Lindel ”Ofness Spaces in NTH Topological Spaces

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Abstract

In this study, the lindel”of property of spaces will be examined across n^{th} topologies, referred to as n^{th}-lindel”of spaces. Furthermore, the characteristics of these spaces will be analyzed in relation to lindelö]f spaces and tri-Lindelf spaces. Several theoretical results have been presented and proven, and various well-known theorems concerning Lindel?f spaces have been extended to accommodate n^{th} topologies. An illustrative examples are provided to support the findings.

Keywords: lindel”of spaces; Hausdorff; bitopological spaces; tri-topological spaces

1 Introduction

A central focus in set-theoretic topology is the study and exploration of relationships between various classes of topological spaces that fall between Lindel?f spaces. In this context, the class of Lindel?f spaces plays a crucial role, as it naturally occupies an intermediate position between these classes. Where the lindelöf space known as every open cover of (Q, τ) has a countable subcover of Q. but a tri-topological space $(Q, \tau_1, \tau_2, \tau_3)$ is called tri-lindelöf space if every τ_i-open cover of Q has a tripartite countable subcover of Q, where i≠1,2,3. This paper containing the concept of n^{th} topological spaces where n^{th} topological space is $(Q, \tau_1, \tau_2, ..., \tau_n)$ such that $\tau_{j's}$ are topologies on Q, where j=1,2,...,n.
A notable characteristic of these classes of spaces is that several important separation axioms, such as normality and the Hausdorff condition, coincide within these classes. This renders them highly significant, both theoretically and practically, whether addressing problems from a purely topological perspective or those emerging from other areas of mathematics.

2 Literature review

The concept of a lindel”of space in topological space (Q, τ) was presented by [7]. Recent research [2], [3], [6] has delved deeper into these areas. This paper explores the concept of n^{th}-lindelöf and n^{th}-metalindelöf spaces and presents associated conclusions. In the next section, we introduce the concept of topological space, n^{th}-topological space and some important concept in n^{th}-topological space like: open and closed sets, derived set, closure set, interior and exterior sets, separation axioms, etc... . Then we talk about the concept of lindelöf

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space in n^{th}-topological spaces, discuss its features, and apply it to other spaces. We examine well-known definitions that will be applied in the sequel. Finally, we explore n^{th}-metalindelöfness spaces and demonstrate several features of these spaces.

The terms τ_u, τ_{dis}, τ_{cof} and τ_{coc} represent the ordinary or usual topology, discrete topology, co-finite topology, and co-countable topology, respectively. The concept of bitopological spaces can be represented as $Q = (Q, \tau_1, \tau_2)$ where τ_1, τ_2 are two topologies on Q and the concept of tri-topological space $(Q, \tau_1, \tau_2, \tau_3)$ where τ_1, τ_2 and τ_3 are three topologies on Q . This is connected to prior research on tri-topological spaces, where each topology is a set of points that meet a set of axioms. [4] explained tripartite Hausdorff, tripartite regular, and tripartite normal spaces using a set of standard results known as Tietze extintion. The primary goal of this paper is to introduce and investigate a novel sort of $n^{th}-$lindelöf space, the n^{th}-metalindelöf space. N^{th}-topological spaces are sets containing n topologies, represented as $Q = (Q, \tau_1, \tau_2, ..., \tau_n)$, where $\tau_{i's}$ are topologies on Q, where i=1,2,...,n. N^{th}-topological spaces have variations that correspond to well-known topological space features.

3 Preliminaries

In this section ,we will show some concept in n^{th}-topological space.

Definition 2.1 [1] Let $Q \neq \varphi$, $\tau \subset$ P(Q)= {O: O $\subseteq Q$}, then τ is called topology on X if the following conditions are satisfied:
(i) $\varphi, Q \in \tau$.
(ii) Closed under intersection.
(iii) Closed under union.

Definition 2.2 Let $Q \neq \varphi$ be a set and τ_i is a topology on Q for all i=1,2,...,n, then $(Q, \tau_1, \tau_2, ..., \tau_n)$ is called n^{th}-topological space.

Definition 2.3 Let $(Q, \tau_1, \tau_2, \tau_3)$ be a n^{th}-topological space and O $\subset Q$, then if O $\in \bigcup\limits_{i=1}^{n} \tau_i$ we say that O is n^{th}-open set and Q-O is called n^{th}-closed set and if O and Q-O in $\bigcup\limits_{i=1}^{n} \tau_i$, then O is called n^{th}-clopen set.

Example 1. let Q=$\mathbb{N}$ be a set of all nutral numbers and define τ_i={ $\mathbb{R}$,φ,{i},{i+1},{i,i+1}} for all i=1,2,...,n. Then Q=$(Q, \tau_1, \tau_2, ..., \tau_n)$ is n^{th}-topological space and O={i,i+1} is n^{th}-open set.
Since O$\in \bigcup\limits_{i=1}^{n} \tau_i$ and F = $\mathbb{Z}$ - {i,i+1} is n^{th}-closed set because $\mathbb{Z}$ - F = {i,i+1} $\in \bigcup\limits_{i=1}^{n} \tau_i$ and $\mathbb{Z}$,φ are n^{th}-clopen sets because they are belongs to $\bigcup\limits_{i=1}^{n} \tau_i$.

