# Estimation of Population Mean using Neutrosophic Exponential Estimators with Application to Real Data

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**Abstract**

One of the traditional problems in survey sampling is to estimate the population parameter like mean variance etc. This article investigates the mathematical derivations and application of neutrosophic statistics to address the challenges posed by imprecise, indeterminacies or ambiguous data, such as daily stock prices, weather forecast, social media sentiment and temperatures. The suggested estimators are highly useful for computing results while working with unclear, hazy, and neutrosophic-type data. These estimators produce answers that are interval-form rather than single-valued, which may give our population parameter a better chance of being off. We propose three novel neutrosophic exponential ratio-type estimators for the population mean, utilizing information of neutrosophic auxiliary variables. Expressions for bias and mean square error (MSE) of these estimators are derived using first-order approximations to assess their performance in terms of accuracy. To demonstrate their effectiveness, we apply the proposed estimators to real-life neutrosophic data sets. Additionally, a simulation study shows that our estimators outperform existing methods in terms of MSEs and percentage relative efficiency (PREs). This study further expands its originality by including pre-existing estimators into the neutrosophic framework, showcasing its versatility and adaptability. The results suggest that neutrosophic statistics provide a robust framework for analyzing uncertain data, facilitating more reliable decision-making in various applications.

**Keywords:** Auxiliary information; Bias; Estimation; Mean square error(MSE); Neutrosophic estimators; Percentage relative efficiency(PRE); Simple Random Sampling; Simulation

## 1 Introduction

In sampling theory, the primary goal of the researchers is to create estimation techniques that estimate the population parameters resulting in achieving greater efficiency with the minimal sampling errors[27] . The efficiency of the estimators can be enhanced by improving sampling techniques, increasing the sampling size, or utilizing subsidiary information available[10] . Over the past decades, it has been observed that the use of auxiliary information has enhanced the efficiency of estimation techniques.

Auxiliary information, or subsidiary information, is a known variable that is approximately related to the variable of interest (study variable). For instance, if it is known that a forest has N trees and we want to determine the volume of the trees, the diameter of the trees can be used as an auxiliary variable. Cochran[9] was the first to utilize auxiliary information to enhance the precision of ratio method of estimation. Ratio

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estimator is employed when there is a positive correlation between the study variable and the auxiliary variable. Murthy[14] proposed the product method of estimation, which serves as a complement to the ratio estimator. One can utilize the product method of estimation when there is negative correlation between the study variable and the auxiliary variable. Regression methods of estimation are applicable in both the scenarios. Bahl[8] proposed exponential method of estimation by incorporating an exponential function of a subsidiary variable. Using information on auxiliary attributes, Sharma[18] proposed some exponential ratio-product type estimators under second order approximation. Singh[21,23,35] and Singh[34] proposed estimators using auxiliary information under various scenarios.

In classical statistics, the data we analyze is typically crisp, deterministic, and results in a point value. When the data are unclear, ambiguous, or indeterminate, or when they are in the form of intervals, classical statistics may not be applicable. For instance, observing daily stock prices or daily temperatures of a city fall into this category. In such cases, we turn to fuzzy statistics, which is often more reliable than classical statistics. The concept of fuzzy statistics was introduced by Zadeh[33] . One disadvantage of fuzzy statistics is that it does not quantify the measure of indeterminacy. To address the challenge of indeterminacy in data Smarandache[24] , Smarandache[25] , Khan[12] , Rani,[16] Smarandache[26] worked in this field. Neutrosophic statistics serves as an extension of both classical statistics and fuzzy statistics, specifically applied when dealing with indeterminate data Vishwakarma[29] .

Recently, researchers have developed some neutrosophic estimators. Aslam[3] designed a new sampling plan for the process loss function using the neutrosophic approach. Aslam[4] introduced neutrosophic analysis of variance (NANOVA) for testing of mean. Aslam[5] designed a control chart for neutrosophic exponentially weighted moving average (NEWMA) using repetitive sampling and it can be applied effectively to control RTC(Road Traffic Crashes). Aslam[6] have analyzed solar energy data, which is in the form of interval by proposing a new Anderson-Darling-Test under neutrosophic statistics. Tahir[28] proposed some neutrosophic ratio-type estimators for estimating the population mean using auxiliary variable. Vishwakarma[29] proposed a neutrosophic ranked set sampling (NeRSS) method and generalized estimator for estimating the population means. Kumar[13] proposed neutrosophic exponential-type estimator for estimating the population mean in the presence of uncertainty by using neutrosophic study and auxiliary variable. Alomair[2] proposed neutrosophic Hartley-Ross-type ratio estimators for estimating the population mean of neutrosophic data, even in the presence of outliers and when study variable is both sensitive and non-sensitive. Singh[19] proposed a neutrosophic regression cum ratio type estimators for the estimation of population mean under uncertain environment along with application in medical science.For elevated estimation of population mean Yadav[30] developed a generalized neutrosophic sampling strategy. Yadav[32] developed factor type exponential neutrosophic estimator in two phase survey sampling. Yadav[31] proposed novel neutrosophic factor type exponential estimators that use auxiliary information to improve population mean precision. Saha[17] and Singh[20] proposed estimators under neutrosophic framework.

In this article, we propose three neutrosophic estimators. The first two estimators combine regression and exponential methods to form a modified difference-cum-exponential neutrosophic estimator. Third estimator is a modified ratio cum exponential estimator. We compared our proposed estimators with the existing ones. First, we discussed the properties of our proposed estimators in section 4. Then, we compared the efficiency of the suggested estimators with the existing neutrosophic simple mean estimator, neutrosophic ratio estimator, neutrosophic modified ratio estimator, neutrosophic exponential estimator, neutrosophic improved exponential ratio estimator, neutrosophic generalized exponential ratio estimator using two real data sets and simulation in section 5. Finally, we discovered with the simulation study and the empirical study that the difference-cum-exponential type neutrosophic estimators and modified ratio cum exponential estimator outperform the existing neutrosophic exponential estimators.

## 2 Notation

Let us consider a finite population $T_1, T_2, ..., T_N$ of size N, from which a neutrosophic random sample of size $n_N \in [n_L, n_U]$ is drawn by the neutrosophic SRS method. Let $y_N(i)$ is the $i_{th}$ sample observation of neutrosophic data,which is of the form $y_N(i) \in [y_L, y_U]$ and similarly for subsidiary variable $x_N(i)$.Let $\bar{Y}_N(i) \in$

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$[\bar{Y}_L, \bar{Y}_U]$ and $\bar{X}_N(i) \in [\bar{X}_L, \bar{X}_U]$ be the population mean of the study variable and subsidiary variable, respectively. Also, let $C_{yN} \in [C_{yL}, C_{yU}]$ and $\rho_{xyN} \in [\rho_{xyL}, \rho_{xyU}]$ are the neutosophic coefficient of variation and neutrosophic correlation between $Y_N$ and $X_N$. Moreover, the parameter $\beta_{2(x)N} \in [\beta_{2(x)L}, \beta_{2(x)U}]$ is the neutrosophic subsidiary variable $X_N$. Let $\bar{e}_{yN} \in [\bar{e}_{yL}, \bar{e}_{yU}]$ and $\bar{e}_{xN} \in [\bar{e}_{xL}, \bar{e}_{xU}]$ are the neutrosophic mean error, where,

$$\bar{e}_{yN} = \bar{y}_{N(i)} - \bar{Y}_N \tag{1}$$
$$\bar{e}_{xN} = \bar{x}_{N(i)} - \bar{X}_N \tag{2}$$

such that

$$E(\bar{e}_{yN}) = E(\bar{e}_{xN}) = 0 \tag{3}$$
$$E(\bar{e}^2_{yN}) = \theta_N \bar{Y}^2_N C^2_{yN} \tag{4}$$
$$E(\bar{e}^2_{xN}) = \theta_N \bar{X}^2_N C^2_{xN} \tag{5}$$
$$E(\bar{e}_{yN}\bar{e}_{xN}) = \theta_N \bar{X}_N \bar{Y}_N C_{yN} C_{xN} \rho_{xyN} \tag{6}$$

