An Exploration of Hesitant Fuzzy Normed Linear Space

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Abstract

In this paper, the notion of hesitant fuzzy norm based on the Bag-Samanta's Type Fuzzy Norm on linear space has been introduced. Further the concepts of ascending family of semi-norms, convergence and fuzzy continuous linear operators are studied in hesitant fuzzy normed linear space.

Keywords: Hesitant fuzzy norm; Hesitant fuzzy normed linear spaces; Hesitant fuzzy continuous linear operators

1. Introduction

Uncertainty occurs in practically all aspects of our everyday lives, a wide range of methods have been constructed to define, illustrate, organize, and handle it. The most widely used theories to deal with uncertainty are fuzzy set theory and probability hypothesis, which are meant to explain statistical and numerical uncertainty in turn. These two categories of models contain information that differs conceptually. Probability theory provides insight into relative frequencies, but fuzzy set theory depicts how things are comparable to precisely defined properties (Bezdek 1993).The fuzzy set, which was first presented by Zadeh in 1965, has shown being among the most effective decision support strategies, offering the capacity to handle ambiguity and uncertainties. The most notable extension of fuzzy set theory following Zadeh's groundbreaking work in 1965 has to do with representing the grades of membership of the underneath fuzzy set in Yager (2014).A novel generalized kind of fuzzy set termed hesitant fuzzy set (HFS) was recently developed by Torra (2010) based on the extensional forms of fuzzy sets. This opens up new avenues for study on decision making in hesitant contexts. The concept of a fuzzy norm, of the Minkowsky kind, was initially developed by Katsaras in 1984.It was related to a fuzzy set that is absorbing and perfectly convex. Felbin presented an additional concept of fuzzy norm in 1992, wherein each member of the vector space was given a real number which is fuzzy. After in 2003, Cheng and Mordeson, Bag and Samanta provided an additional fuzzy norm notion that will be shown most adequately, despite the fact that it can still be enhanced, made simpler, or used more broadly. In a more extended scenario, Nadaban and Dzitac's paper yielded fuzzy norms' decomposition theorems into a family of semi-norms. We further observe that Saadati and Vaezpour presented a novel definition of the fuzzy norm in 2005. In 2010, Golet extended the idea of fuzzy norm to continuous t-norm. As a result of ongoing development, two significant issues with fuzzy normed linear space are the study of fuzzy continuous mapping in fuzzy normed linear space and fuzzy bounded linear operators. HFS has several benefits over standard fuzzy sets and their other expansions, particularly when it comes to anonymous collective decision-making. Initially, Torra (2010) introduced the notion of hesitant fuzzy set and clarified the intersection, union and complement, of HFSs. Moreover, In 2009 Torra and Narukawa introduced a principle of extension that allowed for the expansion of current fuzzy set operations to hesitant fuzzy sets and explained how

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to use this novel set type within the decision-making framework. The original mathematical formulations for HFS were provided by Xu and Xia (2011a,b), who also looked into the measurements of distance, similarity, and correlation for HFSs.

2. Preliminaries

In this section, already defined definition on fuzzy set (some basic definitions), are given.

Consider the nonempty set $\mathbb{X}$.

Definition 2.1 [43]: In $\mathbb{X}$, a function $\xi: \mathbb{X} \to [0,1]$ is a fuzzy set. Denote $F(\mathbb{X})$ is the family of all fuzzy sets.

Definition 2.2 [43]: Fuzzy set union and intersection of $\{\xi_i\}_{i\in I}$ are described by

$(\vee_{i\in I} \xi_i)(x) = \sup\{\xi_i(x): i \in I\}$, $(\wedge_{i\in I} \xi_i)(x) = \inf\{\xi_i(x): i \in I\}$.

Definition 2.3:

The set V which is nonempty, together with two algebraic operations, satisfies,

$(V, +)$ is a group and with respect to scalar multiplication,

(i) k(a+b)= $ka + kb$

$(ii)(k + l)a = ka + kb$

$(iii)k(la) = (kl)a$

$(iv)1. a = a$ $\quad ,\forall a, b, c \in V$ & $\in k, l \in R^*$

The triple $(V, +,\cdot)$ is called vector space or linear space.

Definition 2.4 [32]: On a linear space $\mathbb{X}$, a fuzzy set $\mathcal{N}$ in $\mathbb{X} \times [0, \infty)$ is known as a fuzzy norm on $\mathbb{X}$ if it fulfills,

(FN1) $\mathcal{N}(x_1, 0) = 0, \forall x_1 \in \mathbb{X}$

(FN2) $\mathcal{N}(x_1, t) = 1, \forall t > 0 \Leftrightarrow x_1 = 0$.

(FN3) $(\lambda x_1, t) = \mathcal{N}\left(x_1, \frac{t}{|\lambda|}\right), \forall x_1 \in \mathbb{X}, \forall t > 0, \forall \lambda \in \mathbb{K}^*$,($\mathbb{K}^*$ $is\ non\ negative\ real\ numbers$)

(FN4) $\mathcal{N}(x_1 + y_1, t + s) \geq \mathcal{N}(x_1, t) * \mathcal{N}(y_1, s), \forall x_1, y_1 \in X,, \forall t, s > 0$

(FN5) $\forall x_1 \in X, \mathcal{N}(x_1,\cdot)$ is left continuous and $\lim_{n\to\infty} \mathcal{N}(x_1, t) = 1$.

Thus,$(\mathbb{X}, \mathcal{N},*)$ is called fuzzy normed linear space.

Definition 2.5 [44]: If $\mathbb{X}$ is a fixed set, then an hesitant fuzzy set on $\mathbb{X}$ is expressed regarding a function h that when applied to $\mathbb{X}$ produces a subset of [0,1].

Xu and Xia (2011 a) denoted the hesitant fuzzy set as, $\mathcal{H} = \{< x, h(x) >: x \in \mathbb{X}\}$

3. Hesitant fuzzy norm

In this section, hesitant fuzzy norm has been introduced, example and some theorems on linear space based on hesitant fuzzy set are discussed, based on this definition.

