Accurate Numerical Method Using Exponential Spline for solving boundary Value Problems

Ahmed R Khlefha[1,*]

[1] Department of Mathematics, College of Education, University of Sumer, Thi-Qar, Iraq

Email: arkdsh85@gmail.com

Abstract

This study introduces a precise numerical technique employing exponential splines for singly perturbed singularity boundary values problems. A numerical scheme is devised to address issues encountered in diverse scientific and engineering domains. The framework consists of a triad of nonlinear equations. The approach is employed in several test cases to demonstrate accuracy and implementation.

Keywords: Finite difference; Absolute errors; Exponential Spline

1. Introduction

Considering the nonlinear singular perturbation problems,

$$\epsilon \dddot{y}^{(2)}(\hat{z}) - \hat{\mathcal{P}}\left(\dddot{y}^{(1)}(\hat{z}), \dddot{y}_{\iota}, \hat{z}\right) = 0. \tag{1}$$

Moreover, with regard to boundary conditions

$$\dddot{y}(0) = \mathcal{B}_1, \dddot{y}(1) = \mathcal{B}_2.$$

Where $a \leq \hat{z} \leq b$, $0 < \hat{z} < 1$, $\mathcal{B}_1$ and $\mathcal{B}_2$, are finite constants, provided that $\hat{\mathcal{P}}$ Is a constrained and continuous function.

Various applications in science and engineering address challenges that depict intricate physical and chemical models. The resolution of solitary perturbation problems demonstrates a multi-scale nature. Numerous techniques, including finite difference, boundary element, and collocation approaches, are available for addressing linear singular perturbation problems [1–14]. Dag at al [15] proposed a numerical solution of singularly boundary value problems via the finite difference approach. The collocation technique employed parabolic and square B-spline basis functions on a mathematically evaluated mesh of the solution domain. Rashidina at el [16] analyzed self-adjoint singular perturbed boundary value problems. Recently, Tirmizi [17] Implemented the spline approach for straight singularity perturbations problems, achieving both the second and fourth levels of convergence based on the selection of empty parameters. Kadalbajoo at el [13] analyzed a second degree converging smoothing within an expansion technique for nonlinear single perturbation scenarios. Nonetheless, their methodologies are only pertinent to nonsingular issues. Challenges were encountered historically in the numerical resolution of boundary value. Solution typically degrades around the singularity. This study aims to provide a computationally efficient numerical method utilizing exponential spline, ensuring fourth-order convergence for decreasing values of ε while eliminating griding size restrictions in the presence of singularities. The document is structured as outlined below: Section two provides a concise overview of the derivation of exponential spline. In section three, we propose the application of the exponential spline method to singular problems. Section four presents the numerical results. Finely, presented the conclusions in Section five.

DOI: https://doi.org/10.54216/PMTCS.040204

Received: January 28, 2024 Revised: May 20, 2024 Accepted: August 17, 2024

2. Derivation of Exponential Spline

We examine an identical mesh Λ with nodal points $\tilde{z}_i$ on the interval $[a, b]$ such that,$a = \hat{z}_0 < \tilde{z}_i < \hat{z}_i < \cdots \hat{z}_{n-1} < \hat{z}_n = \dot{b}, \hat{z}_i = \dot{a} +, ih, i = 0,1,2, \dots, \dots, n$. To $\dot{a} = \hat{z}_0, \dot{b} = \hat{z}_n$ and $h = \frac{\dot{b}-\dot{a}}{n}$.

the exponential splin $\dddot{\mathcal{K}}(\hat{z})$ that estimates $\dddot{y}(\hat{z})$ at the mesh points $\hat{z}_i, i = 0,1,2, \dots, n-1, n$ depends on a parameter ω, For every segment $[\hat{z}_i, \hat{z}_{i+1}]$, $i = 0,1,2, \dots, n - ,1, n$ the exponential spline functions, $\widehat{\mathcal{K}}(\tilde{z})$, has the following form:

$$\dddot{\mathcal{K}}(\hat{z}_i) = \hat{a} e^{\omega(\hat{z}-\hat{z}_i)} + \hat{b} e^{2\omega(\hat{z}-\hat{z}_i)} + \hat{c} e^{3\omega(\hat{z}-\hat{z}_i)} + \hat{d} e^{4\omega(\hat{z}-\hat{z}_i)} \qquad (2)$$