where,
$\theta_N = (\frac{1}{n_N} - \frac{1}{N_N})$ , $\theta_N \in [\theta_L, \theta_U]$, $\bar{e}_{yN} \in [\bar{e}_{yL}, \bar{e}_{yU}]$, $\bar{e}_{yN}\bar{e}_{xN} \in [\bar{e}_{yL}\bar{e}_{xL}, \bar{e}_{yU}\bar{e}_{xU}]$, $\bar{e}^2_{yN} \in [\bar{e}^2_{yL}, \bar{e}^2_{yU}]$, $\bar{e}^2_{xN} \in [\bar{e}^2_{xL}, \bar{e}^2_{xU}]$, $C^2_{yN} = \frac{\sigma^2_{yN}}{\bar{Y}^2_N}$, $C^2_{yN} \in [C^2_{yL}, C^2_{yU}]$, $\sigma^2_{yN} \in [\sigma^2_{yL}, \sigma^2_{yU}]$, $C^2_{xN} = \frac{\sigma^2_{xN}}{\bar{X}^2_N}$, $C^2_{xN} \in [C^2_{xL}, C^2_{xU}]$, $\sigma^2_{xN} \in [\sigma^2_{xL}, \sigma^2_{xU}]$, $\rho_{xyN} = \frac{\sigma_{xyN}}{\sigma_{xN}\sigma_{yN}}$, $\rho_{xyN} \in [\rho_{xyL}, \rho_{yxU}]$, $\sigma_{xyN} \in [\sigma_{xyL}, \sigma_{xyU}]$.

To evaluate the performance of our proposed estimators, we utilized following existing neutrosophic estimators:

## 3 Existing Estimators

### 3.1 Neutrosophic simple mean estimator

The Neutrosophic simple mean is $\bar{y}_{nN}$, where $\bar{y}_{nN} \in (\bar{y}_{nL}, \bar{y}_{nU})$ and variance of neutrosophic simple mean estimator is given as

$$V(\bar{y}_N) = f_N \bar{Y}^2_N C^2_{yN} \tag{7}$$

### 3.2 Neutrosophic ratio estimator

In the presence of subsidiary information and when the data have indeterminacy Tahir[28] introduced a Neutrosophic ratio estimator for estimating the mean of a finite population, which is defined as follows:

$$\bar{y}_{rN} = \frac{\bar{y}_N}{\bar{x}_N}\bar{X}_N \tag{8}$$

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where $\bar{y}_{rN} \in [\bar{y}_{rL}, \bar{y}_{rU}]$.
The bias and MSE of $\bar{y}_{rN}$ up to first order of approximation are given by

$$Bias(\bar{y}_{rN}) = \theta_N \bar{Y}_N [C_{xN}^2 - C_{xN} C_{yN} \rho_{xyN}] \tag{9}$$

$$MSE(\bar{y}_{rN}) = \theta_N \bar{Y}_N^2 [C_{yN}^2 + C_{xN}^2 - 2C_{xN} C_{yN} \rho_{xyN}] \tag{10}$$

where,
$Bias(\bar{y}_{rN}) \in [Bias(\bar{y}_{rL}), Bias(\bar{y}_{rU})]$; $MSE(\bar{y}_{rN}) \in [MSE(\bar{y}_{rL}), MSE(\bar{y}_{rU})]$.

### 3.3 Neutrosophic modified ratio type estimator

Using coefficient of variation as a subsidiary variable Tahir[28] proposed a neutrosophic modified ratio type estimator, given as

$$\bar{y}_{MrN} = \bar{y}_N \Big[\frac{\bar{X}_N + C_{xN}}{\bar{x}_N + C_{xN}}\Big] \tag{11}$$

Bias and MSE of $\bar{y}_{MrN}$ up to the first order of approximation are given as

$$Bias(\bar{y}_{MrN}) = \theta_N \bar{Y}_N \left[\left(\frac{\bar{X}_N}{\bar{X}_N + C_{xN}}\right)^2 C_{xN}^2 - \left(\frac{\bar{X}_N}{\bar{X}_N + C_{xN}}\right) C_{xN} C_{yN} \rho_{xyN}\right] \tag{12}$$

$$MSE(\bar{y}_{MrN}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \left(\frac{\bar{X}_N}{\bar{X}_N + C_{xN}}\right)^2 C_{xN}^2 - 2\left(\frac{\bar{X}_N}{\bar{X}_N + C_{xN}}\right) C_{xN} C_{yN} \rho_{xyN}\right] \tag{13}$$

where,
$\bar{y}_{MrN} \in [\bar{y}_{MrL}, \bar{y}_{MrL}]; Bias(\bar{y}_{MrN}) \in [Bias(\bar{y}_{MrL}), Bias(\bar{y}_{MrU})]$ and $MSE(\bar{y}_{MrN}) \in [MSE(\bar{y}_{MrL}), MSE(\bar{y}_{MrU})]$

### 3.4 Neutrosophic exponential type estimators

A neutrosophic exponential type estimator for estimating population mean suggested by Tahir[28] is as follows:

$$\bar{y}_{expN} = \bar{y}_N exp\left(\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N}\right) \tag{14}$$

The bias and MSE of $\bar{y}_{expN}$ up to first-order of approximation are given by

$$Bias(\bar{y}_{expN}) = \theta_N \bar{Y}_N \left[\frac{3}{8} C_{xN}^2 - \frac{1}{2} C_{xN} C_{yN} \rho_{xyN}\right] \tag{15}$$

$$MSE(\bar{y}_{expN}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \frac{1}{4} C_{xN}^2 - C_{xN} C_{yN} \rho_{xyN}\right] \tag{16}$$

where,
$\bar{y}_{expN} \in [\bar{y}_{expL}, \bar{y}_{expU}], Bias(\bar{y}_{expN}) \in [Bias(\bar{y}_{expL}), Bias(\bar{y}_{expU})]; MSE(\bar{y}_{expN}) \in [MSE(\bar{y}_{expL}), MSE(\bar{y}_{expU})]$

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### 3.5 Neutrosophic improved exponential ratio type estimator

A Neutrosophic improved exponential ratio-type estimator developed by Tahir[28] , given as follows:

$$\bar{y}_{IexpN} = \bar{y}_N exp\left[\frac{(l\bar{X}_N + m) - (l\bar{x}_N + m)}{(l\bar{X}_N + m) + (l\bar{x}_N + m)}\right] \tag{17}$$

where $\bar{y}_{IexpN} \in [\bar{y}_{Iexp_L}, \bar{y}_{Iexp_U}]$ and $l$ & $m(-\infty < l, m < \infty)$ are two real constant and supposed to be estimated.

The bias and MSE of $\bar{y}_{IexpN}$ up to first-order of approximation are given by

$$Bias(\bar{y}_{IexpN}) = \theta_N \bar{Y}_N \left[\frac{3}{8}\left(\frac{\bar{X}_N}{l\bar{X}_N + m}\right)^2 C_{xN}^2 - \frac{1}{2}\left(\frac{\bar{X}_N}{l\bar{X}_N + m}\right) C_{xN} C_{yN} \rho_{xyN}\right] \tag{18}$$

$$MSE(\bar{y}_{IexpN}) = \theta_N \bar{Y}_N^2 \left[C_N^2 + \frac{3}{8}\left(\frac{l\bar{X}_N}{2(l\bar{X}_N + m)}\right)^2 C_{xN}^2 - \frac{1}{2}\left(\frac{2l\bar{X}_N}{2(l\bar{X}_N + m)}\right) C_{xN} C_{yN} \rho_{xyN}\right] \tag{19}$$

where,
$Bias(\bar{y}_{IexpN}) \in [Bias(\bar{y}_{IexpL}), Bias(\bar{y}_{IexpU})]; MSE(\bar{y}_{IexpN}) \in [MSE(\bar{y}_{IexpL}), MSE(\bar{y}_{IexpU})]$

### 3.6 Neutrosophic generalized exponential ratio type estimator

a Neutrosophic generalized exponential ratio-type estimator developed by Tahir[28] , given as follows:

$$\bar{y}_{GexpN} = \bar{y}_N exp\left[\eta\left(\frac{\bar{X}_N^{1/g} - \bar{x}_N^{1/g}}{\bar{X}_N^{1/g} + (a-1)\bar{x}_N^{1/g}}\right)\right] \tag{20}$$

where, $\bar{y}_{GexpN} \in [\bar{y}_{GexpL}, \bar{y}_{GexpU}]$, $\eta(-\infty < \eta < \infty)$ and $g(g > 0)$are two real constants and assumed to be known,and the other constant $a(a > 0)$ is supposed to be estimated.

