



Neutrosophic fuzzy metric spaces and fixed points results with integral contraction type

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Abstract

In this study, we introduce fixed point theorems related to integral type contractions, framed within the advanced context of neutrosophic fuzzy metric spaces. Additionally, we derive multiple fixed point results that are relevant to this particular setting.

Keywords: Fixed point; Neutrosophic fuzzy metric; Integral contraction

1 Introduction

The notion of Fuzzy Sets (FSs), first introduced by Zadeh,¹ has significantly influenced a wide range of scientific fields since its emergence. While this framework is highly pertinent to practical applications, it has not consistently offered satisfactory solutions to various problems over the years. As a result, there has been a renewed focus on research aimed at resolving these difficulties. In this context, Atanassov² proposed Intuitionistic Fuzzy Sets (IFSs) to tackle such challenges. Furthermore, the Neutrosophic Set (NS), created by Smarandache,³ serves as a sophisticated extension of traditional set theory. Other important generalizations include interval-valued FS,⁴ interval-valued IFS,⁵ as well as paraconsistent, dialetheist, paradoxist, and tautological sets,⁶ along with Pythagorean fuzzy sets.⁷ Neutrosophic sets demonstrate a wide array of applications across various fields. For example, Barbosa and Smarandache⁸ introduced the Neutrosophic One-Round Zero-Knowledge Proof protocol (N-1-R) ZKP, which expands upon the One-Round (1-R) ZKP framework by incorporating Neutrosophic numbers. Additionally, the authors in⁹ offer a detailed characterization of efficient and optimally suitable solutions pertaining to scalar optimization challenges. They also delineate the Kuhn-Tucker conditions that are pertinent to both efficiency and proper efficiency. For a more in-depth investigation into the applications of neutrosophic sets and their extensive uses, it is advisable to refer to the literature cited in^{2,7,14-16} and the related references therein. We are looking to integrate our work with new concepts, as in the application of^{2,18-20} The Banach fixed-point theorem,¹⁷ commonly known as the Banach contraction principle, represents a pivotal concept in mathematics, especially within the realm of metric spaces. This theorem guarantees both the existence and uniqueness of fixed points for certain self-maps in metric spaces, thereby offering a structured approach to locating these fixed points. Essentially, the Banach fixed-point theorem provides a comprehensive foundation for Picard's method of successive approximations. Introduced by Stefan Banach (1892–1945) in 1922, this theorem has inspired a multitude of mathematicians to explore various extensions and generalizations across diverse mathematical fields, as indicated by the references in^{2,23-26} A notable illustration of this is the concept of neutrosophic metric space (NMS), which was first introduced by Kirisci and Simsek.²⁹ This framework has been employed to examine a variety of fixed point theorems.

2 Preliminary

In this framework, the interval $]0-, 1 + [$ is characterized as a non-standard unit interval. Within this context, non-standard finite numbers are articulated as $(1+) = 1 + \epsilon$, where "1" represents the standard component and ϵ denotes the non-standard element. In a similar manner, $(0-) = 0 - \epsilon$, with "0" indicating the standard component and ϵ as the non-standard element. The numbers 0 and 1 can be interpreted as non-standard values that are infinitesimally small yet less than 0 or infinitesimally small yet greater than 1, respectively, and these values are encompassed within the non-standard unit interval $]0-, 1 + [$.

Definition 2.1.¹ In relation to a universal set U , a fuzzy set F is defined by the notation $F = \{ \langle a, \mu_F(\xi) \rangle : 0 \leq \mu_F(\xi) \leq 1, \xi \in U \}$. In this context, $\mu_F(\xi)$ represents the degree of membership of the element ξ within the fuzzy set F .

Definition 2.2.³ A neutrosophic set V relative to a universal set U is defined as $V = \{ \langle \xi, (T_N(\xi), I_N(\xi), F_N(\xi)) \rangle : \xi \in U, T_N(\xi), I_N(\xi), F_N(\xi) \in]0-, 1 + [\}$. In this context, $T_N(\xi)$, $I_N(\xi)$, and $F_N(\xi)$ represent the membership degrees of truth, indeterminacy, and falsity for an element ξ within the set V , respectively, while $]0-, 1 + [$ signifies a non-standard unit interval.

Definition 2.3.²⁷ A neutrosophic fuzzy set B within a universal set U is characterized as follows: $B = \{ \langle x, (\mu_B(\xi), T_B(\xi, \mu), I_B(\xi, \mu), F(\xi, \mu)) \rangle : \xi \in U, \mu_B(\xi) \in [0, 1], T_B(\xi, \mu), I_B(\xi, \mu), F(\xi, \mu) \in]0-, 1 + [\}$. In this framework, the membership degree $\mu_B(\xi)$ is represented by three distinct components: the truth membership grade $T_B(\xi, \mu)$, the indeterminacy membership grade $I_B(\xi, \mu)$, and the falsity membership grade $F(\xi, \mu)$. The notation $]0-, 1 + [$ signifies a nonstandard unit interval.

Triangular norms (shortly TN), initially introduced by Menger,²⁸ are a fundamental concept in mathematical analysis. Menger's innovative approach involved using probability distributions to assess the distance between two elements within a specific space, moving beyond the traditional reliance on numerical values. This technique facilitates the extension of the triangle inequality in metric spaces through the application of triangular norms. In contrast, triangular conorms (shortly CN) serve as the dual counterparts to t-norms. Both TN and CN are essential in fuzzy operations, especially concerning intersections and unions.

Through this manuscript, $\mathbb{R}^+ = [0, \infty)$, $I = [0, 1]$.

Definition 2.4. Consider an operation $\diamond : I \times I \rightarrow I$. This operation is classified as continuous TN (CTN) if it meets the following criteria: for any elements $\sigma, \sigma', t, t' \in I$.

