On The Group of Units Problem of the Non-Commutative Logical Extension of the Rings Z_p and Z_{2^n}

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Abstract

This paper is dedicated to studying the group of units problem of the non-commutative logical extension of two different rings Z_p and Z_{2^n}, where we classify the group of units of these rings as semi-direct products of well-known abelian groups as follows:

$$U(NCR)_{Z_p} \cong Z_{P-1} \propto (Z_P \propto Z_{P-1})$$

$$U(NCR)_{2^n} \cong (Z_2 \times Z_{2^{n-2}}) \ltimes \left(Z_{2^n} \ltimes (Z_2 \times Z_{2^{n-2}})\right).$$

Keywords: Group of units; NCR ring; Rings modulo integers; Prime number

1. Introduction

The group of unit's classification of some modern ring extensions is considered as an active research problem studied by many authors, we can see some clear examples of this classifications in [2-8].

In [1], a novel non-commutative ring extension called non-commutative logical extension of a ring as a similar structure to refined neutrosophic rings.

The non-commutative logical extension of a ring R is defined as follows [1]:

Let $(R, +, .)$ be any ring, then:

$$(NCR)_R = \{a + bN_1 + cN_2 \ ; a, b, c \in R, {N_1}^2 = N_1N_2 = N_1, {N_2}^2 = N_2N_1 = N_2\}.$$

$(NCR)_R$ is called the non-commutative logical extension of the ring R.

In [1], the following interesting open question was asked:

How can we classify the group of units of $(NCR)_R$?.

In other words, how can we express the group of units of $(NCR)_R$ as direct products or semi-direct products of well-known groups. This question is open in general, and we will handle it in two special cases.

The first one is the group of units of $U(NCR)_{Z_p}$, where we will show that it is classified as:

$$U(NCR)_{Z_p} \cong Z_{P-1} \propto (Z_P \propto Z_{P-1})$$

The second one is the group of units of $U(NCR)_{2^n}$, where we will show that it is classified as:

$$U(NCR)_{2^n} \cong (Z_2 \times Z_{2^{n-2}}) \ltimes \left(Z_{2^n} \ltimes (Z_2 \times Z_{2^{n-2}})\right).$$

2. Main Results

Definition: [1]

Let $(Z_p, +, .)$ be the ring of integers modulo p, where p is a prime number.

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Then: $(NCR)_{Z_p} = \{a + bN_1 + cN_2 \ ; a, b, c \in Z_p, N_1{}^2 = N_1N_2 = N_1, N_2{}^2 = N_2N_1 = N_2\}$.

$(NCR)_{Z_p}$ is called the non-commutative logical extension of the ring Z_p.

In this work, we answer the following special case of the open question [1].

How can we classify the group of units of $(NCR)_{Z_p}$?

How can we classify the group of units of $(NCR)_{Z_{2^n}}$?

Theorem:

Let $(NCR)_{Z_p}$ be the non-commutative logical extension of the ring Z_p, then:

1] $x = a + bN_1 + cN_2 \in U(NCR)_{Z_p}$ if and only if:

$a \in U(Z_p), a + b + c \in U(Z_p)$

2] If $x \in U(NCR)_{Z_p}$, then:

$x^{-1} = a^{-1} - ba^{-1}(a + b + c)^{-1}N_1 - ca^{-1}(a + b + c)^{-1}N_2$.

3] If $p \geq 3$, then $(NCR)_{Z_p}$ is a non- abelian group.

Proof:

1] Let $x = a + bN_1 + cN_2 \in U(NCR)_{Z_p}$, then there exists $y = a' + b'N_1 + c'N_2 \in U(NCR)_{Z_p}$ such that: $xy = 1$.