The bias and MSE of $\bar{y}_{GexpN}$ up to first-order of approximation are given by

$$Bias(\bar{y}_{GexpN}) = \theta_N \bar{Y}_N \left[\frac{\eta C_{xN}^2}{ag^2} - \frac{\eta C_{xN}^2}{a^2 g^2} + \frac{\eta^2 C_{xN}^2}{2a^2 g^2} - \frac{\eta C_{xN} C_{yN} \rho_{xy_N}}{ag}\right] \tag{21}$$

$$MSE(\bar{y}_{GexpN}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \frac{\eta^2 C_{xN}^2}{a^2 g^2} - \frac{2\eta C_{xN} C_{yN} \rho_{xy_N}}{ag}\right] \tag{22}$$

where, $Bias(\bar{y}_{GexpN}) \in [Bias(\bar{y}_{GexpL}), Bias(\bar{y}_{GexpU})]$ and $MSE(\bar{y}_{GexpN}) \in [MSE(\bar{y}_{GexpL}), MSE(\bar{y}_{GexpU})]$

Minimum $MSE(\bar{y}_{GexpN})$ expression, using optimum value of 'a' obtained after differentiating eq. (22) with respect to 'a' and setting it to zero is given as:

$$MSE(\bar{y}_{GexpN})_{min} = \theta_N \bar{Y}_N^2 C_{yN}^2 (1 - \rho_{xyN}^2) \tag{23}$$

$MSE(\bar{y}_{GexpN})_{min}$ is obatined for the optimum value of $\hat{a} = \frac{\eta C_{xN}^2}{g C_{xN} \rho_{xyN}}$

Minimum MSE of estimator $\bar{y}_{GexpN}$ is equal to the MSE of the linear regression estimator.

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## 4 Neutrosophic Proposed Estimators

In this article, we have proposed three Neutrosophic estimators. Among these, first two estimators are the combination of regression and exponential methods of estimation and the third one is modified exponential ratio estimator. The proposed Neutrosophic estimators are as follows:

### 4.1 Proposed Estimator- I

Inspired by the work of Grover[11] , we have developed a Neutrosophic difference-cum-exponential estimator under conditions of indeterminacy, formulated as follows:

$$\bar{y}_{P1_N} = \left[\alpha_1 \bar{y}_N + \alpha_2(\bar{X}_N - \bar{x}_N)\right] exp\left[\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N}\right] \tag{24}$$

where $\alpha_1$ and $\alpha_2(-\infty < \alpha_1, \alpha_2 < \infty)$ are two real constant and supposed to be estimated and $\bar{y}_{P1_N} \in [\bar{y}_{P1_L}, \bar{y}_{P1_U}]$
Using approximations given in equation (1) and (2), equation (24) is written as:

$$\bar{y}_{P1_N} = [\alpha_1(\bar{Y}_N + \bar{e}_{yN}) + \alpha_2\{\bar{X}_N - (\bar{X}_N + \bar{e}_{xN})\}]exp\left[\frac{\bar{X}_N - (\bar{X}_N + \bar{e}_{xN})}{\bar{X}_N + (\bar{X}_N + \bar{e}_{xN})}\right]$$

$$\bar{y}_{P1_N} = \left[\alpha_1 \bar{Y}_N + \alpha_1 \bar{e}_{yN} - \alpha_2 \bar{e}_{xN}\right] exp\left[-\frac{\bar{e}_{xN}}{2\bar{X}_N}\left(1 + \frac{\bar{e}_{xN}}{2\bar{X}_N}\right)^{-1}\right]$$

or

$$\bar{y}_{P1_N} = \alpha_1 \bar{Y}_N + \alpha_1 \bar{e}_{yN} - \alpha_2 \bar{e}_{xN} - \alpha_1 \frac{\bar{Y}_N}{2\bar{X}_N}\bar{e}_{xN} - \theta_1 \frac{\bar{Y}_N}{2\bar{X}_N}\bar{e}_{yN}\bar{e}_{xN} + \frac{\alpha_2}{2\bar{X}_N}\bar{e}^2_{xN} + \frac{3}{8}\bar{e}^2_{xN}\alpha_1 \frac{\bar{Y}_N}{\bar{X}^2_N}$$

or

$$\bar{y}_{P1_N} - \bar{Y}_N = (\alpha_1 - 1)\bar{Y}_N + \alpha_1 \bar{e}_{yN} - \alpha_2 \bar{e}_{xN} - \alpha_1 \frac{\bar{Y}_N}{2\bar{X}_N}\bar{e}_{xN} - \alpha_1 \frac{\bar{Y}_N}{2\bar{X}_N}\bar{e}_{yN}\bar{e}_{xN} + \frac{\alpha_2}{2\bar{X}_N}\bar{e}^2_{xN} + \frac{3}{8}\bar{e}^2_{xN}\alpha_1 \frac{\bar{Y}_N}{\bar{X}^2_N} \tag{25}$$

Taking expectation on both side of the equation (25) we get bias of the proposed estimator $\bar{y}_{P1_N}$

$$Bias(\bar{y}_{P1_N}) = E(\bar{y}_{P1_N} - \bar{Y}_N) = (\alpha_1 - 1)\bar{Y}_N - \frac{\alpha_1}{2}\bar{Y}_N C_{xN} C_{yN} \rho_{xy_N} + \frac{1}{2}(\alpha_2 + \frac{3}{4}\alpha_1 \bar{Y}_N) C^2_{xN} \tag{26}$$

where,$Bias(\bar{y}_{P1_N}) \in [Bias(\bar{y}_{P1_L}), Bias(\bar{y}_{P1_U})]$
For MSE, squaring and taking expectation of equation (25), we have

$$MSE(\bar{y}_{P1_N}) = \bar{Y}^2_N + \alpha^2_1\left(\bar{Y}^2_N + \delta^2_y + \frac{\bar{Y}^2_N}{\bar{X}^2_N}\delta^2_x - 2\frac{\bar{Y}_N}{\bar{X}^2_N}\delta_{xy}\right) + \alpha^2_2\delta^2_x + 2\alpha_1\alpha_2\left(\frac{\bar{Y}_N}{\bar{X}_N}\delta^2_x - \delta_{xy}\right) \tag{27}$$

$$-2\alpha_1\left(\bar{Y}^2_N - \frac{\bar{Y}_N}{2\bar{X}_N}\delta xy + \frac{3}{8}\frac{\bar{Y}^2_N}{\bar{X}^2_N}\delta^2_x\right) - \alpha_2\frac{\bar{Y}_N}{2\bar{X}_N}\delta^2_x \tag{28}$$

where,
$\delta_x = \theta_N \bar{X}^2_N C^2_{xN}$
$\delta_y = \theta_N \bar{Y}^2_N C^2_{yN}$

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$\delta_{xy} = \theta_N \bar{X}_N \bar{Y}_N C_{xN} C_{yN} \rho xy_N$

$$MSE(\bar{y}_{P1_N}) = \bar{Y}_N^2 + \alpha_1^2 A_1 + \alpha_2^2 B_1 + 2\alpha_1\alpha_2 C_1 - 2\alpha_1 D_1 - 2\alpha_2 E_1 \tag{29}$$

where,
$A_1 = \bar{Y}_N^2 + \delta_y^2 + \frac{\bar{Y}_N^2}{\bar{X}_N^2}\delta_x^2 - 2\frac{\bar{Y}_N}{\bar{X}_N^2}\delta_{xy}$
$B_1 = \delta_x^2$
$C_1 = \frac{\bar{Y}_N}{\bar{X}_N}\delta_x^2 - \delta_{xy}$
$D_1 = \bar{Y}_N^2 - \frac{\bar{Y}_N}{2\bar{X}_N}\delta xy + \frac{3}{8}\frac{\bar{Y}_N^2}{\bar{X}_N^2}\delta_x^2$
$E_1 = \frac{\bar{Y}_N}{2\bar{X}_N}\delta_x^2$

