Fekete-Szegö and Second Hankel Determinant for a Certain Subclass of Bi-Univalent Functions associated with Lucas-Balancing Polynomials

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Abstract

In this paper, a new subclass of bi-univalent functions linked to Lucas-Balancing polynomials is introduced. Bounds for the coefficients in the Taylor-Maclaurin series, denoted as $|a_2|$ and $|a_3|$, are determined for these functions. The Fekete-Szegö functional problems are also addressed, and bounds for the second Hankel determinant for functions in this specific subclass are established. Additionally, it is shown that by adjusting the parameters in the main findings, several new results can be derived.

Keywords: Lucas-Balancing polynomials; bi-univalent functions; analytic functions; Taylor-Maclaurin coefficients; Fekete-Szegö functional; second Hankel determinant

1 Introduction

Let $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, and $\mathcal{A}$ be denoting the class of all analytic functions f that are defined in $\mathbb{U}$. Consider the condition $f(0) = f'(0) - 1 = 0$. Consequently, each $f \in \mathcal{A}$ can be expressed using a Taylor-Maclaurin series expansion given by:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{U}). \tag{1}$$

Additionally, a single-valued analytic function f in the simply connected domain $\mathbb{U}$ is called univalent (Schlicht or simple) if it is injective. Let $\mathcal{S} = \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{U}\}$ In mathematics, subordination is a relationship between two functions, let's call them f and g, where f is said to be subordinate to g. This means that there exists another function $h(z)$ with specific properties defined within $\mathbb{U}$, such that $f(z) = g(h(z))$, where $h(0) = 0$ and $|h(z)| \leq 1$ for all z in $\mathbb{U}$. In [1], it was stated that if the function g is univalent in $\mathbb{U}$, then

$$f(z) \prec g(z) \quad \text{if and only if} \quad f(0) = g(0)$$

and

$$f(\mathbb{U}) \subset g(\mathbb{U}).$$

According to the Koebe one-quarter theorem [2], it is well known that every function $f \in S$ has an inverse function f^{-1} satisfying $f^{-1}(f(z)) = z, (z \in \mathbb{U})$ and $f\left(f^{-1}(w)\right) = w,\ \left(|w| < r_0(f),\, r_0(f) \geq \frac{1}{4}\right)$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots . \tag{2}$$

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A function f belonging to a certain class $\mathcal{A}$ is called bi-univalent in $\mathbb{U}$ if both the function f and its inverse $g = f^{-1}$ are univalent in $\mathbb{U}$. Denote by Σ the class of bi-univalent functions in $\mathbb{U}$ as given by equation (1). Recent research has seen the introduction of various subclasses of the bi-univalent function class Σ. These studies have mainly focused on finding bounds for the first two coefficients, $|a_2|$ and $|a_3|$, in the Taylor-Maclaurin series expansion of these functions (see [3–12]).

The most significant and thoroughly investigated subclasses of $\mathcal{S}$ are the class $\mathcal{S}^*(\epsilon)$ of starlike functions of order ϵ and the class, $\mathcal{K}(\epsilon)$ of convex functions of order ϵ in $\mathbb{U}$, with $0 \le \epsilon < 1$, which are defined by

$$\mathcal{S}^*(\epsilon) := \left\{ f : f \in \mathcal{S} \text{ and } \operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \epsilon, \ (z \in \mathbb{U};\ 0 \le \epsilon < 1) \right\},$$

and

$$\mathcal{K}(\epsilon) := \left\{ f : f \in \mathcal{S} \text{ and } \operatorname{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \epsilon, \ (z \in \mathbb{U};\ 0 \le \epsilon < 1) \right\}.$$

In 1976, the n-th Hankel determinant of function f, as given in (1), was formally defined by Noonan and Thomas [13] for integers $m \ge 1$ and $n \ge 1$. This determinant, denoted as $H_n(m)$, takes the form:

$$\mathrm{H}_n(m) = \begin{vmatrix} b_m & b_{m+1} & \cdots & b_{m+n-1} \\ b_{m+1} & b_{m+2} & \cdots & b_{m+n-2} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m+n-1} & b_{m+n-2} & \cdots & b_{m+2n-2} \end{vmatrix}, \qquad b_1 := 1.$$

Various authors have discussed This determinant extensively, particularly with $m = 1, 2$ and $n = 2$ (referenced in [14–16]. For instance, the Hankel determinants can be shown as:

$$\mathrm{H}_2(1) = \begin{vmatrix} b_1 & b_2 \\ b_2 & b_3 \end{vmatrix} = b_3 - b_2^2$$

and

$$\mathrm{H}_2(2) = \begin{vmatrix} b_2 & b_3 \\ b_3 & b_4 \end{vmatrix} = b_2 b_4 - b_3^2$$

These determinants are commonly known as the Fekete-Szegö and the second Hankel determinant functionals, respectively. The further generalized functional often considered is $b_3 - \eta b_2^2$, where $0 \le \eta < 1$, is referred to as the Fekete-Szegö problem, which can be found in reference [17]. In 1969, Keogh and Merkes investigated the Fekete-Szegö problem, specifically focusing on the classes of starlike and convex functions [18]. Moreover, the bounds for the second Hankel determinant $H_2(2)$ were obtained for the classes of starlike and convex functions [19].

In this paper, we utilize Lucas-Balancing polynomials in which its Balancing numbers, notated as (B_n), $n \ge 0$, were originally introduced by Behera and Panda [20]. These numbers are defined by the recurrence relation $B_{n+1} = 6B_n - B_{n-1}$ for $n \ge 1$, with initial values set at $B_0 = 0$ and $B_1 = 1$. A related sequence of the Lucas-Balancing numbers, denoted as $C_n = \sqrt{8B_n^2 + 1}$, has garnered significant attention. Its recurrence relation is given by $C_{n+1} = 6C_n - C_{n-1}$ for $n \ge 1$, and have initial terms $C_0 = 1$ and $C_1 = 3$. For more details, we refer readers to [21–29].

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Definition 1.1 (Lucas-Balancing Polynomials, [30])**.** Given any complex number x and an integer $n \geq 2$, Lucas-Balancing polynomials are recursively defined as follows:

$$C_n(x) = 6xC_{n-1}(x) - C_{n-2}(x), \tag{3}$$

where the initial conditions are given by:

$$C_0(x) = 1, \quad C_1(x) = 3x. \tag{4}$$

Using the recurrence relation (3), we can derive the following expressions:

$$C_2(x) = 18x^2 - 1 \quad C_3(x) = 108x^3 - 9x. \tag{5}$$

Lucas-Balancing polynomials, similar to other polynomial families, can be derived using specific generating functions. One example of such a generating function is given by:

Lemma 1.2. *[30] The generating function for Balancing polynomials is given by*

$$\mathcal{B}(x, z) = \sum_{n=0}^{\infty} C_n(x)\xi^n = \frac{1 - 3xz}{1 - 6xz + z^2}, \tag{6}$$

where x is within the interval $[-1, 1]$, and z lies in the open unit disk $\mathbb{U}$.

Recently, many authors have investigated upper bounds for the coefficients and the Hankel determinant of functions belonging to various subclasses of univalent functions (see [31–37]).

