Regular-Closed Functions Between Soft Topological Spaces

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Abstract

The object of the present paper is to introduce a new class of soft functions called soft regular-closed functions. This class contains the class of soft closed functions. Numerous theorems that give properties of such soft functions are presented. Moreover, sufficient conditions for a soft function to be soft regular-closed are given. In addition, several preservation theorems of soft separations axioms using soft regular-closed are given. Finally, the correspondence between this class of soft functions and the class of regular-closed functions in classical topology is studied.

Keywords: Soft closed functions; Soft compactness; Soft separation axioms; Generated soft topology

1 Introduction and Preliminaries

The majority of our conventional modeling, reasoning, and computing tools are characterized by crispness, determinism, and precision. However, there are a lot of complex problems in social science, medical science, engineering, economics, and the environment that include facts that aren't always clear-cut. Because these challenges involve a variety of uncertainties, we are not always able to apply the classical methodologies. Mathematical tools for handling uncertainties can be found in the major current theories, such as the theory of probability, the theory of fuzzy sets,[1–3] the theory of intuitionistic fuzzy sets,[4,5] the theory of vague sets,[6] the theory of interval mathematics,[5,7] and the theory of rough sets.[8] But as[9] makes clear, each of these hypotheses has its own set of challenges. These problems may have arisen from the theories' inadequate parametrization tools; as a result, Molodtsov[9] introduced the idea of soft set theory in 1999 as a new mathematical technique for handling ambiguity or uncertainties that is unaffected by the problems mentioned above.

The concept of soft topological spaces was presented in.[10] The subject of soft topology is an area of mathematics concerned with the study of topological spaces using soft sets, which have fuzzy bounds. Recently, it received a lot of attention because of its possible applications in domains including computer science, engineering, and economics. Also, it offers a more realistic and adaptable framework for modeling and analyzing complex systems where the boundaries between areas are not always evident. Soft sets enable the representation of uncertainty and imprecision, which are inherent in many real-world systems.

Mathematicians extended many concepts of classical topological spaces to include soft topological spaces in[1,11,13–22] and other works.

Soft closed functions between soft topological spaces are employed to define and investigate numerous critical properties and invariants of soft topological spaces. These functions have a wide range of applications, including soft optimization theory, soft approximation theory, soft control theory, soft data analysis, soft image processing, and soft decision-making. This motivated us to write this paper.

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In this paper, we introduce and investigate soft regular-closed functions as a new type of soft closed functions.

Let $\mathcal{A}$ be a set of parameters, and let Z be a non-empty set. A soft set over Z relative to $\mathcal{A}$ is a function $K : Z \longrightarrow \mathcal{P}(\mathcal{A})$. $SS(Z,\mathcal{A})$ denotes the family of all soft sets over Z relative to $\mathcal{A}$. If $M \in SS(Z,\mathcal{A})$ such that $M(a) = Y$ for any $a \in \mathcal{A}$ (resp. $M(d) = Y$ and $M(a) = \varnothing$ for each $a \in \mathcal{A} - \{d\}$), then M is represented by C_Y (resp. d_Y). $1_{\mathcal{A}}$ and $0_{\mathcal{A}}$ will denote C_Z and $C_{\varnothing}$, respectively. If $M \in SS(Z,\mathcal{A})$, then M is a soft point over X relative to $\mathcal{A}$ and represented as d_x if $M(d) = \{x\}$ and $M(a) = \varnothing$ for every $a \in \mathcal{A} - \{d\}$. The collection of all soft points over X with respect to $\mathcal{A}$ will be represented by $SP(Z,\mathcal{A})$. If $d_x \in SP(Z,\mathcal{A})$ and $M \in SS(Z,\mathcal{A})$, then d_x is said to belong to M (notation: $d_x \tilde{\in} M$) if $x \in M(d)$. Let $M, N \in SS(Z,\mathcal{A})$. Then M is a soft subset of N, denoted by $M \tilde{\subseteq} N$, if $M(a) \subseteq N(a)$ for each $a \in \mathcal{A}$. The soft union (resp. intersection, difference) of M and N is denoted by $M \tilde{\cup} N$ (resp. $M \tilde{\cap} N$, $M - N$) and defined by $(M \tilde{\cup} N)(a) = M(a) \cup N(a)$ (resp. $(M \tilde{\cap} N)(a) = M(a) \cap N(a)$, $(M - N)(a) = M(a) - N(a)$) for each $a \in \mathcal{A}$. For any sub-collection $\mathcal{H} \subseteq SS(X,\mathcal{A})$, the soft union (resp. soft intersection) of the members of $\mathcal{H}$ is denoted by $\tilde{\cup}_{H \in \mathcal{H}} H$ (resp. $\tilde{\cap}_{H \in \mathcal{H}} H$) and defined by $(\tilde{\cup}_{H \in \mathcal{H}} H)(a) = \cup_{H \in \mathcal{H}} H(a)$ (resp. $(\tilde{\cap}_{H \in \mathcal{H}} H)(a) = \cap_{H \in \mathcal{H}} H(a)$) for each $a \in \mathcal{A}$. Let $SS(Z,\mathcal{A})$ and $SS(Z,\mathcal{S})$ be two families of soft sets, and $r : X \longrightarrow W$, $w : \mathcal{A} \longrightarrow S$ be two functions. Then a soft mapping $f_{rw} : SS(Z,\mathcal{A}) \longrightarrow SS(W,\mathcal{S})$ is defined as follows: For each $M \in SS(Z,\mathcal{A})$ and $N \in SS(W,\mathcal{S})$, $(f_{rw}(M))(b) = \varnothing$ if $w^{-1}(b) = \varnothing$, $(f_{rw}(M))(b) = \cup_{a \in w^{-1}(b)} r(H(a))$ if $w^{-1}(b) \neq \varnothing$, and $\left(f_{rw}^{-1}(N)\right)(a) = r^{-1}(N(w(a)))$. A sub-collection $\sigma \subseteq SS(Z,\mathcal{A})$ is called a soft topology on Z relative to $\mathcal{A}$, and the triplet $(Z,\sigma,\mathcal{A})$ is called a soft topological space if $\{0_{\mathcal{A}}, 1_{\mathcal{A}}\} \subseteq \sigma$, $M \tilde{\cap} N \in \sigma$ for any $\{M,N\} \subseteq \sigma$, and $\tilde{\cup}_{H \in \mathcal{H}} H \in \sigma$ for any $\mathcal{H} \subseteq \sigma$. Let $(Z,\sigma,\mathcal{A})$ be a soft topological space and $M \in SS(Z,\mathcal{A})$. If $M \in \sigma$, then M is called a soft open set in $(Z,\sigma,\mathcal{A})$, and if $1_{\mathcal{A}} - M \in \sigma$, then M is called a soft closed set in $(Z,\sigma,\mathcal{A})$.

In this paper, we will adhere to the terminology and concepts of[23,24] and refer to topological space as TS and soft topological space as STS.

Let $(Z,\sigma,\mathcal{A})$ and $(Z,\Im)$ be STS and TS, respectively. Let $M \in SS(Z,\mathcal{A})$ and $Y \subseteq Z$. Then the soft interior of M in $(Z,\sigma,\mathcal{A})$, the soft closure of M in $(Z,\sigma,\mathcal{A})$, the interior of Y in $(Z,\Im)$, and the closure of Y in $(Z,\Im)$ are represented by $Int_{\sigma}(M)$, $Cl_{\sigma}(M)$, $Int_{\Im}(Y)$, and $Cl_{\Im}(Y)$, respectively, and the family of all soft closed sets in $(Z,\sigma,\mathcal{A})$ (resp. closed sets in $(Z,\Im)$) will be denoted by σ^c (resp. $\Im^c$).

The following definitions will be used in the sequel:

Definition 1.1. [25] *Let $(Z,\Im)$ be a TS and let $Y \subseteq Z$. Then Y is called a regular-open (resp. regular-closed) set in $(Z,\Im)$ if $Y = Int_{\Im}(Cl_{\Im}(Y))$ (resp. $Y = Cl_{\Im}(Int_{\Im}(Y))$). The collection of all regular-open sets (resp. regular-open sets) in $(Z,\Im)$ will be denoted by $RO(\Im)$ (resp. $RC(\Im)$).*

Definition 1.2. [26]*A soft function $g : (Z,\Im) \longrightarrow (W,\aleph)$ is called regular-closed if $g(R) \in \aleph^c$ for each $R \in RC(\Im)$.*

Definition 1.3. [27] *Let $(Z,\varphi,\mathcal{A})$ be a STS and let $K \in SS(Z,\mathcal{A})$. Then K is called a soft regular-open (resp. soft regular-closed) set in $(Z,\varphi,\mathcal{A})$ if $K = Int_{\varphi}(Cl_{\varphi}(K))$ (resp. $K = Cl_{\varphi}(Int_{\varphi}(K))$). The collection of all regular-open sets (resp. regular-open sets) in $(Z,\varphi,\mathcal{A})$ will be denoted by $RO(\varphi)$ (resp. $RC(\varphi)$).*

For a STS $(Z,\varphi,\mathcal{A})$, the soft topology on Z relative to $\mathcal{A}$ that has $RO(\varphi)$ as a soft base will be denoted by φ_{δ}.

Definition 1.4. *Let $f_{rw} : (Z,\varphi,\mathcal{P}) \longrightarrow (W,\sigma,\mathcal{S})$ be a soft function. Then f_{rw} is called*

(a)[30] soft θ-continuous if for each $a_z \in SP(Z,\mathcal{P})$ and each $K \in \sigma$ such that $f_{rw}(a_z) \tilde{\in} K$, there exists $H \in \varphi$ such that $a_z \tilde{\in} H$ and $f_{rw}(Cl_{\varphi}(H)) \tilde{\subseteq} Cl_{\sigma}(K)$.

(b)[29] soft almost-open if $f_{rw}(G) \in \sigma$ for each $G \in RO(\varphi)$.

(c)[28] soft almost-continuous for each $a_z \in SP(Z,\mathcal{P})$ and each $K \in \sigma$ such that $f_{rw}(a_z) \tilde{\in} K$, there exists $H \in \varphi$ such that $a_z \tilde{\in} H$ and $f_{rw}(H) \tilde{\subseteq} Int_{\sigma}(Cl_{\sigma}(K))$.

(d)[31] soft δ-continuous if $f_{rw}^{-1}(K) \in \varphi_{\delta}$ for each $K \in \sigma$.