Diffrentiating equation (29) with respect to $\alpha_1$ and $\alpha_2$ and equating it to zero, we get

$$\alpha_1 A_1 + \alpha_2 C_1 = D_1 \tag{30}$$

$$\alpha_2 B_1 + \alpha_1 C_1 = E_1 \tag{31}$$

Solving equation (30)and (31), we get the value of $\alpha_1$ and $\alpha_2$,

$$\alpha_1 = \frac{C_1E_1 - B_1D_1}{C_1^2 - A_1B_1} \tag{32}$$

$$\alpha_2 = \frac{C_1D_1 - A_1E_1}{C_1^2 - A_1B_1} \tag{33}$$

Subsituting these value of $\alpha_1$ and $\alpha_2$ in equation (29) we get the minimum MSE of the estimator $\bar{y}_{P1_N}$,

$$MSE(\bar{y}_{P1_N})_{min} = \bar{Y}_N^2 + \frac{B_1D_1^2 - 2C_1D_1E_1 + A_1E_1^2}{C_1^2 - A_1B_1} \tag{34}$$

where,$MSE(\bar{y}_{P1_N})_{min} \in [MSE(\bar{y}_{P1_L})_{min}, MSE(\bar{y}_{P1_U})_{min}]$

**Particular case: when $\alpha_1$+$\alpha_2$=1**

Taking $\alpha_1$+$\alpha_2$=1 in equation(24),estimator $\bar{y}_{P1_N}$ can be written as

$$\bar{y}^{\star}_{P1_N} = \left[\alpha_1\bar{y}_N + (1-\alpha_1)(\bar{X}_N - \bar{x}_N)\right] exp\left[\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N}\right] \tag{35}$$

Putting $\alpha_2 = 1 - \alpha_1$ in equation (25) and (29),bias and MSE expression of the estimator $\bar{y}^{\star}_{P1_N}$ upto the first order of approximation is obatined as follows:

$$Bias(\bar{y}^{\star}_{P1_N}) = (\alpha_1 - 1)\bar{Y}_N - \frac{\alpha_1}{2}\bar{Y}_N C_{xN} C_{yN} \rho_{xy_N} + \frac{1}{2}\left\{(1-\alpha_1) + \frac{3}{4}\alpha_1\bar{Y}_N\right\} C_{xN}^2 \tag{36}$$

where,$Bias(\bar{y}^{\star}_{P1_N}) \in [Bias(\bar{y}^{\star}_{P1_L}), Bias(\bar{y}^{\star}_{P1_U})]$
and

$$MSE(\bar{y}^{\star}_{P1_N}) = \bar{Y}_N^2 + (B_1 - 2E_1) + \alpha_1^2(A_1 + B_1 - 2C_1) - 2\alpha_1(B_1 - C_1 + D_1 - E_1) \tag{37}$$

Diffrentiating equation(37) with respect to $\alpha_1$ and equating it to zero we get the optimum value of $\alpha_1$,as

$$\alpha_1 = \frac{(B_1 - C_1 + D_1 - E_1)}{(A_1 + B_1 - 2C_1)} = \alpha_{opt}(say) \tag{38}$$

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Then the resulting minimum MSE of estimator $\bar{y}^{\star}_{P1}$ is

$$MSE(\bar{y}_{P1_N})_{min} = \bar{Y}_N^2 + (B_1 - 2E_1) - \frac{(B_1 - C_1 + D_1 - E_1)^2}{(A_1 + B_1 - 2C_1)} \tag{39}$$

where, $MSE(\bar{y}^{\star}_{P1_N})_{min} \in [MSE(\bar{y}^{\star}_{P1_L})_{min}, MSE(\bar{y}^{\star}_{P1_U})_{min}]$

### 4.2 Proposed Estimator- II

We have proposed modified Neutrosophic difference cum exponential estimator under indeterminacy given as follows:

$$\bar{y}_{P2_N} = \left[k_1\bar{y}_{nN} + k_2(\bar{X}_N - \bar{x}_{nN}) + \frac{\bar{y}_N}{2}\left\{exp\left(\frac{\bar{X}_N - \bar{x}_{nN}}{\bar{X}_N + \bar{x}_{nN}}\right) + exp\left(\frac{\bar{x}_{nN} - \bar{X}_N}{\bar{X}_N + \bar{x}_{nN}}\right)\right\}\right] exp\left[\frac{\bar{X}_N - \bar{x}_{nN}}{\bar{X}_N + \bar{x}_{nN}}\right] \tag{40}$$

where, $k_1$ and $k_2(-\infty < k_1, k_2 < \infty)$ are two real constant and supposed to be estimated and $\bar{y}_{P2_N} \in [\bar{y}_{P2_L}, \bar{y}_{P2_U}]$.
Using approximation given in equation (1)and (2), equation (40) is written as:

$$\bar{y}_{P2_N} = \left[k_1(\bar{Y}_N + \bar{e}_{yN}) + k_2\left\{\bar{X}_N - (\bar{X}_N + \bar{e}_{xN})\right\} + \left(\frac{\bar{Y}_N + \bar{e}_{yN}}{2}\right)\left\{exp\left(\frac{\bar{X}_N - (\bar{X}_N + \bar{e}_{xN})}{\bar{X}_N + (\bar{X}_N + \bar{e}_{xN})}\right) + exp\left(\frac{(\bar{X}_N + \bar{e}_{xN}) - \bar{X}_N}{\bar{X}_N + (\bar{X}_N + \bar{e}_{xN})}\right)\right\}\right] exp\left[\frac{\bar{X}_N - (\bar{X}_N + \bar{e}_{xN})}{\bar{X}_N + (\bar{X}_N + \bar{e}_{xN})}\right]$$

or

$$\bar{y}_{P2_N} = \left[k_1(\bar{Y}_N + \bar{e}_{yN}) - k_2\bar{e}_{xN} + \frac{\bar{Y}_N + \bar{e}_{yN}}{2}\left\{2 + \frac{\bar{e}^2_{xN}}{4\bar{X}^2_N}\right\}\right]\left[1 - \frac{\bar{e_{xN}}}{2\bar{X}_N} + \frac{3}{8}\frac{\bar{e}^2_{xN}}{\bar{X}^2_N}\right]$$

or

$$\bar{y}_{P2_N} - \bar{Y}_N = K_1\bar{Y}_N + (k_1+1)\bar{e}_{yN} - (k_1+1)\,\bar{Y}_N\frac{\bar{e}_{xN}}{2\bar{X}_N} - k_2\bar{e}_{xN} - (k_1+1)\frac{1}{2\bar{X}_N}\bar{e}_{xN}\bar{e}_{yN} + \left(\frac{\bar{Y}_N}{2\bar{X}^2_N} + \frac{k_2}{2\bar{X}_N} + k_1\frac{3}{8}\frac{\bar{Y}_N}{\bar{X}^2_N}\right)\bar{e}^2_{xN} \tag{41}$$