2 Fekete–Szegö Functional Estimations

In this section, we introduce a subclass of bi-univalent functions. Initially, we present a lemma crucial to our results, followed by the definition of this newly proposed subclass. Subsequently, we will present the coefficient estimates for the subclass denoted as $\boldsymbol{\mathcal{M}_{\Sigma}(\mu, \delta, \mathcal{B}(x, z))}$, as defined in Definition 2.2.

Lemma 2.1. *[2] Suppose Ω be the class of all analytic functions, and let $\omega \in \Omega$ with $\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n, \ z \in \mathbb{U}$. Then*

$$|\omega_1| \leq 1, \quad |\omega_n| \leq 1 - |\omega_1|^2 \quad for \quad n \in \mathbb{N}\backslash\{1\}.$$

Definition 2.2. Let $\mu \geq 1, \delta \in [0, 1]$ and $x \in (\frac{1}{2}, 1]$. We say that a function $f \in \Sigma$ given by (1) is in the class $\mathcal{M}_{\Sigma}(\mu, \delta, \mathcal{B}(x, z))$ if the following subordinations are satisfied

$$\frac{z\,(f'(z))^{\mu}}{(1-\delta)z + \delta f(z)} \prec \mathcal{B}(x, z) \tag{7}$$

and

$$\frac{w\,(g'(w))^{\mu}}{(1-\delta)w + \delta g(w)} \prec \mathcal{B}(x, w), \tag{8}$$

where the function $g(w) = f^{-1}(w)$ is defined by (2) and $\mathcal{B}(x, z)$ is the generating function of the Lucas-Balancing polynomials given by (6).

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Example 2.3. Let the function f be in the class Σ and $g = f^{-1}$ is defined by (2). We say that f in the class $\mathcal{M}_{\Sigma}(1, \delta, \mathcal{B}(x, z))$, if the following subordination conditions hold:

$$\frac{zf'(z)}{(1-\delta)z + \delta f(z)} \prec \mathcal{B}(x, z) \tag{9}$$

and

$$\frac{wg'(g)}{(1-\delta)w + \delta g(w)} \prec \mathcal{B}(x, w), \tag{10}$$

Example 2.4. Let the function f be in the class Σ and $g = f^{-1}$ is defined by (2). We say that f in the class $\mathcal{M}_{\Sigma}(1, 1, \mathcal{B}(x, z))$, if the following subordination conditions hold:

$$\frac{zf'(z)}{f(z)} \prec \mathcal{B}(x, z) \tag{11}$$

and

$$\frac{wg'(g)}{g(w)} \prec \mathcal{B}(x, w), \tag{12}$$

Example 2.5. Let the function f be in the class Σ and $g = f^{-1}$ is defined by (2). We say that f in the class $\mathcal{M}_{\Sigma}(2, 1, \mathcal{B}(x, z))$, if the following subordination conditions hold:

$$f'(z)\frac{zf'(z)}{f(z)} \prec \mathcal{B}(x, z) \tag{13}$$

and

$$g'(w)\frac{wg'(g)}{g(w)} \prec \mathcal{B}(x, w), \tag{14}$$

Example 2.6. Let the function f be in the class Σ and $g = f^{-1}$ is defined by (2). We say that f in the class $\mathcal{M}_{\Sigma}(\mu, 0, \mathcal{B}(x, z))$, if the following subordination conditions hold:

$$(f'(z))^{\mu} \prec \mathcal{B}(x, z) \tag{15}$$

and

$$(g'(w))^{\mu} \prec \mathcal{B}(x, w), \tag{16}$$

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Example 2.7. Let the function f be in the class Σ and $g = f^{-1}$ is defined by (2). We say that f in the class $\mathcal{M}_\Sigma(1,0,\mathcal{B}(x,z))$, if the following subordination conditions hold:

$$f'(z) \prec \mathcal{B}(x,z) \tag{17}$$

and

$$g'(w) \prec \mathcal{B}(x,w), \tag{18}$$

Theorem 2.8. *Let $f \in \Sigma$ of the form (1) be in the class $\mathcal{M}_\Sigma(\mu,\delta,\mathcal{B}(x,z))$. Then*

$$|a_2| \le \frac{3x\sqrt{3x}}{\sqrt{\left|9x^2\left(2\mu^2+\mu(1-2\delta)-\delta(1-\delta)\right)-\left(18x^2-1\right)(2\mu-\delta)^2\right|}}, \tag{19}$$

and

$$|a_3| \le \frac{27x^3}{\left|9x^2\left(2\mu^2+\mu(1-2\delta)-\delta(1-\delta)\right)-\left(18x^2-1\right)(2\mu-\delta)^2\right|}+\frac{3x}{3\mu-\delta}. \tag{20}$$

Proof. Since $f \in \mathcal{M}_\Sigma(\mu,\delta,\mathcal{B}(x,z))$. Then

$$\frac{z\left(f'(z)\right)^\mu}{(1-\delta)z+\delta f(z)} = \mathcal{B}(x,u(z)) \tag{21}$$

and

$$\frac{w\left(g'(w)\right)^\mu}{(1-\delta)w+\delta g(w)} = \mathcal{B}(x,v(w)), \tag{22}$$

where $u,\ v \in \Omega$ are defined by

$$u(z) = \sum_{n=1}^{\infty} c_n\, z^n \qquad and \qquad v(w) = \sum_{n=1}^{\infty} d_n\, w^n. \tag{23}$$

Using Lemma 2.1, we obtain

$$|c_n| \le 1 \text{ and } |d_n| \le 1,\ n \in \mathbb{N}. \tag{24}$$

Then, from (21) and (22) we have

$$\begin{aligned} &1+(2\mu-\delta)a_2z+\left[(2\mu^2-2\mu(\delta+1)+\delta^2)a_2^2-(\delta-3\mu)a_3\right]z^2+\Big[\left(2\delta^2-5\delta\mu+6\mu\left(\mu-1\right)\right)a_2a_3 \\ &\quad -\frac{1}{3}\left(3\delta^3-6\delta^2\mu+6\delta\mu\left(\mu-1\right)-4\mu\left(\mu^2-3\mu+2\right)\right)a_2^3+\left(4\mu-\delta\right)a_4\Big]z^3+\cdots \\ &\quad = 1+C_1(x)c_1z+\left[C_1(x)c_2+C_2(x)c_1^2\right]z^2+\left[C_1(x)c_3+2C_2(x)c_1c_2+C_3(x)c_1^3\right]z^3+\cdots, \end{aligned} \tag{25}$$

and

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$$1-(2\mu-\delta)a_2w+\left[\left(2\mu^2+(2\mu-\delta)(2-\delta)\right)a_2^2+(\delta-3\mu)a_3\right]w^2+\Big[\left(2\delta^2-5\delta\left(\mu+1\right)+2\mu\left(3\mu+7\right)\right)a_2a_3$$
$$+\frac{1}{3}\Big(3\delta^3-6\delta^2\left(\mu+2\right)+3\delta\left(2\mu^2+8\mu+5\right)-4\mu\left(\mu^2+6\mu+8\right)\Big)a_2^3-\left(4\mu-\delta\right)a_4\Big]w^3+\cdots$$
$$=1+C_1(x)d_1w+\left[C_1(x)d_2+C_2(x)d_1^2\right]w^2+\left[C_1(x)d_3+2C_2(x)d_1d_2+C_3(x)d_1^3\right]w^3+\cdots. \tag{26}$$

