n}{3} \ + \frac{n}{3} + 4 \ \ if\ n \equiv 0(mod\ 3)\ and\ m \equiv 1,2(mod\ 3); \\ \left\lfloor \frac{m}{3} \right\rfloor \left\lfloor \frac{n}{3} \right\rfloor \ + \left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + 5 \ \ otherwise. \end{cases}$$

**Proof:** This Theorem is essentially a generalization of Theorem 1. We briefly mention the adjustments we need to apply to the proof of Theorem 1. We consider the following cases:

Case 1. $m,n \equiv 0(mod\ 3)$.

We divide $P_m \boxtimes P_n$ into $\frac{mn}{9}$ blocks each of which consists of nine vertices. We denote these blocks by $B_{i,j} : 1 \leq i \leq \frac{m}{3} ; 1 \leq j \leq \frac{n}{3}$ so that:

$$B_{i,j} = \{(3i-2,3j-2),(3i-2,3j-1),(3i-2,3j),(3i-1,3j-2),$$

$(3i-1,3j-1),(3i-1,3j),(3i,3j-2),(3i,3j-1),(3i,3j)\}$.

The rest of the proof of the lower bound is similar to Case 1 of Theorem 1, we conclude that:

$$\chi_d(P_m \boxtimes P_n) \geq \frac{mn}{9} + 4 \tag{7}$$

Now we establish the upper bound by applying a proper dominator coloring on the strong grid $P_m \boxtimes P_n$ using $\frac{mn}{9} + 4$ colors. We define this dominator coloring as $C = (V_1, V_2, \ldots, V_{\frac{mn}{9}+3}, V_{\frac{mn}{9}+4})$ with:

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For $1 \leq i \leq \left\lceil \frac{m}{2} \right\rceil$:

If $2i - 1 = 1 + 6l; 3 + 6l: 0 \leq l \leq \left\lfloor \frac{m}{6} \right\rfloor$:

$V_1 = \left\{(2i - 1,2j - 1): ; 1 \leq j \leq \left\lceil \frac{n}{2} \right\rceil\right\}$;

$V_3 = \left\{(2i - 1,2j): 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor\right\}$;

If $2i - 1 = 5 + 6l;\ 0 \leq l \leq \left\lfloor \frac{m}{6} \right\rfloor$:

$V_1 = \left\{(2i - 1,1 + 6j), (2i - 1,3 + 6j): 0 \leq j \leq \frac{n}{3} - 2\right\}$;

$V_3 = \left\{(2i - 1,1 + 6j), (2i - 1,3 + 6j): 0 \leq j \leq \frac{n}{3} - 2\right\}$;

If $2i = 2 + 6l; 4 + 6l: 0 \leq l \leq \left\lfloor \frac{m}{6} \right\rfloor$:

$V_2 = \left\{(2i, 1 + 6j), (2i, 3 + 6j): 0 \leq j \leq \frac{n}{3} - 2\right\}$;

$V_4 = \left\{(2i, 4 + 6j), (2i, 6 + 6j): 0 \leq j \leq \frac{n}{3} - 2\right\}$;

If $2i = 6 + 6l;\ 0 \leq l \leq \left\lfloor \frac{m}{6} \right\rfloor$:

$V_2 = \left\{(2i, 2j - 1): 0 \leq j \leq \frac{n}{3} - 2\right\}$;

$V_4 = \left\{(2i, 2j): 0 \leq j \leq \frac{n}{3} - 2\right\}$;

For every block $B_{i,j}: 1 \leq i \leq \frac{m}{3}; 1 \leq j \leq \frac{n}{3}$ we include an entire color class by assigning a unique color to $\{(3i - 1,3j - 1)\}$; thus we add $\frac{mn}{9}$ color classes which makes the total number of used color classes equal $\frac{mn}{9} + 4$. Therefore:

$$\chi_d(P_m \boxtimes P_n) \leq \frac{mn}{9} + 4 \tag{8}$$

From (7) and (8) we include that $\chi_d(P_m \boxtimes P_n) = \frac{mn}{9} + 4$ if $m, n \equiv 0(mod\ 3)$.