Definition 2.4 Let Q=$\mathbb{R}$ the set of all real numbers and define some toplogies like:
i) τ_{ind} = { $\mathbb{R}$,φ } is the indiscrete topology defined on $\mathbb{R}$ and it is the smallest topology on $\mathbb{R}$.
ii) τ_{dis} = $P(\mathbb{R})$ is the discrete topology on $\mathbb{R}$ and it is the biggest topology on $\mathbb{R}$.
iii) τ_u is the usual topology define on $\mathbb{R}$.
iv) τ_{cof} is the co-finite topology defined on $\mathbb{R}$.
v) τ_{coc} is the co-countable topology on $\mathbb{R}$.
vi) τ_L is the leftray topology defined on $\mathbb{R}$.
vii) τ_r is the rightray topology defined on $\mathbb{R}$.

Example 2. Consider Q=$\mathbb{R}$ the set of real numbers and define τ_{ind}, τ_{dis}, τ_u, τ_L, τ_r, τ_{cof} and τ_{coc} on $\mathbb{R}$, then $(\mathbb{R}, \tau_{ind}, \tau_{dis}, \tau_u, \tau_L, \tau_r, \tau_{cof}, \tau_{coc})$ form a 7^{th}-topological space and clearly the open interval (a, b) such that $a < b$ in $\mathbb{R}$ is an n^{th}-open set since it is belong to τ_u and $\tau_u \subset (\tau_{ind} \cup \tau_{dis} \cup \tau_u \cup \tau_L \cup \tau_r \cup \tau_{cof} \cup \tau_{coc})$.

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And (a, ∞) for some a $\in \mathbb{R}$ in n^{th}-clopen set since $(a, \infty) \in \tau_L \subset (\tau_{ind} \cup \tau_{dis} \cup \tau_u \cup \tau_L \cup \tau_r \cup \tau_{cof} \cup \tau_{coc})$ and $\mathbb{R}$-(a, ∞)=$(-\infty, a] \in \tau_{dis} \subset (\tau_{ind} \cup \tau_{dis} \cup \tau_u \cup \tau_L \cup \tau_r \cup \tau_{cof} \cup \tau_{coc})$.

Definition 2.5 Let $(Q, \tau_1, \tau_2, ..., \tau_n)$ be a n^{th}-topological space, $Q \neq \varphi$ and O $\subset Q$, then q $\in Q$ is called n^{th}-Limit point of O if for all τ_i-open set u_q such that $u_q \cap (\mathrm{O} - \{q\}) \neq \varphi$, i=1,2,...,n.
The set of all limit points in n^{th}-topological space is called n^{th}-derived set and it is denoted by
O$'$ ={q : q is n^{th}-limit point of O}.
Properties of n^{th}-derived set: Let $(Q, \tau_1, \tau_2, ..., \tau_n)$ be a n^{th}-topological space and let O, B $\subset Q$, then:
(i) $\varphi' = \varphi$.
(ii) If O $\subset$ B, then O' $\subset$ B'.
(iii) (O $\cup$ B)' = O' $\cup$ B'.
(iv) (O $\cap$ B)' $\subset$ O' $\cap$ B'.

Proof: (i) By contradiction :
Assume that $\varphi' \neq \varphi$, then there exist z $\in \varphi'$, then for all τ_i-open set u_z we have $u_z \cap (\varphi - \{z\}) \neq \varphi$, but $(\varphi - \{z\})$=φ, then $u_z \cap \varphi \neq \varphi$, thus $\varphi \neq \varphi$ and that is contradiction. So, φ'=φ.

Definition 2.6 Let $(Q, \tau_1, \tau_2, ..., \tau_n)$ be a n^{th}-topological space, $Q \neq \varphi$ and O $\subset Q$, then the n^{th}-closure set is denoted by $\overline{\mathrm{O}}$ = O $\cup$ O$'$.
Properties of n^{th}-closure set: Let $(Q, \tau_1, \tau_2, ..., \tau_n)$ be a n^{th}-topological space and let O, B $\subset Q$, then:
(i) $\overline{\varphi} = \varphi$ and $\overline{Q} = Q$.
(ii) $\overline{\mathrm{O} \cup \mathrm{B}} = \overline{\mathrm{O}} \cup \overline{\mathrm{B}}$ and $\overline{O \cap \mathrm{B}} \subset \overline{\mathrm{O}} \cap \overline{\mathrm{B}}$.
(iii)$\overline{\mathrm{O}}$ is n^{th}-closed set.
(iv) O = $\overline{\mathrm{O}}$ if and only if O is n^{th}-closed set.
(v) q $\in \overline{\mathrm{O}}$ if and only if for all n^{th}-open set u_q such that q $\in u_q$ we have $u_q \cap$ O $\neq \varphi$.

Proof: (ii) Let O, B $\subset Q$, then
$\overline{\mathrm{O} \cup B} = (\mathrm{O} \cup \mathrm{B}) \cup (\mathrm{O} \cup \mathrm{B})'$ = $(\mathrm{O} \cup \mathrm{B}) \cup (\mathrm{O}' \cup \mathrm{B}')$ = $(\mathrm{O} \cup \mathrm{O}')(\mathrm{B} \cup \mathrm{B}')$ = $\overline{\mathrm{O}} \cup \overline{B}$.
and $\overline{\mathrm{O} \cap B} = (\mathrm{O} \cap \mathrm{B}) \cap (\mathrm{O} \cap \mathrm{B})' \subset (\mathrm{O} \cap \mathrm{B}) \cap (\mathrm{O}' \cap \mathrm{B}')$ = $(\mathrm{O} \cup \mathrm{O}')(\mathrm{B} \cap \mathrm{B}')$ = $\overline{\mathrm{O}} \cap \overline{B}$.