Equating the coefficient in equations (25) and (26) , implies

$$(2\mu-\delta)a_2=C_1(x)c_1, \tag{27}$$

$$\left(2\mu^2-2\mu(\delta+1)+\delta^2\right)a_2^2-(\delta-3\mu)a_3=C_1(x)c_2+C_2(x)c_1^2, \tag{28}$$

$$\left(2\delta^2-5\delta\mu+6\mu\left(\mu-1\right)\right)a_2a_3-\frac{1}{3}\left(3\delta^3-6\delta^2\mu+6\delta\mu\left(\mu-1\right)-4\mu\left(\mu^2-3\mu+2\right)\right)a_2^3$$
$$+\left(4\mu-\delta\right)a_4=C_1(x)c_3+2C_2(x)c_1c_2+C_3(x)c_1^3, \tag{29}$$

$$-(2\mu-\delta)a_2=C_1(x)d_1, \tag{30}$$

$$\left(2\mu^2+(2\mu-\delta)(2-\delta)\right)a_2^2+(\delta-3\mu)a_3=C_1(x)d_2+C_2(x)d_1^2, \tag{31}$$

and

$$\left(2\delta^2-5\delta\left(\mu+1\right)+2\mu\left(3\mu+7\right)\right)a_2a_3+\frac{1}{3}\Big(3\delta^3-6\delta^2\left(\mu+2\right)+3\delta\left(2\mu^2+8\mu+5\right)$$
$$-4\mu\left(\mu^2+6\mu+8\right)\Big)a_2^3-\left(4\mu-\delta\right)a_4=C_1(x)d_3+2C_2(x)d_1d_2+C_3(x)d_1^3. \tag{32}$$

Utilizing (27) and (30), we can deduce the following equations

$$c_1=-d_1 \tag{33}$$

and

$$c_1^2+d_1^2=\frac{2(2\mu-\delta)^2a_2^2}{\left(C_1(x)\right)^2}. \tag{34}$$

Moreover, from equations (28), (31) and (34) we obtain

$$a_2^2=\frac{\left(C_1(x)\right)^3(c_2+d_2)}{(C_1(x))^2\left(4\mu^2+2\mu(1-2\delta)-2\delta(1-\delta)\right)-2C_2(x)\left(2\mu-\delta\right)^2}. \tag{35}$$

From equations (27) and (33) and Lemma 2.1, and , we have

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$$|a_2|^2 \leq \frac{|C_1(x)|^3}{\left|(C_1(x))^2\left(2\mu^2+\mu(1-2\delta)-\delta(1-\delta)\right)-C_2(x)\left(2\mu-\delta\right)^2\right|}, \tag{36}$$

thus

$$|a_2| \leq \frac{|C_1(x)|\sqrt{|C_1(x)|}}{\sqrt{\left|(C_1(x))^2\left(2\mu^2+\mu(1-2\delta)-\delta(1-\delta)\right)-C_2(x)\left(2\mu-\delta\right)^2\right|}}. \tag{37}$$

Substituting the values of $C_1(x)$ and $C_2(x)$ given by (21) and (22) into equation (37) implies

$$|a_2| \leq \frac{3x\sqrt{3x}}{\sqrt{\left|9x^2\left(2\mu^2+\mu(1-2\delta)-\delta(1-\delta)\right)-(18x^2-1)\left(2\mu-\delta\right)^2\right|}}.$$

Subtracting equation (31) from equation (28), yields

$$a_3 = a_2^2 + \frac{C_1(x)(c_2-d_2)}{2(3\mu-\delta)}. \tag{38}$$

Hence

$$|a_3| \leq |a_2|^2 + \frac{|C_1(x)|\,|c_2-d_2|}{2(3\mu-\delta)}. \tag{39}$$

Referring to equation (21), (22), and Lemma 2.1 and we have

$$|a_3| \leq \frac{27x^3}{\left|9x^2\left(2\mu^2+\mu(1-2\delta)-\delta(1-\delta)\right)-(18x^2-1)\left(2\mu-\delta\right)^2\right|} + \frac{3x}{3\mu-\delta}. \tag{40}$$

Hence, the proof of Theorem 2.8 is achieved. □

Theorem 2.9. *Let $f \in \Sigma$ given by the form (1) be in the class $\mathcal{M}_\Sigma(\mu,\delta,\mathcal{B}(x,z))$. Then*

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \frac{3x}{3\mu-\delta} & \text{if} \quad 0 \leq |h(\eta)| \leq \frac{1}{2(3\mu-\delta)} \\ 6x\,|h(\eta)| & \text{if} \quad |h(\eta)| \geq \frac{1}{2(3\mu-\delta)}, \end{cases}$$

where

$$h(\eta) = \frac{9x^2(1-\eta)}{9x^2\left(4\mu^2+2\mu(1-2\delta)-2\delta(1-\delta)\right)-2(18x^2-1)\left(2\mu-\delta\right)^2}.$$

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Proof. From equations (35) and (38), we obtain

$$\begin{aligned} a_3-\eta a_2^2 &= a_2^2+\frac{C_1(x)(c_2-d_2)}{2(3\mu-\delta)}-\eta a_2^2 \\ &=(1-\eta)a_2^2+\frac{C_1(x)(c_2-d_2)}{2(3\mu-\delta)} \\ &=(1-\eta)\frac{\big(C_1(x)\big)^3(c_2+d_2)}{(C_1(x))^2\left(4\mu^2+2\mu(1-2\delta)-2\delta(1-\delta)\right)-2C_2(x)\left(2\mu-\delta\right)^2}+\frac{C_1(x)(c_2-d_2)}{2(3\mu-\delta)} \\ &=\big(C_1(x)\big)\left(\left[h(\eta)+\frac{1}{2(3\mu-\delta)}\right]c_2+\left[h(\eta)-\frac{1}{2(3\mu-\delta)}\right]d_2\right), \end{aligned}$$

where

$$h(\eta)=\frac{\big(C_1(x)\big)^2(1-\eta)}{(C_1(x))^2\left(4\mu^2+2\mu(1-2\delta)-2\delta(1-\delta)\right)-2C_2(x)\left(2\mu-\delta\right)^2}.$$

Moreover, utilizing (21), (22) and (24), we obtain

$$\left|a_3-\eta a_2^2\right|\leq\begin{cases}\frac{3x}{3\mu-\delta} & \text{if} \quad 0\leq|h(\eta)|\leq\frac{1}{2(3\mu-\delta)}\\ 6x\,|h(\eta)| & \text{if} \quad |h(\eta)|\geq\frac{1}{2(3\mu-\delta)}.\end{cases}$$