Case 2. $m \equiv 0(mod\ 3)$ and $n \equiv 1(mod\ 3)$.

In a similar way to Case 1, we divide $P_m \boxtimes P_n$ into $\frac{m}{3}\left\lfloor \frac{n}{3} \right\rfloor$ blocks each of which consists of nine vertices, denoted by $B_{i,j}: 1 \leq i \leq \frac{m}{3}; 1 \leq j \leq \left\lfloor \frac{n}{3} \right\rfloor$ so that:

$$B_{i,j} = \{(3i - 2,3j - 2), (3i - 2,3j - 1), (3i - 2,3j), (3i - 1,3j - 2),$$
$$(3i - 1,3j - 1), (3i - 1,3j), (3i, 3j - 2), (3i, 3j - 1), (3i, 3j)\}.$$

While the remaining vertices (vertices of the last column) can be divided into $\frac{m}{3}$ sub-blocks each of which consists of three vertices, denoted by $SB_i: 1 \leq i \leq \frac{m}{3}$ so that: $SB_i = \{(3i - 2, n), (3i - 1, n), (3i, n)\}$. By applying the same argument of Case 2 of Theorem 1, we conclude that each block $B_{i,j}: 1 \leq i \leq \frac{m}{3}; 1 \leq j \leq \left\lfloor \frac{n}{3} \right\rfloor$ must contain an entire color class. Furthermore, each sub-block $SB_i: 1 \leq i \leq \frac{m}{3}$ must contain at least one entire color class. Otherwise, none of its vertices dominates an entire color class and that is a contradiction. This means:

$$\chi_d(P_m \boxtimes P_n) \geq \frac{m}{3}\left\lfloor \frac{n}{3} \right\rfloor + \frac{m}{3} + 4 \tag{9}$$

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We conduct a proper dominator coloring on $P_3 \boxtimes P_n$ using $\frac{m}{3}\left\lfloor\frac{n}{3}\right\rfloor+\frac{m}{3}+4$ colors. We define this dominator coloring as $C=(V_1,V_2,\dots,V_{\left\lfloor\frac{mn}{9}\right\rfloor+\frac{m}{3}+4},V_{\left\lfloor\frac{mn}{9}\right\rfloor+\frac{m}{3}+4})$ with:

For $1 \le i \le \left\lceil\frac{m}{2}\right\rceil$:

If $2i-1=1+6l;3+6l$: $0 \le l \le \left\lfloor\frac{m}{6}\right\rfloor$:

$V_1=\left\{(2i-1,2j-1):;1 \le j \le \left\lceil\frac{n}{2}\right\rceil\right\}$;

$V_3=\left\{(2i-1,2j):1 \le j \le \left\lfloor\frac{n}{2}\right\rfloor\right\}$;

If $2i-1=5+6l;\ 0 \le l \le \left\lfloor\frac{m}{6}\right\rfloor$:

$V_1=\left\{(2i-1,1+6j),(2i-1,3+6j):0 \le j \le \left\lfloor\frac{n}{3}\right\rfloor-2\right\}$;

$V_3=\left\{(2i-1,1+6j),(2i-1,3+6j):0 \le j \le \left\lfloor\frac{n}{3}\right\rfloor-2\right\}$;

If $2i=2+6l;4+6l$: $0 \le l \le \left\lfloor\frac{m}{6}\right\rfloor$:

$V_2=\left\{(2i,1+6j),(2i,3+6j):0 \le j \le \left\lfloor\frac{n}{3}\right\rfloor-2\right\}$;

$V_4=\left\{(2i,4+6j),(2i,6+6j):0 \le j \le \left\lfloor\frac{n}{3}\right\rfloor-2\right\}$;

If $2i=6+6l;\ 0 \le l \le \left\lfloor\frac{m}{6}\right\rfloor$:

$V_2=\left\{(2i,2j-1):0 \le j \le \left\lfloor\frac{n}{3}\right\rfloor-2\right\}$;

$V_4=\left\{(2i,2j):0 \le j \le \left\lfloor\frac{n}{3}\right\rfloor-2\right\}$; For every block $B_{i,j}:1 \le i \le \frac{m}{3};1 \le j \le \left\lfloor\frac{n}{3}\right\rfloor$ we include an entire color class by assigning a unique color to $\{(3i-1,3j-1)\}$; thus we add $\frac{m}{3}\left\lfloor\frac{n}{3}\right\rfloor$ color classes.