1. $\sigma \diamond 1 = \sigma$,
2. If $\sigma \leq \sigma'$ and $t \leq t'$, then $\sigma \diamond t \leq \sigma' \diamond t'$,
3. \diamond is continuous,
4. \diamond is commutative and associate.

Definition 2.5. Consider an operation $\bullet : I \times I \rightarrow I$. This operation is classified as continuous TN (CTN) if it meets the following criteria: for all elements $\sigma, \sigma', t, t' \in I$.

1. $\sigma \bullet 0 = s$,
2. If $\sigma \leq \sigma'$ and $t \leq t'$, then $\sigma \bullet t \leq \sigma' \bullet t'$,
3. \bullet is continuous,
4. \bullet is commutative and associate.

Definition 2.6.²⁹ A 6-tuple $(F, \Pi, \mathcal{U}, \Theta, \diamond, \bullet)$ is referred to as a Neutrosophic Metric Space (NMS) if the set F is a non-empty arbitrary collection, \diamond signifies a continuous t-norm, \bullet indicates a continuous t-conorm, and the elements Π, \mathcal{U} , and Θ are three fuzzy sets established on the Cartesian product $F^2 \times (0, \infty)$. These components must satisfy the following specific conditions for all elements $\xi, \omega, c \in F$ and for all positive real numbers λ, ρ .

1. $0 \leq \Pi(\xi, \omega, \lambda) \leq 1, 0 \leq \mathcal{U}(\xi, \omega, \lambda) \leq 1, 0 \leq \Theta(\xi, \omega, \lambda) \leq 1,$
2. $0 \leq \Pi(\xi, \omega, \lambda) + \mathcal{U}(\xi, \omega, \lambda) + \Theta(\xi, \omega, \lambda) \leq 3,$
3. $\Pi(\xi, \omega, \lambda) = 1,$ for $\lambda > 0$ iff $\xi = \omega$
4. $\Pi(\xi, \omega, \lambda) = H(\omega, \xi, \lambda),$ for $\lambda > 0$
5. $\Pi(\xi, \omega, \lambda) \diamond \Pi(\omega, c, \rho) \leq \Pi(\xi, c, \lambda + \rho)$
6. $\Pi(\xi, \omega, \cdot) : \mathbb{R}^+ \rightarrow I$ is continuous
7. $\lim_{\lambda \rightarrow \infty} \Pi(\xi, \omega, \lambda) = 1$
8. $\mathcal{U}(\xi, \omega, \lambda) = 0$ iff $\xi = \omega$
9. $\mathcal{U}(\xi, \omega, \lambda) = \mathcal{U}(\omega, \xi, \lambda),$
10. $\mathcal{U}(\xi, \omega, \lambda) \bullet \mathcal{U}(\omega, c, \rho) \geq \mathcal{U}(\xi, c, \lambda + \rho),$
11. $\mathcal{U}(\xi, \omega, \cdot) : \mathbb{R}^+ \rightarrow I$ is continuous
12. $\lim_{\lambda \rightarrow \infty} \mathcal{U}(\xi, \omega, \lambda) = 0$
13. $\Theta(\xi, \omega, \lambda) = 0,$ for $\lambda > 0$ iff $\xi = \omega$
14. $\Theta(\xi, \omega, \lambda) = \Theta(\omega, \xi, \lambda),$
15. $\Theta(\xi, \omega, \lambda) \bullet \Theta(\omega, c, \rho) \geq S(\xi, c, \lambda + \rho),$
16. $\Theta(\xi, \omega, \cdot) : \mathbb{R}^+ \rightarrow I$ is continuous
17. $\lim_{\lambda \rightarrow \infty} \Theta(\xi, \omega, \lambda) = 0$
18. If $\lambda \leq 0,$ then $\Pi(\xi, \omega, \lambda) = 0, \mathcal{U}(\xi, \omega, \lambda) = \Theta(\xi, \omega, \lambda) = 1$

The functions $\Pi(\xi, \omega, \lambda), \mathcal{U}(\xi, \omega, \lambda),$ and $\Theta(\xi, \omega, \lambda)$ represent the degrees of nearness, neutralness, and non-nearness between the elements ξ and ω in relation to the parameter $\lambda,$ respectively.

Recently, Ghosh et al.³⁰ presented the notion of neutrosophic fuzzy metric spaces and examined various topological characteristics associated with this concept.

Definition 2.7.³⁰ A 7-tuple $(F, \Pi, \Psi, \mathcal{U}, \Theta, \diamond, \bullet)$ is known as a Neutrosophic Fuzzy Metric Space (NFMS) if F is an arbitrary set, \diamond is a continuous t-norm, \bullet is a continuous t-conorm, and $\Pi, \Psi, \mathcal{U},$ and Θ are fuzzy sets on $F^2 \times (0, \infty)$ satisfying the following conditions for all $\xi, \omega, c, \in F$ and $\lambda, \rho > 0.$

1. $0 \leq \Pi(\xi, \omega, \lambda) \leq 1, 0 \leq \Psi(\xi, \omega, \lambda) \leq 1, 0 \leq \mathcal{U}(\xi, \omega, \lambda) \leq 1, 0 \leq \Theta(\xi, \omega, \lambda) \leq 1,$
2. $0 \leq \Pi(\xi, \omega, \lambda) + \Psi(\xi, \omega, \lambda) + \mathcal{U}(\xi, \omega, \lambda) + \Theta(\xi, \omega, \lambda) \leq 4,$
3. $\Pi(\xi, \omega, \lambda) = 1,$
4. $\Pi(\xi, \omega, \lambda) = H(\omega, \xi, \lambda),$
5. $\Pi(\xi, \omega, \lambda) \diamond \Pi(\omega, c, \rho) \leq \Pi(\xi, c, \lambda + \rho),$ for $\rho, \lambda > 0$
6. $\Pi(\xi, \omega, \cdot) : \mathbb{R}^+ \rightarrow I$ is continuous
7. $\lim_{\lambda \rightarrow \infty} \Pi(\xi, \omega, \lambda) = 1$
8. $\Psi(\xi, \omega, \lambda) = 1,$ iff $\xi = \omega$