Hence, the proof of Theorem 2.9 is achieved. □

Corollary 2.10. *Let $f\in\Sigma$ given by the form (1) be in the class $\mathcal{M}_\Sigma(1,\delta,\mathcal{B}(x,z))$. Then*

$$|a_2|\leq\frac{3x\sqrt{3x}}{\sqrt{|(1-9x^2)\delta^2+(45x^2-4)\delta-45x^2+4|}},$$

$$|a_3|\leq\frac{27x^3}{|(1-9x^2)\delta^2+(45x^2-4)\delta-45x^2+4|}+\frac{3x}{3-\delta}.$$

and

$$\left|a_3-\eta a_2^2\right|\leq\begin{cases}\frac{3x}{3-\delta} & \textit{if} \quad 0\leq|h_3(\eta)|\leq\frac{1}{2(3-\delta)}\\ 6x\,|h_3(\eta)| & \textit{if} \quad |h_3(\eta)|\geq\frac{1}{2(3-\delta)},\end{cases}$$

where

$$h_3(\eta)=\frac{9x^2(1-\eta)}{2\big((1-9x^2)\delta^2+(45x^2-4)\delta-45x^2+4\big)}.$$

Corollary 2.11. *Let $f\in\Sigma$ given by the form (1) be in the class $\mathcal{M}_\Sigma(1,1,\mathcal{B}(x,z))$. Then*

$$|a_2|\leq\frac{3x\sqrt{3x}}{\sqrt{|1-9x^2|}},$$

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$$|a_3| \leq \frac{27x^3}{|1-9x^2|} + \frac{3x}{2}.$$

and

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{3x}{2} & \textit{if} \quad 0 \leq |h_3(\eta)| \leq \frac{1}{4} \\ 6x\,|h_3(\eta)| & \textit{if} \quad |h_3(\eta)| \geq \frac{1}{4}, \end{cases}$$

where

$$h_3(\eta) = \frac{9x^2(1-\eta)}{2\big(1-9x^2\big)}.$$

Corollary 2.12. *Let* $f \in \Sigma$ *given by the form (1) be in the class* $\mathcal{M}_\Sigma(2, 1, \mathcal{B}(x, z))$. *Then*

$$|a_2| \leq \frac{3x\sqrt{3x}}{\sqrt{|9-108x^2|}},$$

$$|a_3| \leq \frac{27x^3}{|9-108x^2|} + \frac{3x}{5},$$

and

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{3x}{5} & \textit{if} \quad 0 \leq |h_4(\eta)| \leq \frac{1}{10} \\ 6x\,|h_4(\eta)| & \textit{if} \quad |h_4(\eta)| \geq \frac{1}{10}, \end{cases}$$

where

$$h_4(\eta) = \frac{9x^2(1-\eta)}{18-216x^2}.$$

Corollary 2.13. *Let* $f \in \Sigma$ *given by the form (1) be in the class* $\mathcal{M}_\Sigma(\mu, 0, \mathcal{B}(x, z))$. *Then*

$$|a_2| \leq \frac{3x\sqrt{3x}}{\sqrt{|(4-54x^2)\mu^2 + 9\mu x^2|}},$$

$$|a_3| \leq \frac{27x^3}{|(4-54x^2)\mu^2 + 9\mu x^2|} + \frac{x}{\mu},$$

and

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{x}{\mu} & \textit{if} \quad 0 \leq |h_5(\eta)| \leq \frac{1}{6\mu} \\ 6x\,|h_5(\eta)| & \textit{if} \quad |h_5(\eta)| \geq \frac{1}{6\mu}, \end{cases}$$

where

$$h_5(\eta) = \frac{9x^2(1-\eta)}{2\mu\big((4-54x^2)\mu + 9x^2\big)}.$$

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Corollary 2.14. *Let* $f \in \Sigma$ *given by the form (1) be in the class* $\mathcal{M}_\Sigma(1, 0, \mathcal{B}(x, z))$*. Then*

$$|a_2| \leq \frac{3x\sqrt{3x}}{\sqrt{|4 - 54x^2|}},$$

$$|a_3| \leq \frac{27x^3}{|4 - 54x^2|} + x,$$

and

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} x & \text{if} \quad 0 \leq |h_6(\eta)| \leq \frac{1}{6} \\ 6x\,|h_6(\eta)| & \text{if} \quad |h_6(\eta)| \geq \frac{1}{6}, \end{cases}$$

where

$$h_6(\eta) = \frac{9x^2(1-\eta)}{8 - 90x^2}.$$

3 Second Hankel Determinant

In this section, we aim to determine the upper bound for the second Hankel determinant for functions $f \in \Sigma$ of the form (1) belonging to the class $\mathcal{M}_\Sigma(\mu, \delta, \mathcal{B}(x, z))$ by using Lucas-Balancing polynomials. Also, we present significant consequences associated with the class $\mathcal{M}_\Sigma(\mu, \delta, \mathcal{B}(x, z))$. We will begin by examining the following lemmas that play a crucial role in constructing our main results.

Lemma 3.1. *(see [38]) If* $p \in \mathcal{P}$ *be the function defined by*

$$p(z) = 1 + \sum_{i=1}^{\infty} c_i\, z^i, \tag{41}$$

then

$$|c_k| \leq 2, \qquad k = 1,\, 2,\, ...$$

Lemma 3.2. *(see [39]) If* $p \in \mathcal{P}$ *be the function defined by (41), then*

$$2c_2 = c_1^2 + (4 - c_1^2)\,x \tag{42}$$

and

$$4c_3 = c_1^3 + 2(4 - c_1^2)\,c_1\,x - c_1(4 - c_1^2)\,x^2 + 2(4 - c_1^2)(1 - |x|^2)\,z \tag{43}$$

for some x *and* z *with* $|x| \leq 1$ *and* $|z| \leq 1$.

Theorem 3.3. *Let* $f \in \Sigma$ *given by the form (1) be in the class* $\mathcal{M}_\Sigma(\mu, \delta, \mathcal{B}(x, z))$*. Then*

$$\left|a_2 a_4 - a_3^2\right| \leq \begin{cases} K(2^-, x) & M_1 \geq 0 \quad \text{and} \quad M_2 \geq 0; \\ \max\left\{\dfrac{36x^2}{(3\mu-\delta)^2}, K(2^-, x)\right\} & M_1 > 0 \quad \text{and} \quad M_2 < 0; \\ \dfrac{36x^2}{(3\mu-\delta)^2} & M_1 \leq 0 \quad \text{and} \quad M_2 \leq 0; \\ \max\left\{K(c_0, x), K(2^-, x)\right\} & M_1 < 0 \quad \text{and} \quad M_2 > 0, \end{cases} \tag{44}$$