The equation $xy = 1$ implies:

$$\begin{cases} aa' = 1 \\ ab' + ba' + bb' + bc' = 0 \\ ac' + ca' + cb' + cc' = 0 \end{cases} \Rightarrow \begin{cases} aa' = 1 \\ (a + b + c)(a' + b' + c') = 1 \end{cases} \Rightarrow a, a + b + c \in U(Z_p).$$

Conversely, assume that $a, a + b + c \in U(Z_p)$, then we can find

$y = a^{-1} - ba^{-1}(a + b + c)^{-1}N_1 - ca^{-1}(a + b + c)^{-1}N_2 \in U(N \subset R)_{Z_p}$.

$x.y = aa^{-1} - N_1(ba^{-1} - b^2a^{-1}(a + b + c)^{-1} - bca^{-1}(a + b + c)^{-1} - b(a + b + c)^{-1}) + N_2(ca^{-1} - bca^{-1}(a + b + c)^{-1} - c^2a^{-1}(a + b + c)^{-1} - c(a + b + c)^{-1})$.

Put $k_1 = ba^{-1} - b^2a^{-1}(a + b + c)^{-1} - bca^{-1}(a + b + c)^{-1} - b(a + b + c)^{-1}, k_2 = ca^{-1} - bca^{-1}(a + b + c)^{-1} - c^2(a^{-1})(a + b + c)^{-1} - c(a + b + c)^{-1}$,

We have:

$k_1 = ba^{-1} - b(a + b + c)^{-1}[ba^{-1} + ca^{-1} + 1] = ba^{-1} - b(a + b + c)^{-1}(ba^{-1} + ca^{-1} + aa^{-1}) = ba^{-1} - b(a + b + c)^{-1}.a^{-1}(a + b + c) = ba^{-1} - ba^{-1} = 0$.

Also,

$k_2 = ca^{-1} - c(a + b + c)^{-1}(ba^{-1} + ca^{-1} + 1) = ca^{-1} - c(a + b + c)^{-1}(ba^{-1} + ca^{-1} + aa^{-1}) = ca^{-1} - c(a + b + c)^{-1}.a^{-1}(a + b + c) = ca^{-1} - ca^{-1} = 0$,

Hence $x = a + bN_1 + cN_2 \in U(NCR)_{Z_p}$.

2] It holds directly from 1.

3] Let $a_0 \in U(Z_p)$ with $a_0 \neq 1$, then:

$x = a_0 + a_1N_1 - a_2N_2 \in U(NCR)_{Z_p}$; $a_0 + a_1 \neq a_0, a_0 + a_1 \in U(Z_p), a_1{}^2 \neq 0$.

Take $y = a_0 + a_1N_1$, then $y \in U(NCR)_{Z_p}$.

$xy = a_0{}^2 + N_1(2a_0a_1 + a_1{}^2) + N_2(-a_1a_0 - a_1{}^2)$,

$yx = a_0{}^2 + N_1(2a_0a_1) + N_2(-a_1a_0)$,

If $xy = yx$, then $a_1{}^2 = 0$ a contradiction.

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Hence $xy \neq yx$, and $U(NCR)_{Z_p}$ is a non- abelian.

Theorem:

Let $G = U(NCR)_{Z_p}$, then $Z(G) = U\left(Z_p\right) \cong Z_{p-1}$.

Proof:

Let $A = a_0 + a_1N_1 + a_2N_2 \in Z(G)$, then:

$\forall$ $B = b_0 + b_1N_1 + b_2N_2 \in G$, we have $AB = BA$.

The equation $AB = BA$ equivalents: $a_1b_2 = a_2b_1$.

Take $b_2 = 0, b_1 = 1 - b_0$, then:

$a_2(1 - b_0) = 0 \Longrightarrow a_2 = a_2b_0$ for all $b_0 \in U\left(Z_p\right)$, hence a_2 is a zero divisor, thus $a_2 = 0$.

Take $b_1 = 0, b_2 = 1 - b_0$, then $a_1(1 - b_0) = 0$ for all $b_0 \in U\left(Z_p\right)$, hence a_1 is a zero divisor, and $a_1 = 0$.