Definition 2.7 Let $(Q, \tau_1, \tau_2, ..., \tau_n)$ be a n^{th}-topological space , $Q \neq \varphi$ and O is subset of Q, then a point q $\in$ O is said to be n^{th}-Interior point of O if there exist at least one neighborhood of q (N(q,ε)) such that N(q,ε) $\subseteq$ O.
The set of all n^{th}-interior point is called the n^{th}-Interior set and it is denoted by O$^\circ$ $\equiv INT(A) = \overline{O^C}^C$.
Properties of n^{th}interior set: Let $(Q, \tau_1, \tau_2, ..., \tau_n)$ be a n^{th}-topological space and let O, B $\subset Q$, then:
(i) $\varphi^\circ = \varphi$ and $Q^\circ = Q$.
(ii) $(\mathrm{O} \cap \mathrm{B})^\circ = \mathrm{O}^\circ \cap B^\circ$ and $\mathrm{O}^\circ \cup B^\circ \subset (\mathrm{O} \cup \mathrm{B})^\circ$.
(iii) O$^\circ$ is n^{th}-open set.
(iv) n $\in$ O$^\circ$ if and only if there exist n^{th}-open set u_n such that n $\in u_n \subset$ A.

Definition 2.8 Let $(Q, \tau_1, \tau_2, ..., \tau_n)$ be a n^{th}-topological space, $Q \neq \varphi$ and O is subset of Q, then the point q is said to be n^{th}-Exterior point of O if there exist at least one neighborhood of q such that N (q, ε) $\cap$ O= φ.
The set of all n^{th}-Exterior point is called n^{th}-Exterior set and it is denoted by
EX(O)=Int(O^c)= $\overline{\mathrm{O}}^C$.
Properties of n^{th}-exterior set: Let $(Q, \tau_1, \tau_2, ..., \tau_3)$ be a n^{th}-topological space and let O, B $\subset Q$, then:
(i) EX(φ) =X and EX(X)=φ.
(ii) If O $\subset$ B, then EX(B) $\subset$ EX(O).
(iii) EX(O) is τ_i-open set.
(iv) e $\in$ EX(O) if and only if there exist n^{th}-open set u_e such that e$\in u_e \subset$O^c.

Proof: (iii)

since EX(O) = $INT(\mathrm{O}^c)$, then EX(O) =$\overline{\mathrm{O}^{cc}}^{c}$, but O^{cc} = O thus, $EX(O) = \overline{\mathrm{O}}^{c}$ and by definition of n^{th}-closure set we have $\overline{\mathrm{O}}$ is n^{th}-closed set so, the complement of n^{th}-closed set is n^{th}-open set, therefor EX(O) is n^{th}-open set. As a result the interior set is also n^{th}-open set.

Definition 2.9 Let $(Q, \tau_1, \tau_2, ..., \tau_3)$ be a n^{th}-topological space, $Q \neq \varphi$ and O is subset of Q, then the point q is said to be n^{th}-Boundary point of O, If every neighborhood of q satisfy that N(q,ε) $\cap$ O $\neq \varphi$ and N(q, ε) $\cap$ $\mathrm{O}^c \neq \varphi$.
The set of all n^{th}-boundary point is called n^{th}-Boundary set, and it is denoted by
Bd(O) = $\overline{\mathrm{O}} - \mathrm{O}^\circ$ = $\overline{\mathrm{O}} \cap \overline{\mathrm{O}^c}$.
Properties of n^{th}-boundary set: Let $(Q, \tau_1, \tau_2, ..., \tau_3)$ be a n^{th}-topological space and let O, B $\subset Q$, then:
(i) Bd(φ) = Bd(Q)= φ.
(ii) Bd(O) is n^{th}-closed set.
(iii) b $\in$ Bd(O) if and only if for all n^{th}-open set u_b such that b $\in u_b$ we have $u_b \cap$ O $\neq \varphi$ and $u_b \cap \mathrm{O}^c \neq \varphi$.

Proof: (iii)

Let s $\in$ Bd(O) and u_s be a n^{th}-open set such that s u_s, then s $\in$ $(\overline{\mathrm{O}} \cap \overline{\mathrm{O}^C})$ if and only if s $\in \overline{\mathrm{O}}$ and s $\in \overline{\mathrm{O}^c}$ if and only if s $\in$ (O $\cup$ O$'$) and s $\in \mathrm{O}^c \cup (\mathrm{O}^c)'$ if and only if (s $\in$ O or s $\in$ O') and (s $\in \mathrm{O}^c$ or s $\in (\mathrm{O}^c)'$ if and only if s $\in$ O' and s $\in \mathrm{O}^c$ if and only if $u_s \cap$ (O$/\{s\}$) $\neq \varphi$ and $s \cap \mathrm{O}^c \neq \varphi$, but $s \subset u_s$, so we have $u_s \cap$ O $\neq \varphi$ and $u_S \cap \mathrm{O}^c \neq \varphi$.

Now we will show some separation axioms below: **Definition 2.10** [5] If for all q $\neq$ s in Q, there exist open set u_q such that q $\in u_q$ and s $\notin u_s$,or there exist open set v_s such that s $\in v_s$ and q $\notin v_q$, then a topological space (Q, τ) is called T_0-space.

Definition 2.11 If for all q $\neq$ s in Q, there exist n^{th}-open set u_q such that q $\in u_q$ and s $\notin u_s$, or there exist n^{th}-open set v_s such that s $\in v_s$ and q $\notin v_s$, where i $\neq$ j and i,j = 1,2,...,n then a n^{th}-topological space $(Q, \tau_1, \tau_2, ..., \tau_n)$ is called n^{th}-T_0-space.