For a field $\mathbb{R}$ or $\mathbb{C}$. Let * be a t-norm that is continuous and X be a vector space Hesitant fuzzy set $\mathcal{H}$ in $\mathbb{X} \times [0, \infty)$ is known as hesitant fuzzy norm on in $\mathbb{X}$, if it fulfills,

HFN1. $\mathcal{H}(x_{11}, 0) = \emptyset^*, \forall x_1 \in \mathbb{X}$.

HFN2.$\mathcal{H}(x_{11}, t_1) = U^*, \ \forall t_1 > 0, if\ and\ only\ if\ x_{11} = 0$.

HFN3. $\mathcal{H}(\lambda x_{11}, t_1) = \mathcal{H}\left(x_{11}, \frac{t_1}{|\lambda|}\right), \forall x_1 \in \mathbb{X}, \forall t_1 \geq 0, \forall \lambda \in \mathbb{R}^*$

$(where\ \mathbb{R}^* is\ non\ negative\ real\ numbers)$

HFN4. $\mathcal{H}(x_{11} + y_{11}, t_1 + u_1) \supseteq \mathcal{H}(x_{11}, t_1) \cap \mathcal{H}(y_{11}, u_1)\ \forall x_{11}, y_{11} \in X, \forall t_1, u_1 \geq 0$.

HFN5. $\forall x_{11} \in \mathbb{X}, \mathcal{H}(x_{11},\cdot)$ is left continuous and $\lim_{n\to\infty} \mathcal{H}(x_{11}, t_1) = U^*$.

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The triple $(\mathbb{X}, \mathcal{H}, *)$ is known as hesitant fuzzy normed linear space.

Example 3.1:

Suppose X is a linear normed space. $\mathcal{H}: \mathbb{X} \times [0, \infty) \to P[0,1]$ is defined by,

$$\mathcal{H}(x_{11}, t_1) = \begin{cases} \emptyset^* , & if \ \ t_1 = 0 \ and \ \forall x_{11} \in \mathbb{X} \\ U^* , & if \ \ x_{11} = 0 \ and \ \forall t_1 > 0 \\ s \in P[0,1] \ where \ s \ is \ an \ arbitrary \ subset \ of \ [0,1] & , otherwise \end{cases}$$

Proof :

HFN1. $\mathcal{H}(x_{11}, 0) = \emptyset^*, \ \forall x_{11} \in \mathbb{X}$.

HFN2. $\mathcal{H}(x_{11}, t_1) = U^*, \ \forall t_1 > 0, if \ and \ only \ if \ \ x_{11} = 0$.

HFN3. By the definition of $\mathcal{H}(x_{11}, t_1), \mathcal{H}(\lambda x_{11}, t_1) = s \ \ and \ \mathcal{H}\left(x_{11}, \frac{t_1}{|\lambda|}\right) = s$

$$\Rightarrow \mathcal{H}(\lambda x_{11}, t_1) = \mathcal{H}\left(x_{11}, \frac{t_1}{|\lambda|}\right).$$

HFN4. **Case (i):** If $y_{11} \neq 0, x_{11} = 0, \ then$ $(a)\ t_1 = 0, u_1 = 0$

$(b)\ t_1 = 0, u_1 \neq 0$

$(c) t_1 \neq 0, u_1 = 0$

$(d) t_1 \neq 0, u_1 \neq 0.$

Subcase (a): $If \ u_1 = 0, \ t_1 = 0,$

Then $\mathcal{H}(0,0) = \emptyset^*, \mathcal{H}(y_{11}, 0) = \emptyset^*, \ \mathcal{H}(y_{11}, 0) = \emptyset^*$

$$\Rightarrow \emptyset^* \supseteq \emptyset^* \cap \emptyset^* \Rightarrow \mathcal{H}(y_{11}, 0) \supseteq \mathcal{H}(0,0) \cap \mathcal{H}(y_{11}, 0)$$

Subcase (b): $If \ \ u_1 \neq 0, t_1 = 0,$

Then $H(0,0) = \emptyset^* , \mathcal{H}(y_{11}, u_1) = s \ , \ \mathcal{H}(y_{11}, u_1) = s.$

$$\Rightarrow s \supseteq \emptyset^* \cap s \Rightarrow \mathcal{H}(y_{11}, u_1) \supseteq \mathcal{H}(0,0) \cap \mathcal{H}(y_{11}, u_1)$$

Subcase (c): $If \ t_1 \neq 0, u_1 = 0$

Then $\mathcal{H}(y_{11}, t_1) = s \ , \mathcal{H}(0, t_1) = U^* , \mathcal{H}(y_{11}, 0) = \emptyset^*.$

$$\Rightarrow s \supseteq U^* \cap \emptyset^* \Rightarrow \mathcal{H}(y_{11}, t_1) \supseteq \mathcal{H}(0, t_1) \cap \mathcal{H}(y_{11}, 0).$$

Subcase (d) : If $t_1 \neq 0, u_1 \neq 0.$

Then $\mathcal{H}(0, t_1) = U^*, \mathcal{H}(y_{11}, t_1 + u_1) = s \ , \ \mathcal{H}(y_{11}, u_1) = s.$

$$\Rightarrow s \supseteq U^* \cap s \Rightarrow \mathcal{H}(y_{11}, t_1 + u_1) \supseteq \mathcal{H}(0, t_1) \cap \mathcal{H}(y_{11}, u_1).$$

Case (ii): If $y_{11} \neq 0, x_{11} = 0, then$

$(a)\ t_1 = 0, u_1 = 0$

$(b)\ t_1 = 0, u_1 \neq 0$

$(c) t_1 \neq 0, u_1 = 0$

$(d) t_1 \neq 0, u_1 \neq 0.$

Subcase (a): $If \ u_1 = 0, t_1 = 0,$

Then $\mathcal{H}(0,0) = \emptyset^*, \mathcal{H}(x_{11}, 0) = \emptyset^*, \mathcal{H}(x_{11}, 0) = \emptyset^*$

$$\Rightarrow \emptyset^* \supseteq \emptyset^* \cap \emptyset^* \Rightarrow \mathcal{H}(x_{11}, 0) \supseteq \mathcal{H}(x_{11}, 0) \cap \mathcal{H}(0,0).$$

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Subcase (b): $If\ u_1 \neq 0\ t_1 = 0$,

Then $\mathcal{H}(y_{11}, u_1) = s$, $\mathcal{H}(x_{11}, 0) = \emptyset^*$, $\mathcal{H}(0, u_1) = U^*$.

$$\Rightarrow s \supseteq \emptyset^* \cap U^* \Rightarrow \mathcal{H}(x_{11}, u_1) \supseteq \mathcal{H}(x_{11}, 0) \cap \mathcal{H}(0,0)$$

Subcase (c): $If u_1 = 0$, $t_1 \neq 0$,

Then $\mathcal{H}(x_{11}, t_1) = s$, $\mathcal{H}(x_{11}, t_1) = s^*$, $\mathcal{H}(0,0) = \emptyset^*$.

$$\Rightarrow s \supseteq s \cap \emptyset^* \Rightarrow \mathcal{H}(x_{11}, t_1) \supseteq \mathcal{H}(x_{11}, t_1) \cap \mathcal{H}(0,0).$$

Subcase (d) : If $t_1 \neq 0, u_1 \neq 0$.