Now taking expectation of the equation(41) on both sides we get the bias of the proposed estimator $\bar{y}_{P2_N}$

$$Bias(\bar{y}_{P2_N}) = K_1\bar{Y}_N - (k_1+1)\frac{1}{2\bar{X}_N}\delta_{xyN} + (\frac{\bar{Y}_N}{2\bar{X}^2_N} + \frac{k_2}{2\bar{X}_N} + k_1\frac{3}{8}\frac{\bar{Y}_N}{\bar{X}^2_N})\delta^2_{xN} \tag{42}$$

where,$Bias(\bar{y}_{P2_N}) \in [Bias(\bar{y}_{P2_L}), Bias(\bar{y}_{P2_U})]$
Squaring and taking expectation of equation(41), we get the MSE of the proposed estimator $\bar{y}_{P2_N}$

$$MSE(\bar{y}_{P2_N}) = k_1^2\left(\bar{Y}_N^2 + \delta_y^2 + \frac{\bar{Y}_N^2}{\bar{X}_N^2}\delta x^2 - \frac{2\bar{Y}_N}{\bar{X}_N}\delta_{xy}\right) + k_2^2\delta_x^2 + 2k_1\left(\delta_y^2 + \frac{3}{4}\frac{\bar{Y}_N^2}{\bar{X}_N^2}\delta x^2 - \frac{\bar{Y}_N}{\bar{X}_N}\delta_{xy}\right) - 2k_2\left(\delta_{xy} - \frac{\bar{Y}_N}{2\bar{X}_N}\delta x^2\right) - 2k_1k_2\delta_{xy} + \left(\delta_y^2 + \frac{\bar{Y}_N^2}{4\bar{X}_N^2}\delta x^2 - \frac{\bar{Y}_N}{\bar{X}_N}\delta_{xy}\right) \tag{43}$$

$$MSE(\bar{y}_{P2_N}) = k_1^2A_2 + k_2^2B_2 - 2k_1C_2 - 2k_2D_2 + 2k_1k_2E_2 + F_2 \tag{44}$$

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where,
$A_2 = \bar{Y}_N^2 + \delta_y^2 + \frac{\bar{Y}_N^2}{\bar{X}_N^2}\delta x^2 - \frac{2\bar{Y}_N}{\bar{X}_N}\delta_{yx}$
$B_2 = \delta_x^2$
$C_2 = -\left(\delta_y^2 + \frac{3}{4}\frac{\bar{Y}_N^2}{\bar{X}_N^2}\delta x^2 - \frac{\bar{Y}_N}{\bar{X}_N}\delta_{xy}\right)$
$D_2 = \delta_{xy} - \frac{\bar{Y}_N}{2\bar{X}_N}\delta x^2$
$E_2 = -\delta_{xy}$
$F_2 = \delta_y^2 + \frac{\bar{Y}_N^2}{4\bar{X}_N^2}\delta x^2 - \frac{\bar{Y}_N}{\bar{X}_N}\delta_{xy}$

Diffrentiating equation (44) with respect to $k_1$ and $k_2$ and equating it to zero, we get

$$k_1 A_2 + k_2 E_2 = C_2 \tag{45}$$

$$k_2 B_2 + k_1 E_2 = D_2 \tag{46}$$

Solving equation (45)and (46), we get the value of $k_1$ and $k_2$

$$k_1 = \frac{B_2 C_2 - E_2 D_2}{A_2 B_2 - E_2^2} \tag{47}$$

$$k_2 = \frac{A_2 D_2 - C_2 E_2}{A_2 B_2 - E_2^2} \tag{48}$$

Subsituting these value of $k_1$ and $k_2$ in equation (44) we get the minimum MSE of the estimator $\bar{y}_{P2_N}$,

$$MSE(\bar{y}_{P2_N})_{min} = \frac{2C_2 D_2 E_2 - A_2 D_2^2 - B_2 C_2^2}{A_2 B_2 - E_2^2} + F_2 \tag{49}$$

where,$MSE(\bar{y}_{P2_N})_{min} \in [MSE(\bar{y}_{P2_L})_{min}, MSE(\bar{y}_{P2_U})_{min}]$

### 4.3 Proposed Estimator- III

We have proposed modified Neutrosophic exponential estimator under indeterminacy given as follows:

$$\bar{y}_{P3_N} = [\gamma_1 \bar{y}_{nN} + \gamma_2]\, exp\left[\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N}\right] \tag{50}$$

where $\gamma_1$ and $\gamma_2(-\infty < \gamma_1, \gamma_2 < \infty)$ are two real constant and supposed to be estimated and $\bar{y}_{P3_N} \in [\bar{y}_{P3_L}, \bar{y}_{P3_U}]$.

Using approximations given in equation (1) and (2), equation (50) is framed as:

$\bar{y}_{P3_N} = \left[\gamma_1(\bar{Y}_N + \bar{e}_{yN}) + \gamma_2\right] exp\left[\frac{\bar{X}_N - (\bar{X}_N + \bar{e}_{xN})}{\bar{X}_N + (\bar{X}_N + \bar{e}_{xN})}\right]$
or

$\bar{y}_{P3_N} = \{\gamma_1 \bar{Y}_N + \gamma_1 \bar{e}_{yN} + \gamma_2\}[1 - \frac{\bar{e}_{xN}}{2\bar{X}_N} + \frac{3}{8}\frac{\bar{e}_{xN}^2}{\bar{X}_N^2}]$
or

$$\bar{y}_{P3_N} - \bar{Y}_N = \gamma_2 + (\gamma_1 - 1)\bar{Y}_N + \gamma_1 \bar{e}_{yN} - \gamma_2 \frac{\bar{e_x}N}{2\bar{X}_N} - \gamma_1 \bar{Y}_N \frac{\bar{e_x}N}{2\bar{X}_N} - \gamma_1 \frac{\bar{e_y}N\bar{e_y}N}{2\bar{X}_N} + (\gamma_2 + \gamma_1 \bar{Y}_N)\frac{3}{8}\frac{\bar{e}_{xN}^2}{\bar{X}_N^2} \tag{51}$$

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Taking expectation of equation (51) on both the sides we get bias of proposed estimator $\bar{y}_{P3_N}$

$$Bias(\bar{y}_{P3_N}) = E(\bar{y}_{P3_N} - \bar{Y}_N) = \gamma_2 + (\gamma_1 - 1)\bar{Y}_N - \gamma_1 \frac{\delta_{xy}}{2\bar{X}_N} + (\gamma_2 + \gamma_1 \bar{Y}_N)\frac{3}{8}\frac{\delta_x^2}{\bar{X}_N^2} \tag{52}$$

where,$Bias(\bar{y}_{P3_N}) \in [Bias(\bar{y}_{P3_L}), Bias(\bar{y}_{P3_U})]$
For MSE, squaring and taking expectation of the equation (51), we have

$$MSE(\bar{y}_{P3_N}) = \gamma_1^2\left(\bar{Y}_N^2 + \delta_y^2 + \frac{\bar{Y}_N^2}{\bar{X}_N^2}\delta x^2 - 2\frac{\bar{Y}_N}{\bar{X}_N}\delta_{xy}\right) + \gamma_2^2\left(1 + \frac{\delta_x^2}{\bar{X}_N^2}\right) + 2\gamma_1\gamma_2\left(\bar{Y}_N - \frac{\delta_{xy}}{\bar{X}_N} + \frac{\bar{Y}_N}{\bar{X}_N^2}\delta_x^2\right) - 2\gamma_1\left(\bar{Y}_N^2 + \frac{3}{8}\frac{\bar{Y}_N^2}{\bar{X}_N^2}\delta x^2 - \frac{\bar{Y}_N}{2\bar{X}_N}\delta_{xy}\right) - 2\gamma_2\left(\bar{Y}_N + \frac{3}{8}\frac{\bar{Y}_N}{\bar{X}_N^2}\delta x^2\right) + \bar{Y}_N^2 \tag{53}$$

where,
$\delta_x = \theta_N \bar{X}_N^2 C_{xN}^2$
$\delta_y = \theta_N \bar{Y}_N^2 C_{yN}^2$
$\delta_{xy} = \theta_N \bar{X}_N \bar{Y}_N C_{xN} C_{yN} \rho xyN$

$$MSE(\bar{y}_{P3_N}) = \bar{Y}_N^2 + \gamma_1^2 A_3 + \gamma_2^2 B_3 + 2\gamma_1\gamma_2 C_3 - 2\gamma_1 D_3 - 2\gamma_2 E_3 \tag{54}$$

where,
$A_3 = \bar{Y}_N^2 + \delta_y^2 + \frac{\bar{Y}_N^2}{\bar{X}_N^2}\delta x^2 - 2\frac{\bar{Y}_N}{\bar{X}_N}\delta_{xy}$
$B_3 = 1 + \frac{\delta_x^2}{\bar{X}_N^2}$
$C_3 = \bar{Y}_N - \frac{\delta_{xy}}{\bar{X}_N} + \frac{\bar{Y}_N}{\bar{X}_N^2}\delta_x^2$
$D_3 = \bar{Y}_N^2 + \frac{3}{8}\frac{\bar{Y}_N^2}{\bar{X}_N^2}\delta x^2 - \frac{\bar{Y}_N}{2\bar{X}_N}\delta_{xy}$
$E_3 = \bar{Y}_N + \frac{3}{8}\frac{\bar{Y}_N}{\bar{X}_N^2}\delta x^2$