Thus $Z(G) = \left\{a_0; a_0 \in U\left(Z_p\right)\right\} = U\left(Z_p\right) \cong Z_{p-1}$.

Theorem:

Let $G = U(NCR)_{Z_p}$, then:

1] $K = \left\{1 + a(N_1 - N_2)\,; a \in Z_p\right\}$ is a normal subgroup of G.

2] $K \cong (Z_p, +)$

3] $S = \{1 + aN_1\,; 1 + a \in U(Z_P)\}$ is a subgroup of G.

4] $S \cong U(Z_P) \cong Z_{p-1}$.

5] $T = \{1 + a_1N_1 + a_2N_2\,; 1 + a_1 + a_2 \in U(Z_P)\}$ is a subgroup of G.

6] $T = K \propto S \cong Z_P \propto Z_{p-1}$.

Proof:

1] Let $x = 1 + a(N_1 - N_2), y = 1 + b(N_1 - N_2) \in K$, then:

$xy = 1 + a(N_1 - N_2) + b(N_1 - N_2) + ab(N_1 - N_2)(N_1 - N_2) = 1 + (a + b)(N_1 - N_2) \in K$

$x^{-1} = 1 - aN_1 + aN_2 = 1 - a(N_1 - N_2) \in K$ and $x \in U(NCR)_{Z_p}$, thus $K \leq G$.

$\forall$ $z = z_0 + z_1N_1 + z_2N_2 \in U(NCR)_{Z_p}$, then:

$z^{-1} = z_0^{-1} - z_0^{-1}z_1z'N_1 - z_0^{-1}z_2z'N_2\,; z' = (z_0 + z_1 + z_2)^{-1}$.

$z^{-1} \times z = (z_0^{-1} - z_0^{-1}z_1z'N_1 - z_0^{-1}z_2z'N_2)\big(1 + a(N_1 - N_2)\big)(z_0 + z_1N_1 + z_2N_2) = z_0^{-1}(1 - z_1z'N_1 - z_2z'N_2)[z_0 + N_1(z_1 + a(z_0 + z_1 + z_2)) + N_2(z_2 - a(z_0 + z_1 + z_2))] = z_0^{-1}\big[z_0 + N_1(z_1 + az'^{-1} - z_1z_0z' - az_1 - z_1z_2z' + az_1 - z_1^2z') + N_2(z_2 - az'^{-1} - z_2z'z_0 - z_2z_1z' - az_2 - z_2^2z' + az_2)\big] = 1 + N_1\big[z_0^{-1}\big(z_1z'(-z_0 - z_1 - z_2) + z_1 + az'^{-1}\big)\big] + N_2\big[z_0^{-1}\big(z_2z'(-z_0 - z_1 - z_2) + z_2 + az'^{-1}\big)\big] = 1 + N_1\big(az_0^{-1}z'^{-1}\big) + N_2\big(-az'^{-1}z_0^{-1}\big) = 1 + az_0^{-1}z'^{-1}(N_1 - N_2) \in K$, hence K is normal.

2] Define $f: K \to Z_p$ such that:

$f\big(1 + a(N_1 - N_2)\big) = a$.

It is clear that (f) is a well-defined bijection.

Let $x = 1 + a(N_1 - N_2), y = 1 + b(N_1 - N_2) \in K$, then:

$f(xy) = f\big(1 + (a + b)(N_1 - N_2)\big) = a + b = f(x) + f(y)$, so that f is a group isomorphism, and:$(K, x) \cong (Z_p, +)$.

3] Let $x = 1 + a_1N_1, y = 1 + a_2N_1 \in S$, then:

$xy = 1 + (a_2 + a_1 + a_1a_2)N_1 \in S$.

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$x^{-1} = 1 - a_1(1 + a_1)^{-1}N_1 \in S$, thus, S is a subgroup of G.

4] Define $f: S \to U(Z_P); f(1 + aN_1) = 1 + a$.

It is clear that (f) is a well-defined mapping.