For every sub-block $SB_i:1 \le i \le \frac{m}{3}$ we also include an entire color class by assigning a unique color to $\{(3i-1,n)\}$; thus we add $\frac{m}{3}$ color classes, which makes the total number of used color classes equal $\frac{m}{3}\left\lfloor\frac{n}{3}\right\rfloor+\frac{m}{3}+4$. Therefore:

$$\chi_d(P_m \boxtimes P_n) \le \frac{m}{3}\left\lfloor\frac{n}{3}\right\rfloor+\frac{m}{3}+4 \tag{10}$$

From (9) and (10) we prove that $\chi_d(P_m \boxtimes P_n)=\frac{m}{3}\left\lfloor\frac{n}{3}\right\rfloor+\frac{m}{3}+4$ if $m \equiv 0(mod\ 3)$ and $n \equiv 1(mod\ 3)$.

Case 3. $m \equiv 0(mod\ 3)$ and $n \equiv 2(mod\ 3)$.

similarly to Case 2, the $\frac{m}{3}\left\lfloor\frac{n}{3}\right\rfloor$ blocks are $B_{i,j}:1 \le i \le \frac{m}{3};1 \le j \le \left\lfloor\frac{n}{3}\right\rfloor$ when:

$$B_{i,j}=\{(3i-2,3j-2),(3i-2,3j-1),(3i-2,3j),(3i-1,3j-2),$$
$$(3i-1,3j-1),(3i-1,3j),(3i,3j-2),(3i,3j-1),(3i,3j)\}.$$

While the $\frac{m}{3}$ sub-blocks are $SB_i:1 \le i \le \frac{m}{3}$ so that:

$SB_i=\{(3i-2,n-1),(3i-1,n-1),(3i,n-1),(3i-2,n),(3i-1,n),(3i,n)\}$.

For every block and every sub-block we need to include an entire color class which means:

$$\chi_d(P_m \boxtimes P_n) \ge \frac{m}{3}\left\lfloor\frac{n}{3}\right\rfloor+\frac{m}{3}+4 \tag{11}$$

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Now we conduct a proper dominator coloring on $P_3 \boxtimes P_n$ using $\frac{m}{3}\left\lfloor\frac{n}{3}\right\rfloor + \frac{m}{3} + 4$ colors defined as $C = (V_1, V_2, \dots, V_{\left\lfloor\frac{mn}{9}\right\rfloor + \frac{m}{3} + 4}, V_{\left\lfloor\frac{mn}{9}\right\rfloor + \frac{m}{3} + 4})$ with:

For $1 \le i \le \left\lceil\frac{m}{2}\right\rceil$:

If $2i - 1 = 1 + 6l; 3 + 6l: 0 \le l \le \left\lfloor\frac{m}{6}\right\rfloor$:

$V_1 = \left\{(2i - 1, 2j - 1): ; 1 \le j \le \left\lceil\frac{n}{2}\right\rceil\right\}$;

$V_3 = \left\{(2i - 1, 2j): 1 \le j \le \left\lfloor\frac{n}{2}\right\rfloor\right\}$;

If $2i - 1 = 5 + 6l;\ 0 \le l \le \left\lfloor\frac{m}{6}\right\rfloor$:

$V_1 = \left\{(2i - 1, 1 + 6j), (2i - 1, 3 + 6j): 0 \le j \le \left\lfloor\frac{n}{3}\right\rfloor - 2\right\} \cup \{(2i - 1, n - 1)\}$;

$V_3 = \left\{(2i - 1, 1 + 6j), (2i - 1, 3 + 6j): 0 \le j \le \left\lfloor\frac{n}{3}\right\rfloor - 2\right\}$;