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where

$$K(2^{-},x)=\frac{4\left(C_{1}\left(x\right)\right)^{2}}{\left(3\mu-\delta\right)^{2}}+\frac{4M_{1}c^{4}+12M_{2}c^{2}}{3\left(2\mu-\delta\right)^{4}\left(4\mu-\delta\right)^{3}\left(3\mu-\delta\right)^{2}}, \tag{45}$$

$$K(c_{0},x)=\frac{4(C_{1}(x))^{2}}{(3\mu-\delta)^{2}}-\frac{3M_{2}^{2}}{M_{1}\left(2\mu-\delta\right)^{4}\left(4\mu-\delta\right)^{3}\left(3\mu-\delta\right)^{2}},\quad c_{0}=\sqrt{\frac{-6M_{2}}{M_{1}}} \tag{46}$$

$$\begin{aligned}M_{1}&=\left(3\mu-\delta\right)^{2}\left|\Omega\left(\mu,b\right)\right|C_{1}\left(x\right)-6\left(4\mu-\delta\right)^{2}\left(2\mu-\delta\right)^{3}\left(3\mu-\delta\right)^{2}\left(C_{1}\left(x\right)\right)^{2}\\&-27\left(2\mu-\delta\right)^{2}\left(4\mu-\delta\right)^{3}\left(3\mu-\delta\right)\left(C_{1}\left(x\right)\right)^{3}-12\left(2\mu-\delta\right)^{3}\left(4\mu-\delta\right)^{2}\left(3\mu-\delta\right)^{2}C_{1}\left(x\right)C_{2}\left(x\right)\\&-3\left(2\mu-\delta\right)^{3}\left(4\mu-\delta\right)^{2}\left(3\mu-\delta\right)^{2}\left(C_{1}\left(x\right)\right)^{2}+3\left(2\mu-\delta\right)^{4}\left(4\mu-\delta\right)^{3}\left(C_{1}\left(x\right)\right)^{2}\end{aligned} \tag{47}$$

and

$$\begin{aligned}M_{2}&=2\left(2\mu-\delta\right)^{3}\left(4\mu-\delta\right)^{2}\left(3\mu-\delta\right)^{2}\left(C_{1}\left(x\right)\right)^{2}+9\left(2\mu-\delta\right)^{2}\left(4\mu-\delta\right)^{3}\left(3\mu-\delta\right)\left(C_{1}(x)\right)^{3}\\&+4\left(2\mu-\delta\right)^{3}\left(4\mu-\delta\right)^{2}\left(3\mu-\delta\right)^{2}C_{1}\left(x\right)C_{2}\left(x\right)+\left(2\mu-\delta\right)^{3}\left(4\mu-\delta\right)^{2}\left(3\mu-\delta\right)^{2}\left(C_{1}\left(x\right)\right)^{2}\\&-2\left(2\mu-\delta\right)^{4}\left(4\mu-\delta\right)^{3}\left(C_{1}\left(x\right)\right)^{2}\end{aligned} \tag{48}$$

where $C_1(x)$, $C_2(x)$, and $C_3(x)$ are defined in equation (21) and (22).

Proof. Subtracting equation (32) from equation (29) and applying equations (27), (33), and (38), we obtain:

$$\begin{aligned}a_{4}=&\left[\frac{Q(\mu,\delta)\cdot\left(C_{1}\left(x\right)\right)^{3}}{12\left(2\mu-\delta\right)^{3}\left(4\mu-\delta\right)^{3}}-\frac{C_{3}(x)}{4\mu-\delta}\right]c_{1}^{3}-\frac{C_{1}\left(x\right)}{2\left(4\mu-\delta\right)}\left(c_{3}-d_{3}\right)\\&-\frac{5\left(C_{1}\left(x\right)\right)^{2}}{4\left(3\mu-\delta\right)\left(2\mu-\delta\right)}c_{1}\left(c_{2}-d_{2}\right)-\frac{C_{2}\left(x\right)}{4\mu-\delta}c_{1}\left(c_{2}+d_{2}\right)\end{aligned} \tag{49}$$

where

$$Q(\mu,\delta)=-30\left(2\mu-\delta\right)^{3}+2\left(4\mu-\delta\right)^{2}\left(-6\delta^{3}+12\delta^{2}\left(\mu+1\right)-3\delta\left(4\mu^{2}+6\mu+5\right)+4\mu\left(2\mu^{2}+3\mu+10\right)\right)$$

Thus from (27), (38), and (49) we can simply establish that

$$\begin{aligned}a_{2}a_{4}-a_{3}^{2}=&\left[\frac{\left(Q\left(\mu,\delta\right)-12\left(4\mu-\delta\right)^{3}\right)\left(C_{1}\left(x\right)\right)^{4}}{12\left(2\mu-\delta\right)^{4}\left(4\mu-\delta\right)^{3}}-\frac{C_{1}\left(x\right)C_{3}\left(x\right)}{\left(4\mu-\delta\right)\left(2\mu-\delta\right)}\right]c_{1}^{4}\\&-\frac{\left(C_{1}\left(x\right)\right)^{2}}{2\left(4\mu-\delta\right)\left(2\mu-\delta\right)}c_{1}\left(c_{3}-d_{3}\right)-\frac{9\left(C_{1}\left(x\right)\right)^{3}}{4\left(3\mu-\delta\right)\left(2\mu-\delta\right)^{2}}c_{1}^{2}\left(c_{2}-d_{2}\right)\\&-\frac{C_{1}\left(x\right)C_{2}\left(x\right)}{\left(4\mu-\delta\right)\left(2\mu-\delta\right)}c_{1}^{2}\left(c_{2}+d_{2}\right)-\frac{\left(C_{1}\left(x\right)\right)^{2}\left(c_{2}-d_{2}\right)^{2}}{4\left(3\mu-\delta\right)^{2}}\end{aligned} \tag{50}$$

According to Lemma (3.2), we have

$$c_{2}-d_{2}=\frac{4-c^{2}}{2}\left(x-y\right), \tag{51}$$

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$$c_2 + d_2 = c_1^2 + \frac{4-c^2}{2}(x+y), \tag{52}$$

and

$$c_3 - d_3 = \frac{c_1^3}{2} + \frac{(4-c^2)c_1}{2}(x+y) - \frac{(4-c_1^2)c_1}{4}(x^2+y^2) + \frac{4-c_1^2}{2}\Big[(1-|x|^2)z - (1-|y|^2)w\Big], \tag{53}$$

for some $x,\ y,\ z,$ and w with $|x| \le 1,\ |y| \le 1,\ |z| \le 1,$ and $|w| \le 1$. Thus, by plugging equations (51), (52) and (53) into equation (50), we obtain that

$$\begin{aligned}\left|a_2a_4 - a_3^2\right| \le\ & \frac{C_1(x)\,|\Omega(\mu,\delta)|\,c_1^4}{12\,(2\mu-\delta)^4\,(4\mu-\delta)^3} + \frac{(C_1(x))^2\left(4-c_1^2\right)c_1}{2\,(4\mu-\delta)\,(2\mu-\delta)} \\ & + \left[\frac{(C_1(x))^2\left(4-c_1^2\right)c_1^2}{4\,(4\mu-\delta)\,(2\mu-\delta)} + \frac{9\,(C_1(x))^3\,(4-c_1^2)c_1^2}{8\,(3\mu-\delta)\,(2\mu-\delta)^2} + \frac{C_1(x)\,C_2(x)\left(4-c_1^2\right)c_1^2}{2\,(4\mu-\delta)\,(2\mu-\delta)}\right](|x|+|y|) \\ & + \left[\frac{(C_1(x))^2\left(4-c_1^2\right)c_1^2}{8\,(4\mu-\delta)\,(2\mu-\delta)} - \frac{(C_1(x))^2\left(4-c_1^2\right)c_1}{4\,(4\mu-\delta)\,(2\mu-\delta)}\right]\left(|x|^2+|y|^2\right) + \frac{(C_1(x))^2(4-c_1^2)^2}{16(3\mu-\delta)^2}\,(|x|+|y|)^2\end{aligned} \tag{54}$$

where

$$\begin{aligned}\Omega(\mu,\delta) =\ & \Big(Q(\mu,\delta) - 12\,(4\mu-\delta)^3\Big)\,(C_1(x))^3 - 12\,(4\mu-\delta)^3\,(2\mu-\delta)^2\,C_3(x) \\ & - 3\,(4\mu-\delta)^3\,(2\mu-\delta)^2\,(C_1(x)) - 12\,(4\mu-\delta)^3\,(2\mu-\delta)^2\,C_2(x)\end{aligned} \tag{55}$$