9. $\Psi(\xi, \omega, \lambda) = \Psi(\omega, \xi, \lambda)$, for $\lambda > 0$
10. $\Psi(\xi, \omega, \lambda) \diamond \Psi(\omega, c, \rho) \leq \Psi(\xi, c, \lambda + \rho)$,
11. $\Psi(\xi, \omega, \cdot) : \mathbb{R}^+ \rightarrow I$ is continuous
12. $\lim_{\lambda \rightarrow \infty} \Psi(\xi, \omega, \lambda) = 1$
13. $\mathcal{U}(\xi, \omega, \lambda) = 0$, iff $\xi = \omega$
14. $\mathcal{U}(\xi, \omega, \lambda) = \mathcal{U}(\omega, \xi, \lambda)$,
15. $\mathcal{U}(\xi, \omega, \lambda) \bullet \mathcal{U}(\omega, c, \rho) \geq \mathcal{U}(\xi, c, \lambda + \rho)$,
16. $\mathcal{U}(\xi, \omega, \cdot) : \mathbb{R}^+ \rightarrow I$ is continuous
17. $\lim_{\lambda \rightarrow \infty} \mathcal{U}(\xi, \omega, \lambda) = 0$
18. $\Theta(\xi, \omega, \lambda) = 0$, iff $\xi = \omega$
19. $\Theta(\xi, \omega, \lambda) = \Theta(\omega, \xi, \lambda)$,
20. $\Theta(\xi, \omega, \lambda) \bullet \Theta(\omega, c, \rho) \geq S(\xi, c, \lambda + \rho)$,
21. $\Theta(\xi, \omega, \cdot) : \mathbb{R}^+ \rightarrow I$ is continuous
22. $\lim_{\lambda \rightarrow \infty} \Theta(\xi, \omega, \lambda) = 0$
23. If $\lambda \leq 0$, then $\Pi(\xi, \omega, \lambda) = 0, \mathcal{U}(\xi, \omega, \lambda) = \Theta(\xi, \omega, \lambda) = 1$

In this context, $\Pi(\xi, \omega, \lambda)$ represents the certainty that distance between ξ and ω is less than λ , $\Psi(\xi, \omega, \lambda)$ represents the degree of nearness, $\mathcal{U}(\xi, \omega, \lambda)$ stands for the degree of neutralness, and $\Theta(\xi, \omega, \lambda)$ denotes the degree of non-nearness between ξ and ω with respect to λ , respectively.

The convergence, Cauchyness, completeness are given as follows.

Definition 2.8.³⁰ Let (ξ_n) be a sequence in a NFMS $(F, \Pi, \Psi, \mathcal{U}, \Theta, \diamond, \bullet)$. Then

1. (ξ_n) converges to $\xi \in F$ iff for a given $\epsilon \in (0, 1)$, $\lambda > 0$ there is $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$

$$\Pi(\xi_n, \xi, \lambda) > 1 - \epsilon, \mathcal{U}(\xi_n, \xi, \lambda) < \epsilon, \Theta(\xi_n, \xi, \lambda) < \epsilon$$

i.e.,

$$\lim_{n \rightarrow \infty} \Pi(\xi_n, \xi, \lambda) = 1, \lim_{n \rightarrow \infty} \Psi(\xi_n, \xi, \lambda) = 1, \lim_{n \rightarrow \infty} \mathcal{U}(\xi_n, \xi, \lambda) = 0, \lim_{n \rightarrow \infty} \Theta(\xi_n, \xi, \lambda) = 0$$

2. (ξ_n) is called Cauchy iff for a given $\epsilon \in (0, 1)$, $\lambda > 0$ there is $n_0 \in \mathbb{N}$ such that for each $n, m \geq n_0$

$$\Pi(\xi_n, \xi_m, \lambda) > 1 - \epsilon, \Psi(\xi_n, \xi_m, \lambda) > 1 - \epsilon, \mathcal{U}(\xi_n, \xi_m, \lambda) < \epsilon, \Theta(\xi_n, \xi_m, \lambda) < \epsilon$$

i.e.,

$$\lim_{n, m \rightarrow \infty} \Pi(\xi_n, \xi_m, \lambda) = 1, \lim_{n, m \rightarrow \infty} \Psi(\xi_n, \xi_m, \lambda) = 1, \lim_{n, m \rightarrow \infty} \mathcal{U}(\xi_n, \xi_m, \lambda) = 0, \lim_{n, m \rightarrow \infty} \Theta(\xi_n, \xi_m, \lambda) = 0$$

3. $(F, \Pi, \Psi, \mathcal{U}, \Theta, \diamond, \bullet)$ is called complete if each Cauchy sequence is convergent to an element in F .

3 Main Result

Definition 3.1. In this context, we define a real-valued function of three variables, denoted as f , to possess the property (UC) if, for any sequences (ξ_n) and (ω_n) , the following equality holds:

$$\lim_{\lambda \rightarrow \lambda_0} \lim_{n \rightarrow \infty} f(\xi_n, \omega_n, \lambda) = \lim_{n \rightarrow \infty} \lim_{\lambda \rightarrow \lambda_0} f(\xi_n, \omega_n, \lambda).$$

whenever the two limits are exist.