where $\hat{a}, \hat{b}, \hat{c}$ and $\hat{d}$ are arbitrary real and finite constants, and ω is a free parameter. In order to find the four coefficients of integration for equation (2) using $\dddot{y}_i, \dddot{y}_{i+1}$, $\ddot{\Psi}_i$, and $\ddot{\Psi}_{i+1}$ .We define the interpolator conditions for the variable for $\hat{\mathrm{z}}_i$ and $\hat{z}_{i+1}$,

$$\dddot{\mathcal{K}}(\hat{z}_i) = \dddot{y}_i,$$

$$\dddot{\mathcal{K}}(\hat{z}_{i+1}) = \dddot{y}_{i+1},$$

$$\dddot{\mathcal{K}}(\hat{z}_i) = \ddot{\Psi}_i,$$

$$\dddot{\mathcal{K}}^{(2)}(\hat{z}_i) = \ddot{\Psi}_{i+1},$$

A spline can be defined in terms of $\dddot{y}_i, \dddot{y}_{i+1}$, $\ddot{\Psi}_i$, and $\ddot{\Psi}_{i+1}$, the coefficients presented in Equation (2) are computed as follows:

$$\hat{a} = e^{-\vartheta} - e^{3\vartheta}(-5 + 7e^{\vartheta})\ddot{\Psi}_i + (7 - 5e^{\vartheta})\ddot{\Psi}_{i+1} + 4\lambda^2 + 4\lambda^2 e^{3\theta}(-20 + 7e^{\vartheta})\dddot{y}_i$$

$$(20e^{2\vartheta} - 7)\dddot{y}_{i+1},$$

$$\hat{b} = e^{3\vartheta}\left((7e^{2\vartheta} + 7e^{\vartheta} - 8)\,\ddot{\Psi}_i + (8e^{2\vartheta} - 7e^{\vartheta} - 7)\ddot{\Psi}_{i+1} - \lambda^2\left(e^{3\vartheta}(7e^{2\vartheta} + 7e^{\vartheta} - 128)\dddot{y}_i + \right.\right.$$

$$\left.\left.(128e^{2\vartheta} - 7 - 7e^{\vartheta})\dddot{y}_{i+1}\right)\right),$$

$$\hat{c} = (e^{2\vartheta} + e^{3\vartheta} - 4e^{4\vartheta})\ddot{\Psi}_i - (e^{\vartheta} + e^{2\vartheta})\ddot{\Psi}_{i+1} + \lambda^2(e^{2\vartheta}(e^{2\theta} - 4e^{\vartheta})\dddot{y}_i + (4e^{\vartheta} + 4e^{2\vartheta} - 1)\dddot{y}_{i+1}$$

$$\hat{d} = e^{-2\vartheta}\left(e^{2\vartheta}(5e^{\vartheta} - 3)\ddot{\Psi}_i + (5e^{\vartheta} - 3)\ddot{\Psi}_{i+1} - \lambda^2\left(e^{2\vartheta}(5e^{\vartheta} - 27)\dddot{y}_{i+1} + (27e^{\vartheta} - 5)\dddot{y}_{i+1}\right)\right).$$

Where $\vartheta = h\omega$. Applying the continuity of the first derivative of $\mathcal{K}_i^{(1)}(\hat{z}) = \mathcal{K}_{i-1}^{(1)}(\tilde{z})$ at $\hat{z} = \hat{z}_i$ for $i = 1, \dots, n - 1$, we get the following consistency relation:

$$h^2(\phi_1 M_{i-1} + \phi_2 M_i + \phi_3 M_{i+1}) = \phi_4 \dddot{y}_{i+1} + \phi_5 \dddot{y}_i + \phi_6 \dddot{y}_{i-1} \qquad (3)$$

