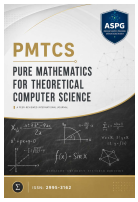
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FULL LENGTH ARTICLE

Accurate Numerical Method Using Exponential Spline for Solving Boundary Value Problems

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ABSTRACT

This study introduces a precise numerical technique employing exponential splines for singly perturbed singularity boundary value problems. A numerical scheme is devised to address issues encountered in diverse scientific and engineering domains. The framework consists of a triad of nonlinear equations. The approach is employed in several test cases to demonstrate accuracy and implementation.

Keywords: Finite difference ▪ Absolute errors ▪ Exponential Spline

1. INTRODUCTION

Considering the nonlinear singular perturbation problems,

$$\varepsilon\, \ddot{y}^{(2)}(\hat{z}) - \hat{\mathscr{P}}\left(\ddot{y}^{(1)}(\hat{z}), \ddot{y}_i, \hat{z} \right) = 0. \qquad (1)$$

Moreover, with regard to the boundary conditions,

$$\ddot{y}(0) = \mathscr{B}_1, \qquad \ddot{y}(1) = \mathscr{B}_2.$$

Here, $a \leq \hat{z} \leq b$, $0 < \hat{z} < 1$, and $\mathscr{B}_1$ and $\mathscr{B}_2$ are finite constants, provided that $\hat{\mathscr{P}}$ is a constrained and continuous function.

Various applications in science and engineering address challenges that depict intricate physical and chemical models. The resolution of solitary perturbation problems demonstrates a multi-scale nature. Numerous techniques, including finite difference, boundary element, and collocation approaches, are available for addressing linear singular perturbation problems [1–14]. Dag et al. [15] proposed a numerical solution of singularly perturbed boundary value problems via the finite difference approach. The collocation technique employed parabolic and square B-spline basis functions on a mathematically evaluated mesh of the solution domain. Rashidina et al. [16] analyzed self-adjoint singularly perturbed boundary value problems. Recently, Tirmizi [17] implemented the spline approach for straight singularity perturbation problems, achieving both the second and fourth levels of convergence based on the selection of empty parameters. Kadalbajoo et al. [13] analyzed a second-degree converging smoothing within an expansion technique for nonlinear single-perturbation scenarios. Nonetheless, their methodologies are only pertinent to nonsingular issues.

Challenges were encountered historically in the numerical resolution of boundary value problems. A solution typically degrades around the singularity. This study aims to provide a computationally efficient numerical method utilizing exponential splines, ensuring fourth-order convergence for decreasing values of ε while eliminating grid-size restrictions in the presence of singularities. The document is structured as outlined below. Section 2 provides a concise overview of the derivation of the exponential spline. In Section 3, we propose the application of the exponential spline method to singular problems. Section 4 presents the numerical results. Finally, the conclusions are presented in Section 5.

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2. DERIVATION OF EXPONENTIAL SPLINE

We examine a uniform mesh Λ with nodal points $\tilde{z}_i$ on the interval $[a,b]$ such that

$$a = \hat{z}_0 < \tilde{z}_i < \hat{z}_i < \cdots < \hat{z}_{n-1} < \hat{z}_n = \dot{b},$$
$$\hat{z}_i = \dot{a} + ih, \qquad i = 0,1,2,\ldots,n,$$

where $\dot{a} = \hat{z}_0$, $\dot{b} = \hat{z}_n$, and $h = (\dot{b}-\dot{a})/n$. The exponential spline $\ddot{\mathscr{K}}(\hat{z})$, which estimates $\ddot{y}(\hat{z})$ at the mesh points $\hat{z}_i$, $i = 0,1,2,\ldots,n-1,n$, depends on a parameter ω. For every segment $[\hat{z}_i,\hat{z}_{i+1}]$, $i = 0,1,2,\ldots,n-1$, the exponential spline function has the following form:

$$\ddot{\mathscr{K}}(\hat{z}_i) = \hat{a}e^{\omega(\hat{z}-\hat{z}_i)} + \hat{b}e^{2\omega(\hat{z}-\hat{z}_i)} + \hat{c}e^{3\omega(\hat{z}-\hat{z}_i)} + \hat{d}e^{4\omega(\hat{z}-\hat{z}_i)}. \tag{2}$$