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Definition 1.5. *A STS $(Z, \varphi, \mathcal{P})$ is called*

(a)[32] *soft compact if for each $\mathcal{K} \subseteq \varphi$ such that $\tilde{\cup}_{K \in \mathcal{K}} K = 1_{\mathcal{P}}$, there exists a finite subcollection $\mathcal{K}_1 \subseteq \mathcal{K}$ such that $\tilde{\cup}_{K \in \mathcal{K}_1} K = 1_{\mathcal{P}}$.*

(b)[33] *soft Hausdorff if for any a_x, $b_y \in SP(Z, \mathcal{P})$ with $a_x \neq b_y$, there exist $H, K \in \varphi$ such that $a_x \tilde{\in} H$, $b_y \tilde{\in} K$, and $H \tilde{\cap} K = 0_{\mathcal{P}}$.*

(c)[33] *soft regular if for any $a_x \in SP(Z, \mathcal{P})$ and any $G \in \varphi$ such that $a_x \tilde{\in} G$, there exists $K \in \varphi$ such that $a_z \tilde{\in} K \tilde{\subseteq} Cl_{\varphi}(K) \tilde{\subseteq} G$.*

(d)[33] *soft normal if for any $A, B \in \varphi^c$ such that $A \tilde{\cap} B = 0_{\mathcal{P}}$, there exist $H, K \in \varphi$ such that $A \tilde{\subseteq} H$, $B \tilde{\subseteq} K$, and $H \tilde{\cap} K = 0_{\mathcal{P}}$.*

(e)[34] *soft almost-regular if for each $D \in RC(\varphi)$ and each $a_x \tilde{\in} 1_{\mathcal{P}} - D$, there are $M, N \in \varphi$ such that $a_x \tilde{\in} M$, $D \tilde{\subseteq} N$, and $M \tilde{\cap} N = 0_{\mathcal{P}}$.*

(f)[35] *soft Urysohn if for any a_x, $b_y \in SP(Z, \mathcal{P})$ with $a_x \neq b_y$, there exist $H, K \in \varphi$ such that $a_x \tilde{\in} H$, $b_y \tilde{\in} K$, and $Cl_{\varphi}(H) \tilde{\cap} Cl_{\varphi}(K) = 0_{\mathcal{P}}$.*

Definition 1.6. [36] *Let $(Z, \varphi, \mathcal{P})$ be a STS, and let $H \in SS(Z, \mathcal{P})$. Then H is called soft compact relative to $(Z, \varphi, \mathcal{P})$ if for each $\mathcal{K} \subseteq \varphi$ such that $H \tilde{\subseteq} \tilde{\cup}_{K \in \mathcal{K}} K$, there exists a finite subcollection $\mathcal{K}_1 \subseteq \mathcal{K}$ such that $H \tilde{\subseteq} \tilde{\cup}_{K \in \mathcal{K}_1} K$.*

Definition 1.7. [32] *Let $(P, \varphi, \mathcal{A})$ and $(S, \sigma, \mathcal{B})$ be two STSs. Then the soft topology on $P \times S$ relative to $\mathcal{A} \times \mathcal{B}$ having $\{H \times K : H \in \varphi \text{ and } K \in \sigma\}$ as a soft base will be denoted by $pr(\varphi \times \sigma)$.*

2 Soft Regular-Closed Functions

Definition 2.1. *A soft function $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ is called soft regular-closed if $f_{rw}(H) \in \sigma^c$ for each $H \in RC(\varphi)$.*

Theorem 2.2. *Let $\{(P, \varphi_a) : a \in \mathcal{A}\}$ and $\{(S, \sigma_b) : b \in \mathcal{B}\}$ be two collections of TSs. Consider the functions $r : P \longrightarrow S$ and $w : \mathcal{A} \longrightarrow \mathcal{B}$, where w is a bijection. Then $f_{rw} : (P, \oplus_{a \in \mathcal{A}} \varphi_a, \mathcal{A}) \longrightarrow (S, \oplus_{b \in \mathcal{B}} \sigma_b, \mathcal{B})$ is soft closed iff $r : (P, \varphi_a) \longrightarrow \left(S, \sigma_{w(a)}\right)$ is closed for all $a \in \mathcal{A}$.*

Proof. *Necessity*. Let $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ be soft closed. Let $d \in \mathcal{A}$. To show that $r : (P, \varphi_d) \longrightarrow \left(S, \sigma_{w(d)}\right)$ is closed, let $V \in (\varphi_d)^c$. Then $d_V \in \left(\oplus_{a \in \mathcal{A}} \varphi_a\right)^c$ and so $f_{rw}(d_V) \in \left(\oplus_{b \in \mathcal{B}} \sigma_b\right)^c$. Since w is bijective, then $f_{rw}(d_V) = (w(d))_{r(V)}$. Thus, $\left((w(d))_{r(V)}\right)(w(d)) = r(V) \in \left(\sigma_{w(d)}\right)^c$. It follows that $r : (P, \varphi_d) \longrightarrow \left(S, \sigma_{w(d)}\right)$ is soft closed.

Sufficiency. Let $r : (P, \varphi_a) \longrightarrow \left(S, \sigma_{w(a)}\right)$ be closed for all $a \in \mathcal{A}$. Let $H \in \left(\oplus_{a \in \mathcal{A}} \varphi_a\right)^c$. We will show that $(f_{rw}(H))(b) \in (\sigma_b)^c$ for every $b \in \mathcal{B}$. Let $b \in \mathcal{B}$. Since $H \in \left(\oplus_{a \in \mathcal{A}} \varphi_a\right)^c$, then $H\left(w^{-1}(b)\right) \in \left(\varphi_{w^{-1}(b)}\right)^c$. Since r is injective, then $(f_{rw}(H))(b) = H\left(w^{-1}(b)\right)$. Since $r : \left(P, \varphi_{w^{-1}(b)}\right) \longrightarrow \left(S, \sigma_{w(w^{-1}(b))=b}\right)$ is closed, then $H\left(w^{-1}(b)\right) \in (\sigma_b)^c$. ☐

Corollary 2.3. *Consider the functions $r : (P, \Im) \longrightarrow (S, \aleph)$ and $w : \mathcal{A} \longrightarrow \mathcal{B}$, where w is a bijection. Then $r : (P, \Im) \longrightarrow (S, \aleph)$ is closed iff $f_{rw} : (P, \tau(\Im), \mathcal{A}) \longrightarrow (S, \tau(\aleph), \mathcal{B})$ is soft closed.*

Proof. For each $a \in \mathcal{A}$ and $b \in \mathcal{B}$, put $\varphi_a = \Im$ and $\sigma_b = \aleph$. Then $\tau(\Im) = \oplus_{a \in \mathcal{A}} \varphi_a$ and $\tau(\aleph) = \oplus_{b \in \mathcal{B}} \sigma_b$. Theorem 2.2 ends the proof. ☐

Theorem 2.4. *Let $\{(P, \varphi_a) : a \in \mathcal{A}\}$ and $\{(S, \sigma_b) : b \in \mathcal{B}\}$ be two collections of TSs. Consider the functions $r : P \longrightarrow S$ and $w : w : \mathcal{A} \longrightarrow \mathcal{B}$, where w is a bijection. Then $f_{rw} : (P, \oplus_{a \in \mathcal{A}} \varphi_a, \mathcal{A}) \longrightarrow (S, \oplus_{b \in \mathcal{B}} \sigma_b, \mathcal{B})$ is soft regular-closed iff $r : (P, \varphi_a) \longrightarrow \left(S, \sigma_{w(a)}\right)$ is regular-closed for all $a \in \mathcal{A}$.*

Proof. Necessity. Let $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ be soft regular-closed. Let $d \in \mathcal{A}$. To show that $r : (P, \varphi_d) \longrightarrow \left(S, \sigma_{w(d)}\right)$ is regular-closed, let $V \in RC(\varphi_d)$. Then by Theorem 14 of,[37] $d_V \in RC(\oplus_{a\in\mathcal{A}}\varphi_a)$ and so $f_{rw}(d_V) \in (\oplus_{b\in\mathcal{B}}\sigma_b)^c$. Since w is bijective, then $f_{rw}(d_V) = (w(d))_{r(V)}$. Thus, $\left((w(d))_{r(V)}\right)(w(d)) = r(V) \in \left(\sigma_{w(d)}\right)^c$. It follows that $r : (P, \varphi_d) \longrightarrow \left(S, \sigma_{w(d)}\right)$ is soft regular-closed.

Sufficiency. Let $r : (P, \varphi_a) \longrightarrow \left(S, \sigma_{w(a)}\right)$ be regular-closed for all $a \in \mathcal{A}$. Let $H \in RC(\oplus_{a\in\mathcal{A}}\varphi_a)$. We will show that $(f_{rw}(H))(b) \in (\sigma_b)^c$ for every $b \in \mathcal{B}$. Let $b \in \mathcal{B}$. Since $H \in RC(\oplus_{a\in\mathcal{A}}\varphi_a)$, then by Theorem 14 of,[37] $H\left(w^{-1}(b)\right) \in \left(\varphi_{w^{-1}(b)}\right)^c$. Since r is injective, then $(f_{rw}(H))(b) = H\left(w^{-1}(b)\right)$. Since $r : \left(P, \varphi_{w^{-1}(b)}\right) \longrightarrow \left(S, \sigma_{w(w^{-1}(b))=b}\right)$ is regular-closed, then $H\left(w^{-1}(b)\right) \in (\sigma_b)^c$. □

Corollary 2.5. *Consider the functions* $r : (P, \Im) \longrightarrow (S, \aleph)$ *and* $w : \mathcal{A} \longrightarrow \mathcal{B}$*, where* w *is a bijection. Then* $r : (P, \Im) \longrightarrow (S, \aleph)$ *is regular-closed iff* $f_{rw} : (P, \tau(\Im), \mathcal{A}) \longrightarrow (S, \tau(\aleph), \mathcal{B})$ *is soft regular-closed.*

Proof. For each $a \in \mathcal{A}$ and $b \in \mathcal{B}$, put $\varphi_a = \Im$ and $\sigma_b = \aleph$. Then $\tau(\Im) = \oplus_{a\in\mathcal{A}}\varphi_a$ and $\tau(\aleph) = \oplus_{b\in\mathcal{B}}\sigma_b$. Theorem 2.4 ends the proof. □

Theorem 2.6. *Every soft closed function is soft regular-closed.*

Proof. Let $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ be soft closed. Let $H \in RC(\varphi) \subseteq \varphi$. Then $H \in \varphi^c$ and so $f_{rw}(H) \in \sigma^c$. Therefore, f_{rw} is soft regular-closed. □

In general, the opposite of Theorem 2.6 is not true:

Example 2.7. *Let* $\Im$ *and* $\aleph$ *be the cocountable and the usual topologies on* $\mathbb{R}$*, respectively. Consider the identities functions* $r : (\mathbb{R}, \Im) \longrightarrow (\mathbb{R}, \aleph)$ *and* $w : \mathbb{Z} \longrightarrow \mathbb{Z}$*. Since* $RC(\Im) = \{\emptyset, \mathbb{R}\}$*,* $r(\emptyset) = \emptyset \in \aleph^c$*, and* $r(\mathbb{R}) = \mathbb{R} \in \aleph^c$*, then* r *is regular-closed. On the other hand, since* $\mathbb{Q} \in \Im$ *while* $r(\mathbb{Q}) = \mathbb{Q} \notin \aleph^c$*, then* r *is not closed. Therefore, by Corollaries 2.3 and 2.5,* $f_{rw} : (\mathbb{R}, \tau(\Im), \mathbb{Z}) \longrightarrow (\mathbb{R}, \tau(\aleph), \mathbb{Z})$ *is soft regular-closed but not soft closed.*

Theorem 2.8. *Let* $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ *be soft regular-closed and surjective. Then for any* $b_y \in SP(S, \mathcal{B})$ *and any* $G \in RO(\varphi)$ *such that* $f_{rw}^{-1}(b_y) \tilde{\subseteq} G$*, we have* $b_y \tilde{\in} Int_\sigma(f_{rw}(G))$*.*

Proof. Since $G \in RO(\varphi)$, then $1_\mathcal{A} - G \in RC(\varphi)$. Since f_{rw} is soft regular-closed, then $f_{rw}(1_\mathcal{A} - G) \in \sigma^c$. Since $f_{rw}^{-1}(b_y) \tilde{\subseteq} G$, then $b_y \tilde{\in} 1_\mathcal{B} - f_{rw}(1_\mathcal{A} - G) \in \sigma$. Since f_{rw} is surjective, then $1_\mathcal{B} = f_{rw}(1_\mathcal{A}) = f_{rw}(G \tilde{\cup} (1_\mathcal{A} - G)) = f_{rw}(G) \tilde{\cup} f_{rw}(1_\mathcal{A} - G)$; hence, $1_\mathcal{B} - f_{rw}(1_\mathcal{A} - G) \tilde{\subseteq} f_{rw}(G)$. This shows that $b_y \tilde{\in} Int_\sigma(f_{rw}(G))$. □

Corollary 2.9. *Let* $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ *be soft regular-closed and surjective. Then for any* $K \in SS(P, \mathcal{A})$ *and any* $a_x \in SP(P, \mathcal{A})$ *such that* $f_{rw}^{-1}(f_{rw}(a_x)) \tilde{\subseteq} K$*, we have* $f_{rw}(a_x) \tilde{\in} Int_\sigma(f_{rw}(Int_\varphi(Cl_\varphi(K))))$*.*

Proof. Let $K \in SS(P, \mathcal{A})$ and $a_x \in SS(P, \mathcal{A})$. Let $b_y = f_{rw}(a_x)$ and $G = Int_\varphi(Cl_\varphi(K))$. Then we have $b_y \in SP(S, \mathcal{B})$, $G \in RO(\varphi)$, and $f_{rw}^{-1}(b_y) = f_{rw}^{-1}(f_{rw}(a_x)) \tilde{\subseteq} K \tilde{\subseteq} Int_\varphi(Cl_\varphi(K)) = G$. Thus, by Theorem 2.8, $f_{rw}(a_x) = b_y \tilde{\in} Int_\sigma(f_{rw}(G)) = Int_\sigma(f_{rw}(Int_\varphi(Cl_\varphi(K))))$. □

Lemma 2.10. *For any STS* $(P, \varphi, \mathcal{A})$*,* $\{Cl_\varphi(G) : G \in \varphi\} = RC(\varphi)$*.*

Proof. Let $G \in \varphi$. Since $Int_\varphi(Cl_\varphi(G)) \tilde{\subseteq} Cl_\varphi(G) \in \varphi^c$, then $Cl_\varphi(Int_\varphi(Cl_\varphi(G))) \tilde{\subseteq} Cl_\varphi(G)$. On the other hand, since $G \in \varphi$, then $G = Int_\varphi(G)$ and so, $Cl_\varphi(G) \tilde{\subseteq} Cl_\varphi(Int_\varphi(G)) \tilde{\subseteq} Cl_\varphi(Int_\varphi(Cl_\varphi(G)))$. Therefore, $Cl_\varphi(G) = Cl_\varphi(Int_\varphi(Cl_\varphi(G)))$. Hence, $Cl_\varphi(G) \in RC(\varphi)$.