Let $x = 1 + aN_1, y = 1 + bN_1 \in S$, then:

$f(xy) = f\big(1 + N_1(a + b + ab)\big) = 1 + a + b + ab = (1 + a)(1 + b) = f(x)f(y)$.

If $f(x) = 1$, then $1 + a = 1 \Longrightarrow a = 0$, and $Ker(f) = \{1\}$.

For every $t \in U(Z_P)$, there exists $x = 1 + (t - 1)N_1 \in S$ such that $f(x) = t$.

Hence (f) is a group isomorphism, and $(S, x) \cong (U(Z_P), x) \cong (Z_{P-1}, +)$.

5] Let $x = 1 + a_1N_1 + a_2N_2, y = 1 + b_1N_1 + b_2N_2 \in T$, then:

$x^{-1} = 1 - a_1(1 + a_1 + a_2)^{-1}N_1 - a_2(1 + a_1 + a_2)^{-1}N_2 \in T$, that is because:

$$\begin{aligned}1 - a_1(1 + a_1 + a_2)^{-1} - a_2(1 + a_1 + a_2)^{-1} \\ &= (1 + a_1 + a_2)(1 + a_1 + a_2)^{-1} - a_1(1 + a_1 + a_2)^{-1} - a_2(1 + a_1 + a_2)^{-1} \\ &= (1 + a_1 + a_2)^{-1}(1 + a_1 + a_2 - a_1 - a_2) = (1 + a_1 + a_2)^{-1} \in U(Z_P).\end{aligned}$$

$xy = 1 + N_1\big(b_1 + a_1(1 + b_1 + b_2)\big) + N_2\big(b_2 + a_2(1 + b_1 + b_2)\big) \in T$, that is because:

$1 + b_1 + a_1(1 + b_1 + b_2) + b_2 + a_2(1 + b_1 + b_2) = (1 + b_1 + b_2)(1 + a_1 + a_2) \in U(Z_P)$.

Thus, T is a subgroup of G.

6] It is clear that $K \leq T, S \leq T$,and:

$|K| = p, |S| = p - 1 \Longrightarrow K \cap S = \{1\}$.

Also, $K \lhd T$(because $K \lhd G$).

$|T| = |\{1 + a_1N_1 + a_2N_2 \,; 1 + a_1 + a_2 \in U(Z_P)\}| = p(p - 1)$.

This means that:

$|T| = |K|.|S| = |K.S| \Longrightarrow T = K \propto S$, thus:

$T \cong Z_P \propto Z_{P-1}$.

Remark:

Since $Z(G) = U(Z_P) = \{a \,; a \in Z_P\}$, then:

$Z(G) \cap T = \{1\}$.

Theorem:

$U(NCR)_{Z_p} \cong Z_{P-1} \propto (Z_P \propto Z_{P-1})$

Proof:

We will show that $G = Z(G) \propto T$.

We know that $x = a_0 + a_1N_1 + a_2N_2 \in G$ if and only if:

$a_0 \in U(Z_P), a_0 + a_1 + a_2 \in U(Z_P)$.

We have $(p - 1)$ different values of a_0, and $(p - 1)$ different values of $a_0 + a_1 + a_2$.

For a fixed value of $a_0 = t_1$, and a fixed value of $a_0 + a_1 + a_2 = t_2$, we get $a_1 + a_2 = t_2 - t_1 \in \{0,1, \dots, p - 1\}$.

This means that we have (p) different values of $a_1 + a_2$, so that:

$|G| = (p - 1) \times (p - 1) \times p = p(p - 1)^2 = |Z(G).T| \Longrightarrow G = Z(G) \propto T \cong Z_{P-1} \propto (Z_P \propto Z_{P-1})$.

Remark:

Since G is a semi- direct product of $(Z(G))$ and T, we can get the following results:

1] $G/Z(G) \cong Inn\ (G) \cong T \cong Z_P \propto Z_{P-1}$.

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2] $G$is meta- abelian group, that is because it is a semi- direct product of two abelian groups.