Here, $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$ are arbitrary real and finite constants, and ω is a free parameter. To find the four coefficients of integration in Equation (2) using $\ddot{y}_i$, $\ddot{y}_{i+1}$, $\ddot{\Psi}_i$, and $\ddot{\Psi}_{i+1}$, we define the interpolation conditions at $\hat{z}_i$ and $\hat{z}_{i+1}$ as

$$\ddot{\mathscr{K}}(\hat{z}_i) = \ddot{y}_i, \qquad \ddot{\mathscr{K}}(\hat{z}_{i+1}) = \ddot{y}_{i+1},$$
$$\ddot{\mathscr{K}}(\hat{z}_i) = \ddot{\Psi}_i, \qquad \ddot{\mathscr{K}}^{(2)}(\hat{z}_i) = \ddot{\Psi}_{i+1}.$$

A spline can be defined in terms of $\ddot{y}_i$, $\ddot{y}_{i+1}$, $\ddot{\Psi}_i$, and $\ddot{\Psi}_{i+1}$. The coefficients presented in Equation (2) are computed as follows:

$$\begin{aligned}
\hat{a} &= e^{-\vartheta} - e^{3\vartheta}(-5+7e^{\vartheta})\ddot{\Psi}_i + (7-5e^{\vartheta})\ddot{\Psi}_{i+1} + 4\lambda^2 \\
&\quad + 4\lambda^2 e^{3\vartheta}(-20+7e^{\vartheta})\,\ddot{y}_i + (20e^{2\vartheta}-7)\,\ddot{y}_{i+1}, \\
\hat{b} &= e^{3\vartheta}\Big((7e^{2\vartheta}+7e^{\vartheta}-8)\ddot{\Psi}_i + (8e^{2\vartheta}-7e^{\vartheta}-7)\ddot{\Psi}_{i+1} \\
&\quad - \lambda^2\big(e^{3\vartheta}(7e^{2\vartheta}+7e^{\vartheta}-128)\,\ddot{y}_i \\
&\quad + (128e^{2\vartheta}-7-7e^{\vartheta})\,\ddot{y}_{i+1}\big)\Big), \\
\hat{c} &= (e^{2\vartheta}+e^{3\vartheta}-4e^{4\vartheta})\ddot{\Psi}_i - (e^{\vartheta}+e^{2\vartheta})\ddot{\Psi}_{i+1} \\
&\quad + \lambda^2\Big(e^{2\vartheta}(e^{2\vartheta}-4e^{\vartheta})\,\ddot{y}_i + (4e^{\vartheta}+4e^{2\vartheta}-1)\,\ddot{y}_{i+1}\Big), \\
\hat{d} &= e^{-2\vartheta}\Big(e^{2\vartheta}(5e^{\vartheta}-3)\ddot{\Psi}_i + (5e^{\vartheta}-3)\ddot{\Psi}_{i+1} \\
&\quad - \lambda^2\big(e^{2\vartheta}(5e^{\vartheta}-27)\,\ddot{y}_{i+1} + (27e^{\vartheta}-5)\,\ddot{y}_{i+1}\big)\Big).
\end{aligned}$$

Here, $\vartheta = h\omega$. Applying continuity of the first derivative, $\mathscr{K}_i^{(1)}(\hat{z}) = \mathscr{K}_{i-1}^{(1)}(\tilde{z})$ at $\hat{z} = \hat{z}_i$ for $i = 1,\ldots,n-1$, gives the consistency relation

$$h^2(\phi_1 M_{i-1} + \phi_2 M_i + \phi_3 M_{i+1}) = \phi_4\,\ddot{y}_{i+1} + \phi_5\,\ddot{y}_i + \phi_6\,\ddot{y}_{i-1}. \tag{3}$$