Theorem 2.1 Let $(Q, \tau_1, \tau_2, ..., \tau_3)$ be a n^{th}-topological space, then the following are equivalent:
(i) Q is n^{th}-T_0-space.
(ii) For all q $\notin$ s, we have q $\notin \overline{\{q\}}$ or s $\notin \overline{\{q\}}$.
(iii) For all q $\neq$ s, we have $\overline{\{q\}} \neq \overline{\{s\}}$.

Proof: (**(i) implies (ii)**)

Let q $\neq$ s, then there exist τ_i-open set such that q $\in u_q$ and s $\notin u - q$ or there exist τ_j-open set v_s such that s $\in v_s$ and q $\notin v_s$ where i=1,2,3. So, we have a $\in u_q$ and $u_q \cap \{s\}$ = φ or s in v_s and $v_s \cap \{q\}$ = φ. Thus, q $\notin \overline{\{s\}}$ or s $\notin \overline{\{q\}}$.

(**(ii) implies (iii)**)

Let q $\neq$ s, then if q $\notin \overline{\{s\}}$ and q $\in \overline{\{q\}}$, then we have $\overline{\{q\}} \neq \overline{\{s\}}$. Additionally, if s $\notin \overline{\{q\}}$ and s $\in \overline{\{s\}}$, then we have $\overline{\{q\}} \neq \overline{\{q\}}$.

(**(iii) implies (i)**)

Let q $\neq$ s and by given $\overline{\{q\}} \neq \overline{\{s\}}$, but q $\in \overline{\{q\}}$ and s $\in \overline{\{s\}}$, then q $\notin Q - \overline{\{q\}}$=$v_s$ which is n^{th}-open set in Q since $\overline{\{q\}}$ is n^{th}-closed set in Q and s$\in Q - \overline{\{q\}}$=v_s, where i=1,2,...,n. Thus, Q is n^{th}-T_0-space.

Definition 2.12 If for all q $\neq$ s in Q, there exist n^{th}-open set u_q such that q $\in u_q$ and s $\notin u_q$, and there exist n^{th}-open set v_s such that s $\in v_s$ and q $\notin v_s$, where i$\neq$j and i, j = 1,2,3, then a tri-topological space $(Q, \tau_1, \tau_2, ..., \tau_n)$ is called n^{th}-T_1-space.

Definition 2.13 If for all q $\neq$ s in Q, there exist n^{th}-open set u_q such that q $\in u_q$ and there exist n^{th}-open set v_s such that s $\in v_s$ and $u_q \cap v_s = \varphi$, where i$\neq$j and i, j=1,2,...,n, then a n^{th}-topological space $(Q, \tau_1, \tau_2, ..., \tau_n)$ is called n^{th}-T_2-space.

Definition 2.14 If for all a $\neq$ b in Q, there exist τ_i-closed set O$_q$ and B_s such that q $\in$ O$_q$, s $\in B_s$ and O$_q$ $\cap$ $B_s = \varphi$, i=1,2,3, then a tri-topological space $(Q, \tau_1, \tau_2, \tau_3)$ is called tri-$T_{2\frac{1}{2}}$-space.

Definition 2.15 If for all q $\notin$ O and O is n^{th}-closed set, there exist n^{th}-open set u_a and n^{th}-open set v_{O} such that q $\in u_q$, O $\subset v_{\mathrm{O}}$ and $u_q \cap v_{\mathrm{O}}$ =φ, where i$\neq$j, i, j=1,2,...,n, then a n^{th}-topological space $(Q, \tau_1, \tau_2, ..., \tau_n)$ is n^{th}-regular space.

Theorem 2.2 A space $(Q, \tau_1, \tau_2, ..., \tau_n)$ is n^{th}-regular space if and only if for all q $\in u_q$,where u_q is n^{th}-open set, there exist n^{th}-open set w_q such that q$\in w_q \subset \overline{w_q} \subset u_q$.

Proof: $(\rightarrow)$
Let q $\in u_q$, then q $\notin {u_q}^c$,but ${u_q}^c$ is n^{th}-closed set, then we can say that ${u_q}^c$=A. So, by definition of n^{th}-regular space, there exist n^{th}-open set w_q and v_{O} such that q $\in w_q$, 0 $\subset v_{\mathrm{O}}$ and $w_q \cap v_{\mathrm{O}}$ =φ, but clearly $w_q \subset \overline{w_q}$.
It is enough to show $\overline{w_q} \subset u_q$, now $w_q \cap v_{\mathrm{O}}$=φ, then we can say that $w_q \subset {v_{\mathrm{O}}}^c$, then $\overline{w_q} \subset \overline{{v_{\mathrm{O}}}^c} = {v_{\mathrm{O}}}^c$. So, we have $\overline{w_a} \subset {v_{\mathrm{O}}}^c$, but ${u_q}^c$= O $\subset v_{\mathrm{O}}$, then ${u_q}^c \subset v_{\mathrm{O}}$, then ${v_{\mathrm{O}}}^c \subset u_q$, thus $\overline{w_q} \subset u_q$. We are done.
$(\leftarrow)$
Let q $\notin$ O and O is n^{th}-open set, then q $\in$ O^c and O^c is n^{th}-open set, then by given there exist n^{th}-open set w_q such that q $\in w_q \subset \overline{w_q} \subset$ O^c. Now we have two givens, q $\in w_q$ and O $\subset \overline{w_q}^c$, where w_q and $\overline{w_q}^c$ is n^{th}-open sets, i=1,2,...,n ..... (i).
it is enough to show that $w_q \cap \overline{w_q}^c = \varphi$, suppose not, then there exist z such that z $\in (w_q \cap \overline{w_q}^c)$, that is implies z $\in w_q$ and z $\in \overline{w_q}^c$, then z $\in w_q$ and z $\notin w_q$ and z $\in w_q'$, then we have z $\in$ ($w_q \cap {w_q}^c$) that is contradiction. So, $w_q \cap \overline{w_q}^c = \varphi$ ..... (ii).
By (i) and (ii) we have, a space $(Q, \tau_1, \tau_2, ..., \tau_n)$ is n^{th}-regular space.