Throughout the remainder of this study, we will assume that each of the fuzzy sets $\Pi, \Psi, \mathcal{U}, \Theta$ exhibits the UC property.

We will commence with several pertinent lemmas.

Lemma 3.2. Let $(F, \Pi, \Psi, \mathcal{U}, \Theta, \diamond, \bullet)$ be a NFMS. Then

1. $\Pi(\xi, \omega, \cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is non-decreasing
2. $\Psi(\xi, \omega, \cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is non-decreasing
3. $\mathcal{U}(\xi, \omega, \cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is non-increasing
4. $\Theta(\xi, \omega, \cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is non-increasing

Proof. (1) Let $\lambda_1, \lambda_2 > 0$, with $\lambda_1 > \lambda_2$. Then, there is $\delta > 0$ such that $\lambda_1 = \lambda_2 + \delta$. From (5), we get

$$\begin{aligned} \Pi(\xi, \omega, \lambda_1) &= \Pi(\xi, \omega, \lambda_2 + \delta) \\ &\geq \Pi(\xi, \omega, \lambda_2) \diamond \Pi(\omega, \omega, \delta) \\ &= \Pi(\xi, \omega, \lambda_2). \end{aligned}$$

The proofs for (2),(3) and (4) are identical to that of (1). □

Lemma 3.3. Let $(F, \Pi, \Psi, \mathcal{U}, \Theta, \diamond, \bullet)$ be a NFMS, and let (ξ_n) be a sequence such that for $\lambda > 0$

$$\begin{aligned} \Pi(\xi_p, \xi_q, \lambda) &\geq \Pi(\xi_{p-1}, \xi_{q-1}, \lambda) \\ \Psi(\xi_p, \xi_q, \lambda) &\geq \Psi(\xi_{p-1}, \xi_{q-1}, \lambda) \\ \mathcal{U}(\xi_p, \xi_q, \lambda) &\leq \mathcal{U}(\xi_{p-1}, \xi_{q-1}, \lambda) \\ \Theta(\xi_p, \xi_q, \lambda) &\leq \Theta(\xi_{p-1}, \xi_{q-1}, \lambda) \end{aligned} \tag{1}$$

and

$$\lim_{n \rightarrow \infty} \Pi(\xi_n, \xi_{n+1}, \lambda) = 1, \quad \lim_{n \rightarrow \infty} \Psi(\xi_n, \xi_{n+1}, \lambda) = 1, \quad \lim_{n \rightarrow \infty} \mathcal{U}(\xi_n, \xi_{n+1}, \lambda) = 0, \quad \text{and} \quad \lim_{n \rightarrow \infty} \Theta(\xi_n, \xi_{n+1}, \lambda) = 0. \tag{2}$$

If (ξ_n) is not Cauchy, then there exist an $1 > \epsilon > 0$ and $\lambda > 0$ along with two subsequences (ξ_{n_k}) and (ξ_{m_k}) derived from (ξ_n) , where (m_k) such that one at least of the following holds.

$$\begin{aligned} \lim_{k \rightarrow \infty} \Pi(\xi_{n_k}, \xi_{m_k}, \lambda) &= 1 - \epsilon, \\ \lim_{k \rightarrow \infty} \Psi(\xi_{n_k}, \xi_{m_k}, \lambda) &= 1 - \epsilon, \\ \lim_{k \rightarrow \infty} \mathcal{U}(\xi_{n_k}, \xi_{m_k}, \lambda) &= \epsilon, \\ \lim_{k \rightarrow \infty} \Theta(\xi_{n_k}, \xi_{m_k}, \lambda) &= \epsilon. \end{aligned}$$

Proof. If (ξ_n) is not Cauchy, then for each $\lambda > 0$

$$\lim_{n,m \rightarrow \infty} \Pi(\xi_n, \xi_m, \lambda) \neq 1, \text{ or } \lim_{n,m \rightarrow \infty} \Psi(\xi_n, \xi_m, \lambda) \neq 1, \text{ or } \lim_{n,m \rightarrow \infty} \mathcal{U}(\xi_n, \xi_m, \lambda) \neq 0, \text{ or } \lim_{n,m \rightarrow \infty} \Theta(\xi_n, \xi_m, \lambda) \neq 0.$$

Case 1: If $\lim_{n,m \rightarrow \infty} \Pi(\xi_n, \xi_m, \lambda) \neq 1$, then there are $\lambda > 0$, and $\epsilon > 0$ along with two subsequences (ξ_{n_k}) and (ξ_{m_k}) derived from (ξ_n) , where (m_k) is selected as the smallest index satisfying the condition.

$$\Pi(\xi_{n_k}, \xi_{m_k}, \lambda) \leq 1 - \epsilon, \quad m_k > n_k > k. \tag{3}$$

This implies that

$$\Pi(\xi_{n_k}, \xi_{m_{k-1}}, \lambda) > 1 - \epsilon. \tag{4}$$

chose $\delta > 0$. Then

$$\begin{aligned} \Pi(\xi_{n_k}, \xi_{m_k}, \lambda + \delta) &\geq \Pi(\xi_{n_k}, \xi_{m_{k-1}}, \lambda) \diamond \Pi(\xi_{m_{k-1}}, \xi_{m_k}, \delta) \\ &> (1 - \epsilon) \diamond \Pi(\xi_{m_{k-1}}, \xi_{m_k}, \delta) \end{aligned}$$

Using Equation 2, we get

$$\liminf_{k \rightarrow \infty} \Pi(\xi_{n_k}, \xi_{m_k}, \lambda + \delta) \geq (1 - \epsilon).$$