Then $\mathcal{H}(x_{11}, t_1 + u_1) = s$, $\mathcal{H}(x_{11}, t_1) = U^*$, $\mathcal{H}(0, u_1) = s$.

$$\Rightarrow s \supseteq s \cap U^* \Rightarrow \mathcal{H}(x_{11}, t_1 + u_1) \supseteq \mathcal{H}(x_{11}, t_1) \cap \mathcal{H}(0, u_1)$$

HFN5. $\forall x_1 \in \mathbb{X}, \mathcal{H}(x_{11}, \cdot)$ is left continuous and $\lim_{n\to\infty} \mathcal{H}(x_{11}, t_1) = U^*$.

Hence, $\mathcal{H}$ is a hesitant fuzzy norm on $\mathbb{X}$ and $(\mathbb{X}, \mathcal{H}, *)$ is a hesitant fuzzy normed linear space.

Theorem 3.2:

Let $(\mathbb{X}, \mathcal{H}, *)$ be a hesitant fuzzy normed linear space. Let

$p_s(x_1) := \inf\{t > 0 : \mathcal{H}(x_{11}, t_1) \supset s\}, s \in P[0,1]$.

Then

(i) $\wp = \{p_s\}_{s\in P[0,1]}$ is a family of semi-norms on $\mathbb{X}$, that ascends.
(ii) For $x_{11} \in \mathbb{X}, t_1 > 0, s \in P[0,1]$ we possess $p_s(x_{11}) < t_2$ if and only if $\mathcal{H}(x_{11}, t_2) \supset s$.
(iii) For any decreasing subsets $\{s_n\}\ in\ P[0,1], \cap s_n = s \in P[0,1]$,
Since $p_{s_n}(x_{11_n}) \to p_s(x_{11}), \forall x_{11} \in X$.that is $\wp = \{p_s\}_{s\in P[0,1]}$ is right continuous.

Proof:

(i) As $\mathcal{H}(0, t_1) = U^*$,$\forall t_1 > 0$. Which implies that

$$p_s(x_{11}) := \inf\{t_1 > 0 : \mathcal{H}(0, t_1) \supset s\} = 0$$

We show that $p_s(\lambda x_{11}) = |\lambda|\, p_s(x_{11}), \forall x_{11} \in \mathbb{X}, \forall \lambda \in \mathbb{R}^*$.

First observe that, for $\lambda \neq 0$, we have

$$\begin{aligned} p_s(\lambda x_1) &= \inf\{t_1 > 0 : \mathcal{H}(\lambda x_{11}, t_1) \supset s\} \\ &= \inf\left\{t_1 > 0 : \mathcal{H}\left(x_{11}, \frac{t_1}{|\lambda|}\right) \supset s\right\} \\ &= \inf\left\{t_1 |\lambda| > 0 : \mathcal{H}\left(x_{11}, \frac{|\lambda| t_1}{|\lambda|}\right) \supset s\right\} \\ &= \inf\{t_1 |\lambda| > 0 : \mathcal{H}(x_{11}, t_1) \supset s\} \\ &= |\lambda| \inf\{t_1 > 0 : \mathcal{H}(x_{11}, t_1) \supset s\} \\ &= |\lambda|\, p_s(x_{11}) \end{aligned}$$

Thus,

$$\begin{aligned} p_s(x_{11}) + p_s(y_{11}) &= \inf\{t_1 > 0 : \mathcal{H}(x_{11}, t_1) \supset s\} + \inf\{u_1 > 0 : \mathcal{H}(y_{11}, u_1) \supset s\} \\ &= \inf\{t_1 + u_1 > 0 : \mathcal{H}(x_{11}, t_1) \supset s, \mathcal{H}(y_{11}, u_1) \supset s\} \\ &= \inf\{t_1 + u_1 > 0 : \mathcal{H}(x_{11}, t_1) \cap \mathcal{H}(y_{11}, u_1) \supset s\} \\ &\supseteq \inf\{t_1 + u_1 > 0 : \mathcal{H}(x_{11} + y_{11}, t_1 + u_1) \supset s\} \\ &= p_s(x_{11} + y_{11}) \end{aligned}$$

It remains to be proven that $\wp = \{p_s\}_{s\in P[0,1]}$ is an ascending family.

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Let $s_1 \subseteq s_2$

Then $\{t_1 > 0: \mathcal{H}(x_{11}, t_1) \supset s_2\} \subseteq \{t_1 > 0: \mathcal{H}(x_{11}, t_1) \supset s_1\}$

Thus $inf\{t_1 > 0: \mathcal{H}(x_{11}, t_1) \supset s_2\} \supseteq inf\{t_1 > 0: \mathcal{H}(x_{11}, t_1) \supset s_1\}$

$$p_{s_2} \supseteq p_{s_1}, \forall x_{11} \in \mathbb{X} .$$

(ii) " $\Rightarrow$ " To show that $t_2 \in \{ t_1 > 0: \mathcal{H}(x_{11}, t_1) \supset s\}$

We suppose that $t_2 \notin \{ t_1 > 0: \mathcal{H}(x_{11}, t_1) \supset s\}$,

then there exists $t_0 \in \{ t_1 > 0: \mathcal{H}(x_{11}, t_1) \supset s\}$ such that $t_0 < t_2$.