Conversely, let $U \in RC(\varphi)$. Then $Cl_\varphi(Int_\varphi(U)) = U$ and so, $U = Cl_\varphi(Int_\varphi(U))$, where $Int_\varphi(U) \in \varphi$. Hence, $U \in \{Cl_\varphi(G) : G \in \varphi\}$. □

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Theorem 2.11. *Let $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ be soft regular-closed and surjective. If $(P, \varphi, \mathcal{A})$ is soft almost-regular, $(S, \sigma, \mathcal{B})$ is soft compact, and $f_{rw}^{-1}(b_y) \in RC(\varphi)$ for each $b_y \in SP(S, \mathcal{B})$, then f_{rw} is soft continuous.*

Proof. Suppose to the contrary that there exists $a_x \in SP(P, \mathcal{A})$ such that f_{rw} is not soft continuous at a_x. Then there exists $G \in \sigma$ such that $f_{rw}(a_x) \tilde{\in} G$ and $f_{rw}(K) \tilde{\cap} (1_{\mathcal{B}} - G) \neq 0_{\mathcal{B}}$ for every $K \in \varphi$ with $a_x \tilde{\in} K$. Let $\delta = \{K \in \varphi : a_x \tilde{\in} K\}$. Since f_{rw} is soft regular-closed and by Lemma 2.10, $\{Cl_{\varphi}(K) : K \in \delta\} \subseteq RC(\varphi)$, then $\{f_{rw}(Cl_{\varphi}(K)) : K \in \delta\} \subseteq \sigma^c$. Thus, $\{f_{rw}(Cl_{\varphi}(K)) \tilde{\cap} (1_{\mathcal{B}} - G) : K \in \delta\} \subseteq \sigma^c - \{0_{\mathcal{B}}\}$.

Claim. $\{f_{rw}(Cl_{\varphi}(K)) \tilde{\cap} (1_{\mathcal{B}} - G) : K \in \delta\}$ has the finite soft intersection property.

Proof of claim. Suppose to the contrary that a finite sub-collection $\delta_{\circ} \subseteq \delta$ such that

$$\tilde{\cap}_{K \in \delta_{\circ}} (f_{rw}(Cl_{\varphi}(K)) \tilde{\cap} (1_{\mathcal{B}} - G)) = (1_{\mathcal{B}} - G) \tilde{\cap} (\tilde{\cap}_{K \in \delta_{\circ}} f_{rw}(Cl_{\varphi}(K))) = 0_{\mathcal{B}}.$$

Let $T = \tilde{\cap}_{K \in \delta_{\circ}} K$. Since $\delta_{\circ}$ is finite, then $T \in \varphi$. Since $a_x \tilde{\in} K$ for each $K \in \delta_{\circ}$, then $a_x \tilde{\in} T$. Thus, we have $T \in \delta$, and so $f_{rw}(Cl_{\varphi}(T)) \tilde{\cap} (1_{\mathcal{B}} - G) \neq 0_{\mathcal{B}}$. On the other hand,

$$\begin{array}{rcl} f_{rw}(Cl_{\varphi}(T)) \tilde{\cap} (1_{\mathcal{B}} - G) & = & f_{rw}(Cl_{\varphi}(\tilde{\cap}_{K \in \delta_{\circ}} K)) \tilde{\cap} (1_{\mathcal{B}} - G) \\ & \tilde{\subseteq} & f_{rw}(\tilde{\cap}_{K \in \delta_{\circ}} Cl_{\varphi}(K)) \tilde{\cap} (1_{\mathcal{B}} - G) \\ & \tilde{\subseteq} & (1_{\mathcal{B}} - G) \tilde{\cap} (\tilde{\cap}_{K \in \delta_{\circ}} f_{rw}(Cl_{\varphi}(K))) \\ & = & 0_{\mathcal{B}}, \end{array}$$

and hence $f_{rw}(Cl_{\varphi}(T)) \tilde{\cap} (1_{\mathcal{B}} - G) = 0_{\mathcal{B}}$. This is a contradiction.

Since $(S, \sigma, \mathcal{B})$ is soft compact, $\{f_{rw}(Cl_{\varphi}(K)) \tilde{\cap} (1_{\mathcal{B}} - G) : K \in \delta\} \subseteq \sigma^c$ and $\{f_{rw}(Cl_{\varphi}(K)) \tilde{\cap} (1_{\mathcal{B}} - G) : K \in \delta\}$ has the finite soft intersection property, then $\tilde{\cap}_{K \in \delta} (f_{rw}(Cl_{\varphi}(K)) \tilde{\cap} (1_{\mathcal{B}} - G)) \neq 0_{\mathcal{B}}$. Let $b_y \tilde{\in} \tilde{\cap}_{K \in \delta} (f_{rw}(Cl_{\varphi}(K)) \tilde{\cap} (1_{\mathcal{B}} - G$ Since $b_y \tilde{\in} 1_{\mathcal{B}} - G$ while $f_{rw}(a_x) \tilde{\in} G$, then $b_y \neq f_{rw}(a_x)$. So, $a_x \tilde{\in} 1_{\mathcal{A}} - f_{rw}^{-1}(b_y)$. Since $(P, \varphi, \mathcal{A})$ is soft almost-regular and $f_{rw}^{-1}(b_y) \in RC(\varphi)$, then there exist $L, M \in \varphi$ such that $a_x \tilde{\in} L$, $f_{rw}^{-1}(b_y) \tilde{\subseteq} M$, and $L \tilde{\cap} M = 0_{\mathcal{A}}$; hence $Cl_{\varphi}(L) \tilde{\cap} M = 0_{\mathcal{A}}$. Since $f_{rw}^{-1}(b_y) \tilde{\subseteq} M$, then $f_{rw}^{-1}(b_y) \tilde{\cap} Cl_{\varphi}(L) \tilde{\subseteq} M \tilde{\cap} Cl_{\varphi}(L) = 0_{\mathcal{A}}$, and so $b_y \tilde{\notin} f_{rw}(Cl_{\varphi}(L))$. On the other hand, since $a_x \tilde{\in} L \in \varphi$, then $L \in \delta$ and so, $b_y \tilde{\in} f_{rw}(Cl_{\varphi}(L)) \tilde{\cap} (1_{\mathcal{B}} - G) \tilde{\subseteq} f_{rw}(Cl_{\varphi}(L))$. This is a contradiction. ☐

Theorem 2.12. *Let $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ be a soft function. If $f_{rw} : (P, \varphi_{\delta}, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ is soft closed, then $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ is soft regular-closed.*

Proof. Let $K \in RC(\varphi)$. Since $RO(\varphi) \subseteq \varphi_{\delta}$, then $RC(\varphi) \subseteq (\varphi_{\delta})^c$ and so $K \in (\varphi_{\delta})^c$. Since $f_{rw} : (P, \varphi_{\delta}, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ is soft closed, then $f_{rw}(K) \in \sigma^c$. This shows that $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ is soft regular-closed. ☐

The following example shows that the converse of Theorem 2.12 is not true in general:

Example 2.13. *Let $P = (0, 2)$, $S = (0, 1]$, $\mathcal{A} = \{a\}$, and $\mathcal{B} = \{b\}$. Let φ be the soft topology on P relative to $\mathcal{A}$ having $\{a_U : U \subseteq P \text{ and } P - U \text{ is finite}\} \cup \left\{a_{(0,1)}, a_{(1,2)}\right\}$ as a soft subbase. Let $\sigma = \{0_{\mathcal{B}}, 1_{\mathcal{B}}\} \cup \left\{b_{(t,1]} : 0 \leqslant t < 1\right\}$. Define $r : P \longrightarrow S$ and $w : \mathcal{A} \longrightarrow \mathcal{B}$ by*

$$r(x) = \begin{cases} x & \text{if } 0 < x \leqslant 1 \\ x - 1 & \text{if } 1 < x < 2 \end{cases} \text{ and } w(a) = b.$$

Since $RC(\varphi) = \left\{0_{\mathcal{A}}, 1_{\mathcal{A}}, a_{(0,1]}, a_{[1,2)}\right\}$, $f_{rw}\left(a_{(0,1]}\right) = f_{rw}\left(a_{[1,2)}\right) = 1_{\mathcal{B}}$, then $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ is soft regular-closed. Since $\left\{a_{(0,1]}, a_{[1,2)}\right\} \subseteq RC(\varphi)$, then $a_{(0,1]} \tilde{\cap} a_{[1,2)} = a_{\{1\}} \in (\varphi_{\delta})^c$. Since $a_{\{1\}} \in (\varphi_{\delta})^c$ while $f_{rw}\left(a_{\{1\}}\right) = b_{\{1\}} \notin \sigma^c$, then $f_{rw} : (P, \varphi_{\delta}, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ is not soft closed.

Theorem 2.14. *A soft function $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ is soft regular-closed iff for any $T \in SS(S, \mathcal{B})$ and any $U \in RO(\varphi)$ such that $f_{rw}^{-1}(T) \tilde{\subseteq} U$, there exists $V \in \sigma$ such that $T \tilde{\subseteq} V$ and $f_{rw}^{-1}(V) \tilde{\subseteq} U$.*

Proof. Necessity. Let $T \in SS(S,\mathcal{B})$ and let $U \in RO(\varphi)$ such that $f_{rw}^{-1}(T)\tilde{\subseteq}U$. Since $U \in RO(\varphi)$, $1_{\mathcal{A}} - U \in RC(\varphi)$. Since f_{rw} is soft regular-closed, then $f_{rw}(1_{\mathcal{A}} - U) \in \sigma^c$ and so, $1_{\mathcal{B}} - f_{rw}(1_{\mathcal{A}} - U) \in \sigma$. Let $V = 1_{\mathcal{B}} - f_{rw}(1_{\mathcal{A}} - U)$. Then $V \in \sigma$. Since $f_{rw}^{-1}(T)\tilde{\subseteq}U$, then $1_{\mathcal{A}} - U \tilde{\subseteq} 1_{\mathcal{A}} - f_{rw}^{-1}(T) = f_{rw}^{-1}(1_{\mathcal{B}} - T)$, and so, $f_{rw}(1_{\mathcal{A}} - U)\tilde{\subseteq} f_{rw}\left(f_{rw}^{-1}(1_{\mathcal{B}} - T)\right)\tilde{\subseteq}1_{\mathcal{B}} - T$. Hence, $T\tilde{\subseteq}1_{\mathcal{B}} - f_{rw}(1_{\mathcal{A}} - U) = V$. On the other hand,

$$\begin{array}{rcc} f_{rw}^{-1}(V) & = & f_{rw}^{-1}(1_{\mathcal{B}} - f_{rw}(1_{\mathcal{A}} - U)) \\ & = & f_{rw}^{-1}(1_{\mathcal{B}}) - f_{rw}^{-1}(f_{rw}(1_{\mathcal{A}} - U)) \\ & = & 1_{\mathcal{A}} - f_{rw}^{-1}(f_{rw}(1_{\mathcal{A}} - U)) \\ & \tilde{\subseteq} & 1_{\mathcal{A}} - (1_{\mathcal{A}} - U) \\ & = & U. \end{array}$$

Sufficiency. Let $K \in RC(\varphi)$. We will show that $1_{\mathcal{B}} - f_{rw}(K) \in \sigma$. Let $b_y \tilde{\in} 1_{\mathcal{B}} - f_{rw}(K)$. Then, we have $f_{rw}^{-1}(b_y)\tilde{\subseteq}1_{\mathcal{A}} - K \in RO(\varphi)$. By assumptions, there exists $V \in \sigma$ such that $b_y\tilde{\in}V$ and $f_{rw}^{-1}(V)\tilde{\subseteq}1_{\mathcal{A}} - K$. Thus, we have $b_y\tilde{\in}V\tilde{\subseteq}1_{\mathcal{B}} - f_{rw}(K)$. Hence, $1_{\mathcal{B}} - f_{rw}(K) \in \sigma$. □

Theorem 2.15. *Let $f_{rw} : (P,\varphi,\mathcal{A}) \longrightarrow (S,\sigma,\mathcal{B})$ be a soft function. Then the following are equivalent:*

(a) f_{rw} is soft regular-closed.