The coefficients are

$$\begin{aligned}
\phi_1 &= \frac{2e^{-2\vartheta}(e^{\vartheta}-1)^2}{3\vartheta^2}, \\
\phi_2 &= \frac{4(e^{\vartheta}+1)(e^{\vartheta}-1)^2}{3\vartheta^2}, \\
\phi_3 &= \frac{2e^{3\vartheta}(e^{\vartheta}-1)^2}{3\vartheta^2}, \\
\phi_4 &= \frac{2}{3}e^{-2\vartheta}(-11e^{\vartheta}+16e^{2\vartheta}+1), \\
\phi_5 &= \frac{2}{3}(e^{\vartheta}+1)(-28e^{\vartheta}+11e^{2\vartheta}+11), \\
\alpha_6 &= \frac{2}{3}e^{3\vartheta}(e^{2\vartheta}-11e^{\vartheta}+16).
\end{aligned}$$

Consider the following approximations:

$$\begin{aligned}
\ddot{y}_{i+1}^{(1)} &= \frac{-\ddot{y}_{i+1} + 4\,\ddot{y}_i - 3\,\ddot{y}_{i+1}}{2h}, \\
\ddot{y}_i^{(1)} &= \frac{\ddot{y}_{i+1} - \ddot{y}_{i-1}}{2h}, \\
\ddot{y}_{i-1}^{(1)} &= \frac{3\,\ddot{y}_{i+1} - 4\,\ddot{y}_i + \ddot{y}_{i-1}}{2h}.
\end{aligned} \tag{4}$$

3. APPLICATION OF EXPONENTIAL SPLINE METHOD

We examine the implementation of method (3) for the numerical resolution of model equations across diverse categories of single-perturbation issues. Consider the following singly perturbed model problem:

$$\begin{aligned}
&-\varepsilon\,\ddot{y}^{(2)}(\hat{z}) + \Psi_1(\hat{z})\,\ddot{y}^{(1)}(\hat{z}) + \Psi_2(\hat{z}) \\
&+ \mathscr{S}_1 e^{\ddot{y}(\hat{z})} + \mathscr{S}_2\,\ddot{y}^{(2)}(\hat{z}) + \mathscr{S}_3\left(\ddot{y}^{(1)}(\hat{z})\right)^2 \\
&+ \mathscr{S}_4\,\ddot{y}(\hat{z})\,\ddot{y}^{(1)}(\hat{z}) + \mathscr{G}(\hat{z}) = 0.
\end{aligned} \tag{5}$$

We now examine the utilization of this exponential spline technique and the finite-difference estimates (4) in relation to the nonlinear singly perturbed singularity equation (5), resulting in the following scheme:

$$\begin{aligned}
\varepsilon\mathscr{S}_{\tilde{z}}^2\,\ddot{y}_k &= 2h^2\widetilde{\mathscr{T}}\phi_2\Psi_{1k}\Psi_{1k-1}(\mathscr{S}_{\tilde{z}}^2 - \tilde{m}_{\tilde{z}}\mathscr{S}_{\tilde{z}})\,\ddot{y}_k \\
&\quad + 2h^2\widetilde{\mathscr{T}}\phi_2\Psi_{1k}\Psi_{1k+1}(\mathscr{S}_{\tilde{z}}^2 + \tilde{m}_{\tilde{z}}\mathscr{S}_{\tilde{z}})\,\ddot{y}_k \\
&\quad + h^3\widetilde{\mathscr{T}}\phi_2\Psi_{1k}(2\tilde{m}_{\tilde{z}}\mathscr{S}_{\tilde{z}} - \mathscr{S}_{\tilde{z}}^2 - 2)\,\ddot{y}_k \\
&\quad + h^3\ddot{\gamma}\widetilde{\mathscr{T}}\phi_2\Psi_{1k}\Psi_{2k+1}(2\tilde{m}_{\tilde{z}}\mathscr{S}_{\tilde{z}} + \mathscr{S}_{\tilde{z}}^2 + 2)\,\ddot{y}_k \\
&\quad + 2h^3\ddot{\gamma}\phi_2\Psi_{1k}(\mathscr{G}_{k+1} - \mathscr{G}_{k-1}) \\
&\quad + 2h^3\ddot{\gamma}\phi_2\Psi_{1k}\vartheta_2(2+\mathscr{S}_{\tilde{z}}^2)(2\tilde{m}_{\tilde{z}}\mathscr{S}_{\tilde{z}})\,\ddot{y}_k \\
&\quad + 2h^3\ddot{\gamma}\phi_2\Psi_{1k} + \vartheta_1\left(e^{\tilde{m}_{\tilde{z}}\mathscr{S}_{\tilde{z}}\ddot{y}_k} - e^{-\mu_r\mathscr{S}_{\tilde{z}}\ddot{y}_k}\right) \\
&\qquad \times e^{(1+\frac{1}{2}\mathscr{S}_{\tilde{z}}^2)\ddot{y}_k} + h^2\ddot{\gamma}\phi_2\vartheta_4\Psi_{1k} \\
&\quad + h^2\ddot{\gamma}\phi_2\vartheta_4\Psi_{1k}\left(\ddot{y}_k\mathscr{S}_{\tilde{z}}^2\,\ddot{y}_k + (2\tilde{m}_{\tilde{z}}\mathscr{S}_{\tilde{z}}\,\ddot{y}_k)^2 + 2\mathscr{S}_r^4\,\ddot{y}_k\right) \\
&\quad + h\phi_2\Psi_{1k}(4\ddot{\gamma}\vartheta_3\mathscr{S}_{\tilde{z}}^2 + 1)\,\ddot{y}_k(2\tilde{m}_{\tilde{z}}\mathscr{S}_{\tilde{z}})\,\ddot{y}_k \\
&\quad + h\phi_1\Psi_{1k}(\tilde{m}_{\tilde{z}}\mathscr{S}_{\tilde{z}} - \mathscr{S}_{\tilde{z}}^2)\,\ddot{y}_k \\
&\quad + \phi_1\Psi_{1k+1}(\tilde{m}_{\tilde{z}}\mathscr{S}_{\tilde{z}} + \mathscr{S}_{\tilde{z}}^2)\,\ddot{y}_k \\
&\quad + 2h^2\phi_2\Psi_{2k}\ddot{y}_k + h^2\phi_1\Psi_{2k-1}\left(1 - \tilde{m}_{\tilde{z}}\mathscr{S}_{\tilde{z}} + \tfrac{1}{2}\mathscr{S}_{\tilde{z}}^2\right)\,\ddot{y}_k \\
&\quad + h^2\phi_1\Psi_{2k+1}\left(1 + \tilde{m}_{\tilde{z}}\mathscr{S}_{\tilde{z}} + \tfrac{1}{2}\mathscr{S}_{\tilde{z}}^2\right)\,\ddot{y}_k \\
&\quad + h^2(\mathscr{G}_{k+1} + \mathscr{G}_{k-1}) + 2h^2\phi_2\mathscr{G}_k + 2\phi_1\vartheta_3\mathscr{S}_{\tilde{z}}^4\,\ddot{y}_k \\
&\quad + h^2\phi_1\vartheta_2\left(2\ddot{y}_k\mathscr{S}_{\tilde{z}}^2\,\ddot{y}_k + \tfrac{1}{2}\vartheta_3(\alpha+\phi_2)(2\tilde{m}_{\tilde{z}}\mathscr{S}_{\tilde{z}}\,\ddot{y}_k)^2\right).
\end{aligned} \tag{6}$$

The exponential spline (6) is of fourth order for $\omega = -1/(20\varepsilon)$. Nonetheless, the technique is ineffective when the coefficients $\Psi_1(\tilde{z})$, $\Psi_2(\tilde{z})$, and $\mathscr{G}(\tilde{z})$ exhibit singularities, particularly when solutions are sought at $k = \pm 1$. We address this challenge by altering the scheme to ensure that solutions maintain their order and precision, even near the singularity. We use the following Taylor approximation:

$$\begin{aligned}
\Psi_{1,k\pm1} &= \Psi_{1k} \pm h\Psi_{1k}^{(1)} + \frac{h^2}{2}\Psi_{1k}^{(2)} \pm \frac{h^3}{6}\Psi_{1k}^{(3)} \\
&\quad + \frac{h^4}{24}\Psi_{1k}^{(4)} \pm \frac{h^5}{120}\Psi_{1k}^{(5)} + \mathscr{O}(h^6).
\end{aligned} \tag{7}$$