The use of Lemma (3.2) enables us to consider, without any limitations, that $c \in [0,2]$, where $c_1 = c$. Consequently, by defining $\gamma_1 = |x|$ and $\gamma_2 = |y|$, we can straightforwardly deduce that

$$\left|a_2a_4 - a_3^2\right| \le T_1 + T_2(\gamma_1+\gamma_2) + T_3(\gamma_1^2+\gamma_2^2) + T_4(\gamma_1+\gamma_2)^2 = \psi(\gamma_1,\gamma_2), \tag{56}$$

where

$$T_1 = \frac{C_1(x)\,|\Omega(\mu,\delta)|\,c^4}{12\,(2\mu-\delta)^4\,(4\mu-\delta)^3} + \frac{(C_1(x))^2\left(4-c^2\right)c}{2\,(4\mu-\delta)\,(2\mu-\delta)} \ge 0, \tag{57}$$

$$T_2 = \left[\frac{(C_1(x))^2\left(4-c^2\right)c^2}{4\,(4\mu-\delta)\,(2\mu-\delta)} + \frac{9\,(C_1(x))^3\,(4-c^2)c^2}{8\,(3\mu-\delta)\,(2\mu-\delta)^2} + \frac{C_1(x)\,C_2(x)\left(4-c^2\right)c^2}{2\,(4\mu-\delta)\,(2\mu-\delta)}\right] \ge 0, \tag{58}$$

$$T_3 = \left[\frac{(C_1(x))^2(4-c^2)(c-2)c}{8(4\mu-\delta)(2\mu-\delta)}\right] \le 0, \tag{59}$$

and

$$T_4 = \left[\frac{(C_1(x))^2(4-c^2)^2}{16(3\mu-\delta)^2}\right] \ge 0. \tag{60}$$

The essential target now is to maximize the function $\psi(\gamma_1,\gamma_2)$ defined by (56) within the closed square delineated by

$$\mathbb{S} = \{(\gamma_1,\gamma_2) : 0 \le \gamma_1 \le 1, 0 \le \gamma_2 \le 1\}$$

Since $T_3 < 0$ and $T_3 + 2T_4 > 0$ for all $x \in (\frac{1}{2}, 1)$ and $c \in (0,2)$, we can deduce that

$$\psi_{\gamma_1\,\gamma_1}\psi_{\gamma_2\,\gamma_2} - (\psi_{\gamma_1\,\gamma_2})^2 < 0, \quad \text{for all} \quad \gamma_1, \gamma_2 \in \mathbb{S}.$$

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As a consequence of this, it is impossible for the function ψ to attain a local maximum in the interior of the square $\mathbb{S}$. Therefore, we will now explore the maximum value of ψ on the boundary of $\mathbb{S}$.
(1) When $\gamma_1 = 0$ and $0 \leq \gamma_2 \leq 1$ (similarly $\gamma_2 = 0$ and $0 \leq \gamma_1 \leq 1$), we obtain that

$$\psi(0, \gamma_2) = \varphi_1(\gamma_2) = T_1 + T_2\gamma_2 + (T_3 + T_4)\gamma_2^2.$$

The next step is to handle separately the following two cases.

Case I: If $T_3 + T_4 \geq 0$, for $0 < \gamma_2 < 1$, for any given $c \in [0, 2)$, and for all $x \in (\frac{1}{2}, 1)$, it is obvious that $\varphi_1'(\gamma_2) = 2(T_3+T_4)\gamma_2+T_2 > 0$, indicating that the function $\varphi_1(\gamma_2)$ is increasing.
Case II: If $T_3 + T_4 < 0$ and since $T_2 + 2(T_3 + T_4) \geq 0$, for $0 < \gamma_2 < 1$, for any given $c \in [0, 2)$, and for all $x \in (\frac{1}{2}, 1)$, it is obvious that $T_2 + 2(T_3 + T_4) < 2(T_3 + T_4)\gamma_2 + T_2 < T_2$, and thus $\varphi_1'(\gamma_2) > 0$. Therefore for any given $c \in [0, 2)$ and $x \in (\frac{1}{2}, 1)$, the maximum of $\varphi_1(\gamma_2)$ in both cases occurs when $\gamma_2 = 1$ and

$$\max \varphi_1(\gamma_2) = \varphi_1(1) = T_1 + T_2 + T_3 + T_4. \tag{61}$$

Also for $c = 2$, we obtain that

$$\psi\left(\gamma_1, \gamma_2\right) = T_1\Big|_{c=2} = \frac{4C_1(x)\left|\Omega(\mu, \delta)\right|}{3\left(2\mu - \delta\right)^4\left(4\mu - \delta\right)^3}. \tag{62}$$

Taking in consideration the equation (62) and the above two cases, for $0 \leq \gamma_2 < 1$, for any fixed $c \in [0, 2]$, and for all $x \in (\frac{1}{2}, 1)$, the maximum of $\varphi_1(\gamma_2)$ is

$$\max \varphi_1\left(\gamma_2\right) = \varphi_1(1) = T_1 + T_2 + T_3 + T_4.$$

(2) When $\gamma_1 = 1$ and $0 \leq \gamma_2 \leq 1$ (similarly $\gamma_2 = 1$ and $0 \leq \gamma_1 \leq 1$), we obtain that

$$\psi(1, \gamma_2) = \varphi_2(\gamma_2) = \left(T_3 + T_4\right)\gamma_2^2 + \left(T_2 + 2T_4\right)\gamma_2 + T_1 + T_2 + T_3 + T_4.$$