If $2i = 2 + 6l; 4 + 6l: 0 \le l \le \left\lfloor\frac{m}{6}\right\rfloor$:

$V_2 = \left\{(2i, 1 + 6j), (2i, 3 + 6j): 0 \le j \le \left\lfloor\frac{n}{3}\right\rfloor - 2\right\}$;

$V_4 = \left\{(2i, 4 + 6j), (2i, 6 + 6j): 0 \le j \le \left\lfloor\frac{n}{3}\right\rfloor - 2\right\}$;

If $2i = 6 + 6l;\ 0 \le l \le \left\lfloor\frac{m}{6}\right\rfloor$:

$V_2 = \left\{(2i, 2j - 1): 0 \le j \le \left\lfloor\frac{n}{3}\right\rfloor - 2\right\} \cup \{(2i, n - 1)\}; ;$

$V_4 = \left\{(2i, 2j): 0 \le j \le \left\lfloor\frac{n}{3}\right\rfloor - 2\right\}$;

$$\chi_d(P_m \boxtimes P_n) \le \frac{m}{3}\left\lfloor\frac{n}{3}\right\rfloor + \frac{m}{3} + 4 \tag{12}$$

From (11) and (12) we conclude that $\chi_d(P_m \boxtimes P_n) = \frac{m}{3}\left\lfloor\frac{n}{3}\right\rfloor + \frac{m}{3} + 4$ if $m \equiv 0 (mod\ 3)$ and $n \equiv 2 (mod\ 3)$.

The remaining cases are repetitive. In all of the remaining cases, we can notice that $P_m \boxtimes P_n[(i,j): 1 \le i \le \left\lfloor\frac{m}{3}\right\rfloor;\ 1 \le j \le \left\lfloor\frac{n}{3}\right\rfloor]$ are divided into $\left\lfloor\frac{m}{3}\right\rfloor\left\lfloor\frac{n}{3}\right\rfloor$ blocks each of which contains nine vertices. If $m \equiv 1,2 (mod\ 3)$. the last row (two rows) are divided into $\left\lfloor\frac{n}{3}\right\rfloor$ sub-blocks, each of which contains three vertices (six vertices) respectively. In a similar way, if $n \equiv 1,2 (mod\ 3)$. the last column (two columns) are divided into $\left\lfloor\frac{m}{3}\right\rfloor$ sub-blocks, each of which contains three vertices (six vertices) respectively. Every block and every sub-block must contain a unique color class and this distribution forms a proper dominator coloring on $P_m \boxtimes P_n$.

**Observation 3:** If $m \equiv k_1 (mod\ 3)$ and $n \equiv k_2 (mod\ 3)$ where $k_1, k_2 \in \{1,2\}$, we notice that $V(P_m \boxtimes P_n)$ can be divided into:

- $\left\lfloor\frac{m}{3}\right\rfloor\left\lfloor\frac{n}{3}\right\rfloor$ blocks $B_{i,j}: 1 \le i \le \left\lfloor\frac{m}{3}\right\rfloor; 1 \le j \le \left\lfloor\frac{n}{3}\right\rfloor$ as defined earlier.
- $\left\lfloor\frac{m}{3}\right\rfloor$ sub-blocks $SB_i: 1 \le i \le \left\lfloor\frac{m}{3}\right\rfloor$ which belong to column $n$ if $k_2 = 1$ and to columns $n - 1, n$ if $k_2 = 2$.
- $\left\lfloor\frac{n}{3}\right\rfloor$ sub-blocks $SB_j: 1 \le j \le \left\lfloor\frac{n}{3}\right\rfloor$ which belong to row $n$ if $k_1 = 1$ and to rows $n - 1, n$ if $k_1 = 2$.
- One mini sub-block denoted by $MSB$, so that:
    - ❖ $MSB = \{(m, n)\}$ if $k_1 = k_2 = 1$.
    - ❖ $MSB = \{(m, n - 1), (m, n)\}$ if $k_1 = 2; k_2 = 1$.
    - ❖ $MSB = \{(m - 1, n), (m, n)\}$ if $k_1 = 1; k_2 = 2$.
    - ❖ $MSB = \{(m - 1, n - 1), (m - 1, n), (m, n - 1), (m, n)\}$ if $k_1 = k_2 = 2$.