Also,

$$\begin{aligned} (1 - \epsilon) &\leq \lim_{\delta \rightarrow 0^+} \liminf_{k \rightarrow \infty} \Pi(\xi_{n_k}, \xi_{m_k}, \lambda + \delta) \\ &= \liminf_{k \rightarrow \infty} \lim_{\delta \rightarrow 0^+} \Pi(\xi_{n_k}, \xi_{m_k}, \lambda + \delta) \\ &= \liminf_{k \rightarrow \infty} \Pi(\xi_{n_k}, \xi_{m_k}, \lambda). \end{aligned}$$

Also, from 3, it follows

$$\limsup_{k \rightarrow \infty} \Pi(\xi_{n_k}, \xi_{m_k}, \lambda) \leq (1 - \epsilon).$$

So, we get

$$\lim_{k \rightarrow \infty} \Pi(\xi_{n_k}, \xi_{m_k}, \lambda) = (1 - \epsilon).$$

Again, we have

$$\begin{aligned} \Pi(\xi_{n_{k-1}}, \xi_{m_{k-1}}, \lambda + \delta) &\geq \Pi(\xi_{n_{k-1}}, \xi_{n_k}, \delta) \diamond \Pi(\xi_{n_k}, \xi_{m_{k-1}}, \lambda) \\ &> \Pi(\xi_{n_{k-1}}, \xi_{n_k}, \delta) \diamond (1 - \epsilon). \end{aligned}$$

Using Equation 2, we get $\liminf_{k \rightarrow \infty} \Pi(\xi_{n_{k-1}}, \xi_{m_{k-1}}, \lambda + \delta) \geq (1 - \epsilon)$.

Also,

$$\begin{aligned} (1 - \epsilon) &\leq \lim_{\delta \rightarrow 0^+} \liminf_{k \rightarrow \infty} \Pi(\xi_{n_{k-1}}, \xi_{m_{k-1}}, \lambda + \delta) \\ &= \liminf_{k \rightarrow \infty} \lim_{\delta \rightarrow 0^+} \Pi(\xi_{n_{k-1}}, \xi_{m_{k-1}}, \lambda + \delta) \\ &= \liminf_{k \rightarrow \infty} \Pi(\xi_{n_{k-1}}, \xi_{m_{k-1}}, \lambda). \end{aligned}$$

From Eq 3, we get

$$\Pi(\xi_{n_k-1}, \xi_{m_k-1}, \lambda) \leq \Pi(\xi_{n_k}, \xi_{m_k}, \lambda) \leq (1 - \epsilon).$$

So,

$$\limsup_{k \rightarrow \infty} \Pi(\xi_{n_k-1}, \xi_{m_k-1}, \lambda) \leq (1 - \epsilon).$$

Hence,

$$\lim_{k \rightarrow \infty} \Pi(\xi_{n_k-1}, \xi_{m_k-1}, \lambda) = (1 - \epsilon).$$

The demonstration for the remaining cases is Similar to that of Case (1). □

To support our primary findings, we require the following two categories of functions:

$$\Phi = \{ \varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ : \varphi \text{ is Lebesgue integrable with } 0 < \int_0^c \varphi(l) dl < \infty, \forall c > 0 \}$$

$$\Gamma = \{ \gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+ : \gamma \text{ is continuous with } \lim_{n \rightarrow \infty} \gamma^n(s) = 0, \forall s > 0 \}$$

Remark 3.4. If $\gamma \in \Gamma$, then

1. $\gamma(s) < s$, for $s > 0$,
2. $\gamma(0) = 0$.

Additionally, the following essential lemma is required:

Lemma 3.5. ³¹ Consider $\varphi \in \Phi$ and a sequence (t_n) consisting of positive real numbers for which $\lim_{n \rightarrow \infty} t_n = t$. It follows that

$$\lim_{n \rightarrow \infty} \int_0^{t_n} \varphi(l) dl = \int_0^t \varphi(l) dl.$$

Definition 3.6. Let $(F, \Pi, \Psi, \Upsilon, \Theta, \diamond, \bullet)$ be a NFMS, $\varphi \in \Phi$, and $\gamma \in \Gamma$. A mapping $f : F \rightarrow F$ is called (φ, γ) -neutrosophic contraction $((\varphi, \gamma)$ -NC) if for each $\xi, \omega \in F$ and each $\lambda > 0$, we have

$$\int_0^{\Pi(f\xi, f\omega, \lambda)^{-1}} \varphi(l) dl \leq \gamma \left(\int_0^{\Pi(\xi, \omega, \lambda)^{-1}} \varphi(l) dl \right),$$

$$\int_0^{\Psi(f\xi, f\omega, \lambda)^{-1}} \varphi(l) dl \leq \gamma \left(\int_0^{\Psi(\xi, \omega, \lambda)^{-1}} \varphi(l) dl \right),$$

$$\int_0^{\Upsilon(f\xi, f\omega, \lambda)} \varphi(l) dl \leq \gamma \left(\int_0^{\Upsilon(\xi, \omega, \lambda)} \varphi(l) dl \right),$$

and

$$\int_0^{\Theta(f\xi, f\omega, \lambda)} \varphi(l) dl \leq \gamma \left(\int_0^{\Theta(\xi, \omega, \lambda)} \varphi(l) dl \right).$$

Theorem 3.7. Let $(F, \Pi, \Psi, \mathcal{U}, \Theta, \diamond, \bullet)$ be a complete NFMS, Suppose that there is $\zeta^* \in \mathcal{Z}$ such that $f : F \rightarrow F$ is ζ^* -neutrosophic contraction. Consequently, the function f possesses a unique fixed point.