3] G' is abelian subgroup of G.

Definition:

The non-commutative logical extension of Z_{2^n} is:

$(NCR)_{Z_{2^n}} = \{x + yN_1 + +zN_2\ ; x, y, z \in Z_{2^n}, {N_1}^2 = N_1N_2 = N_1, {N_2}^2 = N_2N_1 = N_2\}$.

We denote by $G = U(NCR)_{Z_{2^n}}$.

Theorem:

1] $K = \{1 + x(N_1 - N_2)\ ; x \in Z_{2^n}\}$ is a normal subgroup of G.

2] $K \cong Z_{2^n}$

3] $S = \{1 + xN_1\ ; 1 + x \in U(Z_{2^n})\}$ is a subgroup of G.

4] $S \cong U(Z_{2^n}) \cong Z_2 \times Z_{2^{n-2}}$

5] $T = \{1 + xN_1 + yN_2\ ; 1 + x + y \in U(Z_{2^n})\}$ is a subgroup of G.

6] $\cap S = \{1\}, K \lhd T, S \leq T$

Proof:

1] Let $x = 1 + a(N_1 - N_2), y = 1 + b(N_1 - N_2) \in K$, then:

$xy = 1 + a(N_1 - N_2) + b(N_1 - N_2) + ab(N_1 - N_2)(N_1 - N_2) = 1 + (a + b)(N_1 - N_2) \in K$

$x^{-1} = 1 - aN_1 + aN_2 = 1 - a(N_1 - N_2) \in K$ and $x \in U(NCR)_{Z_{2^n}}$, thus $K \leq G$.

$\forall\ z = z_0 + z_1N_1 + z_2N_2 \in U(NCR)_{Z_{2^n}}$, then:

$z^{-1} = z_0^{-1} - z_0^{-1}z_1z'N_1 - z_0^{-1}z_2z'N_2\ ; z' = (z_0 + z_1 + z_2)^{-1}$.

$z^{-1} \times z = (z_0^{-1} - z_0^{-1}z_1z'N_1 - z_0^{-1}z_2z'N_2)\big(1 + a(N_1 - N_2)\big)(z_0 + z_1N_1 + z_2N_2) = z_0^{-1}(1 - z_1z'N_1 - z_2z'N_2)[z_0 + N_1(z_1 + a(z_0 + z_1 + z_2)) + N_2(z_2 - a(z_0 + z_1 + z_2))] = z_0^{-1}\big[z_0 + N_1(z_1 + az'^{-1} - z_1z_0z' - az_1 - z_1z_2z' + az_1 - z_1^2z') + N_2(z_2 - az'^{-1} - z_2z'z_0 - z_2z_1z' - az_2 - z_2^2z' + az_2)\big] = 1 + N_1\big[z_0^{-1}\big(z_1z'(-z_0 - z_1 - z_2) + z_1 + az'^{-1}\big)\big] + N_2\big[z_0^{-1}\big(z_2z'(-z_0 - z_1 - z_2) + z_2 + az'^{-1}\big)\big] = 1 + N_1\big(az_0^{-1}z'^{-1}\big) + N_2\big(-az'^{-1}z_0^{-1}\big) = 1 + az_0^{-1}z'^{-1}(N_1 - N_2) \in K$, hence K is normal.

2] Define $f: K \to Z_{2^n}$such that:

$f\big(1 + a(N_1 - N_2)\big) = a$.

It is clear that (f) is a well-defined bijection.

Let $x = 1 + a(N_1 - N_2), y = 1 + b(N_1 - N_2) \in K$, then:

$f(xy) = f\big(1 + (a + b)(N_1 - N_2)\big) = a + b = f(x) + f(y)$, so that f is a group isomorphism, and:

$(K, .) \cong (Z_{2^n}, +)$.

3] Let $x = 1 + a_1N_1, y = 1 + a_2N_1 \in S$, then:

$xy = 1 + (a_2 + a_1 + a_1a_2)N_1 \in S$.