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Regardless of $|MSB|$, we notice that $MSB$ must contain at least one entire color class or else a contradiction occurs since $(m, n)$ would not dominate an entire color class. This makes the total number of required unique color classes on $SB_i: 1 \le i \le \left\lfloor\frac{m}{3}\right\rfloor$, $SB_j: 1 \le j \le \left\lfloor\frac{n}{3}\right\rfloor$ and $MSB$ equal $\left\lfloor\frac{m}{3}\right\rfloor + \left\lfloor\frac{n}{3}\right\rfloor + 1$.

Case 4. $m \equiv 1(mod\ 3)$ and $n \equiv 0(mod\ 3)$. Then $\chi_d(P_m \boxtimes P_n) = \left\lfloor\frac{m}{3}\right\rfloor\frac{n}{3} + \frac{n}{3} + 4$.

Case 5. $m \equiv 1(mod\ 3)$ and $n \equiv 1(mod\ 3)$. Then $\chi_d(P_m \boxtimes P_n) = \left\lfloor\frac{m}{3}\right\rfloor\left\lfloor\frac{n}{3}\right\rfloor + \left\lfloor\frac{m}{3}\right\rfloor + \left\lfloor\frac{n}{3}\right\rfloor + 5$.

Case 6. $m \equiv 1(mod\ 3)$ and $n \equiv 2(mod\ 3)$. Then $\chi_d(P_m \boxtimes P_n) = \left\lfloor\frac{m}{3}\right\rfloor\left\lfloor\frac{n}{3}\right\rfloor + \left\lfloor\frac{m}{3}\right\rfloor + \left\lfloor\frac{n}{3}\right\rfloor + 5$. Figure 2 shows that $\chi_d(P_7 \boxtimes P_8) \le 13$.

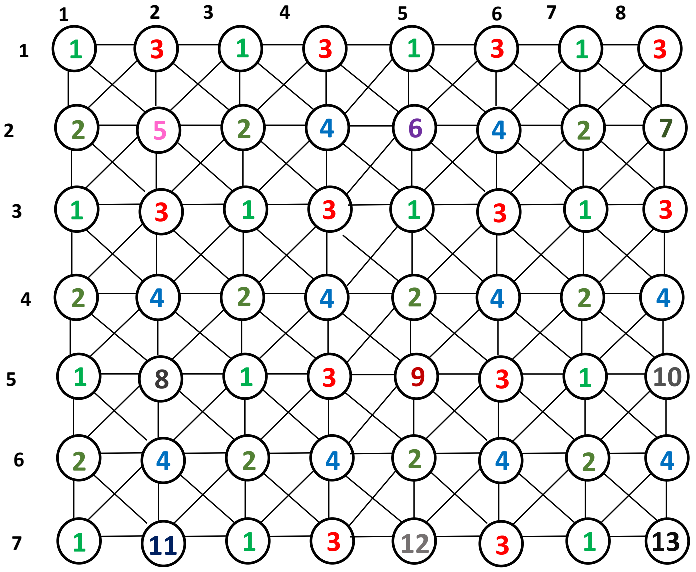

**Figure 2.** $\chi_d(P_7 \boxtimes P_8) \le 13$.

Case 7. $m \equiv 2(mod\ 3)$ and $n \equiv 0(mod\ 3)$. Then $\chi_d(P_m \boxtimes P_n) = \left\lfloor\frac{m}{3}\right\rfloor\frac{n}{3} + \frac{n}{3} + 4$.

Case 8. $m \equiv 2(mod\ 3)$ and $n \equiv 1(mod\ 3)$. Then $\chi_d(P_m \boxtimes P_n) = \left\lfloor\frac{m}{3}\right\rfloor\left\lfloor\frac{n}{3}\right\rfloor + \left\lfloor\frac{m}{3}\right\rfloor + \left\lfloor\frac{n}{3}\right\rfloor + 5$.