Definition 2.16 [3] A topological space (Q, τ) is T_3-space if it is T_1-space and regular space.

Definition 2.17 A tri-topological space $(Q, \tau_1, \tau_2, ..., \tau_n)$ is n^{th}-T_3-space if it is n^{th}-T_1-space and n^{th}-regular space.

Definition 2.18 A space $(Q, \tau_1, \tau_2, ..., \tau_n)$ is n^{th}-normal space if for all two disjoint n^{th}-closed set O and B, there exist n^{th}-open sets u_{O} and v_B such that O $\subset u_{\mathrm{O}}$, B $\subset v_B$ and $u_{\mathrm{O}} \cap v_B$=φ.

Definition 2.19 A n^{th}-topological space $(Q, \tau_1, \tau_2, ..., \tau_n)$ is n^{th}-T_4-space if it is n^{th}-T_1-space and n^{th}-normal space.

Theorem 2.3 If $(Q, \tau_1, \tau_2, ..., \tau_n)$ is n^{th}-T_k-space, then it is n^{th}-T_{k-1}-space.

Example.3 If a space $(Q, \tau_1, \tau_2, ..., \tau_n)$ is n^{th}-T_4-space, then it is n^{th}-T_3-space.

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Proof: Let $(Q, \tau_1, \tau_2, ..., \tau_n)$ is n^{th}-T_4-space, then it is n^{th}-T_1-space and n^{th}-normal space, that is implies for all two disjoint n^{th}-closed set O and B, there exist n^{th}-open sets u_{O} and v_B such that O $\subset u_{\text{O}}$, B $\subset v_B$ and $u_{\text{O}} \cap v_B = \varphi$ .....(i), now let b $\in$ B, then b $\in v_B$ but O $\cap$ B = φ, therefore b $\notin$ A .....(ii),
by (i) and (ii) we have $(Q, \tau_1, \tau_2, ..., \tau_n)$ is n^{th}-regular space.
Thus $(Q, \tau_1, \tau_2, ..., \tau_3)$ is tri-T_1-space and tri-regular space, hence $(Q, \tau_1, \tau_2, \tau_3)$ is n^{th}-T_3-space.

Definition 2.20 [3] Let (Q, τ) be a topological space and W $= \{\text{O}_\alpha : \alpha \in \lambda, \text{O}_\alpha \subset \text{Q}\}$ is called:
(i) cover of Q if and only if $\bigcup\limits_{\alpha \in \lambda} \text{O}\alpha = Q$.
(ii) open cover of Q if and only if W is cover and O_α is open set, where $\alpha \in \lambda$.
(iii) closed cover of Q if and only if W is cover and O_α is closed set, where $\alpha \in \lambda$.
(iv) $C = \{\text{B}_\gamma : \gamma \in \Gamma\}$ is a subcover of W if and only if :
(i) C $\subset$ W (ii) $\bigcup\limits_{\gamma \in \Gamma} \text{B}_\gamma = X$
A space (Q, τ) is called compact space, if every open cover of Q has a finite subcover.

Definition 2.21 Let $(Q, \tau_1, \tau_2, ..., \tau_n)$ be a n^{th}-topological space and W $= \{\text{O}_\alpha : \alpha \in \lambda, \text{O}_\alpha \subset \text{Q}\}$ is called:
(i) n^{th}-cover of Q if and only if $\bigcup\limits_{\alpha \in \lambda} \text{O}\alpha = Q$.
(ii) n^{th}-open cover of Q if and only if W is n^{th}-cover and O_α is n^{th}-open set, where $\alpha \in \lambda$, i=1,2,...,n.
(iii) n^{th}-closed cover of Q if and only if W is n^{th}-cover and O_α is n^{th}-closed set, where $\alpha \in \lambda$, i=1,2,...,n.
(iv) $C = \{\text{B}_\gamma : \gamma \in \Gamma\}$ is a n^{th}-subcover of W if and only if :
(i) C $\subset$ W (ii) $\bigcup\limits_{\gamma \in \Gamma} \text{B}_\gamma = Q$

A space $(Q, \tau_1, \tau_2, ..., \tau_n)$ is called n^{th}-compact space, if every n^{th}-open cover of Q has a finite n^{th}-subcover.

Definition 2.22 If $(Q, \tau_1, \tau_2, ..., \tau_n)$ is a n^{th}-topological space, then τ is said to be locally n^{th}-compact, if each point of Q has an n^{th}-open neighborhood whose n^{th}-closure is n^{th}-compact.
Note that : every n^{th}-compact space is locally n^{th}-compact.

4 LINDELÖFNESS SPACES AND NEARLY LINDELÖFNESS SPACE IN Nth-TOPOLOGICAL SPACE

In this section we will talk about lindeöf in topological space, lindelöf space in n^{th}-topological space, nearly lindelöf space in n^{th}-topological space and some theorems and their properties.

Definition 3.1 [2] Let (Q, τ) be a topological space, then it is called lindelöf space if every open cover of Q has a countable subcover of Q.

Definition 3.2 Let $(Q, \tau_1, \tau_2, ..., \tau_n)$ be a n^{th}-topological space, then it is called n^{th}-lindelöf space if every n^{th}-open cover of Q has a countable n^{th}-subcover of Q.