Proof. Let $\xi_0 \in F$ represent an arbitrary point. We examine the Picard sequence (ξ_n) characterized by the relation $\xi_{n+1} = f(\xi_n)$ for all $n \geq 0$. By Definition 3.6 we have

$$\int_0^{\frac{1}{\Pi(\xi_n, \xi_{n+1}, \lambda)} - 1} \varphi(l) dl \leq \gamma \left(\int_0^{\frac{1}{\Pi(\xi_{n-1}, \xi_n, \lambda)} - 1} \varphi(l) dl \right),$$

$$\int_0^{\frac{1}{\Psi(\xi_n, \xi_{n+1}, \lambda)} - 1} \varphi(l) dl \leq \gamma \left(\int_0^{\frac{1}{\Psi(\xi_{n-1}, \xi_n, \lambda)} - 1} \varphi(l) dl \right),$$

$$\int_0^{\mathcal{U}(\xi_n, \xi_{n+1}, \lambda)} \varphi(l) dl \leq \gamma \left(\int_0^{\mathcal{U}(\xi_{n-1}, \xi_n, \lambda)} \varphi(l) dl \right),$$

and

$$\int_0^{\Theta(\xi_n, \xi_{n+1}, \lambda)} \varphi(l) dl \leq \gamma \left(\int_0^{\Theta(\xi_{n-1}, \xi_n, \lambda)} \varphi(l) dl \right).$$

By induction on n , we get that

$$\int_0^{\frac{1}{\Pi(\xi_n, \xi_{n+1}, \lambda)} - 1} \varphi(l) dl \leq \gamma^n \left(\int_0^{\frac{1}{\Pi(\xi_0, \xi_1, \lambda)} - 1} \varphi(l) dl \right),$$

$$\int_0^{\frac{1}{\Psi(\xi_n, \xi_{n+1}, \lambda)} - 1} \varphi(l) dl \leq \gamma^n \left(\int_0^{\frac{1}{\Psi(\xi_0, \xi_1, \lambda)} - 1} \varphi(l) dl \right),$$

$$\int_0^{\mathcal{U}(\xi_n, \xi_{n+1}, \lambda)} \varphi(l) dl \leq \gamma^n \left(\int_0^{\mathcal{U}(\xi_0, \xi_1, \lambda)} \varphi(l) dl \right),$$

and

$$\int_0^{\Theta(\xi_n, \xi_{n+1}, \lambda)} \varphi(l) dl \leq \gamma^n \left(\int_0^{\Theta(\xi_0, \xi_1, \lambda)} \varphi(l) dl \right).$$

By taking the limit, we get

$$\lim_{n \rightarrow \infty} \int_0^{\frac{1}{\Pi(\xi_n, \xi_{n+1}, \lambda)} - 1} \varphi(l) dl = 0,$$

$$\lim_{n \rightarrow \infty} \int_0^{\frac{1}{\Psi(\xi_n, \xi_{n+1}, \lambda)} - 1} \varphi(l) dl = 0,$$

$$\lim_{n \rightarrow \infty} \int_0^{\mathcal{U}(\xi_n, \xi_{n+1}, \lambda)} \varphi(l) dl = 0,$$

and

$$\lim_{n \rightarrow \infty} \int_0^{\Theta(\xi_n, \xi_{n+1}, \lambda)} \varphi(l) dl = 0.$$

Which implies that

$$\lim_{n \rightarrow \infty} \Pi(\xi_n, \xi_{n+1}, \lambda) = 1, \lim_{n \rightarrow \infty} \Psi(\xi_n, \xi_{n+1}, \lambda) = 1, \lim_{n \rightarrow \infty} \mathcal{U}(\xi_n, \xi_{n+1}, \lambda) = 0, \text{ and } \lim_{n \rightarrow \infty} \Theta(\xi_n, \xi_{n+1}, \lambda) = 0.$$

Now, we aim to demonstrate that (ξ_n) is Cauchy. If (ξ_n) isn't, then by Lemma 3.3, there exist an $\epsilon > 0$ and $\lambda > 0$ along with two subsequences (ξ_{n_k}) and (ξ_{m_k}) derived from (ξ_n) , where (m_k) such that one at least of the following holds.

$$\begin{aligned} \lim_{k \rightarrow \infty} \Pi(\xi_{n_k}, \xi_{m_k}, \lambda) &= 1 - \epsilon, \\ \lim_{k \rightarrow \infty} \Psi(\xi_{n_k}, \xi_{m_k}, \lambda) &= 1 - \epsilon, \\ \lim_{k \rightarrow \infty} \mathcal{U}(\xi_{n_k}, \xi_{m_k}, \lambda) &= \epsilon, \\ \lim_{k \rightarrow \infty} \Theta(\xi_{n_k}, \xi_{m_k}, \lambda) &= \epsilon. \end{aligned}$$

Using Definition 3.6, we get

$$\int_0^{\frac{1}{\Pi(\xi_{n_k}, \xi_{m_k}, \lambda)} - 1} \varphi(l) dl \leq \gamma \left(\int_0^{\frac{1}{\Pi(\xi_{n_{k-1}}, \xi_{m_{k-1}}, \lambda)} - 1} \varphi(l) dl \right).$$

By taking the limit, we get

$$\lim_{k \rightarrow \infty} \int_0^{\frac{1}{\Pi(\xi_{n_k}, \xi_{m_k}, \lambda)} - 1} \varphi(l) dl \leq \lim_{k \rightarrow \infty} \gamma \left(\int_0^{\frac{1}{\Pi(\xi_{n_{k-1}}, \xi_{m_{k-1}}, \lambda)} - 1} \varphi(l) dl \right).$$

Hence,

$$\int_0^{\frac{1}{1-\epsilon} - 1} \varphi(l) dl \leq \gamma \left(\int_0^{\frac{1}{1-\epsilon} - 1} \varphi(l) dl \right) < \int_0^{\frac{1}{1-\epsilon} - 1} \varphi(l) dl,$$

which is a contradiction, and hence, $\lim_{n, m \rightarrow \infty} \Pi(\xi_n, \xi_m, \lambda) = 1$.

a contradiction.