(b) For any $U \in RO(\varphi)$, $\tilde{\cup}\left\{b_y : f_{rw}^{-1}(b_y)\tilde{\subseteq}U\right\} \in \sigma$.

(c) For any $G \in RC(\varphi)$, $\tilde{\cup}\left\{b_y : f_{rw}^{-1}(b_y)\tilde{\cap}G \neq 0_{\mathcal{A}}\right\} \in \sigma^c$.

Proof. (a) ⟶ (b): Let $U \in RO(\varphi)$. Let $H = \tilde{\cup}\left\{b_y : f_{rw}^{-1}(b_y)\tilde{\subseteq}U\right\}$. Let $b_y\tilde{\in}H$. Then $f_{rw}^{-1}(b_y)\tilde{\subseteq}U$. So, by Theorem 2.13, there exists $V \in \sigma$ such that $b_y\tilde{\in}V$ and $f_{rw}^{-1}(V)\tilde{\subseteq}U$. Thus, $V\tilde{\subseteq}H$. This shows that $H \in \sigma$.

(b) ⟶ (c): Let $G \in RC(\varphi)$. Then $1_{\mathcal{A}} - G \in RO(\varphi)$. So, by (b), $\tilde{\cup}\left\{b_y : f_{rw}^{-1}(b_y)\tilde{\subseteq}1_{\mathcal{A}} - G\right\} = \tilde{\cup}\left\{b_y : f_{rw}^{-1}(b_y)\tilde{\cap}G = 0_{0_{\mathcal{A}}}\right\} \in \sigma$. Thus, $1_{\mathcal{B}} - \tilde{\cup}\left\{b_y : f_{rw}^{-1}(b_y)\tilde{\cap}G = 0_{\mathcal{A}}\right\} = \tilde{\cup}\left\{b_y : f_{rw}^{-1}(b_y)\tilde{\cap}G \neq 0_{\mathcal{A}}\right\} \in \sigma^c$.

(c) ⟶ (a): Let $G \in RC(\varphi)$. Then by (c), $\tilde{\cup}\left\{b_y : f_{rw}^{-1}(b_y)\tilde{\cap}G \neq 0_{\mathcal{A}}\right\} \in \sigma^c$. Since $f_{rw}(G) = \tilde{\cup}\left\{b_y : f_{rw}^{-1}(b_y)\tilde{\cap}G \neq 0_{\mathcal{A}}\right\}$, then $f_{rw}(G) \in \sigma^c$. Hence, f_{rw} is soft regular-closed. □

Theorem 2.16. *A soft function $f_{rw} : (P,\varphi,\mathcal{A}) \longrightarrow (S,\sigma,\mathcal{B})$ is soft regular-closed iff for each $G \in \varphi \cup RC(\varphi)$, $Cl_\sigma(f_{rw}(G))\tilde{\subseteq}f_{rw}(Cl_\varphi(G))$.*

Proof. Necessity. Suppose that f_{rw} is soft regular-closed. Let $G \in \varphi$. Then, by Lemma 2.10, $Cl_\varphi(G) \in RC(\varphi)$, and so $f_{rw}(Cl_\varphi(G)) \in \sigma^c$. Since $f_{rw}(G)\tilde{\subseteq}f_{rw}(Cl_\varphi(G))$, then $Cl_\sigma(f_{rw}(G))\tilde{\subseteq}f_{rw}(Cl_\varphi(G))$. Let $G \in RC(\varphi)$. Then $f_{rw}(G) \in \sigma^c$. Since $f_{rw}(G)\tilde{\subseteq}f_{rw}(Cl_\varphi(G))$, then $Cl_\sigma(f_{rw}(G)) = f_{rw}(G)\tilde{\subseteq}f_{rw}(Cl_\varphi(G))$.

Sufficiency. Suppose that $Cl_\sigma(f_{rw}(G))\tilde{\subseteq}f_{rw}(Cl_\varphi(G))$ for every $G \in \varphi \cup RC(\varphi)$. Let $G \in RC(\varphi)$. Then $Cl_\varphi(G) = G$. So, by assumption, $Cl_\sigma(f_{rw}(G))\tilde{\subseteq}f_{rw}(Cl_\varphi(G)) = f_{rw}(G)$. This shows that $f_{rw}(G) \in \sigma^c$. □

Lemma 2.17. *Let $(P,\varphi,\mathcal{A})$ be a STS, and let X be a non-empty subset of P.*

(a) If $C_X \in \varphi$, then $RO(\varphi_X) = \{G\tilde{\cap}C_X : G \in RO(\varphi)\}$.

(b) If $C_X \in \varphi$, then $RC(\varphi_X) = \{H\tilde{\cap}C_X : H \in RC(\varphi)\}$.

(c) If $C_X \in RO(\varphi)$, then $RO(\varphi_X) = \left\{H \in RO(\varphi) : H\tilde{\subseteq}C_X\right\}$.

Proof. (a) To show that $RO(\varphi_X) \subseteq \{G\tilde{\cap}C_X : G \in RO(\varphi)\}$, let $K \in RO(\varphi_X)$. Then $K = Int_{\varphi_X}(Cl_{\varphi_X}(K))$. Since $Cl_{\varphi_X}(K) = C_X\tilde{\cap}Cl_{\varphi}(K)$, then $K = Int_{\varphi_X}(C_X\tilde{\cap}Cl_{\varphi}(K))$. Also, since $C_X \in \varphi - \{0_{\mathcal{A}}\}$, then

$$\begin{aligned} K &= Int_{\varphi_X}(C_X\tilde{\cap}Cl_{\varphi}(K)) \\ &= Int_{\varphi}(C_X\tilde{\cap}Cl_{\varphi}(K)) \\ &= Int_{\varphi}(C_X)\tilde{\cap}Int_{\varphi}(Cl_{\varphi}(K)) \\ &= C_X\tilde{\cap}Int_{\varphi}(Cl_{\varphi}(K)). \end{aligned}$$

Let $G = Int_{\varphi}(Cl_{\varphi}(K))$. Then $K = G\tilde{\cap}C_X$ with $G \in RO(\varphi)$.

To show that $\{G\tilde{\cap}C_X : G \in RO(\varphi)\} \subseteq RO(\varphi_X)$, let $G \in RO(\varphi) \subseteq \varphi$. Let $H = G\tilde{\cap}C_X$. Then $H \in \varphi_X$ and so $H\tilde{\subseteq}Int_{\varphi_X}(Cl_{\varphi_X}(H))$.

We are going to show that $Int_{\varphi_X}(Cl_{\varphi_X}(H))\tilde{\subseteq}H = G\tilde{\cap}C_X$. Let $b_y\tilde{\in}Int_{\varphi_X}(Cl_{\varphi_X}(H))\tilde{\subseteq}C_X$. Then $b_y\tilde{\in}C_X$. To show that $b_y\tilde{\in}G$, put $V = Int_{\varphi_X}(Cl_{\varphi_X}(H))$. Then $V \in \varphi_X$. Since $C_X \in \varphi$ and $V \in \varphi_X$, then $V \in \varphi$. Since $b_y\tilde{\in}V\tilde{\subseteq}Cl_{\varphi_X}(H) = C_X\tilde{\cap}Cl_{\varphi}(H)\tilde{\subseteq}Cl_{\varphi}(H) = Cl_{\varphi}(G\tilde{\cap}C_X)\tilde{\subseteq}Cl_{\varphi}(G)$, then $b_y\tilde{\in}Int_{\varphi}(Cl_{\varphi}(G)) = G$. This shows that $Int_{\varphi_X}(Cl_{\varphi_X}(H))\tilde{\subseteq}G\tilde{\cap}C_X = H$.

(b) $RC(\varphi_X) \subseteq \{G\tilde{\cap}C_X : G \in RC(\varphi)\}$, let $H \in RC(\varphi_X)$. Then $C_X - H \in RC(\varphi_X)$. So, by (a), there exists $G \in RO(\varphi)$ such that $C_X - H = G\tilde{\cap}C_X$ and thus, $H = C_X - (G\tilde{\cap}C_X) = C_X - G = C_X\tilde{\cap}(1_{\mathcal{A}} - G)$, where $\in RC(\varphi)$.

To show that $\{G\tilde{\cap}C_X : G \in RC(\varphi)\} \subseteq RC(\varphi_X)$, let $G \in RC(\varphi)$. Then $1_{\mathcal{A}} - G \in RO(\varphi)$. So, by (a), $C_X\tilde{\cap}(1_{\mathcal{A}} - G) \in RO(\varphi_X)$. Since $C_X\tilde{\cap}(1_{\mathcal{A}} - G) = C_X - G = C_X - (G\tilde{\cap}C_X)$, then $G\tilde{\cap}C_X \in RC(\varphi_X)$.

(c) Since $C_X \in RO(\varphi) \subseteq \varphi$, then by (a), $RO(\varphi_X) = \{G\tilde{\cap}C_X : G \in RO(\varphi)\}$. Since $C_X \in RO(\varphi)$, then $\{G\tilde{\cap}C_X : G \in RO(\varphi)\} \subseteq RO(\varphi)$. Hence, $RO(\varphi_X) = \{G\tilde{\cap}C_X : G \in RO(\varphi)\} = \left\{H \in RO(\varphi) : H\tilde{\subseteq}C_X\right\}$. □

Lemma 2.18. *Let $(P, \varphi, \mathcal{A})$ be a STS. Let $\{X_\alpha : \alpha \in \Delta\}$ be a cover of P such that $\{C_{X_\alpha} : \alpha \in \Delta\} \subseteq \varphi - \{0_{\mathcal{A}}\}$. Then*

(a) $G \in \varphi$ iff $G\tilde{\cap}C_{X_\alpha} \in \varphi_{X_\alpha}$ for all $\alpha \in \Delta$.

(b) $G \in \varphi^c$ iff $G\tilde{\cap}C_{X_\alpha} \in (\varphi_{X_\alpha})^c$ for all $\alpha \in \Delta$.

Proof. (a) *Necessity.* Obvious.

Sufficiency. Suppose that $G\tilde{\cap}C_{X_\alpha} \in \varphi_{X_\alpha}$ for all $\alpha \in \Delta$. Since $\{C_{X_\alpha} : \alpha \in \Delta\} \subseteq \varphi$, then $\{G\tilde{\cap}C_{X_\alpha} : \alpha \in \Delta\} \subseteq \varphi$. So, $G = G\tilde{\cap}1_{\mathcal{A}} = G\tilde{\cap}(\tilde{\cup}_{\alpha\in\Delta}C_{X_\alpha}) = \tilde{\cup}_{\alpha\in\Delta}(G\tilde{\cap}C_{X_\alpha}) \in \varphi$.