Properties of n^{th}-lindelöf space:
(i) Every n^{th}-compact space is n^{th}-lindelöf space.
(ii) A n^{th}-lindelöf space is n^{th}-compact if and only if it is countably n^{th}-compact.
Note that: We can say that the set A is countable if either it is finite set or if there exist an injective function from the set A into the natural numbers ($\mathbb{N}$).
(iii) Every n^{th}-closed subspace of a tri-Lindel?f space is n^{th}-Lindelöf.

Definition 3.3 Let $(Q, \tau_1, \tau_2, ..., \tau_n)$ be a n^{th}-topological space ,and $\tilde{C}$ be a cover of Q, thus we can say $\tilde{C}$ is a n^{th}-open cover if $\tilde{C} \subset \bigcup\limits_{i=1}^{3} v_i$.

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Definition 3.4 A n^{th}-topological space $(Q, \tau_1, \tau_2, ..., \tau_n)$ is called 2^{nd} countable base with respect to τ_1 ,τ_2,.., τ_n.

Definition 3.5 A n^{th}-topological space $(Q, \tau_1, \tau_2, ..., \tau_n)$ is called n^{th}-S-lindelöf space if and only if it is lindelöf space, pairwise lindelöf space, tripartite lindelöf space, ...,n^{th}-lindelöf.

Theorem 3.1 If a n^{th}-topological space X is 2^{nd} countable space, then it is n^{th}-lindelöf space.

Remark.1 Every n^{th}-compact space is n^{th}-lindelöf space but the converse need not be true.

Example.1 Let ($\mathbb{R}$,τ_{u_1},τ_{u_2},...,τ_{u_n}) be n^{th}-topological space, then ($\mathbb{R}$,τ_{u_1},τ_{u_2},...,τ_{u_n}) is n^{th}-lindelöf space but not n^{th}-compact space.

Proof: since O = $\{(q, s) : a, s \in \mathbb{Q}\}$ is countable base with respect to τ_i of $\mathbb{R}$, i=1,2,...,n, then $\mathbb{R}$ is 2^{nd} countable space, so by previous theorem we have ($\mathbb{R}$,τ_{u_1},τ_{u_2},...,τ_{u_n}) is n^{th}-lindelöf space. But (q,s) for all q $<$ s in $\mathbb{Q}$ is not closed, then it is not n^{th}-compact therefore, (R,ϑ_{u_1},ϑ_{u_2},ϑ_{u_n}) is n^{th}-compact space.

Corollary.1 Every 2^{nd} countable topological space is n^{th}-topological space (X, $\vartheta_1, \vartheta_2, ..., \vartheta_n$), thus X is a n^{th}-S-lindelöf space.

Remark.2 A compact subset of n^{th}-T_2-space is n^{th}-closed but a n^{th}-lindelöf subset of n^{th}-T_2-space need not be n^{th}-closed.

Definition 3.6 A space $(Q, \tau_1, \tau_2, ..., \tau_n)$ is called n^{th}-topological space if and only if the countable intersection of n^{th}-open sets is open.

Theorem 3.2 If L is a lindelöf subset of n^{th}-T_2 space Q, then for each m $\notin$ L, we can separated m and L in two disjoint n^{th}-open sets in Q.

Definition 3.7 [5] Suppose (Q, τ) be a topological space, thus we can say that Y$\subset Q$ is nearly open set if Y = $(\overline{Y})^{\circ}$ in τ.

Definition 3.8 suppose $(Q, \tau_1, \tau_2, ..., \tau_n)$ be a n^{th}-topological space, then Y $\subset Q$ is called nearly n^{th}-open set if $(\overline{Y})^{\circ}$ in τ_1 and $(\overline{Y})^{\circ}$ in τ_2 and $(\overline{Y})^{\circ}$ in τ_3.

Definition 3.9 A $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ is called nearly open cover of $(Q, \tau_1, \tau_2, \tau_3)$ if:
(i) U_α is tri-nearly open set for all $\alpha \in \Delta$.
(ii) $\bigcup\limits_{\alpha\in\Delta} U_\alpha$= Q.

Definition 3.10 Suppose $(Q, \tau_1, \tau_2, ..., \tau_n)$ be a n^{th}-topological space, and $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ is a nearly n^{th}-open cover of Q, then S_γ=$\{U_{\alpha\gamma}$:$\gamma \in \Gamma\}$ is called nearly n^{th}-subcover of Q if $\bigcup\limits_{\alpha\in\Delta} U_\alpha\gamma$=$Q$.

Definition 3.11 [7] Suppose (Q, τ) be a topological space, then we say that nearly compact space if every nearly open cover of X has a finite nearly subcover.

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Definition 3.12 Suppose $(Q, \tau_1, \tau_2, ..., \tau_n)$ be a n^{th}-topological space, then we say that n^{th}-nearly compact space if every nearly n^{th}-open cover of Q has a finite nearly n^{th}-subcover.

Definition 3.13 [6] Suppose (Q, τ) be a topological space, then Q is called nearly lindelöf space if every nearly open cover of Q has a countably nearly subcover.

Definition 3.14 Suppose $(Q, \tau_1, \tau_2, ..., \tau_n)$ be a n^{th}-topological space, then X is called nearly n^{th}-lindelöf space if every nearly n^{th}-open cover has a countably nearly n^{th}-subcover.

Remark.3 Every nearly n^{th}-compact space is nearly n^{th}-lindelöf space, but the converse need not be true.