(Contrary $t_2 \leq t_1$, $\forall t_1 \in \{ t_1 > 0: \mathcal{H}(x_{11}, t_1) \supset s\}$, hence $t_2 \leq inf\{ t_1 >: \mathcal{H}(x_{11}, t_1) \supset s\}$,

i.e., $t_2 \leq p_s(x_{11})$ which is a contradiction)

As $t_0 \in \{t_1 > 0: \mathcal{H}(x_{11}, t_1) \supset s\}$, $t_0 < t_2$ and $\mathcal{H}(x_{11}, \cdot)$ is non-decreasing, we obtain that

$$s \subset \mathcal{H}(x_{11}, t_0) \subseteq \mathcal{H}(x_{11}, t_2)$$

Hence, $\mathcal{H}(x_{11}, t_2) \supset s$, this leads to a contradiction.

" $\Leftarrow$ "

As $(x_{11}, t_2) \supset s$, which implies that $t_2 \in \{ t_1 > 0: \mathcal{H}(x_{11}, t_1) \supset s\}$ .Thus $p_s(x_{11}) \leq t_2$. suppose that $p_s(x_{11}) = t_2$. As $\mathcal{H}(x_{11}, \cdot)$ is left continuous in t_2, already exists, $\lim_{t_1 \to t_2 t_1 < t_2} \mathcal{H}(x_{11}, t_1) = \mathcal{H}(x_{11}, t_2)$.

Thus, there exists $t_0 < t_2$ such that $\mathcal{H}(x_{11}, t_0) \supset s$.

(Contrary, $\mathcal{H}(x_{11}, t_1) \subseteq s$, $\forall t_0 < t_2$.

Therefore $\lim_{t_1 \to t_2, t_1 < t_2} \mathcal{H}(x_{11}, t_1) \subseteq s$, hence, $\mathcal{H}(x_{11}, t_2) \subseteq s$,which is a contradiction)

But $t_0 < t_2$ and $\mathcal{H}(x_{11}, t_0) \supset s$ are in contradiction with the fact that

$$t_2 = inf\{ t_1 >: \mathcal{H}(x_{11}, t_1) \supset s\}$$

Hence, $p_s(x_{11}) \neq t_2$, thus $p_s(x_{11}) < t_2$.

(iii)

Let $x_{11} \in \mathbb{X}$ and (s_n) be a decreasing subsets in $P[0,1]$

Let $t_2 > p_s(x_{11})$ .then $\mathcal{H}(x_{11}, t_2) \supset s$.

As (s_n) is a decreasing sequence and $\bigcap s_n = s$, there exists $n_0 \in \mathbb{N}$ such that

$s_n \subset \mathcal{H}(x_{11}, t_2), \forall n \geq n_0$. Therefore $p_{s_n}(x_{11}) < t_2, \forall n \geq n_0$, thus $p_{s_n}(x_{11}) \to p_s(x_{11})$.

Theorem 3.3:

Let $\{q_s(x_{11})\}_{s \in P[0,1]}$ represent an ascending and adequate family of semi-norms on the vector space $\mathbb{X}$.

Let $\mathcal{H}' = \mathbb{X} \times [0, \infty] \to P[0,1]$ defined by

$$\mathcal{H}'(x_{11}, t_1) = \begin{cases} \cup \{s \in P[0,1] : q_s(x_{11}) < t_1\} & , \ if \quad t_1 > 0 \\ \oslash^* & , \ if \quad t_1 = 0 \end{cases}$$

Then $(\mathbb{X}, \mathcal{H}', *)$ is hesitant fuzzy normed linear space.

Proof:

First observe that $H'(x_{11}, \cdot)$ is non-decreasing.

Infact $t_{11} < t_{22}$, we possess

$$\cup \{s \in P[0,1] : q_s(x_{11}) < t_{11}\} \subseteq \cup \{s \in P[0,1] : q_s(x_{11}) < t_{22}\}$$

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Hence, $\mathcal{H}'(x_{11}, t_1) \subseteq \mathcal{H}'(x_{11}, t_2)$

HFN 1. $\mathcal{H}'(x_{11}, 0) = \varnothing^*, \forall x_{11} \in X$ is obvious.

HFN2. If $x_{11} = 0$, then $q_s(0) = 0, \forall s \in P[0,1]$ already exists,$q_s(x_{11}) < t_1, \forall t_1 > 0$.

Thus, $\cup \{s \in P[0,1] : q_s(x_{11}) < t_1\} = U^*, \forall t_1 > 0$.

$H'(x_{11}, 0) = U^*, \forall t_1 > 0$.

Conversely,

If $\mathcal{H}'(x_{11}, t_1) = U^*, \forall t_1 > 0$. Then $\cup \{s \in P[0,1] : q_s(x_{11}) < t_1\} = U^*, \forall t_1 > 0$.

Hence, for all $s \in P[0,1]$, already exists, $q_s(x_{11}) < t_1, \forall t_1 > 0$.

Thus for all $s \in P[0,1]$, already exists, $q_s(x_{11}) = 0$.

Due to the sufficiency of the family of semi-norms $\{q_s\}_{s \in P[0,1]}, x_{11} = 0$.

HFN3.

If $t_1 = 0$, then $= \varnothing^* = \mathcal{H}'\left(x_{11}, \frac{t_1}{|\lambda|}\right)$.For $t > 0$, already exists,

$$\mathcal{H}'(\lambda x_{11}, t_1) = \cup \{s \in P[0,1] : q_s(\lambda x_{11}) < t_1\}$$

$$= \cup \{s \in P[0,1] :| \lambda | q_s(x_{11}) < t_1\}$$

$$= \cup \left\{s \in P[0,1] :| q_s(x_{11}) < \frac{t_1}{|\lambda}\right\}$$

$$\mathcal{H}'(\lambda x_1 1, t_1) = \mathcal{H}'\left(x_{11}, \frac{t_1}{| \lambda |}\right)$$

HFN4.

The inequality $H'(x_{11} + y_{11}, t_1 + u_1) \supseteq H'(y_{11}, u_1) \cap H'(x_{11}, t_1)$ is obviously holds for $t_1 = 0 \; or \; u_1 = 0$

suppose that, $\mathcal{H}'(x_{11} + y_{11}, t_1 + u_1) \subseteq \mathcal{H}'(y_{11}, u_1) \cap \mathcal{H}'(x_{11}, t_1)$.