Therefore, from the previously mentioned cases of $T_3 + T_4$, we conclude that

$$\max \varphi_2(\gamma_2) = \varphi_2(1) = T_1 + 2T_2 + 2T_3 + 4T_4. \tag{63}$$

Since $\varphi_1(1) \leq \varphi_2(1)$, for $c \in (0, 2)$ and $x \in (\frac{1}{2}, 1)$, we obtain that

$$\max\left(\psi(\gamma_1, \gamma_2)\right) = \psi(1, 1) \tag{64}$$

on the boundary of $\mathbb{S}$. Thus, the maximum value of ψ occurs when both $\gamma_1 = 1$ and $\gamma_2 = 1$ in the closed square $\mathbb{S}$. Now defining a function $K : [0, 2] \to \mathbb{R}$ by

$$K(c, x) = \max\left(\psi(\gamma_1, \gamma_2)\right) = \psi(1, 1) = T_1 + 2T_2 + 2T_3 + 4T_4 \tag{65}$$

for a fixed value of x and by plugging the expressions of T_1, T_2, T_3, and T_4 into this function K, we will obtain that

$$K\left(c, x\right) = \frac{4\left(C_1\left(x\right)\right)^2}{\left(3\mu - \delta\right)^2} + \frac{M_1c^4 + 12M_2c^2}{12\left(2\mu - \delta\right)^4\left(4\mu - \delta\right)^3\left(3\mu - \delta\right)^2} \tag{66}$$

where

$$\begin{aligned} M_1 = &\left(3\mu - \delta\right)^2\left|\Omega\left(\mu, b\right)\right|C_1\left(x\right) - 6\left(4\mu - \delta\right)^2\left(2\mu - \delta\right)^3\left(3\mu - \delta\right)^2\left(C_1\left(x\right)\right)^2 \\ &- 27\left(2\mu - \delta\right)^2\left(4\mu - \delta\right)^3\left(3\mu - \delta\right)\left(C_1\left(x\right)\right)^3 - 12\left(2\mu - \delta\right)^3\left(4\mu - \delta\right)^2\left(3\mu - \delta\right)^2C_1\left(x\right)C_2\left(x\right) \\ &- 3\left(2\mu - \delta\right)^3\left(4\mu - \delta\right)^2\left(3\mu - \delta\right)^2\left(C_1\left(x\right)\right)^2 + 3\left(2\mu - \delta\right)^4\left(4\mu - \delta\right)^3\left(C_1\left(x\right)\right)^2 \end{aligned} \tag{67}$$

and

$$\begin{aligned} M_2 = &\,2\left(2\mu - \delta\right)^3\left(4\mu - \delta\right)^2\left(3\mu - \delta\right)^2\left(C_1\left(x\right)\right)^2 + 9\left(2\mu - \delta\right)^2\left(4\mu - \delta\right)^3\left(3\mu - \delta\right)\left(C_1(x)\right)^3 \\ &+ 4\left(2\mu - \delta\right)^3\left(4\mu - \delta\right)^2\left(3\mu - \delta\right)^2C_1\left(x\right)C_2\left(x\right) + \left(2\mu - \delta\right)^3\left(4\mu - \delta\right)^2\left(3\mu - \delta\right)^2\left(C_1\left(x\right)\right)^2 \\ &- 2\left(2\mu - \delta\right)^4\left(4\mu - \delta\right)^3\left(C_1\left(x\right)\right)^2 \end{aligned} \tag{68}$$

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If we assume that the function $K(c,x)$ attains a maximum value within the interior range of $0 \leq c \leq 2,$ a simple calculation reveals that

$$K'(x,c) = \frac{M_1 c^3 + 6M_2 c}{3\left(2\mu-\delta\right)^4\left(4\mu-\delta\right)^3\left(3\mu-\delta\right)^2}. \tag{69}$$

Now, in the following discussion, we will examine the sign of $K'(c,x)$ in accordance with different cases of M_1 and M_2 as follows:

(A) If $M_1 \geq 0$ and $M_2 \geq 0,$ then $K'(c,x) \geq 0$; indicating that $K(c,x)$ is increasing. Thus,

$$\max_{0<c<2} K(c,x) = K(2^-,x) = \frac{4\left(C_1(x)\right)^2}{\left(3\mu-\delta\right)^2} + \frac{4M_1c^4 + 12M_2c^2}{3\left(2\mu-\delta\right)^4\left(4\mu-\delta\right)^3\left(3\mu-\delta\right)^2}, \tag{70}$$

which means that:

$$\max_{0<c<2}\left\{\max_{\mathbb{S}}\left\{\psi(\gamma_1,\gamma_1)\right\}\right\} = K(2^-,x).$$

(B) If $M_1 > 0$ and $M_2 < 0,$ then $c_0 = \sqrt{-\frac{6M_2}{M_1}}$ is a critical value of the function $K(c,x)$. By assuming $c_0 \in (0,2),$ $K''(c,x)\,|_{c=c_0} > 0$. That means $c = c_0$ is a local minimum point of $K(c,x)$. Therefore, the function $K(c,x)$ can not possess a local maximum.

(C) If $M_1 \leq 0$ and $M_2 \leq 0,$ then $K'(c,x) \leq 0$; indicating that $K(c,x)$ is decreasing. Thus,

$$\max_{0<c<2} K(c,x) = K(0^+,x) = 4T_4 = \frac{4\left(C_1(x)\right)^2}{\left(3\mu-\delta\right)^2}. \tag{71}$$

(D) If $M_1 < 0$ and $M_2 > 0,$ then c_0 is a critical value of the function $K(c,x)$. By assuming $c_0 \in (0,2)$, $K''(c,x)\,|_{c=c_0} < 0$. That means the function $K(c,x)$ has a local maximum, and its value occurs when $c = c_0$. Thus,

$$\max_{0<c<2} K(c,x) = K(c_0,x), \tag{72}$$

where

$$K(c_0,x) = \frac{4(C_1(x))^2}{(3\mu-\delta)^2} - \frac{3M_2^2}{M_1\left(2\mu-\delta\right)^4\left(4\mu-\delta\right)^3\left(3\mu-\delta\right)^2}$$

Therefore, from (70) to (72), the proof of the Theorem 3.3 is completed. □

Ultimately, we conclude this section by introducing some essential corollaries that are obtained from the class $\mathcal{M}_\Sigma(\mu,\delta,\mathcal{B}(x,z))$.

Corollary 3.4. *Let $f \in \Sigma$ of the form (1) be in the class $\mathcal{M}_\Sigma(1,\delta,\mathcal{B}(x,z))$. Then*

$$\left|a_2a_4 - a_3^2\right| \leq \begin{cases} K(2^-,x) & M_3 \geq 0 \quad and \quad M_4 \geq 0; \\ max\left\{\dfrac{36x^2}{\left(3-\delta\right)^2}, K(2^-,x)\right\} & M_3 > 0 \quad and \quad M_4 < 0; \\ \dfrac{36x^2}{\left(3-\delta\right)^2} & M_3 \leq 0 \quad and \quad M_4 \leq 0; \\ max\left\{K(c_0,x), K(2^-,x)\right\} & M_3 < 0 \quad and \quad M_4 > 0, \end{cases} \tag{73}$$

where

$$K(2^-,x) = \frac{4\left(C_1(x)\right)^2}{\left(3-\delta\right)^2} + \frac{4M_3c^4 + 12M_4c^2}{3\left(2-\delta\right)^4\left(4-\delta\right)^3\left(3-\delta\right)^2}, \tag{74}$$

$$K(c_0,x) = \frac{4(C_1(x))^2}{(3-\delta)^2} - \frac{3M_4^2}{M_3\left(2-\delta\right)^4\left(4-\delta\right)^3\left(3-\delta\right)^2}, \quad c_0 = \sqrt{\frac{-6M_4}{M_3}} \tag{75}$$