Proof: Let Q be a nearly n^{th}-compact space, then every nearly n^{th}-open cover of Q has a finite n^{th}-subcover of Q, thus for each nearly n^{th}-open cover on Q has countable n^{th}-subcover of Q, for all i≠j, i, j=1,2,...,n so Q is nearly n^{th}-lindelöf space. But the converse need not be true, for example a space $(\mathbb{R}, \tau_{u_1}, \tau_{u_2}, ..., \tau_{u_n})$ is nearly n^{th}-lindelöf space but it is not nearly n^{th}-compact space.

Theorem 3.3 A nearly n^{th}-lindelöf space preserved under onto continuous function.

Proof: Let i≠j i, j=1,2,...,n. Let $F : (Q, \tau_1, \tau_2, ..., \tau_n) \to (Y, \vartheta_1, \vartheta_2, ..., \vartheta_m)$ be a surjective continuous function, and q is nearly n^{th}-lindelöf space, we want to show that Y is nearly n^{th}-lindelöf space.
Assume $\tilde{U} = \{U_\alpha : \alpha \in \vartheta\}$ be a nearly n^{th}-open cover of Y ,then U_α is n^{th}-open set for all $\alpha \in \vartheta$. And since f is continuous then $f^{-1}(U_\alpha)$ is n^{th}-open in Q for each $\alpha \in \vartheta\}$. And since f is surjective, then $f^{-1}(U_\alpha)$ =$\{f^{-1}(U_\alpha)$:$\alpha \in \vartheta$ $\}$ is an n^{th}-open cover of Q but Q is nearly n^{th}-lindelöf space, thus we can reduces $f^{-1}(U_\alpha)$ to a countable n^{th}-subcover say $\{f^{-1}(U_\alpha)$:$\alpha \in \vartheta$ $\}$ where $\vartheta \subset \vartheta$ and $|\vartheta| \leq \alpha_0$=$|X|$. Hence $Q \subseteq \bigcup\limits_{\alpha \in \vartheta} f^{-1}(U_\alpha)$ and since f is onto, then Y=$f(o) \subseteq F(\bigcup\limits_{\alpha \in \vartheta} f^{-1}(U_\alpha)) \subseteq \bigcup\limits_{\alpha \in \vartheta} U_\alpha$, so $\tilde{U}$ has countable n^{th}-subcover of Y. So Y is nearly n^{th}-lindelöf space.

Remark.4 We have a compact subset in nearly n^{th}-T_2-space is closed, but a nearly n^{th}-lindelöf space subset in nearly n^{th}-T_2-space need not be closed.

Definition 3.15 A space $(Q, \tau_1, \tau_2, ..., \tau_n$ is called nearly n^{th}-topological space if and only if the countable intersection of nearly n^{th}-open sets is n^{th}-open.

5 Nth-METALINDELÖFNESS SPACES IN Nth-TOPOLOGICAL SPACE

In this section, we explore the concept of locally n^{th}-metalindelöfness in n^{th}-topological spaces and highlight various properties of these spaces.

Definition 4.1 A n^{th}-topological spaces $(Q, \tau_1, \tau_2, ..., \tau_n)$ is called n^{th}-metalindelöf space, if every n^{th}-open cover of the space $(Q, \tau_1, \tau_2, ..., \tau_n)$ has point a countable parallel refinement.

Theorem 4.1 A countable n^{th}-metalindelöf space is n^{th}-compact space.

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Theorem 4.2 A separable n^{th}-metalindelöf space $(Q, \tau_1, \tau_2, ..., \tau_n)$ is n^{th}-Lindelöf.

Definition 4.2 A n^{th}-topological space $(Q, \tau_1, \tau_2, ..., \tau_n)$ is called countably n^{th}-metalindelöf, if every countable n^{th}-open cover of the space $(Q, \tau_1, \tau_2, ..., \tau_n)$ has a point countable parallel refinement.

Example.1 The n^{th}-topological space $(Q, \tau_1, \tau_2, ..., \tau_n)$ is n^{th}-metalindelöf space. It is also countably n^{th}-metacompact.

Theorem 4.3 Every n^{th}-Lindelöf countably n^{th}-metacompact space $(Q, \tau_1, \tau_2, ..., \tau_n)$ is n^{th}-metalindelöf space.

Proof: Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be an n^{th}-open cover of Q. Since Q is n^{th}-Lindelöf, then $\tilde{U}$ has a countable n^{th}-subcover, say $\tilde{A} = \{A_{\alpha i}\}_{i=1}^{\infty}$. Since Q is countably n^{th}-metacompact. Then $\tilde{A}$ has a point countable parallel refinement Ğ of $\tilde{U}$. Hence $(Q, \tau_1, \tau_2, ..., \tau_n)$ is n^{th}-metalindelöf.

Theorem 4.4 Every n^{th}-metalindelöf countably n^{th}-metacompact space $(Q, \tau_1, \tau_2, ..., \tau_n)$ is n^{th}-metacompact space.

Example.2 The n^{th}-topological space $(\mathbb{R}, \tau_{dis_1}, \tau_{dis_2}, ..., \tau_{dis_n})$ is n^{th}-metalindelöf, since $\tau_{dis(i)}$ is n^{th}-open cover $V = \{\{x\} : x \in \mathbb{R}\}$ of $\mathbb{R}$, it is also countably n^{th}-metacompact, where i=1,2,...,n. It is clear that $(\mathbb{R}, \tau_{dis_1}, \tau_{dis_2}, \tau_{dis_n})$ is n^{th}-metaLindelöf space.

Theorem 4.5 Every countable n^{th}-metacompact topological space $(Q, \tau_1, \tau_2, ..., \tau_n)$ is n^{th}-compact.