Where,

$$\phi_1 = \frac{2e^{-2\vartheta}(e^{\vartheta} - 1)^2}{3\vartheta^2},$$

$$\phi_2 = \frac{4(e^{\vartheta}+1)(e^{\vartheta}-1)^2}{3\vartheta^2},$$

$$\phi_3 = \frac{2e^{3\vartheta}(e^{\vartheta}-1)^2}{3\vartheta^2},$$

$$\phi_4 = \frac{2}{3}e^{-2\vartheta}(-11e^{\vartheta} + 16e^{2\vartheta} + 1),$$

$$\phi_5 = \frac{2}{3}(e^{\vartheta} + 1)(-28e^{\vartheta} + 11e^{2\vartheta} + 11),$$

$$\alpha_6 = \frac{2}{3}e^{3\vartheta}(e^{2\vartheta} - 11e^{\vartheta} + 16).$$

Consider the following approximations.

$$\dddot{y}_{i+1}^{(1)} = \frac{-\dddot{y}_{i+1} + 4\dddot{y}_i - 3\dddot{y}_{i+1}}{2h}$$

DOI: https://doi.org/10.54216/PMTCS.040204
Received: January 28, 2024 Revised: May 20, 2024 Accepted: August 17, 2024

$$\dddot{y}_i^{(1)} = \frac{\dddot{y}_{i+1} - \dddot{y}_{i-1}}{2h}$$

$$\dddot{y}_{i-1}^{(1)} = \frac{3\ddot{y}_{i+1} - 4\ddot{y}_i + \ddot{y}_{i-1}}{2h} \tag{4}$$

3. Application of Exponential Spline Method

We examine the implementation of method (3) for the numerical resolution of model equations across diverse categories of single perturbation issues. Examine the subsequent individually perturbed model problem.

$$-\varepsilon\ddot{y}^{(2)}(\hat{z}) + \Psi_1(\hat{z})\ddot{y}^{(1)}(\hat{z}) + \Psi_2(\hat{z}) + \mathcal{S}_1 e^{\ddot{y}(\hat{z})} + \mathcal{S}_2\ddot{y}^{(2)}(\hat{z}) + \mathcal{S}_3\left(\ddot{y}^{(1)}(\hat{z})\right)^2 + \mathcal{S}_4\ddot{y}(\hat{z})\ddot{y}^{(1)}(\hat{z}) + \mathcal{G}(\hat{z}) = 0,$$
(5)

We now examine the utilization of this exponential spline technique and finite differences estimates (4) in relation to the nonlinear singly perturbation singularity equation (5), resulting in the following findings:

$$\begin{aligned}
&\varepsilon\mathcal{S}_{\tilde{z}}^2\ddot{y}_k = 2h^2\tilde{\mathcal{T}}\phi_2{\Psi_1}_k{\Psi_1}_{k-1}(\mathcal{S}_{\tilde{z}}^2 - \tilde{m}_{\tilde{z}}\mathcal{S}_{\tilde{z}})\ddot{y}_k + 2h^2\tilde{\mathcal{T}}\,\phi_2{\Psi_1}_k{\Psi_1}_{k+1}(\mathcal{S}_{\tilde{z}}^2 + \tilde{m}_{\tilde{z}}\mathcal{S}_{\tilde{z}})\ddot{y}_k + h^3\tilde{\mathcal{T}}\,\phi_2{\Psi_1}_k \\
&(2\tilde{m}_{\tilde{z}}\mathcal{S}_{\tilde{z}} - \mathcal{S}_{\tilde{z}}^2 - 2)\ddot{y}_k + h^3\ddot{\gamma}\tilde{\mathcal{T}}\phi_2{\Psi_1}_k{\Psi_2}_{k+1}(2\tilde{m}_{\tilde{z}}\mathcal{S}_{\tilde{z}} + \mathcal{S}_{\tilde{z}}^2 + 2)\ddot{y}_k + 2h^3\ddot{\gamma}\,\phi_2{\Psi_1}_k(\mathcal{G}_{k+1} - \mathcal{G}_{k-1}) + \\
&2h^3\ddot{\gamma}\,\phi_2{\Psi_1}_k\vartheta_2(2 + \mathcal{S}_{\tilde{z}}^2)(2\tilde{m}_{\tilde{z}}\mathcal{S}_{\tilde{z}})\ddot{y}_k + 2h^3\ddot{\gamma}\,\phi_2{\Psi_1}_k + \vartheta_1\left(e^{\tilde{m}_{\tilde{z}}\mathcal{S}_{\tilde{z}}\ddot{y}_k} - e^{-\mu_r\mathcal{S}_{\tilde{z}}\ddot{y}_k}\right)e^{\left(1+\frac{1}{2}\mathcal{S}_{\tilde{z}}^2\right)\ddot{y}_k} + \\
&h^2\ddot{\gamma}\,\phi_2\vartheta_4{\Psi_1}_k + h^2\ddot{\gamma}\,\phi_2\vartheta_4{\Psi_1}_k(\ddot{y}_k\mathcal{S}_{\tilde{z}}^2\ddot{y}_k + (2\tilde{m}_{\tilde{z}}\mathcal{S}_{\tilde{z}}\ddot{y}_k)^2 + 2\mathcal{S}_r^4\ddot{y}_k) + h\,\phi_2{\Psi_1}_k(4\ddot{\gamma}\vartheta_3\mathcal{S}_{\tilde{z}}^2 + 1)\ddot{y}_k \\
&(2\tilde{m}_{\tilde{z}}\mathcal{S}_{\tilde{z}})\ddot{y}_k + h\phi_1{\Psi_1}_k(\tilde{m}_{\tilde{z}}\mathcal{S}_{\tilde{z}} - \mathcal{S}_{\tilde{z}}^2)\ddot{y}_k + \phi_1{\Psi_1}_{k+1}(\tilde{m}_{\tilde{z}}\mathcal{S}_{\tilde{z}} + \mathcal{S}_{\tilde{z}}^2)\ddot{y}_k + 2h^2\,\phi_2{\Psi_2}_k\ddot{y}_k + h^2\phi_1{\Psi_2}_{k-1} \\
&\left(1 - \tilde{m}_{\tilde{z}}\mathcal{S}_{\tilde{z}} + \frac{1}{2}\mathcal{S}_{\tilde{z}}^2\right)\ddot{y}_k + h^2\phi_1{\Psi_2}_{k+1}\left(1 + \tilde{m}_{\tilde{z}}\mathcal{S}_{\tilde{z}} + \frac{1}{2}\mathcal{S}_{\tilde{z}}^2\right)\ddot{y}_k + h^2(\mathcal{G}_{k+1} + \mathcal{G}_{k-1}) + 2h^2\,\phi_2\mathcal{G}_k \\
&+2\phi_1\vartheta_3\mathcal{S}_{\tilde{z}}^4\ddot{y}_k + h^2\phi_1\vartheta_2(2\ddot{y}_k\mathcal{S}_{\tilde{z}}^2\ddot{y}_k + \tfrac{1}{2}\vartheta_3(\alpha + \phi_2)(2\tilde{m}_{\tilde{z}}\mathcal{S}_{\tilde{z}}\ddot{y}_k)^2
\end{aligned} \tag{6}$$

The exponential spline (6) is of fourth order for $\omega = -\frac{1}{20\varepsilon}$. Nonetheless, the technique is ineffective when the coefficients $\Psi_1(\tilde{z}), \Psi_2(\tilde{z})$ and $\mathcal{G}(\tilde{z})$ exhibit singularities, particularly when solutions are sought at $k = \pm 1$. We address this challenge by altering the scheme (6) to ensure that solutions maintain their order and precision, even near the singularity. We utilize the subsequent Taylor approximation.

$${\Psi_1}_{k\pm1} = {\Psi_1}_k \pm h\,\Psi_{1k}^{(1)} + \frac{h^2}{2}\Psi_{1k}^{(2)} \pm \frac{h^3}{6}\Psi_{1k}^{(3)} + \frac{h^4}{24}\Psi_{1k}^{(4)} \pm \frac{h^5}{120}\Psi_{1k}^{(5)} + O(\mathscr{k}^6) \tag{7}$$

By applying Taylor's approximation to ${\Psi_1}_{k\pm1}, {\Psi_2}_{k\pm1}$ and $\mathcal{G}_{k\pm1}$ in equation (7) and disregarding $O(\mathscr{k}^6)$ terme we formulate the ensuing exponential spline methods for operation concise notation.