By applying Taylor's approximation to $\Psi_{1,k\pm1}$, $\Psi_{2,k\pm1}$, and $\mathscr{G}_{k\pm1}$ in Equation (7), and disregarding $\mathscr{O}(h^6)$ terms, we formulate the ensuing exponential spline method in concise notation:

$$
\begin{aligned}
&2h^3\phi_2\ddot{\gamma}\vartheta_1\Psi_{1k}e^{\ddot{y}_k}(2\tilde{m}_{\hat{z}}\mathscr{S}_r)\,\dddot{y}_k\\
&+h^2\vartheta_1 e^{\ddot{y}_k}\left(\tfrac{1}{4}\phi_1(2\tilde{m}_{\hat{z}}\mathscr{S}_r\ddot{y}_k)^2+2\Psi_{2k}+2\phi_1+\phi_1\mathscr{S}_r^2\ddot{y}_k\right)\\
&+h^4\Big(\phi_1\mathscr{G}_k^{(2)}+\phi_1\Psi_{2k}^{(2)}\,\dddot{y}_k\\
&\quad+4\phi_2\ddot{\gamma}\Psi_{1k}\Psi_{2k}^{(1)}\,\dddot{y}_k+4\phi_2\ddot{\gamma}\Psi_{1k}\mathscr{G}_k^{(1)}\Big)\\
&+h^3\Big(\phi_1\Psi_{2k}^{(1)}+4\phi_2\omega\theta_2\Psi_{1k}\,\dddot{y}_k+\tfrac{1}{2}\phi_1\Psi_{1k}^{(2)}\\
&\quad+2\phi_2\ddot{\gamma}\Psi_{1k}(\Psi_{2k}+\Psi_{1k}^{(1)})\Big)(2\tilde{m}_{\hat{z}}\mathscr{S}_r)\,\dddot{y}_k\\
&+h^2\left(\phi_2\ddot{\gamma}\vartheta_4\Psi_{1k}+\tfrac{1}{2}\phi_1\vartheta_2\right)(2\tilde{m}_{\hat{z}}\mathscr{S}_r\ddot{y}_k)^2\\
&+h^2(2\phi_1\vartheta_2\ddot{y}_k+4\phi_2\ddot{\gamma}\vartheta_4\Psi_{1k}\ddot{y}_k\\
&\quad+4\phi_2\ddot{\gamma}\Psi_{1k}+2\phi_1\Psi_{1k}^{(1)}+\phi_1\Psi_{2k}\Big)\mathscr{S}_r^2\,\dddot{y}_k\\
&+h^2\left(2\phi_1\Psi_{2k}\ddot{y}_k+2\phi_1\vartheta_2\ddot{y}_k^2+2\phi_2\vartheta_2\,\ddot{y}_k^2\right.\\
&\quad+2\phi_2\Psi_{2k}\ddot{y}_k+2(\phi_1+\phi_2)\mathscr{G}_k\\
&\quad+h\left[\left(\tfrac{3}{2}\phi_1\vartheta_4+4\phi_2\ddot{\gamma}\vartheta_3\Psi_{1k}\right)\mathscr{S}_r^2\,\dddot{y}_k\right.\\
&\qquad\left.\left.+\phi_1\vartheta_4\ddot{y}_k+\phi_2\vartheta_4\ddot{y}_k+\phi_2\Psi_{1k}\right](2\tilde{m}_{\hat{z}}\mathscr{S}_r)\ddot{y}_k\right).
\end{aligned}
\tag{8}
$$

The exponential spline (8) demonstrates fourth-order precision and lacks the parameters $1/(k+1)$, thereby enabling uncomplicated resolution for $k=1,\ldots,n$.