Diffrentiating equation (54) with respect to $\gamma_1$ and $\gamma_2$ and equating it to zero, we get

$$\gamma_1 A_3 + \gamma_2 C_3 = D_3 \tag{55}$$

$$\gamma_2 B_3 + \gamma_1 C_3 = E_3 \tag{56}$$

Solving equation (55) and (56), we get the value of $\gamma_1$ and $\gamma_2$

$$\gamma_1 = \frac{B_3 D_3 - C_3 E_3}{A_3 B_3 - C_3^2} \tag{57}$$

$$\gamma_2 = \frac{A_3 E_3 - C_3 D_3}{A_3 B_3 - C_3^2} \tag{58}$$

Subsituting these value of $\gamma_1$ and $\gamma_2$ in equation (54) we get the minimum MSE of the estimator $\bar{y}_{P3_N}$,

$$MSE(\bar{y}_{P3_N})_{min} = \bar{Y}_N^2 + \frac{2C_3 D_3 E_3 - B_3 D_3^2 - A_3 E_3^2}{A_3 B_3 - C_3^2} \tag{59}$$

where,$MSE(\bar{y}_{P3_N})_{min} \in [MSE(\bar{y}_{P3_L})_{min}, MSE(\bar{y}_{P3_U})_{min}]$

**Particular case : when $\gamma_1$+$\gamma_2$=1**

Taking $\gamma_1$+$\gamma_2$=1 in equation (50), estimator $\bar{y}_{P3_N}$ can be written as


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$$\bar{y}^{\star}_{P3_N} = [\gamma_1 \bar{y}_{nN} + (1-\gamma_1)]\, exp\left[\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N}\right] \tag{60}$$

Putting $\gamma_2 = 1-\gamma_1$ in equation (52) and (54) , bias and MSE of estimator $\bar{y}^{\star}_{P3_N}$ can be obtained upto the first order of approximation respectively as:

$$Bias(\bar{y}^{\star}_{P3_N}) = (1-\gamma_1) + (\gamma_1 - 1)\bar{Y}_N - \gamma_1 \frac{\delta_{xy}}{2\bar{X}_N} + \left(1 - \gamma_1 + \gamma_1 \bar{Y}_N\right)\frac{3}{8}\frac{\delta_x^2}{\bar{X}_N^2} \tag{61}$$

and

$$MSE(\bar{y}^{\star}_{P3_N}) = \bar{Y}_N^2 + (B_3 - 2E_3) + \gamma_1^2(A_3 + B_3 - 2C_3) - 2\gamma_1(B_3 - C_3 + D_3 - E_3) \tag{62}$$

Diffrentiating equation(62) with respect to $\gamma_1$ and equating it to zero we get the optimum value of $\gamma_1$,as

$$\gamma_1 = \frac{(B_3 - C_3 + D_3 - E_3)}{(A_3 + B_3 - 2C_3)} = \gamma_{opt}(say) \tag{63}$$

Then the resulting minimum MSE of estimator $\bar{y}^{\star}_{P3_N}$ is

$$MSE(\bar{y}^{\star}_{P3_N})_{min} = \bar{Y}_N^2 + (B_3 - 2E_3) - \frac{(B_3 - C_3 + D_3 - E_3)^2}{(A_3 + B_3 - 2C_3)} \tag{64}$$

## 5 Numerical Study

### 5.1 Empirical study:

To demonstrate the properties of our proposed estimators, we have used four data sets given in section 5.1.1 and 5.1.2 below. We have drawn a sample of size $n$ from the population of size $N$ , then we have calculated MSE and PRE for the neutrosophic estimators using the values given in Table 1 and 3. PRE is used to compare the performance of different estimators or statistical methods in terms of their precision and accuracy. The purpose of PRE is to express the relative efficiency of one estimator compared to another, typically a standard or benchmark estimator, in a standardized way. PRE of the estimators is calculated by using the following formula :

$$PRE(ES) = \frac{MSE(\bar{y}_N)}{MSE(ES)} \times 100 \tag{65}$$

where, $ES = \bar{y}_N, \bar{y}_{rN}, \bar{y}_{MrN}, \bar{y}_{expN}, \bar{y}_{IexpN}, \bar{y}_{GexpN}, \bar{y}_{P1N}, \bar{y}^{\star}_{P1N}, \bar{y}_{P2N}, \bar{y}_{P3N}, \bar{y}^{\star}_{P3N}$

#### 5.1.1 An application to COVID-19 scenario

The data set I and II is based on product sales and marketing in field of medical taken from Aleeswari[1] . The data on Networking and product sales collected by the medical professionals for a period of two months (1January -1 March 2021) before and during(1 September- 1 November 2022) lockdown are taken. In this indeterminate data sets, one set consist of two variables, networking with customers before pandemic and product sales before pandemic, and the other set consists of two variables, networking with customers after pandemic and product sales after pandemic.

In the **data set I**, we are considering the percentage of networking before pandemic as subsidiary information and percentage of sales before pandemic as the study variable.

In the same way for the **data set II** we are considering percentage of networking after pandemic as subsidiary information and percentage of sales after pandemic as the study variable. Here, neutrosophic study variable denoted as $Y_N \in [Y_L, Y_U]$ ($Y_L$ is the lowest percentage of sales before/after pandemic) and neutrosophic subsidiary variable denoted as $X_N \in [X_L, X_U]$.The parameters for data set I and II are listed in Table 1.

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Table 1: **Description of the parameters under neutrosophic SRSWOR for Data Set I and II**

| Parameters | Values before pandemic (Data Set I) | Values after pandemic (Data Set II) |
|---|---|---|
| $N$ | [30,30] | [30,30] |
| $n$ | [9,9] | [9,9] |
| $\bar{Y}_N$ | [37.0033,37.3266] | [34.7266,40.040] |
| $\bar{X}_N$ | [34.1600,34.4500] | [34.7266,34.9033] |
| $S_{yN}$ | [12.3153,12.3233] | [12.2704,26.7004] |
| $S_{xN}$ | [15.0789,15.0891] | [15.2288,15.2452] |
| $C_{yN}$ | [0.3328,0.3301] | [0.3533,0.6668] |
| $C_{xN}$ | [0.4414,0.4380] | [0.4385,0.4367] |
| $\rho_{xy}$ | [0.8776,0.8775] | [0.8508,0.8462] |