$$\begin{aligned}
&2h^3\,\phi_2\ddot{\gamma}\vartheta_1{\Psi_1}_k e^{\dddot{y}_k}(2\tilde{m}_{\tilde{z}}\mathcal{S}_r)\ddot{y}_k + h^2\vartheta_1 e^{\dddot{y}_k}\left(\frac{1}{4}\,\phi_1(2\tilde{m}_{\tilde{z}}\mathcal{S}_r\ddot{y}_k)^2 + 2{\Psi_2}_k + 2\phi_1 + \phi_1\mathcal{S}_r^2\dddot{y}_k\right) \\
&+h^4\left(\phi_1\mathcal{G}_k^{(2)} + \phi_1\Psi_{2k}^{(2)}\dddot{y}_k + 4\,\phi_2\ddot{\gamma}{\Psi_1}_k\Psi_{2k}^{(1)}\dddot{y}_k + 4\,\phi_2\ddot{\gamma}{\Psi_1}_k\mathcal{G}_k^{(1)}\right) + h^3\left(\phi_1\Psi_{2k}^{(1)} + 4\,\phi_2\omega\theta_2{\Psi_1}_k\dddot{y}_k\right. \\
&\left.+\frac{1}{2}\phi_1\Psi_{1k}^{(2)} + 2\,\phi_2\ddot{\gamma}{\Psi_1}_k\left({\Psi_2}_k + \Psi_{1k}^{(1)}\right)\right)(2\tilde{m}_{\tilde{z}}\mathcal{S}_r)\ddot{y}_k + h^2\left(\phi_2\ddot{\gamma}\vartheta_4{\Psi_1}_k + \frac{1}{2}\phi_1\vartheta_2\right)(2\tilde{m}_{\tilde{z}}\mathcal{S}_r\ddot{y}_k)^2 \\
&+h^2\left(2\phi_1\vartheta_2\dddot{y}_k + 4\,\phi_2\ddot{\gamma}\vartheta_4{\Psi_1}_k\dddot{y}_k + 4\,\phi_2\ddot{\gamma}{\Psi_1}_k + 2\phi_1\Psi_{1k}^{(1)} + \phi_1{\Psi_2}_k\right)\mathcal{S}_r^2\dddot{y}_k + h^2\left(2\phi_1{\Psi_2}_k\dddot{y}_k +\right. \\
&+2\phi_1\vartheta_2\dddot{y}_k^2 + 2\,\phi_2\vartheta_2\dddot{y}_k^2 + 2\,\phi_2{\Psi_2}_k\dddot{y}_k + 2(\phi_1 + \phi_2)g_k + h\left(\left(\frac{3}{2}\phi_1\vartheta_4 + 4\,\phi_2\ddot{\gamma}\vartheta_3{\Psi_1}_k\right)\mathcal{S}_r^2\dddot{y}_k +\right. \\
&\left.\left.+\phi_1\vartheta_4\ddot{y}_k + \phi_2\vartheta_4\ddot{y}_k + \phi_2{\Psi_1}_k\right)(2\tilde{m}_{\tilde{z}}\mathcal{S}_r)\ddot{y}_k\right.
\end{aligned} \tag{8}$$

The exponential spline 8) demonstrates fourth order precision and lacks the parameters $\frac{1}{(k+1)}$, thereby enabling uncomplicated resolution for for $k = 1, \ldots, \text{n}$.

DOI: https://doi.org/10.54216/PMTCS.040204
Received: January 28, 2024 Revised: May 20, 2024 Accepted: August 17, 2024

4. Numerical solution

To demonstrate the efficacy of the exponential spline Functions method in addressing boundary value problems for both non-singularly perturbed singular. The case studies have been addressed with the suggested approach with differing numbers for n and ϵ. All computations are executed using Maple 22.