4. NUMERICAL SOLUTION

To demonstrate the efficacy of the exponential spline function method in addressing boundary value problems for non-singularly and singularly perturbed cases, the case studies have been addressed with the suggested approach using different values of n and ε. All computations are executed using Maple 22.

Example (1): Consider the nonlinear problem

$$
\begin{aligned}
&\varepsilon\,\dddot{y}^{(2)}(\hat{z})+\frac{2}{\hat{z}}\,\dddot{y}^{(1)}(\hat{z})-\frac{2}{\hat{z}}\,\dddot{y}(\hat{z})+e^{\ddot{y}(\hat{z})}\\
&+(\ddot{y}(\hat{z}))^2+\left(\dddot{y}^{(1)}(\hat{z})\right)^2+\dddot{y}(\hat{z})\,\dddot{y}^{(1)}(\hat{z})\\
&=\varepsilon^2\sinh(\hat{z})+\frac{\varepsilon\cosh(\hat{z})}{\hat{z}}-\frac{\varepsilon\sinh(\hat{z})}{\hat{z}^2}\\
&+\varepsilon^2\sinh^2(\hat{z})+\varepsilon^2\cosh^2(\hat{z})+\varepsilon^2\sinh(\hat{z})\cosh(\hat{z}),
\end{aligned}
$$

with boundary conditions $\ddot{y}(0)=1$ and $\ddot{y}(1)=1$. The analytical solution is

$$\ddot{y}(\hat{z})=\varepsilon\sinh(\hat{z}).$$

Table 1 presents the maximum absolute errors for different values of ε and n.

Example (2): Consider the nonlinear problem

$$\varepsilon\ddot{y}^{(2)}(\hat{z})+(\ddot{y}(\hat{z}))^2=1,$$

with boundary conditions $\ddot{y}(0)=1$ and $\ddot{y}(1)=1$. The analytical solution is

$$\ddot{y}(\hat{z})=1+\hat{z}+\varepsilon\left[\log\left(1+e^{(2\hat{z}-1)/\varepsilon}\right)-\log\left(1+e^{-1/\varepsilon}\right)\right].$$

Table 2 presents the maximum absolute errors for different values of ε and n.

Table 1. The maximum number of absolute mistakes for example (1).

ε	$\mathscr{N}$				
	10	20	40	80	160
0.05000	4.32×10^{-6}	2.51×10^{-6}	1.29×10^{-7}	5.91×10^{-8}	3.31×10^{-8}
0.02500	6.25×10^{-8}	2.16×10^{-7}	1.46×10^{-8}	4.57×10^{-9}	2.73×10^{-9}
0.01250	6.32×10^{-10}	4.13×10^{-10}	2.19×10^{-10}	5.61×10^{-11}	2.74×10^{-11}
0.00625	1.74×10^{-10}	7.34×10^{-11}	3.77×10^{-11}	1.46×10^{-12}	1.03×10^{-12}

Table 2. The maximum number of absolute mistakes for example (2).

ε	$\mathscr{N}$				
	10	20	40	80	160
0.05000	1.64×10^{-7}	8.65×10^{-9}	7.18×10^{-10}	6.43×10^{-11}	3.21×10^{-12}
0.02500	1.19×10^{-7}	6.87×10^{-9}	5.23×10^{-10}	4.71×10^{-11}	2.74×10^{-12}
0.01250	1.04×10^{-8}	4.89×10^{-9}	5.81×10^{-10}	3.01×10^{-11}	1.56×10^{-12}
0.00625	6.13×10^{-9}	5.20×10^{-9}	9.14×10^{-10}	8.52×10^{-11}	7.11×10^{-13}

5. CONCLUSION

We employ a computational method using exponential splines to evaluate the solution range for singularly perturbed boundary issues with undetermined parameters. The numerical findings indicate that the present method closely approximates the exact solution. The suggested technique exhibits almost second-order regular convergence concerning the perturbation variable. We performed a sensitivity evaluation employing several methodologies, illustrating that the perturbed value does not influence the result.

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