(b) *Necessity.* Obvious.

Sufficiency. Suppose that $G\tilde{\cap}C_{X_\alpha} \in (\varphi_{X_\alpha})^c$ for all $\alpha \in \Delta$. Then for each $\alpha \in \Delta$, $C_{X_\alpha} - (G\tilde{\cap}C_{X_\alpha}) = C_{X_\alpha}\tilde{\cap}(1_{\mathcal{A}} - G) \in \varphi_{X_\alpha}$. Thus, by (a), $1_{\mathcal{A}} - G \in \varphi$ and hence $G \in \varphi^c$. □

Theorem 2.19. *Let $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ be a soft function. If there are two covers $\{X_\alpha : \alpha \in \Delta\}$ and $\{Y_\alpha : \alpha \in \Delta\}$ of P and S, respectively, such that $\{C_{X_\alpha} : \alpha \in \Delta\} \subseteq \varphi - \{0_{\mathcal{A}}\}$, $\{C_{Y_\alpha} : \alpha \in \Delta\} \subseteq \sigma - \{0_{\mathcal{B}}\}$, $f_{rw}^{-1}(C_{Y_\alpha}) = C_{X_\alpha}$ and $(f_{rw})_{|C_{X_\alpha}} : (X_\alpha, \varphi_{X_\alpha}, \mathcal{A}) \longrightarrow (Y_\alpha, \sigma_{Y_\alpha}, \mathcal{B})$ is soft regular-closed for all $\alpha \in \Delta$, then $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ is soft regular-closed.*

Proof. Let $G \in RC(\varphi)$. Then, by Lemma 2.17 (b), $G\tilde{\cap}C_{X_\alpha} \in RC(\varphi_{X_\alpha})$, and so $\left((f_{rw})_{|C_{X_\alpha}}\right)(G\tilde{\cap}C_{X_\alpha}) = f_{rw}(G)\tilde{\cap}Y_\alpha \in (\sigma_{Y_\alpha})^c$ for all $\alpha \in \Delta$. Thus, by Lemma 2.18 (b), $f_{rw}(G) \in \sigma^c$. Hence, $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ is soft regular-closed. □

Corollary 2.20. *Let $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ be a soft continuous function. If there are two covers $\{X_\alpha : \alpha \in \Delta\}$ and $\{Y_\alpha : \alpha \in \Delta\}$ of P and S, respectively, such that $\{C_{Y_\alpha} : \alpha \in \Delta\} \subseteq \sigma - \{0_{\mathcal{B}}\}$, $f_{rw}^{-1}(C_{Y_\alpha}) = C_{X_\alpha}$ and $(f_{rw})_{|C_{X_\alpha}} : (X_\alpha, \varphi_{X_\alpha}, \mathcal{A}) \longrightarrow (Y_\alpha, \sigma_{Y_\alpha}, \mathcal{B})$ is soft regular-closed for all $\alpha \in \Delta$, then $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ is soft regular-closed.*

For any function $g : P \longrightarrow S$, the function $h : P \longrightarrow P \times S$ defined by $h(x) = (x, g(x))$ will be denoted by $g^{\#}$.

Theorem 2.21. *Let* $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ *be an injective soft function. If* $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ *is soft regular-closed, then* $f_{r^{\#}w^{\#}} : (P, \varphi, \mathcal{A}) \longrightarrow (P \times S, pr(\varphi \times \sigma), \mathcal{A} \times \mathcal{B})$ *is soft regular-closed.*

Proof. Let $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ be soft regular-closed and injective. Let $G \in RC(\varphi)$. Suppose to the contrary that $f_{r^{\#}w^{\#}}(G) \notin (pr(\varphi \times \sigma))^c$. Then there exists $(b_1, b_2)_{(z_1,z_2)} \tilde{\in} Cl_{pr(\varphi\times\sigma)}(f_{r^{\#}w^{\#}}(G))$ but $(b_1, b_2)_{(z_1,z_2)} \widetilde{\notin} f_{r^{\#}w^{\#}}(G)$. Since $(b_1, b_2)_{(z_1,z_2)} \widetilde{\notin} f_{r^{\#}w^{\#}}(G)$, then either $(b_1)_{z_1} \widetilde{\notin} G$ or $[(b_1)_{z_1} \tilde{\in} G$ and $f_{rw}((b_1)_{z_1}) \neq (b_2)_{z_2}]$. Suppose that $(b_1)_{z_1} \widetilde{\notin} G$. Then $(b_1)_{z_1} \tilde{\in} 1_{\mathcal{A}} - G \in RO(\varphi) \subseteq \varphi$. Since we have $(b_1, b_2)_{(z_1,z_2)} \tilde{\in} (1_{\mathcal{A}} - G) \times 1_{\mathcal{B}} \in pr(\varphi \times \sigma)$ and $(b_1, b_2)_{(z_1,z_2)} \tilde{\in} Cl_{pr(\varphi\times\sigma)}(f_{r^{\#}w^{\#}}(G))$, then $((1_{\mathcal{A}} - G) \times 1_{\mathcal{B}}) \tilde{\cap} f_{r^{\#}w^{\#}}(G) \neq 0_{\mathcal{A}\times\mathcal{B}}$. So, we find $d_z \tilde{\in} G$ such that $f_{r^{\#}w^{\#}}(d_z) = (d, w(d))_{(z,r(z))} \tilde{\in} (1_{\mathcal{A}} - G) \times 1_{\mathcal{B}}$. Thus, we have $d_z \tilde{\in} G \tilde{\cap} (1_{\mathcal{A}} - G)$, a contradiction. Suppose that $[(b_1)_{z_1} \tilde{\in} G$ and $f_{rw}((b_1)_{z_1}) \neq (b_2)_{z_2}]$. Since f_{rw} is injective and $f_{rw}((b_1)_{z_1}) \neq (b_2)_{z_2}$, then $(b_2)_{z_2} \tilde{\in} 1_{\mathcal{B}} - f_{rw}(G)$. Since f_{rw} is soft regular-closed, then $1_{\mathcal{B}} - f_{rw}(G) \in \sigma$. Since we have $(b_1, b_2)_{(z_1,z_2)} \tilde{\in} 1_{\mathcal{A}} \times (1_{\mathcal{B}} - f_{rw}(G)) \in pr(\varphi \times \sigma)$ and $(b_1, b_2)_{(z_1,z_2)} \tilde{\in} Cl_{pr(\varphi\times\sigma)}(f_{r^{\#}w^{\#}}(G))$, then $(1_{\mathcal{A}} \times (1_{\mathcal{B}} - f_{rw}(G))) \tilde{\cap} f_{r^{\#}w^{\#}}(G) \neq 0_{\mathcal{A}\times\mathcal{B}}$. So, we find $d_z \tilde{\in} G$ such that $f_{r^{\#}w^{\#}}(d_z) = (d, w(d))_{(z,r(z))} \tilde{\in} 1_{\mathcal{A}} \times (1_{\mathcal{B}} - f_{rw}(G))$. Thus, $f_{rw}(d_z) = (w(d))_{(r(z))} \tilde{\in} f_{rw}(G) \tilde{\cap} (1_{\mathcal{B}} - f_{rw}(G))$, a contradiction. □

For a given soft function $f_{rw} : SP(P, \mathcal{A}) \longrightarrow SP(S, \mathcal{B})$, the soft set $\tilde{\cup} \left\{ (a, w(a))_{(x,r(x))} : a \in \mathcal{A} \text{ and } x \in P \right\}$ is called the soft graph of f_{rw} and is denoted by $Gr(f_{rw})$. So, $(a, b)_{(x,y)} \tilde{\in} Gr(f_{rw})$ iff $f_{rw}(a_x) = b_y$ iff $r(x) = y$ and $w(a) = b$.

Corollary 2.22. *If* $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ *is soft regular-closed and injective, then* $Gr(f_{rw}) \in (pr(\varphi \times \sigma))^c$.

Proof. By Theorem 2.21, $f_{r^{\#}w^{\#}} : (P, \varphi, \mathcal{A}) \longrightarrow (P \times S, pr(\varphi \times \sigma), \mathcal{A} \times \mathcal{B})$ is soft regular-closed. Since $1_{\mathcal{A}} \in RC(\varphi)$, then $f_{r^{\#}w^{\#}}(1_{\mathcal{A}}) = Gr(f_{rw}) \in (pr(\varphi \times \sigma))^c$. □

The reverse of Theorem 2.21 need not always true.

Example 2.23. *Let* $P = \mathbb{R}$ *and* $\mathcal{A} = \{a\}$. *Let* $\Im$ *be the usual topology on* P *and* $\varphi = \{a_U : U \in \Im\}$. *Define* $r : P \longrightarrow P$ *and* $w : \mathcal{A} \longrightarrow \mathcal{A}$ *by*

$$r(x) = \begin{cases} \frac{1}{x} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases} \quad \text{and } w(a) = a.$$

Then $f_{r^{\#}w^{\#}} : (P, \varphi, \mathcal{A}) \longrightarrow (P \times P, pr(\varphi \times \varphi), \mathcal{A} \times \mathcal{A})$ *is soft closed; hence soft regular-closed. On the other hand, since* $a_{[1.\infty)} \in RC(\varphi)$ *while* $f_{rw}(a_{[1.\infty)}) = a_{(0,1]} \notin \varphi^c$, *then* $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (P, \varphi, \mathcal{A})$ *is not soft regular-closed.*

For any two non-empty sets X and Y, the projections on X and Y will be denoted by π_X and π_Y, respectively. Thus, $\pi_X : X \times Y \longrightarrow X$ and $\pi_Y : X \times Y \longrightarrow Y$ are defined by $\pi_X(x, y) = x$ and $\pi_Y(x, y) = y$ for all $(x, y) \in X \times Y$.

Theorem 2.24. *Let* $(P, \varphi, \mathcal{A})$ *be soft compact and let* $(S, \sigma, \mathcal{B})$ *be any STS. Consider the projections* $\pi_S : P \times S \longrightarrow S$ *and* $\pi_{\mathcal{B}} : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{B}$. *Then* $f_{\pi_S\pi_{\mathcal{B}}} : (P \times S, pr(\varphi \times \sigma), \mathcal{A} \times \mathcal{B}) \longrightarrow (S, \sigma, \mathcal{B})$ *is soft closed.*

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Proof. Let $G \in (pr\,(\varphi \times \sigma))^{c}$. We will show that $1_{\mathcal{B}} - f_{\pi_S \pi_{\mathcal{B}}}(G) \in \sigma$. Let $b_y \tilde{\in} 1_{\mathcal{B}} - f_{\pi_S \pi_{\mathcal{B}}}(G)$. Then $(1_{\mathcal{A}} \times b_y)\,\tilde{\cap} G = 0_{\mathcal{A}\times\mathcal{B}}$ and so, $1_{\mathcal{A}} \times b_y \tilde{\subseteq} 1_{\mathcal{A}\times\mathcal{B}} - G \in pr\,(\varphi \times \sigma)$. Thus, for each $a_x \in SP(P, \mathcal{A})$, there exists $U_{a_x} \in \varphi$ and $V_{a_x} \in \sigma$ such that $a_x \tilde{\in} U_{a_x}$, $b_y \tilde{\in} V_{a_x}$, and $U_{a_x} \times V_{a_x} \tilde{\subseteq} 1_{\mathcal{A}\times\mathcal{B}} - G$. Since $(P, \varphi, \mathcal{A})$ be soft compact and $1_{\mathcal{A}} = \tilde{\cup}_{a_x \in SP(P,\mathcal{A})} U_{a_x}$, then there exists a finite set $\Gamma \subseteq SP(P, \mathcal{A})$ such that $1_{\mathcal{A}} = \tilde{\cup}_{a_x \in \Gamma} U_{a_x}$. Let $H = \tilde{\cap}_{a_x \in \Gamma} V_{a_x}$. Then we have $b_y \tilde{\in} H \in \sigma$.