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$$M_3 = C_1(x)\Big(-27C_1(x)^2(\delta-4)^3(\delta-3)(\delta-2)^2 - 3C_1(x)(\delta-4)^3(\delta-2)^4 + 9C_1(x)(\delta-4)^2(\delta-3)^2(\delta-2)^3 + 12C_2(x)(\delta-4)^2(\delta-3)^2(\delta-2)^3 + (\delta-3)^2\big|\Omega(1,\delta)\big|\Big) \tag{76}$$

$$M_4 = C_1(x)(\delta-4)^2(\delta-2)^2\Big(9(C_1(x))^2(\delta^2-7\delta+12) - C_1(x)(\delta^3-8\delta^2+23\delta-22) - 4C_2(x)(\delta-3)^2(\delta-2)\Big) \tag{77}$$

Corollary 3.5. *Let $f \in \Sigma$ of the form (1) be in the class $\mathcal{M}_\Sigma(1,1,\mathcal{B}(x,z))$. Then*

$$\left|a_2a_4 - a_3^2\right| \leq \begin{cases} K(2^-,x) & M_5 \geq 0 \quad and \quad M_6 \geq 0; \\ max\left\{9x^2, K(2^-,x)\right\} & M_5 > 0 \quad and \quad M_6 < 0; \\ 9x^2 & M_5 \leq 0 \quad and \quad M_6 \leq 0; \\ max\left\{K(c_0,x), K(2^-,x)\right\} & M_5 < 0 \quad and \quad M_6 > 0, \end{cases} \tag{78}$$

where

$$K(2^-,x) = (C_1(x))^2 + \frac{M_3c^4 + 3M_4c^2}{81}, \tag{79}$$

$$K(c_0,x) = (C_1(x))^2 - \frac{M_6^2}{36M_5}, \quad c_0 = \sqrt{\frac{-6M_6}{M_5}} \tag{80}$$

$$M_5 = -C_1(x)\left(1458(C_1(x))^2 + 243C_1(x) + 432C_2(x) - 4\big|\Omega(1,1)\big|\right) \tag{81}$$

$$M_6 = 18C_1(x)\left(27(C_1(x))^2 + 3C_1(x) + 8C_2(x)\right) \tag{82}$$

Corollary 3.6. *Let $f \in \Sigma$ of the form (1) be in the class $\mathcal{M}_\Sigma(2,1,\mathcal{B}(x,z))$. Then*

$$\left|a_2a_4 - a_3^2\right| \leq \begin{cases} K(2^-,x) & M_7 \geq 0 \quad and \quad M_8 \geq 0; \\ max\left\{\frac{36x^2}{25}, K(2^-,x)\right\} & M_7 > 0 \quad and \quad M_8 < 0; \\ \frac{36x^2}{25} & M_7 \leq 0 \quad and \quad M_8 \leq 0; \\ max\left\{K(c_0,x), K(2^-,x)\right\} & M_7 < 0 \quad and \quad M_8 > 0, \end{cases} \tag{83}$$

where

$$K(2^-,x) = \frac{4(C_1(x))^2}{25} + \frac{4M_7c^4 + 12M_8c^2}{2083725}, \tag{84}$$

$$K(c_0,x) = \frac{4(C_1(x))^2}{25} - \frac{M_8^2}{231525M_7}, \quad c_0 = \sqrt{\frac{-6M_8}{M_7}} \tag{85}$$

$$M_7 = -C_1(x)\left(416745(C_1(x))^2 + 214326C_1(x) + 396900C_2(x) - 25\big|\Omega(2,1)\big|\right) \tag{86}$$

$$M_8 = 1323C_1(x)\left(105(C_1(x))^2 + 33C_1(x) + 100C_2(x)\right) \tag{87}$$

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Corollary 3.7. *Let $f \in \Sigma$ of the form (1) be in the class $\mathcal{M}_{\Sigma}(\mu, 0, \mathcal{B}(x,z))$. Then*

$$\left|a_2a_4 - a_3^2\right| \leq \begin{cases} K(2^-, x) & M_9 \geq 0 \quad \text{and} \quad M_{10} \geq 0; \\ max\left\{\dfrac{4x^2}{\mu^2}, K(2^-, x)\right\} & M_9 > 0 \quad \text{and} \quad M_{10} < 0; \\ \dfrac{4x^2}{\mu^2} & M_9 \leq 0 \quad \text{and} \quad M_{10} \leq 0; \\ max\left\{K(c_0, x), K(2^-, x)\right\} & M_9 < 0 \quad \text{and} \quad M_{10} > 0, \end{cases} \tag{88}$$

where

$$K(2^-, x) = \frac{4\left(C_1\left(x\right)\right)^2}{9\mu^2} + \frac{M_9c^4 + 3M_{10}c^2}{6912\mu^9}, \tag{89}$$

$$K(c_0, x) = \frac{4(C_1(x))^2}{9\mu^2} - \frac{M_{10}^2}{3072\mu^9 M_9}, \quad c_0 = \sqrt{\frac{-6M_{10}}{M_9}} \tag{90}$$

$$M_9 = -3C_1(x)\mu^2\left(6912(C_1(x))^2\mu^4 + 2432C_1(x)\mu^5 + 4608C_2(x)\mu^5 - 3\left|\Omega(\mu, 0)\right|\right) \tag{91}$$

$$M_{10} = 128C_1(x)\mu^6\left(54(C_1(x))^2 + 11C_1(x)\mu + 36C_2(x)\mu\right) \tag{92}$$

Corollary 3.8. *Let $f \in \Sigma$ of the form (1) be in the class $\mathcal{M}_{\Sigma}(1, 0, \mathcal{B}(x,z))$. Then*

$$\left|a_2a_4 - a_3^2\right| \leq \begin{cases} K(2^-, x) & M_{11} \geq 0 \quad \text{and} \quad M_{12} \geq 0; \\ max\left\{4x^2, K(2^-, x)\right\} & M_{11} > 0 \quad \text{and} \quad M_{12} < 0; \\ 4x^2 & M_{11} \leq 0 \quad \text{and} \quad M_{12} \leq 0; \\ max\left\{K(c_0, x), K(2^-, x)\right\} & M_{11} < 0 \quad \text{and} \quad M_{12} > 0, \end{cases} \tag{93}$$

where

$$K(2^-, x) = \frac{4\left(C_1\left(x\right)\right)^2}{9} + \frac{M_{11}c^4 + 3M_{12}c^2}{6912}, \tag{94}$$

$$K(c_0, x) = \frac{4(C_1(x))^2}{9} - \frac{M_{12}^2}{3072M_{11}}, \quad c_0 = \sqrt{\frac{-6M_{12}}{M_{11}}} \tag{95}$$

$$M_{11} = -3C_1(x)\left(6912(C_1(x))^2 + 2432C_1(x) + 4608C_2(x) - 3\left|\Omega(1, 0)\right|\right) \tag{96}$$

$$M_{12} = 128C_1(x)\left(54(C_1(x))^2 + 11C_1(x) + 36C_2(x)\right) \tag{97}$$

4 Conclusion

In our current study, we have introduced and examined the coefficient problems associated with the newly introduced subclass, namely, $\mathcal{M}_{\Sigma}(\mu, \delta, \mathcal{B}(x,z))$. We have obtained estimates for the Taylor-Maclaurin coefficients $|a_2|$, $|a_3|$, investigated the Fekete-Szegö functional problems, and determined the bounds of the second Hankel determinant for functions in this subclass.

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