Then there exists $s_0 \in P[0,1]$ such that $\mathcal{H}'(x_{11} + y_{11}, t_1 + u_1) \subset s_0 \subset \mathcal{H}'(y_{11}, u_1) \cap \mathcal{H}'(x_{11}, t_1)$

As,$\mathcal{H}'(x_{11}, t_1) \supset s_0$, there exists $s_1 \in \cup \{s \in P[0,1] : q_s(\lambda x_{11}) < t_1\}$ such that $s_1 \supset s_0$.

[Contrary for all $s^* \in \cup \{s \in P[0,1] : q_s(\lambda x_{11}) < t_1\}$, we have $s^* \subseteq s_0$.

Hence, $\cup \{s \in P[0,1] : q_s(\lambda x_{11}) < t_1\} \subseteq s_0$. which is a contradiction]

As, $\mathcal{H}'(y_{11}, u) \supset s_0$, there exists $s_2 \in \cup \{s \in P[0,1] : q_s(\lambda x_{11}) < t_1\}$, such that $s_2 \supset s_0$.

Let $s_3 = s_1 \cap s_2$. Then $s_3 \supset s_0$, and $q_{s_3}(y_{11}) \leq q_{s_2}(y_{11}) < u_1$, $q_{s_3}(x_{11}) \leq q_{s_2}(x_{11}) < t_1$

Thus, $q_{s_3}(x_{11} + y_{11}) \leq q_{s_3}(y_{11}) + q_{s_3}(y_{11}) < t_1 + u_1$

Therefore, $s_3 \in \cup \{s \in P[0,1] : q_s(x_{11} + y_{11}) < t_1 + u_1\}$

Thus, $\cup \{s \in P[0,1] : q_s(x_{11} + y_{11}) < t_1 + u_1\} \supseteq s_3 \supset s_0$.

Which is a contradiction with the fact that $\mathcal{H}'(x_{11} + y_{11}, t_1 + u_1) \subset s_0$

Hence, $\mathcal{H}'(x_{11} + y_{11}, t_1 + u_1) \supseteq \mathcal{H}'(y_{11}, u_1) \cap \mathcal{H}'(x_{11}, t_1)$

HFN5.

To prove that $\lim_{t \to \infty} \mathcal{H}'(x_{11}, t_1) = U^*$.

Let $s_0 \in P[0,1]$ arbitrary, to show that there exists $t_0 > 0$, such that $\mathcal{H}'(x_{11}, t_0) \supset s_0$.

Let $t_0 > q_{s_1}(x_{11})$, where $s_1 \in P[0,1]$, then $\mathcal{H}'(x_{11}, t_0) = \cup \{s \in P[0,1] : q_s(x_{11}) < t_0\} \supseteq s_1 \supset s_0$.

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Now prove that $\mathcal{H}'(x_{11},\cdot)$ is left continuous in $t_1 > 0$.

Case 1:

$\mathcal{H}'(x_{11},t_1) = \emptyset^*$. Thus for all $u_1 \le t_1$, as $\mathcal{H}'(x_{11},u_1) \subseteq \mathcal{H}'(x_{11},t_1)$, already exists $\mathcal{H}'(x_{11},s) = \emptyset^*$.

Therefore $\lim_{u \to t, u<t} \mathcal{H}'(x_{11},s) = \emptyset^* = \mathcal{H}'(x_{11},t_1)$

Case 2:

$\mathcal{H}'(x_{11},t_1) \ne \emptyset^*$. Let s_0 arbitrary, such that $\emptyset^* \subset s_0 \subset \mathcal{H}'(x_{11},t_1)$. Let (t_{1_n}) be a sequence such that

$t_{1_n} \to t_1, t_{1_n} < t_1$. Prove that there exists $n_0 \in \mathbb{N}$, such that , $\mathcal{H}'(x_{11},t_{1_n}) \supset s_0, \forall n \ge n_0$.

(As $s_0 \in \left(0, H'(x_1,t_1)\right)$ is arbitrary. Which implies that $\lim_{n\to\infty} \mathcal{H}'(x_{11},t_{1_n}) = \mathcal{H}'(x_{11},t_1)$.

If $\emptyset^* \subset s_0 \subset \mathcal{H}'(x_{11},t_1)$. Then, there exists $s_1 \in \cup \{s \in P[0,1] : q_s(x_{11}) < t_1\}$ such that $s_1 \supset s_0$.

Contrary for all $s_2 \in \cup \{s \in P[0,1] : q_s(x_{11}) < t_1\}$, we have $s_2 \subseteq s_0$.

Then $\cup \{s \in P[0,1] : q_s(x_{11}) < t_1\} \subseteq s_0$.,

i.e., $\mathcal{H}'(x_{11},t_1) \subseteq s_0$ which is a contradiction)

As $q_{s_1}(x_{11}) < t$ and $t_{1_n} \to t_1, t_{1_n} < t_1$, there exist $n_0 \in \mathbb{N}$, such that, $\forall n \ge n_0$, we have $t_{1_n} > q_{s_1}(x_1)$, Thus $\mathcal{H}'(x_{11},t_{1_n}) = \cup \left\{s \in P[0,1] : q_s(x_{11}) < t_{1_n}\right\} \supseteq s_1 \supset s_0, \forall n \ge n_0$.

Theorem 3.4:

On a hesitant fuzzy normed linear space $(\mathbb{X},\mathcal{H},*)$,$p_s(x_{11}) := \inf\{t_1 > 0 : \mathcal{H}(x_{11},t_1) \supset s\}, s \in P[0,1]$.

Let $\mathcal{H}' = \mathbb{X} \times [0,\infty] \to P[0,1]$ defined by

$$\mathcal{H}'(x_{11},t_1) = \begin{cases} \cup \{s \in P[0,1] : p_s(x_{11}) < t_1\} & , if \quad t_1 > 0 \\ \emptyset^* & , if \quad t_1 = 0 \end{cases}$$

Then $\mathcal{H}' = \mathcal{H}$.

Proof:

For $t_1 = 0$, already exists $\mathcal{H}'(x_{11},t_1) = \emptyset^* = \mathcal{H}(x_{11},t_1)$.