Table 2: **MSEs and PREs of the existing and proposed estimators under neutrosophic SRSWOR for the data set I and II**

| Estimator | Before pandemic (Data Set I) | | After pandemic (Data Set II) | |
|---|---|---|---|---|
| | MSE | PRE | MSE | PRE |
| $\bar{y}_N$ | [10.7848,10.8809] | [100,100] | [11.3698,11.676] | [100,100] |
| $\bar{y}_{rN}$ | [10.8389,10.8579] | [100.3881,99.3264] | [8.7309,8.4136] | [133.7326,135.1352] |
| $\bar{y}_{MrN}$ | [10.7416,10.1076] | [100.4022,101.1103] | [8.6494,8.3363] | [131.451,140.0622] |
| $\bar{y}_{expN}$ | [4.5861,4.6394] | [237.2588,232.4578] | [5.007,4.9254] | [233.1958,230.8397] |
| $\bar{y}_{IexpN}$ (l=1,m=1) | [13.5853,13.6671] | [79.3856,79.6139] | [12.9933,13.1833] | [87.5047,88.5668] |
| $\bar{y}_{IexpN}$ (l=1,m=0) | [13.9111,13.9901] | [78.2173,77.0883] | [13.2317,13.4201] | [88.2426,84.7217] |
| $\bar{y}_{GexpN}$ | [4.5861,4.6394] | [235.1635,234.5308] | [4.9207,4.7955] | [231.0619,243.4807] |
| $\bar{y}_{P1N}$ | **[4.5482,4.6017]** | **[239.2374,234.3669]** | **[4.8835,4.7601]** | **[239.0901,238.8555]** |
| $\bar{y}^{\star}_{P1N}(\alpha_1+\alpha_2=1)$ | [4.5484,4.6017] | [237.1101,236.4533] | [4.9625,4.881] | [229.114,239.2115] |
| $\bar{y}_{P2N}$ | **[4.4002,4.4558]** | **[247.2837,242.0372]** | **[4.7824,4.6631]** | **[244.146,243.8269]** |
| $\bar{y}_{P3N}$ | **[2.6024,2.6119]** | **[418.1157,412.911]** | **[2.151,2.0728]** | **[542.8266,548.5324]** |
| $\bar{y}^{\star}_{P3N}(\gamma_1+\gamma_2=1)$ | [4.547,4.6007] | [237.182,236.5061] | [4.974,4.8951] | [228.5839,238.5262] |

To study the properties of the suggested estimators and compare them with the existing ones, we have computed the MSEs and PREs of the estimators for the two data sets. Table 2 clearly demonstrates the MSEs and PREs of the estimators. Table 2 shows that the suggested estimators outperform the existing estimators. $\bar{y}_{3P_N}$ is the best performing estimator among them all.

#### 5.1.2 An application to medical science and solar energy data

The **Data Set III** consists of real-life blood pressure data from the Punjab province of Pakistan, which contains indeterminate values, as referenced by Singh[22] . In our study, blood pressure (BPL) is treated as the neutrosophic study variable $Y_N \in [Y_L, Y_U]$. Similarly, pulse rate is considered the neutrosophic auxiliary variable $X_N \in [X_L, X_U]$. The parameters associated with this data are detailed in Table 3.

The **Data Set IV** consists of real-life solar energy data from Aslam[5] , which includes indeterminate values. For the purpose of our study, next-day global horizontal irradiance is treated as the neutrosophic study variable $Y_N \in [Y_L, Y_U]$. Additionally, temperature (T) is considered the neutrosophic auxiliary variable $X_N \in [X_L, X_U]$.The details of the parameters are provided in Table 3. The MSE and PRE for the proposed and existing estimators have been thoroughly calculated to assess their performance. These metrics provide a comparative analysis of the accuracy and efficiency of the estimators. The results are comprehensively presented in Table 4, allowing for a clear comparison between the proposed methods and the traditional approaches.

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Table 3: **Description of the parameters for Data Set III and IV under neutrosophic SRSWOR.**

| Parameters | Data Set III | Data Set IV |
|---|---|---|
| $N$ | [70,70] | [12,12] |
| $n$ | [25,25] | [6,6] |
| $\bar{Y}_N$ | [66.0714,79.7143] | [5209.667,6312.417] |
| $\bar{X}_N$ | [72.3714,79.3142] | [27.200,29.0083] |
| $S_{yN}$ | [7.2678,6.3637] | [1439.122,1292.364] |
| $S_{xN}$ | [2.4327,7.1232] | [8.5090,7.8829] |
| $C_{yN}$ | [0.1099,0.0798] | [0.2762,0.2047] |
| $C_{xN}$ | [0.0336,0.0898] | [0.3128,0.2717] |
| $\rho_{xyN}$ | [-0.0392,-0.1930] | [0.8215,0.8441] |

Table 4: **MSEs and PREs of the existing and proposed estimators under neutrosophic SRSWOR for the data set III and IV**

| **Estimator** | **Data Set III** | | **Data Set IV** | |
|---|---|---|---|---|
| | MSE | PRE | MSE | PRE |
| $\bar{y}_N$ | [1.7176,1.990] | [100,100] | [172589.323,139183.81] | [100,100] |
| $\bar{y}_{rN}$ | [1.5176,2.28115] | [131.1238,61.0912] | [72815.786,72516.891] | [191.145,237.999] |
| $\bar{y}_{MrN}$ | [1.5173,2.8084] | [113.2048,70.8595] | [72038.727,71986.441] | [239.578,193.347] |
| $\bar{y}_{expN}$ | [1.4062,1.597] | [141.5121,107.5547] | [67367.390,44547.369] | [206.604,387.429] |
| $\bar{y}_{IexpN}(l=1,m=1)$ | [1.4126,1.6350] | [121.5953,121.7126] | [172377.933,149740.359] | [100.123,92.950] |
| $\bar{y}_{IexpN}(l=1,m=0)$ | [1.4140,1.6486] | [140.7401,104.183] | [175313.521,153168.571] | [79.391,112.679] |
| $\bar{y}_{GexpN}$ | [1.3562,1.0026] | [126.6518,198.4936] | [56123.682,40016.107] | [307.516,347.820] |
| $\bar{y}_{P1N}$ | [1.3557,1.0023] | [146.7847,171.3596] | [55865.527,39890.904] | [249.141,432.653] |
| $\bar{y}^{\star}\quad(\alpha_1+\alpha_2=1)$ | [1.4058,1.5967] | [122.1789,124.6337] | [67185.512,44461.094] | [256.885,313.046] |
| $\bar{y}_{P2N}$ | [1.3557,1.0019] | [146.7908,171.4420] | [55381.105,39481.125] | [251.320,437.144] |
| $\bar{y}_{P3N}$ | [0.0317,0.3171] | [485.7638,541.6402] | [17845.332,17516.142] | [779.945,985.316] |
| $\bar{y}^{\star}_{P3N}(\gamma_1+\gamma_2=1)$ | [1.4057,1.5962] | [122.1850,124.6727] | [67186.769,44461.517] | [256.879,313.043] |

### 5.2 Simulation Study

For simulation study we have used the concept of Raghav[15] . Following steps are used:

- Generate Neutrosophic auxiliary variable $X_N$ follows neutrosophic normal distribution as $X_N \sim NN(\mu_{xN}, \sigma^2_{xN}); X_N \in (X_L, X_U), \mu_{xN} \in (.2, 1.2), \sigma^2_{xN} \in (1, 1.2)$
- Generate Neutrosophic study variable using the model $Y_N = X_N - 9.8e$,where $e \sim NN(0, 1)$.

The parameters for our simulation study are listed in Table 5.

Table 5: **Description of the parameters for the estimation of means under neutrosophic SRSWOR for simulation study.**

| **Parameters** | **Neutrosophic values** | **Parameters** | **Neutrosophic values** |
|---|---|---|---|
| $N_N$ | [1000,1000], | $S_{yN}$ | [2.2223, 2.3144] |
| $\bar{Y}_N$ | [0.2223, 1.2343] | $C_{yN}$ | [9.9948, 1.8750] |
| $\bar{X}_N$ | [0.2599, 1.2719] | $C_{xN}$ | [3.8978, 0.9560] |
| $S_{xN}$ | [1.0134, 1.2161] | $\rho_{xyN}$ | [0.4185,0.4894] |

Further, we have drawn samples of size $n_N(n_1 = 400, n_2 = 350$ and $n_3 = 300)$ from the population of size $N_N$=1000 by the method of neutrosophic simple random sampling without replacement. With the help of these samples, we have calculated MSEs of the proposed neutrosophic estimators. The whole process of getting MSEs of the neutrosophic estimators through the neutrosophic simple random sampling method is