Hence (ξ_n) is a Cauchy sequence, thus, there is $u \in F$ such that $\xi_n \rightarrow u$.

Definition 3.6 gives that

$$\begin{aligned} \int_0^{\frac{1}{\Pi(fu, \xi_{n+1}, \lambda)}} \varphi(l) dl &\leq \gamma \left(\int_0^{\frac{1}{\Pi(u, \xi_n, \lambda)} - 1} \varphi(l) dl \right) \rightarrow 0 \text{ as } n \rightarrow \infty, \\ \int_0^{\frac{1}{\Psi(fu, \xi_{n+1}, \lambda)}} \varphi(l) dl &\leq \gamma \left(\int_0^{\frac{1}{\Psi(u, \xi_n, \lambda)} - 1} \varphi(l) dl \right) \rightarrow 0 \text{ as } n \rightarrow \infty, \\ \int_0^{\mathcal{U}(fu, \xi_{n+1}, \lambda)} \varphi(l) dl &\leq \gamma \left(\int_0^{\mathcal{U}(u, \xi_n, \lambda)} \varphi(l) dl \right) \rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

and

$$\int_0^{\Theta(fu, \xi_{n+1}, \lambda)} \varphi(l) dl \leq \gamma \left(\int_0^{\Theta(u, \xi_n, \lambda)} \varphi(l) dl \right) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Which implies that ξ_{n+1} converges to fu , hence $u = fu$.

Now, let $v \in F$ with $v = fv$. If $u \neq v$, then from Definition 3.6, it follows that

$$\int_0^{\Theta(u,v,\lambda)} \varphi(l)dl = \int_0^{\Theta(fu,fv,\lambda)} \varphi(l)dl \leq \gamma \left(\int_0^{\Theta(u,v,\lambda)} \varphi(l)dl \right) < \int_0^{\Theta(u,v,\lambda)} \varphi(l)dl,$$

which is a contradiction. So $u = v$.

□

If we let $\gamma(s) = qs$, where q is in the interval $[0, 1)$, we arrive at the subsequent conclusion:

Corollary 3.8. *Let $(F, \Pi, \Psi, \mathcal{U}, \Theta, \diamond, \bullet)$ be a complete NFMS, Suppose that $f : F \rightarrow F$ satisfies the following for each $\xi, \omega \in F$ and each $\lambda > 0$, we have:*

$$\begin{aligned} \int_0^{\frac{1}{\Pi(f\xi, f\omega, \lambda)} - 1} \varphi(l)dl &\leq q \int_0^{\frac{1}{\Pi(\xi, \omega, \lambda)} - 1} \varphi(l)dl, \\ \int_0^{\frac{1}{\Psi(f\xi, f\omega, \lambda)} - 1} \varphi(l)dl &\leq q \int_0^{\frac{1}{\Psi(\xi, \omega, \lambda)} - 1} \varphi(l)dl, \\ \int_0^{\mathcal{U}(f\xi, f\omega, \lambda)} \varphi(l)dl &\leq q \int_0^{\mathcal{U}(\xi, \omega, \lambda)} \varphi(l)dl, \end{aligned}$$

and

$$\int_0^{\Theta(f\xi, f\omega, \lambda)} \varphi(l)dl \leq q \int_0^{\Theta(\xi, \omega, \lambda)} \varphi(l)dl.$$

Consequently, the function f possesses a unique fixed point.

If we establish the function $\gamma(s) = qs$, where q is constrained within the interval $[0, 1)$, and define $\varphi(l) = 1$, we arrive at the subsequent conclusion:

Corollary 3.9. *Let $(F, \Pi, \Psi, \mathcal{U}, \Theta, \diamond, \bullet)$ be a complete NFMS, Suppose that $f : F \rightarrow F$ satisfies the following for each $a, \omega \in F$ and each $\lambda > 0$, we have:*

$$\begin{aligned} \frac{1}{\Pi(f\xi, f\omega, \lambda)} - 1 &\leq q \left(\frac{1}{\Pi(\xi, \omega, \lambda)} - 1 \right), \\ \frac{1}{\Psi(f\xi, f\omega, \lambda)} - 1 &\leq q \left(\frac{1}{\Psi(\xi, \omega, \lambda)} - 1 \right), \\ \mathcal{U}(f\xi, f\omega, \lambda) &\leq q\mathcal{U}(\xi, \omega, \lambda), \end{aligned}$$

and

$$\Theta(f\xi, f\omega, \lambda) \leq q\Theta(\xi, \omega, \lambda).$$

Consequently, the function f possesses a unique fixed point.

4 Conclusion

Fixed point theorems are fundamental results in mathematics, particularly in the fields of analysis, topology, and applied mathematics. These theorems establish conditions under which a function will have a fixed point, which is a point that is mapped to itself by the function. In other hand, neutrosophic fuzzy metric spaces are an advanced concept that combines elements of neutrosophy and fuzzy metric spaces. This framework is useful in dealing with uncertainty and imprecision in various mathematical and practical applications. In this study, we introduced fixed point theorems related to integral type contractions, framed within the advanced context of neutrosophic fuzzy metric spaces. Additionally, we derived multiple fixed point results that are relevant to this particular setting.