For $t_1 > 0$, already exists $\mathcal{H}'(x_{11},t_1) = \cup \{s \in P[0,1] : p_s(x_{11}) < t_1\}$

$$\mathcal{H}'(x_{11},t_1) = \cup \{s \in P[0,1] : \mathcal{H}(x_{11},t_1) < t_1\} \subseteq \mathcal{H}(x_{11},t_1)$$

Suppose that $\mathcal{H}'(x_{11},t_1) \subset \mathcal{H}(x_{11},t_1)$.

Next, there exists $s_0 \in P[0,1]$, so that $\mathcal{H}'(x_{11},t_1) \subset s_0 \subset \mathcal{H}(x_{11},t_1)$.

But $s_0 \subset \mathcal{H}(x_{11},t_1)$ implies that $p_{s_0} < t_1$. Thus $\cup \{s \in P[0,1] : p_s(x_{11}) < t_1\} \supseteq s_0$.

Which is a contradiction. Hence $\mathcal{H}' = \mathcal{H}$.

4. Sequential Convergence in hesitant fuzzy normed linear space

The definitions, hesitant fuzzy convergent, hesitant fuzzy Cauchy sequence, hesitant fuzzy Complete, s^* hesitant fuzzy convergent, s^*- hesitant fuzzy Cauchy, have been introduced, and some propositions are discussed in this section based on these definitions.

Definition 4.1:

Let (x_n) be sequence in X and $(\mathbb{X},\mathcal{H},*)$ be a hesitant fuzzy normed linear space. if there an $x \in \mathbb{X}$ such that and the sequence (x_n) hesitant fuzzy convergent, then

$$\lim_{n\to\infty} \mathcal{H}(x_n - x, t_1) = U^* \quad , \forall t_1 > 0.$$

Here, x is referred to as the sequence's limit. (x_n) as well as we indicate

$\lim_{n\to\infty} x_n = x \quad (or) \;\; x_n \to x$.

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Definition 4.2:

Consider a hesitant fuzzy normed linear space $(\mathbb{X}, \mathcal{H}, *)$. The sequence (x_n) is known as hesitant fuzzy Cauchy sequence, if $\forall\ s \in P[0,1], \forall t_1 > 0, there\ exists\ a\ n_0 \in \mathbb{N}$ such that ,

$\mathcal{H}(x_n - x_m,, t_1) \supset U^* \cap s, \forall n, m \geq n_0.$

Proposition 4.3:

Given a hesitant fuzzy normed linear space$(\mathbb{X}, \mathcal{H}, *)$, all convergent sequences are Cauchy sequences.

Proof:

Let $s \in P[0,1], t_1 > 0.$ Then there exists $s_1, s_2 \in P[0,1]$ such that $s_1 \cap s_2 = U^* \cap s.$

Let $s_1 = \mathcal{H}\left(x_n - x, \frac{t}{2}\right)$ and $s_2 = \mathcal{H}\left(x - x_m, \frac{t}{2}\right)$, Let $s^* = \max\{s_1, s_2\}$.

for $s^* \in P[0,1], t_1 > 0$, (x_n) is a convergent sequence that converges to x,

$\exists\ n_0 \in \mathbb{N}$ so that $\mathcal{H}\left(x_n - x, \frac{t}{2}\right) \supset s^*, \forall n \geq n_0$. Then, for $n, m \geq n_0$, we possess that

$$\mathcal{H}(x_n - x_m,, t_1) = \mathcal{H}\left(x_n - x + x - x_m\ , \frac{t_1}{2} + \frac{t_1}{2}\right)$$

$$\supseteq \mathcal{H}\left(x_n - x, \frac{t_1}{2}\right) \cap H\left(x - x_m\ , \frac{t_1}{2}\right)$$

$$\mathcal{H}(x_n - x_m,, t_1) \supseteq s_1 \cap s_2 = U^* \cap s.$$

Definition 4.4:

A hesitant fuzzy normed linear space is $(\mathbb{X}, \mathcal{H}, *)$, a Cauchy sequence in X is considered hesitant fuzzy complete if it converges to a point in $\mathbb{X}$.

Definition 4.5:

Let $s^* \in P[0,1]$,(x_n) be a sequence in $\mathbb{X}$ and $(\mathbb{X}, \mathcal{H}, *)$ be a hesitant fuzzy normed linear space, if exists $x \in \mathbb{X}$, The sequence (x_n) is known as s^* hesitant fuzzy convergent if $\forall t_1 > 0, there\ exists\ a\ n_0 \in \mathbb{N}$ such that $\mathcal{H}(x_n - x, t_1) \supset s^*$,$\forall n \geq n_0$.

Here x is known as s^* limit of the sequence (x_n) and denote $x_n \overset{s^*}{\rightarrow} x$.

Proposition 4.6:

A sequence in $\mathbb{X}$ is denoted by $(x_n), s^* \in P[0,1]$ and $(\mathbb{X}, \mathcal{H}, *)$ is a hesitant fuzzy normed vector space, if $p_{s^*}(x_n - x) \to 0$ as $n \to \infty$, Then the sequence(x_n) is s^*- convergent to a point x .

Proof :

Since, $p_{s^*}(x_n - x) = \inf\{t_1 > 0: H(x_n - x, t_1) \supset s^*\}$

$p_{s^*}(x_n - x) \to o \Leftrightarrow \forall t_1 > 0, there\ exists\ n_0 \in \mathbb{N} : p_{s^*}(x_n - x) < t_1, \forall n \geq n_0.$

$\Leftrightarrow \forall t_1 > 0, there\ exists\ n_0 \in \mathbb{N} : \mathcal{H}(x_n - x, t_1) \supset s^*, \forall n \geq n_0.$

$\Leftrightarrow x_n \overset{s^*}{\rightarrow} x.$

Definition 4.7:

$(\mathbb{X}, \mathcal{H}, *)$, is a hesitant fuzzy normed linear space, $s^* \in P[0,1]$ and (x_n) is a sequence in $\mathbb{X}$, if

$\forall t_1 > 0, there\ exists\ n_0 \in \mathbb{N} : \mathcal{H}(x_n - x_m, t_1) \supset s^*, \forall n, m \geq n_0$,then the sequence (x_n) is s^*- hesitant fuzzy Cauchy.

Proposition 4.8:

On a hesitant fuzzy normed linear space $(\mathbb{X}, \mathcal{H}, *)$, $s^* \in P[0,1]$. Then every s^*- convergent is s^*-cauchy.