Claim. $H \tilde{\cap} f_{\pi_S \pi_{\mathcal{B}}}(G) = 1_{\mathcal{B}}$.

Proof of Claim. Suppose to the contrary that there exists $(a, d)_{(x,z)} \tilde{\in} G$ such that $f_{\pi_S \pi_{\mathcal{B}}}\left((a, d)_{(x,z)}\right) = d_z \tilde{\in} H$. Choose $h_n \in \Gamma$ such that $a_x \tilde{\in} U_{h_n}$. Then $d_z \tilde{\in} V_{h_n}$ and so, $(a, d)_{(x,z)} \tilde{\in} U_{h_n} \times V_{h_n} \tilde{\subseteq} 1_{\mathcal{A}\times\mathcal{B}} - G$, which is a contradiction.

This cliam finished the proof. □

Theorem 2.25. *Let $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ be a function such that $(P, \varphi, \mathcal{A})$ is soft compact. If $f_{r^{\#}w^{\#}} : (P, \varphi, \mathcal{A}) \longrightarrow (P \times S, pr\,(\varphi \times \sigma), \mathcal{A} \times \mathcal{B})$ is soft regular-closed, f_{rw} is soft regular-closed.*

Proof. Let $G \in RC\,(\varphi)$. Since $f_{r^{\#}w^{\#}}$ is soft regular-closed, then $f_{r^{\#}w^{\#}}(G) \in (pr\,(\varphi \times \sigma))^{c}$. Consider the projections $\pi_S : P \times S \longrightarrow S$ and $\pi_{\mathcal{B}} : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{B}$. Then by Theorem 2.24, $f_{\pi_S \pi_{\mathcal{B}}} : (P \times S, pr\,(\varphi \times \sigma), \mathcal{A} \times \mathcal{B}) \longrightarrow (S, \sigma, \mathcal{B})$ is soft closed. Thus, $f_{\pi_S \pi_{\mathcal{B}}}(f_{r^{\#}w^{\#}}(G)) = f_{rw}(G) \in \sigma^{c}$. Hence, f_{rw} is soft regular-closed. □

A soft restriction of a soft regular-closed function need not be soft regular-closed:

Example 2.26. *Let $P = \mathbb{R}^2$ and $\mathcal{A} = \mathbb{Z}$. Let $\Im$ be the usual topology on P and $\varphi = \{C_U : U \in \Im\}$. Consider the identity functions $r : P \longrightarrow P$ and $w : \mathcal{A} \longrightarrow \mathcal{A}$. Then $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (P, \varphi, \mathcal{A})$ is soft regular-closed. Let $X = [1, 3) \times [1, 3)$. To show that $(f_{rw})_{|C_X} : (X, \varphi_X, \mathcal{A}) \longrightarrow (P, \varphi, \mathcal{A})$ is not soft regular-closed, let $Y = [2, 3) \times [2, 3)$. Then $C_Y \in RC\,(\varphi_X)$ while $(f_{rw})_{|C_X}(C_Y) = C_Y \notin \varphi^{c}$.*

Theorem 2.27. *If $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ is a soft regular-closed function and $X \subseteq P$ such that $C_X \in \varphi - \{0_{\mathcal{P}}\}$ and there exists $Y \subseteq S$ such that $f_{rw}^{-1}(C_Y) = C_X$, then $(f_{rw})_{|C_X} : (X, \varphi_X, \mathcal{A}) \longrightarrow (Y, \sigma_Y, \mathcal{B})$ is soft regular-closed.*

Proof. Let $K \in RC\,(\varphi_X)$. Then $K = Cl_{\varphi_X}(Int_{\varphi_X}(K)) = C_X \tilde{\cap} Cl_{\varphi}(Int_{\varphi_X}(K))$. Since $C_X \in \varphi$, then $Int_{\varphi_X}(K) = Int_{\varphi}(K)$. So, $K = Cl_{\varphi_X}(Int_{\varphi_X}(K)) = C_X \tilde{\cap} Cl_{\varphi}(Int_{\varphi}(K))$. Since f_{rw} is soft regular-closed and $Cl_{\varphi}(Int_{\varphi}(K)) \in RC\,(\varphi)$, then $f_{rw}(Cl_{\varphi}(Int_{\varphi}(K))) \in \sigma^{c}$. Thus, $(f_{rw})_{|C_X}(K) = f_{rw}(Cl_{\varphi}(Int_{\varphi}(K)))\,\tilde{\cap} C_Y \in (\sigma_Y)^{c}$. □

Theorem 2.28. *Let $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ be soft regular-closed and surjective. If $(P, \varphi, \mathcal{A})$ is soft normal and $f_{rw}^{-1}(b_y) \in \varphi^{c}$ for each $b_y \in SP\,(S, \mathcal{B})$, then f_{rw} is soft closed.*

Proof. Suppose to the contrary that there exists $G \in \varphi$ such that $f_{rw}(G) \notin \sigma^{c}$. Then there exists $b_y \tilde{\in} Cl_{\sigma}(f_{rw}(G))$ but $b_y \tilde{\notin} f_{rw}(G)$. Since $b_y \tilde{\notin} f_{rw}(G)$, then $f_{rw}^{-1}(b_y) \tilde{\subseteq} 1_{\mathcal{A}} - G$. Since $f_{rw}^{-1}(b_y) \in \varphi^{c}$, $1_{\mathcal{A}} - G \in \varphi$, and $(P, \varphi, \mathcal{A})$ is soft normal, then there exists $H \in \varphi$ such that $f_{rw}^{-1}(b_y) \tilde{\subseteq} H \tilde{\subseteq} Cl_{\varphi}(H) \tilde{\subseteq} 1_{\mathcal{A}} - G$; hence, $f_{rw}^{-1}(b_y) \tilde{\subseteq} H = Int_{\varphi}(H) \tilde{\subseteq} Int_{\varphi}(Cl_{\varphi}(H)) \tilde{\subseteq} Cl_{\varphi}(H) \tilde{\subseteq} 1_{\mathcal{A}} - G$. Let $K = Int_{\varphi}(Cl_{\varphi}(H))$. Then $K \in RO\,(\varphi)$ such that $f_{rw}^{-1}(b_y) \tilde{\subseteq} K \tilde{\subseteq} 1_{\mathcal{A}} - G$. Since f_{rw} is soft regular-closed, then by Theorem 2.14, there exists $V \in \sigma$ such that $b_y \tilde{\in} V$ and $f_{rw}^{-1}(V) \tilde{\subseteq} K \tilde{\subseteq} 1_{\mathcal{A}} - G$. Thus, $f_{rw}^{-1}(V)\,\tilde{\cap}\,(1_{\mathcal{A}} - G) = 0_{\mathcal{A}}$ and hence, $V \tilde{\cap} f_{rw}(G) = 0_{\mathcal{B}}$. On the other hand, since $b_y \tilde{\in} Cl_{\sigma}(f_{rw}(G))$ and $b_y \tilde{\in} V \in \sigma$, then $V \tilde{\cap} f_{rw}(G) \neq 0_{\mathcal{B}}$, a contradiction. □

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3 Soft Separation Axioms

Theorem 3.1. *Soft normality is preserved under soft continuous regular-closed surjections.*

Proof. Let $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ be a soft continuous soft regular-closed surjection such that $(P, \varphi, \mathcal{A})$ is soft normal. Let $\{M, N\} \subseteq \sigma^c$ such that $M \tilde{\cap} N = 0_{\mathcal{B}}$. Since f_{rw} is soft continuous, then $\left\{f_{rw}^{-1}(M), f_{rw}^{-1}(N)\right\} \subseteq \varphi^c$. Also, we have $f_{rw}^{-1}(M) \tilde{\cap} f_{rw}^{-1}(N) = f_{rw}^{-1}(M \tilde{\cap} N) = f_{rw}^{-1}(0_{\mathcal{B}}) = 0_{\mathcal{A}}$. Since $(P, \varphi, \mathcal{A})$ is soft normal, then there exists $\{G, H\} \subseteq \varphi$ such that $f_{rw}^{-1}(M) \tilde{\subseteq} G$, $f_{rw}^{-1}(N) \tilde{\subseteq} H$, and $G \tilde{\cap} H = 0_{\mathcal{A}}$. Again, by soft normality, there exists $\{L, R\} \subseteq \varphi$ such that $f_{rw}^{-1}(M) \tilde{\subseteq} L \tilde{\subseteq} Cl_{\varphi}(L) \tilde{\subseteq} G$ and $f_{rw}^{-1}(N) \tilde{\subseteq} R \tilde{\subseteq} Cl_{\varphi}(R) \tilde{\subseteq} H$. So, we have $\{Int_{\varphi}(Cl_{\varphi}(L)), Int_{\varphi}(Cl_{\varphi}(R))\} \subseteq RO(\varphi)$, $f_{rw}^{-1}(M) \tilde{\subseteq} Int_{\varphi}(Cl_{\varphi}(L))$, $f_{rw}^{-1}(N) \tilde{\subseteq} Int_{\varphi}(Cl_{\varphi}(R))$, and

$$\begin{array}{rcc} Int_{\varphi}(Cl_{\varphi}(L)) \tilde{\cap} Int_{\varphi}(Cl_{\varphi}(R)) & = & Int_{\varphi}(Cl_{\varphi}(L) \tilde{\cap} Cl_{\varphi}(R)) \\ & \tilde{\subseteq} & Int_{\varphi}(G \tilde{\cap} H) \\ & = & Int_{\varphi}(0_{\mathcal{A}}) \\ & = & 0_{\mathcal{A}}. \end{array}$$

Since f_{rw} is soft regular-closed, then by Theorem 2.14, there exists $\{V, W\} \subseteq \sigma$ such that $M \tilde{\subseteq} V$, $N \tilde{\subseteq} W$, $f_{rw}^{-1}(V) \tilde{\subseteq} Int_{\varphi}(Cl_{\varphi}(L))$, and $f_{rw}^{-1}(W) \tilde{\subseteq} Int_{\varphi}(Cl_{\varphi}(R))$. Since $f_{rw}^{-1}(V) \tilde{\cap} f_{rw}^{-1}(W) \tilde{\subseteq} Int_{\varphi}(Cl_{\varphi}(L)) \tilde{\cap} Int_{\varphi}(Cl_{\varphi}(R)) = 0_{\mathcal{A}}$, then $f_{rw}^{-1}(V) \tilde{\cap} f_{rw}^{-1}(W) = f_{rw}^{-1}(V \tilde{\cap} W) = 0_{\mathcal{A}}$. Since f_{rw} is surjective, then we must have $V \tilde{\cap} W = 0_{\mathcal{B}}$. This completes the proof. □

Theorem 3.2. *Let $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ be soft open, soft regular-closed, and bijective. Then $(S, \sigma, \mathcal{B})$ is soft Hausdorff.*

Proof. Since f_{rw} is soft regular-closed and injective, then by Corollary 2.22, $Gr(f_{rw}) \in (pr(\varphi \times \sigma))^c$. Let $b_y, d_z \in SP(S, \mathcal{B})$ such that $b_y \neq d_z$. Since f_{rw} is bijective, then $f_{rw}(f_{r^{-1}w^{-1}}(b_y)) = f_{rw}\left(\left(w^{-1}(b)\right)_{r^{-1}(y)}\right) \neq d_z$. Thus, $\left(w^{-1}(b), d\right)_{(r^{-1}(y), z)} \tilde{\in} 1_{\mathcal{A} \times \mathcal{B}} - Gr(f_{rw}) \in pr(\varphi \times \sigma)$. So, there exist $G \in \varphi$, $H \in \sigma$, and $\left(w^{-1}(b), d\right)_{(r^{-1}(y), z)} \tilde{\in} G \times H \tilde{\subseteq} 1_{\mathcal{A} \times \mathcal{B}} - Gr(f_{rw})$. Since f_{rw} is soft open, then $f_{rw}(G) \in \sigma$. Thus, we have $b_y \tilde{\in} f_{rw}(G) \in \sigma$, $d_z \tilde{\in} H \in \sigma$, and $f_{rw}(G) \tilde{\cap} H = 1_{\mathcal{B}}$. Therefore, $(S, \sigma, \mathcal{B})$ is soft Hausdorff. □