References

- [1] Zadeh, L.A. (1965) Fuzzy sets, *Inf Comp*, 8, 338–353.
- [2] Atanassov K. (1986) Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 20, 87–96.
- [3] Smarandache, F. (2005) Neutrosophic set, a generalisation of the intuitionistic fuzzy sets, *Inter J Pure Appl Math*, 24, 287–297.
- [4] Turksen, I. (1996) Interval valued fuzzy sets based on normal forms, *Fuzzy Sets and Systems*, 20, 191–210.
- [5] Atanassov K., Gargov G. (1989) Interval valued intuitionistic fuzzy sets, *Inf Comp*, 31, 343–349.
- [6] Smarandache, F. (2003) A unifying field in logics: Neutrosophic logic. Neutrosophy, neutrosophic set, neutrosophic probability and statistics. Phoenix, Xiquan.
- [7] Yager, R.R. (2013) Pythagorean fuzzy subsets. Proc Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, Canada, 2013.
- [8] Barbosa, R.P. and Smarandache, F., 2024. Neutrosophic One-Round Zero-Knowledge Proof. *Plithogenic Logic and Computation*, 2, pp.49-54.
- [9] Abd Elwahed, H., Alburaikan, A. and Smarandache, F., 2024. On characterizing efficient and properly efficient solutions for multi-objective programming problems in a complex space. *Journal of Optimization in Industrial Engineering*, 16(2), pp.369-375.
- [10] M. H. S. R. Arif, A. S. Malik, and M. M. Hassan, "Fuzzy soft sets and their applications in decision making: A comprehensive survey," *J. Intell. Fuzzy Syst.*, vol. 42, no. 5, pp. 4937-4951, 2022.
- [11] S. R. M. K. R. Abed, M. T. Ghanbari, and M. N. El-Basyuni, "Time-dependent fuzzy soft sets and their application in decision-making problems," *Soft Comput.*, vol. 26, no. 4, pp. 1059-1072, 2022.
- [12] Al-Qudah, Y., Al-Sharqi, F., Mishlish, M., & Rasheed, M. M. (2023). Hybrid integrated decision-making algorithm based on AO of possibility interval-valued neutrosophic soft settings. *International Journal of Neutrosophic Science*, 22(3), 84 - 98.
- [13] Al-Qudah, Y. , Al-Sharqi, F. 2023. Algorithm for decision-making based on similarity measures of possibility interval-valued neutrosophic soft setting settings. *International Journal of Neutrosophic Science*, 22(3), pp. 69–83.
- [14] Hazaymeh, A. (2024). Time Effective Fuzzy Soft Set and Its Some Applications with and Without a Neutrosophic. *International Journal of Neutrosophic Science*, (2), 129-29.
- [15] Al-Qudah, Y. (2024). A robust framework for the decision-making based on single-valued neutrosophic fuzzy soft expert setting. *International Journal of Neutrosophic Science*, 23(2), 195-95.
- [16] M U Romdhini, F Al-Sharqi, A Nawawi, A Al-Quran, and H Rashmanlou. Signless Laplacian energy of interval-valued fuzzy graph and its applications. *Sains Malaysiana*, 52(7):2127–2137, 2023.
- [17] Banach, S.(1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *fund. math*, 3, 133-181.
- [18] Hatamleh, R., and Hazaymeh, A. (2024). On Some Topological Spaces Based On Symbolic n-Plithogenic Intervals. *International Journal of Neutrosophic Science*, 25(1), 23-3.
- [19] A. Bataihah. Some fixed point results with application to fractional differential equation via new type of distance spaces. *Results in Nonlinear Analysis* 2024, 7, 202–208.
- [20] Z. N. Khan, A. S. Malik, and S. F. Khan, "Time-dependent fuzzy soft sets and their application in multi-criteria decision-making," *Fuzzy Sets Syst.*, vol. 413, pp. 1-16, 2021.
- [21] Karapınar, E., & fulga, A. (2023). Discussions on Proinov- \mathcal{C}_b -Contraction Mapping on-Metric Space. *Journal of function Spaces*, 2023(9), 1-10.

- [22] Karapınar, E., Romaguera, S., & Tirado, P. (2022). Characterizations of quasi-metric and G -metric completeness involving ω -distances and fixed points. *Demonstratio Mathematica*, 55(1), 939-951.
- [23] A. Bataihah, T. Qawasmeh, (2024). A new type of distance spaces and fixed point Results, *Journal of Mathematical Analysis*, 15(4), 81–90.
- [24] Bataihah, A., Qawasmeh, T., & Shatnawi, M. (2022). Discussion on b-metric spaces and related results in metric and G-metric spaces. *Nonlinear functional Analysis and Applications* 27, no. 2, 233-247.
- [25] Shatanawi, W., Bataihah, A. (2021). Remarks on G-Metric Spaces and Related Fixed Point Theorems, *Thai Journal of Mathematics*, 19(2), 445–455.
<https://thaijmath2.in.cmu.ac.th/index.php/thaijmath/article/view/1168>
- [26] ŞİMŞEK, Necip, and Murat KIRIŞCI. Fixed point theorems in Neutrosophic metric spaces. *Infinite Study*, 2019.
- [27] Das, S., Roy, B.K., Kar, M.B., Kar, S. and Pamučar, D., 2020. Neutrosophic fuzzy set and its application in decision making. *Journal of Ambient Intelligence and Humanized Computing*, 11, pp.5017-5029.
- [28] Menger, K. "Statistical Metrics." *Proceedings of the National Academy of Sciences of the United States of America* 28, no. 12 (1942): 535-537.
- [29] Kirişci, M., & Şimşek, N. (2020). Neutrosophic metric spaces. *Mathematical Sciences*, 14(3), 241-248.
- [30] Ghosh, S., Sonam, Bhardwaj, R. and Narayan, S., 2024. On Neutrosophic Fuzzy Metric Space and Its Topological Properties. *Symmetry*, 16(5), p.613.
- [31] Liu, Z., Li, X., Kang, S.M. and Cho, S.Y., 2011. Fixed point theorems for mappings satisfying contractive conditions of integral type and applications. *Fixed point theory and Applications*, 2011, pp.1-18.