Proof:

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Let (x_n) be a sequence that converges to x in s^*.

Then $\forall t_1 > 0, there\ exists\ n_0 \in \mathbb{N}: \mathcal{H}(x_n - x,, t_1) \supset s^*, \forall n \geq n_0$.

Hence for$t_1 > 0$, already exists that

$$\mathcal{H}(x_n - x_m,, t_1) = \mathcal{H}\left(x_n - x + x - x_m\ , \frac{t_1}{2} + \frac{t_1}{2}\right)$$

$$\supseteq \mathcal{H}\left(x_n - x, \frac{t_1}{2}\right) \cap H\left(x - x_m\ , \frac{t_1}{2}\right)$$

$$\mathcal{H}(x_n - x_m,, t_1) \supseteq s^*, \forall n, m \geq n_0.$$

Then (x_n) is s^*-cauchy.

5. Hesitant Fuzzy continuous linear operators in hesitant fuzzy normed linear spaces

Hesitant fuzzy continuous of a point, a set and its boundedness on hesitant fuzzy norm have been introduced, and some theorems, propositions, corollaries are discussed in this section, based on this definitions.

Definition 5.1:

Consider are hesitant fuzzy normed linear spaces $(\mathbb{X}, \mathcal{H}_1, *_1), (\mathbb{X}, \mathcal{H}_2, *_2)$ with continuous t −norms $*_1, *_2$.

if $\forall \in > 0, \forall\ s_1^* \in P[0,1]$, then $there\ exists\ a\ \ \delta > 0, \exists\ s_2^* \in P[0,1]$ and $\mathbb{T}: \mathbb{X} \to \mathbb{Y}$, is hesitant fuzzy continuous at $x_0 \in \mathbb{X}$. In this way we obtain that $\mathcal{H}_2(\mathbb{T}(x_1) - \mathbb{T}(x_0), \epsilon) \supset s_1^*\ such\ that$

$\forall\ x_1 \in \mathbb{X}:\ \mathcal{H}_1(x_1 -\ x_0, \delta) \supset\ s_2^*$,

If $\mathbb{T}$ is fuzzy continuous on $\mathbb{X}$,then it is hesitantly fuzzy continuous everywhere on $\mathbb{X}$.

Theorem 5.2 :

Consider a linear operator $: \mathbb{X} \to \mathbb{Y}$. Then $\mathbb{T}$ is hesitant fuzzy continuous on $\mathbb{X}$, if and only if $\mathbb{T}$ is fuzzy continuous at a point $x_0 \in X$.

Proof :

Necessary part is obvious.

Sufficient part :

Let $y \in \mathbb{Y}$ be arbitrary. Let $\in > 0,\ s_1^* \in P[0,1]$.

there exist $\delta > 0,\ s_2^* \in P[0,1]$, since $\mathbb{T}$ is hesitant fuzzy continuous at $x_0 \in \mathbb{X}$,

So that, $\forall\ x_1 \in \mathbb{X}:\ \mathcal{H}_1(x_1 -\ x_0, \delta) \supset\ s_2^* \Rightarrow \mathcal{H}_2(\mathbb{T}(x_1) - \mathbb{T}(x_0), \epsilon) \supset s_1^*$.

Replacing x_1 by $x_1 + x_0 - y$, which implies that ,

$\forall\ x_1 \in \mathbb{X}:\ \mathcal{H}_1(x_1 - y, \delta) \supset\ s_2^* \Rightarrow \mathcal{H}_2(\mathbb{T}(x_1) - \mathbb{T}(y), \epsilon) \supset s_1^*$.

Thus $\mathbb{T}$ is hesitant fuzzy continuous at $y \in \mathbb{Y}$.

Corollary 5.3:

Consider the linear operator $\mathbb{T}: \mathbb{X} \to \mathbb{Y}$ if and only if $\forall \in > 0, \forall\ s_1^* \in P[0,1], there\ exists\ a$

$\delta > 0, \exists\ s_2^* \in P[0,1]$ and then T is hesitant fuzzy continuous on X,we possess $\mathcal{H}_2(\mathbb{T}(x_1), \epsilon) \supset s_1^*$ such that $\forall x_1 \in X:\ \mathcal{H}_1(x, \delta) \supset\ s_2^*$.

The proof is obvious based on the earlier theorems.

Theorem 5.4:

A linear operator $\mathbb{T}: \mathbb{X} \to \mathbb{Y}$ is hesitant fuzzy continuous on $\mathbb{X}$ if and only if $\forall\ s_1^* \in P[0,1], there\ exists\ a\ s_2^* \in P[0,1], there\ exists\ a\ M > 0$, such that

$\forall t > 0, \forall\ x_1 \in \mathbb{X}:\ \mathcal{H}_1(x_1, t) \supset\ s_2^* \Rightarrow \mathcal{H}_2(\mathbb{T}(x_1), Mt) \supset s_1^*$.

Proof :

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"⇐" Let $\in > 0, s_1^* \in P[0,1]$ be arbitrary. Then there exists $s_2^* \in P[0,1], M > 0$ such that,

$\forall t > 0, \forall\, x_1 \in \mathbb{X} : \mathcal{H}_1(x_1,\delta) \supset s_2^* \Rightarrow \mathcal{H}_2(\mathbb{T}(x_1), Mt) \supset s_1^*,$

specifically, for $t = \frac{\in}{M}$, we acquire, $\mathcal{H}_1(x_1,\frac{\in}{M}) \supset s_2^* \Rightarrow \mathcal{H}_2(\mathbb{T}(x_1), \in) \supset s_1^*$.

Then for $\delta = \frac{\in}{M} > 0, \Rightarrow$ $\mathbb{T}$ is hesitant fuzzy continuous on $\mathbb{X}$.

"⇒" consider that , there exists a $s_0^* \in P[0,1]$, so that $\forall\, s_2^* \in P[0,1], \forall M > 0$,

there exists a $t_0 > 0$, there exists a $x_0 \in, \mathcal{H}_1(x_0, t_0) \supset \forall s_2^*$ and $\mathcal{H}_2(\mathbb{T}(x_1), Mt_0) \subseteq s_0^*$.