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Table 6: **MSEs and PREs of the existing and proposed estimators from the simulation study.**

| Estimator | $n$=400 | | $n$=350 | | $n$=300 | |
|---|---|---|---|---|---|---|
| | MSE | PRE | MSE | PRE | MSE | PRE |
| $\bar{y}_N$ | [0.0074, 0.0080] | [100.00, 100.00] | [0.0092, 0.0099] | [100.00, 100.00] | [0.0115, 0.0125] | [100.00, 100.00] |
| $\bar{y}_{rN}$ | [0.0063, 0.0061] | [117.46, 131.14] | [0.0078, 0.0076] | [117.94, 130.26] | [0.0099, 0.0095] | [116.16, 131.58] |
| $\bar{y}_{MrN}$ | [0.0073, 0.0064] | [101.37, 125.00] | [0.0090, 0.0080] | [102.22, 123.75] | [0.0113, 0.0100] | [101.77, 125.00] |
| $\bar{y}_{expN}$ | [0.0065, 0.0066] | [113.85, 121.21] | [0.0081, 0.0081] | [113.58, 122.22] | [0.0102, 0.0102] | [112.75, 122.55] |
| $\bar{y}_{IexpN}$ (l=1,m=1) | [0.0073, 0.0077] | [101.37, 103.89] | [0.0090, 0.0096] | [102.22, 103.13] | [0.0114, 0.0120] | [100.88, 104.17] |
| $\bar{y}_{IexpN}$ (l=1,m=0) | [0.0073, 0.0078] | [101.37, 102.56] | [0.0090, 0.0097] | [102.22, 102.06] | [0.0114, 0.0122] | [100.88, 102.46] |
| $\bar{y}_{GexpN}$ | [0.0061, 0.0061] | [121.31, 131.15] | [0.0075, 0.0075] | [122.67, 132.00] | [0.0095, 0.0095] | [121.05, 131.58] |
| $\bar{y}_{P1N}$ | [0.0051, 0.0061] | [145.09, 131.15] | [0.0060, 0.0075] | [153.33, 132.00] | [0.0071, 0.0094] | [161.97, 132.98] |
| $\bar{y}^{\star}_{P1N}$ ($\alpha_1 + \alpha_2 = 1$) | [0.0052, 0.0065] | [142.31, 123.08] | [0.0062, 0.0081] | [148.39, 122.22] | [0.0073, 0.0101] | [157.53, 123.76] |
| $\bar{y}_{P2N}$ | [0.0050, 0.0061] | [148.00, 131.15] | [0.0059, 0.0075] | [155.93, 132.00] | [0.0070, 0.0094] | [164.29, 132.98] |
| $\bar{y}_{P3N}$ | [0.0003, 0.0004] | [2466.6, 2000] | [0.0003, 0.0005] | [3066.67, 1980] | [0.0004, 0.0006] | [2875, 2083.3] |
| $\bar{y}^{\star}_{P3N}$ ($\gamma_1+\gamma_2 = 1$) | [0.0064, 0.0054] | [115.63, 148.15] | [0.0079, 0.0064] | [116.46, 154.69] | [0.0099, 0.0076] | [116.16, 164.47] |

repeated 7000 times. The results of MSEs and PREs for the proposed neutrosophic estimators are shown in Tables 6.

To study the properties of the suggested estimators and compare them with the existing ones, we conducted simulation study and computed the MSEs and PREs of the estimators. The Simulation study yields the same results as the empirical study. Table 4 illustrates that our suggested estimators have the higher PRE as compared to the existing estimators.

## 6 Result and Discussion

Table 2 and Table 4 demonstrates the MSE and PRE of the existing and proposed neutrosophic estimators under neutrosophic SRSWOR for the four real data sets: I, II, III and IV mentioned in Table 1 and 2. The results in Table 2 and 4 demonstrates the following key findings:

• In Data Sets I and II, there exists a pronounced positive correlation between the study variable and the auxiliary variable, which substantially amplifies the efficiency of our proposed estimators $\bar{y}_{P1_N}$, $\bar{y}_{P2_N}$ and $\bar{y}_{P3_N}$ relative to the existing estimators. As evidenced in Table 2, both data sets exhibit a consistent trend, thereby affirming the efficacy of the proposed estimation methodologies for ascertaining the population mean within the neutrosophic framework. This consistency not only highlights the robustness of our proposed approaches but also underscores their relevance in effectively addressing the intricacies associated with indeterminate data.

• In Data Set III, the study variable and auxiliary variable exhibit a negative correlation, which typically results in diminished efficiency for the ratio estimator. Nevertheless, our proposed estimators exhibit superior performance regarding MSE and PRE in comparison to the existing estimators, as delineated in Table 2. This finding underscores the robustness of our estimators, indicating their efficacy even in contexts where the study and auxiliary variables are negatively correlated. In Data Set IV, a similar trend is evident, consistent with the results presented in Tables 2 and 4, reinforcing the reliability of our estimators across various data sets.

• Among the suggested estimators $\bar{y}_{P3_N}$ is the best performing estimator. Its superior performance highlights its robustness in handling neutrosophic data, further validating its use in practical applications where indeterminacy is present.

• The proposed estimators $\bar{y}_{P1_N}$ and $\bar{y}_{P3_N}$ under the condition $\alpha_1 + \alpha_2 = 1$ and $\gamma_1 + \gamma_2 = 1$ respectively, exibit performance that is comparable to that of $\bar{y}_{expN}$ and $\bar{y}_{GexpN}$.


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Table 6 illustrates the MSEs and PREs of the existing and proposed neutrosophic estimators under the framework of neutrosophic SRSWOR. These values are derived from the simulation study we conducted in section 5.2. The simulation study is conducted for sample sizes of 300, 350 and 400, providing a comprehensive comparision of estimator performance. The results in Table 6 clearly demonstrates the following key findings:

• Our suggested estimators $\bar{y}_{P1_N}$, $\bar{y}_{P2_N}$ have demonstrated the highest level of efficiency among all the estimators evaluated in this study. These estimators consistently outperform others, making them the most reliable choices for accurate estimation under the neutrosophic framework.

• Among the proposed estimators $\bar{y}_{P3_N}$ stands out as the most efficient, as it exhibits the highest PRE compared to the other estimators considered in the study. The higher PRE indicates that $\bar{y}_{P3_N}$ delivers more accurate and reliable estimates with less error, making it the preferred estimator when maximizing efficiency is crucial.

• From Table 6, it is clearly evident that as the sample size increases from $n = 300$ to $n = 350$ then $n = 400$, the PRE of the estimators consistently improves. This increase in PRE indicates that the estimators become more efficient as the sample size grows. Simultaneously, the MSE of the estimators decreases with the larger sample sizes, reflecting a reduction in estimation errors. This trend highlights the positive impact of larger sample sizes on the accuracy and efficiency of the estimators under the neutrosophic framework.

## 7 Conclusion

In this article, we have propounded three modified Neutrosophic exponential estimators ($\bar{y}_{P1_N}$, $\bar{y}_{P2_N}$ and $\bar{y}_{P3_N}$) designed for estimating the population mean through the use of auxiliary variables under Neutrosophic SRSWOR in contexts involving indeterminate or interval data. Our proposed estimators exhibit a marked improvement over existing methodologies, including the Neutrosophic simple mean estimator ($\bar{y}_N$), neutrosophic ratio estimator ($\bar{y}_{rN}$), neutrsosophic modified ratio estimator ($\bar{y}_{MrN}$), neutrosophic exponential estimator ($\bar{y}_{expN}$), neutrosophic improved exponential ratio estimator ($\bar{y}_{IexpN}$), neutrosophic generalized exponential ratio estimator ($\bar{y}_{GexpN}$). Our findings indicate that the proposed estimators achieve higher efficiency than their existing exponential counterparts, with $\bar{y}_{P3_N}$ emerging as the most efficient among them. Through comprehensive experiments and analysis based on simulation (presented in Table 6) and real data applications (presented in Table 2) we have demonstrated the effectiveness of the proposed estimators over the other estimators presented in the literature by leveraging the auxiliary information in situation characterized by uncertainty, ambiguity, indeterminacy or interval data.

In the forthcoming studies, we would like to examine the proposed estimators for the other population parameters like Variance, Skewness, Kurtosis etc. under different sampling schemes including, simple random sampling, stratified random sampling, rank set sampling etc.

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