Theorem 4.6 The product of a n^{th}-compact space Q and a n^{th}-metalindelö space Y is n^{th}-metalindelöf, where $(Q, \tau_1, \tau_2, ..., \tau_n)$ and $(Y, \vartheta_1, vartheta_2, ..., \vartheta_n)$ are n^{th}-topological spaces.

Proof: Let $f : Q \times Y \to Y$ be the n^{th}-projection function, such that (q, s). Then $f : Q \times Y \to Y$ is n^{th}-perfect function. Since Y is n^{th}-metalindelöf, then $Q \times Y$ is n^{th}-metalindelöf.

Theorem 4.7 Let $f : (Q, \tau_1, \tau_2, ..., \tau_n) \to (Y, \vartheta_1, \vartheta_2, ..., \vartheta_n)$ be a continuous, n^{th}-closed and onto function. Then Y is n^{th}-metalindelöf if Q is so.

Proof: Let $\tilde{A} = \{U_\alpha : \alpha \in \Delta\} \cup \{V_\beta : \beta \in \Gamma\}$ be any n^{th}-open cover of Y, where $\{U_\alpha : \alpha \in \Delta\}$ consists of n^{th}-ϑ-open members of $\tilde{A}$. Since f is a continuous and onto function, the set $\tilde{U} = \{f^{-1}(U_\alpha) : \alpha \in \Delta\} \cup \{f^{-1}(V_\beta) : \beta \in \Gamma\}$ is a n^{th}-open cover of Q.

Given that Q is a n^{th}-metalindelöf space, there exists a point countable n^{th}-open parallel refinement of $\tilde{U}$, denoted by $\tilde{U}^* = \{f^{-1}(U^*_\alpha) : \alpha \in \Delta\} \cup \{f^{-1}(V^*_\beta) : \beta \in \Gamma\}$. Thus, $\tilde{A}^* = \{U^*_\alpha : \alpha \in \Delta\} \cup \{V^*_\beta : \beta \in \Gamma\}$ is a point countable n^{th}-open parallel refinement of $\tilde{A}$. Therefore, Y is n^{th}-metalindelöf.

Lemma.1 Let f:$(Q, \tau_1, \tau_2, \tau_3) \to (Y, \vartheta_1, \vartheta_2, \vartheta_3)$ be a continuous and onto function. If $\tilde{A} = \{A_\alpha : \alpha \in \Delta\}$ is a point countable family of subsets of Q, then $\{f(A_\alpha) : \alpha \in \Delta\}$ is a point countable family of subsets of Y.

Definition 4.3 A space $(Q, \tau_1, \tau_2, ..., \tau_n)$ is said to be n^{th}-metalindelöf space, if every n^{th}-open cover of Q has an n^{th}-open locally countable refinement.

Definition 4.4 A subset ψ of a space $(Q, \tau_1, \tau_2, ..., \tau_n)$ is said to be n^{th}-paralindelöf relative to Q, if every n^{th}-open cover of ψ by members of τ has a n^{th}-locally countable parallel refinement in X by members of τ.

Corollary.1 Every n^{th}-paralindelöf spaces are n^{th}-metalindelöf.

Theorem 4.8 Every n^{th}-closed subspace of a n^{th}-metalindelöf space $(Q, \tau_1, \tau_2, ..., \tau_n)$ is n^{th}-metalindelöf.

Theorem 4.9 Every n^{th}-metalindelöf subset of a n^{th}-T_2 locally indiscrete space $(Q, \tau_1, \tau_2, ..., \tau_n)$ is n^{th}-closed.

Proof: Let f: $(Q, \tau_1, \tau_2, ..., \tau_n) \to (Y, \vartheta_1, \vartheta_2, ..., \vartheta_n)$ be a bijective and continuous map. If $(Y, \vartheta_1, \vartheta_2, ..., \vartheta_n)$ is a n^{th}-T_2 and n^{th}-locally indiscrete space and $(Q, \tau_1, \tau_2, ..., \tau_n)$ is n^{th}-metalindelöf, then f is a n^{th}-homomorphism. It is sufficient to show that f is n^{th}-closed.

Let A be a n^{th}-closed proper subset of Q. Since A is a n^{th}-metalindelöf subset of X and $f : (Q, \tau_1, \tau_2, ..., \tau_n) \to (Y, \vartheta_1, \vartheta_2, ..., \vartheta_n)$ is continuous, we have that $f(A)$ is a n^{th}-metalindelöf subset of Y. Thus, $f(A)$ is a n^{th}-closed subset of Y. Therefore, $f : (Q, \tau_1, \tau_2, ..., \tau_n) \to (Y, \vartheta_1, \vartheta_2, ..., \vartheta_n)$ is a n^{th}-closed function. Hence the result.

Corollary.2 Let $f : (Q, \tau_1, \tau_2, ..., \tau_n) \to (Y, \vartheta_1, \vartheta_2, ..., \vartheta_n)$ be a n^{th}-bijection continuous map, if $(Y, \vartheta_1, \vartheta_2, ..., \vartheta_n)$ is n^{th}-T_2 and n^{th}-locally indiscrete space and $(Q, \tau_1, \tau_2, ..., \tau_n)$ is n^{th}-paralindelöf, then f is n^{th}-homomorphism.

6 Conclusion

This paper defines the concepts of locally compact and locally metacompact spaces in both topological and n^{th}-topological spaces. Various properties of these spaces and their connections to other topologies are explored. The conclusions drawn may open the door to developing new theorems related to the finite product and mappings of n^{th}-expandable spaces, feebly pairwise expandable spaces, and fuzzy n^{th}-topological spaces, which will be investigated in future research.

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