Case 9. $m \equiv 2(mod\ 3)$ and $n \equiv 2(mod\ 3)$. Then $\chi_d(P_m \boxtimes P_n) = \left\lfloor\frac{m}{3}\right\rfloor\left\lfloor\frac{n}{3}\right\rfloor + \left\lfloor\frac{m}{3}\right\rfloor + \left\lfloor\frac{n}{3}\right\rfloor + 5$.

From all the previous cases we conclude the requested.■

**Theorem 3.** Let $Q_{2,n}$ be a Queen graph with $n \ge 2$;

$$\chi_d(Q_{2,n}) = \begin{cases} 4 \ if \ \ n = 2; \\ 6 \ if \ n = 3; \\ n \ otherwise. \end{cases}$$

**Proof:** We consider the following cases for $n$:

Case 1. $n = 2$. It is obvious that $\chi(Q_{2,2}) = 4 = |V(Q_{2,2})|$. Due to Proposition 1 we conclude that $\chi_d(Q_{2,2}) = |V(Q_{2,2})| = 4$.

Case 2. $n = 3$. In a similar way to Case 1 we notice that $\chi(Q_{2,3}) = 6 = |V(Q_{2,3})|$. As an immediate consequence of Proposition 1, $\chi_d(Q_{2,3}) = |V(Q_{2,3})| = 6$.

Case 3. $n \ge 4$. We notice that the vertices of $Q_{2,n}$ form two sets, which are:

$$R_1 = \{(1, j): 1 \le j \le n\};$$

$$R_2 = \{(2, j): 1 \le j \le n\}.$$

Since every vertex of $R_1$ is adjacent to all other vertices of $R_1$, we immediately conclude that:

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$$\chi(Q_{2,n}) \geq n \tag{13}$$

We apply a proper coloring on $Q_{2,n}$ using $n$ colors. We define this coloring as $C = (V_1, V_2, \dots, V_{n-1}, V_n)$ with:

For $1 \leq i \leq n-2$: $V_i = \{(1,i),\ (2,i+2)\}$;

$V_{n-1} = \{(1,n-1), (2,1)\}$;

$V_n = \{(1,n), (2,2)\}$;

This means:

$$\chi(Q_{2,n}) \leq n \tag{14}$$

From (13) and (14) we prove that:

$$\chi(Q_{2,n}) = n \tag{15}$$

From (15) and due to Proposition 1 we conclude that:

$$\chi_d(Q_{2,n}) \geq n \tag{16}$$

For $1 \leq j \leq n$, since every vertex $(1,j)$ dominates $(2,j)$ and every vertex of $R_1$, then $(1,j)$ dominates the color class that contains $(2,j)$. The same argument applies to every vertex $(2,j)$: $1 \leq j \leq n$. Therefore, the proper coloring defined earlier $(C)$ is a proper dominator coloring of $Q_{2,n}$ with $n$ colors. This means:

$$\chi_d(Q_{2,n}) \leq n \tag{17}$$

Figure 3 illustrates that $\chi_d(Q_{2,5}) \leq 5$. From (16) and (17) we prove that $\chi_d(Q_{2,n}) = n$ if $n \geq 4$. From all the cases we prove the requested.

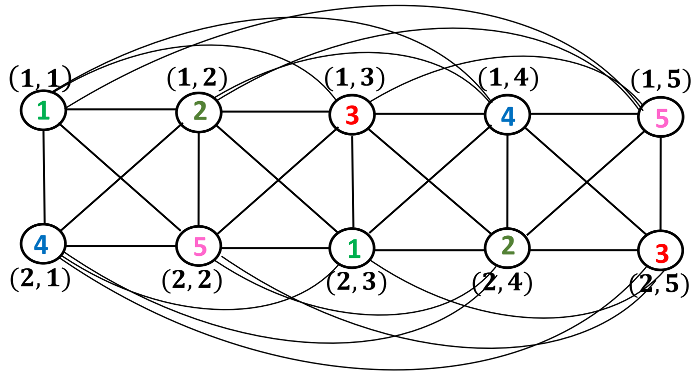

**Figure 3.** $\chi_d(Q_{2,5}) \leq 5$.

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