Theorem 3.3. *Let $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ be soft regular-closed, and bijective. If $(P, \varphi, \mathcal{A})$ is soft Urysohn, then $(S, \sigma, \mathcal{B})$ is soft Hausdorff.*

Proof. Let $b_y, d_z \in SP(S, \mathcal{B})$ such that $b_y \neq d_z$. Since f_{rw} is bijective, then $\left(w^{-1}(b)\right)_{r^{-1}(y)} \neq \left(w^{-1}(d)\right)_{r^{-1}(z)}$. Since $(P, \varphi, \mathcal{A})$ is soft Urysohn, then there exists $\{G, H\} \subseteq \varphi$ such that $\left(w^{-1}(b)\right)_{r^{-1}(y)} \tilde{\in} G$, $\left(w^{-1}(d)\right)_{r^{-1}(z)} \tilde{\in} H$, and $Cl_{\varphi}(G) \tilde{\cap} Cl_{\varphi}(H) = 0_{\mathcal{A}}$. Since $\{G, H\} \subseteq \varphi$, then $G \tilde{\subseteq} Int_{\varphi}(Cl_{\varphi}(G))$ and $H \tilde{\subseteq} Int_{\varphi}(Cl_{\varphi}(H))$. Thus, we have $f_{rw}^{-1}(b_y) \tilde{\subseteq} Int_{\varphi}(Cl_{\varphi}(G)) \in RO(\varphi)$, $f_{rw}^{-1}(d_z) \tilde{\subseteq} Int_{\varphi}(Cl_{\varphi}(H)) \in RO(\varphi)$, and $Int_{\varphi}(Cl_{\varphi}(G)) \tilde{\cap} Int_{\varphi}(Cl_{\varphi}(H)) \tilde{\subseteq}$ $0_{\mathcal{A}}$. Since f_{rw} is soft regular-closed, then by Theorem 2.14, there exists $\{V, W\} \subseteq \sigma$ such that $b_y \tilde{\in} V$, $d_z \tilde{\in} W$, $f_{rw}^{-1}(V) \tilde{\subseteq} Int_{\varphi}(Cl_{\varphi}(G))$, and $f_{rw}^{-1}(W) \tilde{\subseteq} Int_{\varphi}(Cl_{\varphi}(H))$. Since $f_{rw}^{-1}(V) \tilde{\cap} f_{rw}^{-1}(W) \tilde{\subseteq} Int_{\varphi}(Cl_{\varphi}(G)) \tilde{\cap} Int_{\varphi}(Cl_{\varphi}(H)) =$ $0_{\mathcal{A}}$, then $f_{rw}^{-1}(V) \tilde{\cap} f_{rw}^{-1}(W) = f_{rw}^{-1}(V \tilde{\cap} W) = 0_{\mathcal{A}}$. Since f_{rw} is surjective, then we must have $V \tilde{\cap} W = 0_{\mathcal{B}}$. This shows that $(S, \sigma, \mathcal{B})$ is soft Hausdorff. □

Lemma 3.4. *If $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ is soft almost-continuous and soft almost-open, then*

(a) For each $G \in RO(\sigma)$, $f_{rw}^{-1}(G) \in RO(\varphi)$.

(b) For each $K \in RC(\sigma)$, $f_{rw}^{-1}(K) \in RC(\varphi)$.

Proof. (a) Let $G \in RO(\sigma)$. Since f_{rw} is soft almost-continuous, then $f_{rw}^{-1}(G) \in \varphi$ and so $f_{rw}^{-1}(G) \tilde{\subseteq} Int_{\varphi}\left(Cl_{\varphi}\left(f_{rw}^{-1}(G)\right)\right)$. On the other hand, since $Cl_{\varphi}(G) \in RC(\sigma)$ and f_{rw} is soft almost-continuous, then $f_{rw}^{-1}(Cl_{\varphi}(G)) \in \varphi^c$ and so, $Int_{\varphi}\left(Cl_{\varphi}\left(f_{rw}^{-1}(G)\right)\right) \tilde{\subseteq} Cl_{\varphi}\left(f_{rw}^{-1}(G)\right) \tilde{\subseteq} f_{rw}^{-1}(Cl_{\varphi}(G))$. Moreover, since $Int_{\varphi}\left(Cl_{\varphi}\left(f_{rw}^{-1}(G)\right)\right) \in RO(\varphi)$ and f_{rw} is soft almost-open, then $f_{rw}\left(Int_{\varphi}\left(Cl_{\varphi}\left(f_{rw}^{-1}(G)\right)\right)\right) \in \sigma$. Since $Int_{\varphi}\left(Cl_{\varphi}\left(f_{rw}^{-1}(G)\right)\right) \tilde{\subseteq} f_{rw}^{-1}(Cl_{\varphi}(G))$, then $f_{rw}\left(Int_{\varphi}\left(Cl_{\varphi}\left(f_{rw}^{-1}(G)\right)\right)\right) \tilde{\subseteq} f_{rw}\left(f_{rw}^{-1}(Cl_{\varphi}(G))\right) \tilde{\subseteq} Cl_{\varphi}(G)$. Since $f_{rw}\left(Int_{\varphi}\left(Cl_{\varphi}\left(f_{rw}^{-1}(G)\right)\right)\right) \in \sigma$, then $f_{rw}\left(Int_{\varphi}\left(Cl_{\varphi}\left(f_{rw}^{-1}(G)\right)\right)\right) \tilde{\subseteq} Int_{\varphi}(Cl_{\varphi}(G)) = G$. Hence, $Int_{\varphi}\left(Cl_{\varphi}\left(f_{rw}^{-1}(G)\right)\right) \tilde{\subseteq} f_{rw}^{-1} (f_{rw} (Int_{\varphi} (Cl_{\varphi} (f_{rw}^{-1}(G)$ Therefore, $f_{rw}^{-1}(G) = Int_{\varphi}\left(Cl_{\varphi}\left(f_{rw}^{-1}(G)\right)\right)$ and hence, $f_{rw}^{-1}(G) \in RO(\varphi)$. □

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(b) Let $K \in RC(\sigma)$. Then $1_{\mathcal{B}} - K \in RO(\sigma)$, and by (a), $f_{rw}^{-1}(1_{\mathcal{B}} - K) = 1_{\mathcal{A}} - f_{rw}^{-1}(K) \in RO(\varphi)$. Hence, $f_{rw}^{-1}(K) \in RC(\varphi)$.

Lemma 3.5. *If $f_{rw} : (P,\varphi,\mathcal{A}) \longrightarrow (S,\sigma,\mathcal{B})$ is soft θ-continuous and soft almost-open, then f_{rw} is soft almost-continuous.*

Proof. Let $a_x \in SP(P,\mathcal{A})$ and let $G \in \sigma$ such that $f_{rw}(a_x) \in G$. Since f_{rw} is soft θ-continuous, then there exists $H \in \varphi$ such that $a_x \tilde{\in} H$ and $f_{rw}(Cl_{\varphi}(H)) \tilde{\subseteq} Cl_{\sigma}(G)$. Since $Int_{\varphi}(Cl_{\varphi}(H)) \in RO(\varphi)$ and f_{rw} is soft almost-open, then $f_{rw}(Int_{\varphi}(Cl_{\varphi}(H))) \in \sigma$. Since $H \in \varphi$, then $H \tilde{\subseteq} Int_{\varphi}(Cl_{\varphi}(H))$ and so, $f_{rw}(H) \tilde{\subseteq} f_{rw}(Int_{\varphi}(Cl_{\varphi}(H))) \tilde{\subseteq} Int_{\sigma}(f_{rw}(Cl_{\varphi}(H))) \tilde{\subseteq} Int_{\sigma}(Cl_{\sigma}(G))$. This shows that f_{rw} is soft almost-continuous. □

Theorem 3.6. *Soft almost-regularity is invariant under soft θ-continuous, soft almost-open, and soft regular-closed surjections.*

Proof. Let $f_{rw} : (P,\varphi,\mathcal{A}) \longrightarrow (S,\sigma,\mathcal{B})$ be soft θ-continuous, soft almost-open, soft regular-closed, and surjective. Let $(P,\varphi,\mathcal{A})$ be soft almost-regular. Let $b_y \in SS(S,\mathcal{B})$ and let $K \in RC(\sigma)$ such that $b_y \tilde{\notin} K$. By Lemmas 3.4 and 3.5, $f_{rw}^{-1}(K) \in RC(\varphi)$. Since f_{rw} is surjective, then there exists $a_x \in SS(P,\mathcal{A})$ such that $f_{rw}(a_x) = b_y$. Then, we have $a_x \tilde{\notin} f_{rw}^{-1}(K) \in RC(\varphi)$, and by soft almost-regularity of $(P,\varphi,\mathcal{A})$, there exists $\{U,V\} \subseteq \varphi$ such that $a_x \tilde{\in} U$, $f_{rw}^{-1}(K) \tilde{\subseteq} V$, and $U \tilde{\cap} V = 0_{\mathcal{A}}$. It is not difficult to check that $\{Int_{\varphi}(Cl_{\varphi}(U)), Int_{\varphi}(Cl_{\varphi}(V))\} \subseteq RO(\varphi)$ and $Int_{\varphi}(Cl_{\varphi}(U)) \tilde{\cap} Int_{\varphi}(Cl_{\varphi}(V)) = 0_{\mathcal{A}}$. Since f_{rw} is soft almost-open, then $f_{rw}(Int_{\varphi}(Cl_{\varphi}(U))) \in \sigma$. Since f_{rw} is soft regular-closed and $1_{\mathcal{A}} - Int_{\varphi}(Cl_{\varphi}(V)) \in RC(\varphi)$, then $f_{rw}(1_{\mathcal{A}} - Int_{\varphi}(Cl_{\varphi}(V))) \in \sigma^c$ and $1_{\mathcal{B}} - f_{rw}(1_{\mathcal{A}} - Int_{\varphi}(Cl_{\varphi}(V))) \in \sigma$. Since $a_x \tilde{\in} U \tilde{\subseteq} Int_{\varphi}(Cl_{\varphi}(U))$, then $b_y \tilde{\in} f_{rw}(Int_{\varphi}(Cl_{\varphi}(U)))$. Since $f_{rw}^{-1}(K) \tilde{\subseteq} V \tilde{\subseteq} Int_{\varphi}(Cl_{\varphi}(V))$, then $1_{\mathcal{A}} - Int_{\varphi}(Cl_{\varphi}(V)) \tilde{\subseteq} 1_{\mathcal{A}} - f_{rw}^{-1}(K) = f_{rw}^{-1}(1_{\mathcal{B}} - K)$ and so, □

$$f_{rw}(1_{\mathcal{A}} - Int_{\varphi}(Cl_{\varphi}(V))) \tilde{\subseteq} f_{rw}\left(f_{rw}^{-1}(1_{\mathcal{B}} - K)\right) \tilde{\subseteq} 1_{\mathcal{B}} - K;$$

hence, $K \tilde{\subseteq} 1_{\mathcal{B}} - f_{rw}(1_{\mathcal{A}} - Int_{\varphi}(Cl_{\varphi}(V)))$. Since $Int_{\varphi}(Cl_{\varphi}(U)) \tilde{\cap} Int_{\varphi}(Cl_{\varphi}(V)) = 0_{\mathcal{A}}$, then $Int_{\varphi}(Cl_{\varphi}(U)) \tilde{\subseteq} 1_{\mathcal{A}} - Int_{\varphi}(Cl_{\varphi}(V))$ and so,

$$f_{rw}(Int_{\varphi}(Cl_{\varphi}(U))) \tilde{\subseteq} f_{rw}(1_{\mathcal{A}} - Int_{\varphi}(Cl_{\varphi}(V)));$$

this implies that $f_{rw}(Int_{\varphi}(Cl_{\varphi}(U))) \tilde{\cap} (1_{\mathcal{B}} - f_{rw}(1_{\mathcal{A}} - Int_{\varphi}(Cl_{\varphi}(V)))) = 0_{\mathcal{B}}$. This shows that $(S,\sigma,\mathcal{B})$ is soft almost-regular.