The set $V_0 = \{y \in Y : \mathcal{H}_2(y, t_0) \supset s_0^*\}$ is an open neighborhood of $\bar{0}_Y$.

To demonstrate that, for all neighborhood U of $\bar{0}_X$. already exists, $\mathbb{T}(U) \nsubseteq V_0$. This is in opposition to $\mathbb{T}$ is fuzzy continuity at $\bar{0}_X$.

For all $s_2^* \in P[0,1], k > 0$,it is suffices to demonstrate that $\{B(0, s_2^*, k)\}_{s_2^* \in P[0,1].k>0}$ is a fundamental system of neighborhoods of $\bar{0}_X$,

Since, $\mathbb{T}\big(B(0, s_2^*, k)\big) \nsubseteq V_0$. As $M > 0$ is arbitrary, choose $k = \frac{t_0}{M}$, observe that, for $z_0 = \frac{x_0}{M} \in \mathbb{X}$,

we possess $\mathcal{H}_1\left(z_0,\frac{t_0}{M}\right) = H_1\left(\frac{x_0}{M},\frac{t_0}{M}\right) = \mathcal{H}_1(x_0, t_0) \supset s_2^*$

Hence, $z_0 \in B\left(0, s_2^*,\frac{t_0}{M}\right)$. We shall demonstrate that $\mathbb{T}(z_0) \notin V_0$, specifically $\mathcal{H}_2(\mathbb{T}(z_0), t_0) \subseteq s_0^*$.

Indeed $\mathcal{H}_2(\mathbb{T}(z_0), t_0) = \mathcal{H}_2(\mathbb{T}\left(\frac{x_0}{M}, t_0\right) = \mathcal{H}_2(\mathbb{T}(x_0), Mt_0) \subseteq s_0^*$

Corollary 5.5:

Consider $f: (\mathbb{X}, \mathcal{H}_1, *) \to (\mathbb{C}, \mathcal{H}, *)$ is a linear functional and hesitant fuzzy continuous, if and only if $\textit{there exists a}\ \ s_2^* \in P[0,1], \textit{there exists a}\ \ M > 0$, such that $\forall\, t > 0, \forall x_1 \in \mathbb{X}, \mathcal{H}_1(x_1, t) \supset s_2^*$ $\Rightarrow |\, f(x_1)| < Mt$.

Proof :

Based on earlier theorems ,f is hesitant fuzzy continuous if and only if

$\forall\ s_1^* \in P[0,1], \textit{there exists a}\ s_2^* \in P[0,1], \textit{there exists a}\ M > 0$, such that

$\forall t > 0, \forall x_1 \in \mathbb{X} : \mathcal{H}_1(x_1, t) \supset s_2^* \Rightarrow \mathcal{H}(f(x_1), Mt) \supset s_1^*,$

but $\mathcal{H}(f(x_1), Mt) \supset s_1^* \Leftrightarrow \mathcal{H}(f(x_1), Mt) = U^* \Leftrightarrow |f(x_1)| < t$

Henec, $\textit{there exists a}\ s_2^* \in P[0,1], \textit{there exists a}\ \ M > 0$ such that

$\forall\, t > 0, \forall x_1 \in \mathbb{X}, \mathcal{H}_1(x_1, t) \supset s_2^* \Rightarrow |\, f(x_1)| < Mt.$

Corollary 5.6:

Let $(\mathbb{X}, \mathcal{H}_{1,} *_1), (\mathbb{X}, \mathcal{H}_{2,} *_2)$ be HFNLS's and $p_s(x) := \inf\{t > 0: \mathcal{H}_{1,}(x_1, t) \supset s\}, s \in P[0,1]$.

$q_s(x) := \inf\{t > 0: \mathcal{H}_2(x_1, t) \supset s\}, s \in P[0,1]$, consider $T: \mathbb{X} \to \mathbb{Y}$ is linear operator and hesitant fuzzy continuous on X if and only if $\forall\, s_1^* \in P[0,1], \textit{there exists a}\ \ s_2^* \in P[0,1], \textit{there exists a}\ M > 0$ such that

$q_s(Tx_1) \leq M_{p_s(x_1)}, \forall x_1 \in \mathbb{X}.$

Proof :

Based on earlier theorems , $\forall\, s_1^* \in P[0,1], \textit{there exists a}\ \ s_2^* \in P[0,1], \textit{there exists a}\ M > 0$ such that $\forall t > 0, \forall\, x_1 \in \mathbb{X} : \mathcal{H}_1(x_1,t) \supset s_2^* \Rightarrow \mathcal{H}_2(T(x_1), Mt) \supset s_1^*$. Thus ,for $x_1 \in \mathbb{X}$,

we possess $\{t > 0 : \mathcal{H}_1(x_1,t) \supset s_2^*\} \subseteq \{t > 0 : \mathcal{H}_2(T(x_1), Mt) \supset s_1^*\}$

Hence, $inf\{t > 0 : \mathcal{H}_1(x_1,t) \supset s_2^*\} \geq inf\{t > 0 : \mathcal{H}_2(T(x_1), Mt) \supset s_1^*\}$

Namely, $inf\{t > 0 : \mathcal{H}_1(x_1,t) \supset s_2^*\} \geq inf\{\frac{t}{M} > 0 : \mathcal{H}_2(T(x_1), Mt) \supset s_1^*\}$

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Therefore $p_{s_2^*}(x_1) \geq \frac{1}{M} q_{s_1^*}(Tx_1), \forall x_1 \in \mathbb{X} \Rightarrow q_{s_1^*}(Tx_1) \leq M \;\; p_{s_2^*}(x_1), \forall x_1 \in \mathbb{X}$

Corollary 5.7:

Consider $f: (\mathbb{X}, \mathcal{H}_1, *) \to (\mathbb{C}, \mathcal{H}, *)$ is a linear functional and hesitant fuzzy continuous, if and only if

$there\ exists\ a\ \ s_2^* \in P[0,1], there\ exists\ a\ M > 0$, such that $|f(x_1)| \leq M \;\; p_{s_2^*}(x_1), \forall x_1 \in \mathbb{X}$.

The proof is obvious based on the earlier theorems.

6. Conclusion

This work introduced the concept of hesitant fuzzy norm and studied some related results on linear spaces, continuous operators and sequential convergence. In future it can be developed other related topics in metric spaces and bounded linear operators.

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