$x^{-1} = 1 - a_1(1 + a_1)^{-1}N_1 \in S$, thus, S is a subgroup of G.

4] Define $f: S \to U(Z_{2^n}); f(1 + aN_1) = 1 + a$.

It is clear that (f) is a well-defined mapping.

Let $x = 1 + aN_1, y = 1 + bN_1 \in S$, then:

$f(xy) = f\big(1 + N_1(a + b + ab)\big) = 1 + a + b + ab = (1 + a)(1 + b) = f(x)f(y)$.

If $f(x) = 1$, then $1 + a = 1 \Longrightarrow a = 0$, and $Ker(f) = \{1\}$.

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For every $t \in U(Z_{2^n})$, there exists $x = 1 + (t-1)N_1 \in S$ such that $f(x) = t$.

Hence (f) is a group isomorphism, and $(S,.) \cong (U(Z_{2^n}),.) \cong Z_2 \times Z_{2^{n-2}}$.

5] Let $x = 1 + a_1N_1 + a_2N_2,\ y = 1 + b_1N_1 + b_2N_2 \in T$, then:

$x^{-1} = 1 - a_1(1 + a_1 + a_2)^{-1}N_1 - a_2(1 + a_1 + a_2)^{-1}N_2 \in T$, that is because:

$$\begin{aligned}1 - a_1(1 + a_1 + a_2)^{-1} - a_2(1 + a_1 + a_2)^{-1} \\ &= (1 + a_1 + a_2)(1 + a_1 + a_2)^{-1} - a_1(1 + a_1 + a_2)^{-1} - a_2(1 + a_1 + a_2)^{-1} \\ &= (1 + a_1 + a_2)^{-1}(1 + a_1 + a_2 - a_1 - a_2) = (1 + a_1 + a_2)^{-1} \in U(Z_{2^n}).\end{aligned}$$

$xy = 1 + N_1\big(b_1 + a_1(1 + b_1 + b_2)\big) + N_2\big(b_2 + a_2(1 + b_1 + b_2)\big) \in T$, that is because:

$1 + b_1 + a_1(1 + b_1 + b_2) + b_2 + a_2(1 + b_1 + b_2) = (1 + b_1 + b_2)(1 + a_1 + a_2) \in U(Z_{2^n})$.

Thus, T is a subgroup of G.

6] It is clear that $K \leq T, S \leq T$,and:

$K \cap S = \{1\}$.

Also, $K \lhd T$(because $K \lhd G$).

$|T| = |\{1 + a_1N_1 + a_2N_2\ ; 1 + a_1 + a_2 \in U(Z_{2^n})\}| = 2^n(2^{n-1}) = 2^{2n-1}$.

This means that:

$|T| = |K|.|S| = |K.S| \Longrightarrow T = K \propto S$, thus:

$T \cong Z_{2^n} \propto (Z_2 \times Z_{2^{n-2}})$.

3. Result

$H = U(Z_{2^n}) \leq Z(G) \Longrightarrow H \lhd G$, and $|H.T| = 2^{n-1}.2^n.2^{n-1} = 2^{3n-1} = |G|$, hence $G = H.T \cong (Z_2 \times Z_{2^{n-2}}) \ltimes \big(Z_{2^n} \ltimes (Z_2 \times Z_{2^{n-2}})\big)$.

4. Conclusion

In this paper we studied the group of units problem of the non-commutative logical extension of two different rings Z_p and Z_{2^n}, where we classified the group of units of these rings as semi-direct products of well-known abelian groups as follows:

$U(NCR)_{Z_p} \cong Z_{P-1} \propto (Z_P \propto Z_{P-1})$

$U(NCR)_{2^n} \cong (Z_2 \times Z_{2^{n-2}}) \ltimes \big(Z_{2^n} \ltimes (Z_2 \times Z_{2^{n-2}})\big)$.

In the future, we aim to study the classification for all kind of NCR rings.

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