Corollary 3.7. *Soft almost-regularity is invariant under soft continuous, soft open, and soft closed surjections.*

Theorem 3.8. *If $(P,\varphi,\mathcal{A})$ is soft regular and $f_{rw} : (P,\varphi,\mathcal{A}) \longrightarrow (S,\sigma,\mathcal{B})$ is a soft almost-continuous and soft regular-closed surjection such that $f_{rw}^{-1}(b_y)$ is soft compact for each $b_y \in SS(S,\mathcal{B})$, then $(S,\sigma,\mathcal{B})$ is soft almost-regular.*

Proof. Let $b_y \in SS(S,\mathcal{B})$ and let $G \in RO(\sigma)$ such that $b_y \tilde{\in} G$. Since f_{rw} is soft almost-continuous, then $f_{rw}^{-1}(G) \in \varphi$. For each $a_x \tilde{\in} f_{rw}^{-1}(b_y)$, $a_x \tilde{\in} f_{rw}^{-1}(G)$, and by soft regularity of $(P,\varphi,\mathcal{A})$, we find $U_{a_x} \in \varphi$ such that $a_x \tilde{\in} U_{a_x} \tilde{\subseteq} Cl_{\varphi}(U_{a_x}) \tilde{\subseteq} f_{rw}^{-1}(G)$. Since $f_{rw}^{-1}(b_y)$ is soft compact and $f_{rw}^{-1}(b_y) \tilde{\subseteq} \tilde{\cup}_{a_x \tilde{\in} f_{rw}^{-1}(b_y)} U_{a_x}$, then there exists a finite set $\digamma \subseteq SS(P,\mathcal{A})$ such that $a_x \tilde{\in} f_{rw}^{-1}(b_y)$ for each $a_x \in \digamma$ and $f_{rw}^{-1}(b_y) \tilde{\subseteq} \tilde{\cup}_{a_x \in \digamma} U_{a_x}$. We have

$$\begin{aligned} f_{rw}^{-1}(b_y) &\tilde{\subseteq} \tilde{\cup}_{a_x \in \digamma} U_{a_x} \\ &\tilde{\subseteq} \tilde{\cup}_{a_x \in \digamma} Cl_{\varphi}(U_{a_x}) \\ &= Cl_{\varphi}(\tilde{\cup}_{a_x \in \digamma} U_{a_x}) \\ &\tilde{\subseteq} f_{rw}^{-1}(G). \end{aligned}$$

Put $H = \tilde{\cup}_{a_x \in \digamma} U_{a_x}$. Then $H \in \varphi$ and $f_{rw}^{-1}(b_y) \tilde{\subseteq} H \tilde{\subseteq} Cl_{\varphi}(H) \tilde{\subseteq} f_{rw}^{-1}(G)$. So,

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$$\begin{aligned} f_{rw}^{-1}(b_y) &\tilde{\subseteq} H \\ &\tilde{\subseteq} Int_\varphi(Cl_\varphi(H)) \\ &\tilde{\subseteq} Cl_\varphi(Int_\varphi(Cl_\varphi(H))) \\ &\tilde{\subseteq} Cl_\varphi(Cl_\varphi(H)) \\ &= Cl_\varphi(H) \\ &\tilde{\subseteq} f_{rw}^{-1}(G). \end{aligned}$$

Let $K = Cl_\varphi(Cl_\varphi(H))$. Then $K \in RO(\varphi)$ with $f_{rw}^{-1}(b_y) \tilde{\subseteq} K \tilde{\subseteq} Cl_\varphi(K) \tilde{\subseteq} f_{rw}^{-1}(G)$. Since f_{rw} is soft regular-closed, then by Theorem 2.14, there exists $V \in \sigma$ such that $b_y \tilde{\in} V$ and $f_{rw}^{-1}(V) \tilde{\subseteq} K$. Then, we have

$$b_y \tilde{\in} V \tilde{\subseteq} f_{rw}(K) \tilde{\subseteq} f_{rw}(Cl_\varphi(K)) \tilde{\subseteq} f_{rw}\left(f_{rw}^{-1}(G)\right) \tilde{\subseteq} G.$$

Since f_{rw} is soft regular-closed and $Cl_\varphi(K) \in RC(\varphi)$, then $f_{rw}(Cl_\varphi(K)) \in \sigma^c$. Therefore, we have

$$b_y \tilde{\in} V \tilde{\subseteq} Int_\varphi(Cl_\varphi(V)) \tilde{\subseteq} Cl_\varphi(Int_\varphi(Cl_\varphi(V))) = Cl_\varphi(V) \tilde{\subseteq} f_{rw}(Cl_\varphi(K)) \tilde{\subseteq} G.$$

Let $M = Int_\varphi(Cl_\varphi(V))$. Then $M \in RO(\varphi)$ and $b_y \tilde{\in} M \tilde{\subseteq} Cl_\varphi(M) \tilde{\subseteq} G$. It follows that $(S, \sigma, \mathcal{B})$ is soft almost-regular. □

Theorem 3.9. *Soft almost-normality is invariant under soft continuous, soft almost-open, and soft regular-closed surjections.*

Proof. Let $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ be soft continuous, soft almost-open, soft regular-closed and surjective. Let $(P, \varphi, \mathcal{A})$ be soft almost-normal. Let $M \in RO(\sigma)$ and let $K \in \sigma^c$ such that $K \tilde{\subseteq} M$. Since f_{rw} is soft continuous, then $f_{rw}^{-1}(K) \in \varphi^c$. Since f_{rw} is soft continuous (hence soft almost-continuous) and soft almost-open, then by Lemma 3.4, $f_{rw}^{-1}(M) \in RO(\varphi)$. Since $(P, \varphi, \mathcal{A})$ is soft almost-normal and $f_{rw}^{-1}(K) \tilde{\subseteq} f_{rw}^{-1}(M)$, then there exists $U \in \varphi$ such that $f_{rw}^{-1}(K) \tilde{\subseteq} U \tilde{\subseteq} Cl_\varphi(U) \tilde{\subseteq} f_{rw}^{-1}(M)$. Thus,

$$f_{rw}^{-1}(K) \tilde{\subseteq} U \tilde{\subseteq} Int_\varphi(Cl_\varphi(U)) \tilde{\subseteq} Cl_\varphi(Int_\varphi(Cl_\varphi(U))) = Cl_\varphi(U) \tilde{\subseteq} f_{rw}^{-1}(M).$$

Let $N = Int_\varphi(Cl_\varphi(U))$. Then $N \in RO(\varphi)$ and $f_{rw}^{-1}(K) \tilde{\subseteq} N \tilde{\subseteq} Cl_\varphi(N) \tilde{\subseteq} f_{rw}^{-1}(M)$. Since f_{rw} is soft almost-open, then $f_{rw}(N) \in \sigma$. Also, since f_{rw} is soft regular-closed, then $f_{rw}(Cl_\varphi(N)) \in \sigma^c$. Therefore, we have $K \tilde{\subseteq} f_{rw}(N) \tilde{\subseteq} f_{rw}(Cl_\varphi(N)) \tilde{\subseteq} M$. Hence, $(S, \sigma, \mathcal{B})$ is soft almost-normal. □

Theorem 3.10. *Let $f_{rw} : (P, \varphi, \mathcal{A}) \longrightarrow (S, \sigma, \mathcal{B})$ be soft δ-continuous and soft regular-closed surjection such that $f_{rw}^{-1}(b_y)$ is soft N-closed relative to $(P, \varphi, \mathcal{A})$ for each $b_y \in SS(S, \mathcal{B})$. If $(P, \varphi, \mathcal{A})$ is soft almost-regular, then $(S, \sigma, \mathcal{B})$ is so.*

Proof. Let $b_y \in SS(S, \mathcal{B})$ and let $G \in RO(\sigma)$ such that $b_y \tilde{\in} G$. Since f_{rw} is soft δ-continuous, then $f_{rw}^{-1}(G) \in \varphi_\delta$. Then, for each $a_x \tilde{\in} f_{rw}^{-1}(b_y) \tilde{\subseteq} f_{rw}^{-1}(G)$, there exists $U_{a_x} \in RO(\varphi)$ such that $a_x \tilde{\in} U_{a_x} \tilde{\subseteq} f_{rw}^{-1}(G)$. Since $(P, \varphi, \mathcal{A})$ is soft almost-regular, then for each $a_x \tilde{\in} f_{rw}^{-1}(b_y)$, there exists $H_{a_x} \in RO(\varphi)$ such that $a_x \tilde{\in} H_{a_x} \tilde{\subseteq} Cl_\varphi(H_{a_x}) \tilde{\subseteq} U_{a_x} \tilde{\subseteq} f_{rw}^{-1}(G)$. Since $f_{rw}^{-1}(b_y)$ is soft N-closed relative to $(P, \varphi, \mathcal{A})$ and $f_{rw}^{-1}(b_y) \tilde{\subseteq} \tilde{\cup}_{a_x \tilde{\in} f_{rw}^{-1}(b_y)} H_{a_x}$, then there exists a finite set $\digamma \subseteq SS(P, \mathcal{A})$ such that $a_x \tilde{\in} f_{rw}^{-1}(b_y)$ for each $a_x \in \digamma$ and $f_{rw}^{-1}(b_y) \tilde{\subseteq} \tilde{\cup}_{a_x \in \digamma} H_{a_x}$. Put $H = Int_\varphi(Cl_\varphi(\tilde{\cup}_{a_x \in \digamma} H_{a_x}))$. Then $H \in RO(\varphi)$ and $f_{rw}^{-1}(b_y) \tilde{\subseteq} H \tilde{\subseteq} Cl_\varphi(H) \tilde{\subseteq} f_{rw}^{-1}(G)$. Since f_{rw} is soft regular-closed and $Cl_\varphi(H) \in RC(\varphi)$, then $f_{rw}(Cl_\varphi(H)) \in \sigma$, and by Theorem 2.14, there exists $K \in \sigma$ such that $b_y \tilde{\in} K$ and $f_{rw}^{-1}(K) \tilde{\subseteq} H$. Therefore, we have $b_y \tilde{\in} K \tilde{\subseteq} f_{rw}(H) \tilde{\subseteq} f_{rw}(Cl_\varphi(H)) \tilde{\subseteq} G$. Consequently, we obtain $b_y \tilde{\in} K \tilde{\subseteq} Cl_\varphi(K) \tilde{\subseteq} G$. It follows that $(S, \sigma, \mathcal{B})$ is soft almost-regular. □

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4 Conclusion

Soft closed functions between soft topological spaces are employed to define and investigate numerous critical properties and invariants of soft topological spaces. These functions have a wide range of applications, including soft optimization theory, soft approximation theory, soft control theory, soft data analysis, soft image processing, and soft decision making.

In this paper, soft regular-closed functions are defined as an extension of regular-closed functions in general topology. Several characterizations of soft regular-closed functions are given. It is proved that soft regular-closedness is strictly weaker than soft closedness functions. The relationships between these soft functions and their topological analogs are studied. Furthermore, sufficient conditions are provided for the soft regular-closedness of a soft function. Furthermore, utilizing soft regular-closed, many preservation theorems of soft separation axioms are provided. Finally, the study deals with soft restrictions and soft products.

The composition of two regular-closed functions is not necessarily regular-closed. Also, soft regular-closed functions do not necessarily preserve separation axioms. For example, a soft regular-closed function from a soft normal space to another soft topological space may not preserve soft normality. These are two significant limitations.

Future research might look into the following topics: (1) defining soft weakly closed functions; (2) finding a use for this new concept in a ”decision-making problem.”; and (3) extend some topological concepts to include neutrosophic topological spaces as in.[38,39]

5 Author’s contributions

This article was written in collaboration by all of the contributors. The final manuscript was read and approved by all writers.

6 Conflicts of interest

There are no competing interests declared by the authors.

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