Example (1): Consider the nonlinear problems.

$$\epsilon\dddot{y}^{(2)}(\hat{z})+\frac{2}{\tilde{z}}\dddot{y}^{(1)}(\hat{z})-\frac{2}{\tilde{z}}\ddot{y}(\hat{z})+e^{y(\ddot{\hat{z}})}+\left(\ddot{y}(\hat{z})\right)^{2}+\left(\dddot{y}^{(1)}(\hat{z})\right)^{2}+\ddot{y}(\hat{z})\ddot{y}^{(1)}(\hat{z})=\epsilon^{2}\,sinh(\hat{z})$$

$$+\ \frac{\epsilon cosh(\hat{z})}{\tilde{z}}-\frac{\epsilon sinh(\hat{z})}{\tilde{z}^{2}}\ +\ \epsilon^{2}sinh^{2}(\hat{z})+\ \epsilon^{2}cosh^{2}(\hat{z})\ +\ \epsilon^{2}sinh(\hat{z})cosh(\hat{z}),$$

with boundary condition $\ddot{y}(0)=1,\ddot{y}(1)=1$.

The analytical solution is given by,

$\ddot{y}(\hat{z})=\epsilon\,\sinh(\hat{z})$,

Table 1 presents the highest absolute errors for different values of ϵ and n.

Example (2): Consider the nonlinear problems.

$$\epsilon\ddot{y}^{(2)}(\hat{z})+\left(\ddot{y}(\hat{z})\right)^{2}=1,$$

with boundary condition $\ddot{y}(0)=1,\ddot{y}(1)=1$,

The analytical solution is given by,

$$\ddot{y}(\hat{z})=1+\hat{z}+\epsilon\left(\log\left(1+e^{\frac{(2\ddot{\hat{z}}-1)}{\epsilon}}\right)-\log\left(1+e^{\frac{-1}{\epsilon}}\right)\right)$$

Table 2 presents the highest absolute errors for different values of ϵ and n.

Table 1: The maximum number of absolute mistakes for example (1).

ϵ	$\mathcal{N}$				
	10	20	40	80	160
0.05000	4.32×10^{-6}	2.51×10^{-6}	1.29×10^{-7}	5.91×10^{-8}	3.31×10^{-8}
0.02500	6.25×10^{-8}	2.16×10^{-7}	1.46×10^{-8}	4.57×10^{-9}	2.73×10^{-9}
0.01250	6.32×10^{-10}	4.13×10^{-10}	2.19×10^{-10}	5.61×10^{-11}	2.74×10^{-11}
0.00625	1.74×10^{-10}	7.34×10^{-11}	3.77×10^{-11}	1.46×10^{-12}	1.03×10^{-12}

Table 2: The maximum number of absolute mistakes for example (2).

$\boldsymbol{\epsilon}$	$\mathcal{N}$				
	10	20	40	80	160
0.05000	1.64×10^{-7}	8.65×10^{-9}	7.18×10^{-10}	6.43×10^{-11}	3.21×10^{-12}
0.02500	1.19×10^{-7}	6.87×10^{-9}	5.23×10^{-10}	4.71×10^{-11}	2.74×10^{-12}
0.01250	1.04×10^{-8}	4.89×10^{-9}	5.81×10^{-10}	3.01×10^{-11}	1.56×10^{-12}
0.00625	6.13×10^{-9}	5.20×10^{-9}	9.14×10^{-10}	8.52×10^{-11}	7.11×10^{-13}

5. Conclusion

We employ a computational method that employs exponential splines to evaluate the solution range for singularly perturbed boundary issues with undetermined parameters. The numerical findings indicate that the present method closely approximates the exact solution. The suggested technique exhibits almost the second order regular convergence concerning the perturbation variable. We performed a sensitivity evaluation employing several methodologies, illustrating that the perturbed value does not influence the result.

Funding: "This research received no external funding"

Conflicts of Interest: "The authors declare no conflict of interest."

References

[1] Bawa RK. Spline based computational technique for linear singularly perturbed boundary value problems. Appl Math Comput 2005;167:225–36. https://doi.org/10.1016/j.amc.2004.06.112.

[2] Chandru M, Prabha T, Das P, Shanthi V. A Numerical Method for Solving Boundary and Interior Layers Dominated Parabolic Problems with Discontinuous Convection Coefficient and Source Terms. Differ Equations Dyn Syst 2017;27:91–112. https://doi.org/10.1007/s12591-017-0385-3.

[3] Mohapatra J, Shakti D. Numerical treatment for the solution of singularly perturbed pseudo-parabolic problem on an equidistributed grid. Nonlinear Eng 2020;9:169–74. https://doi.org/10.1515/nleng-2020-0006.

[4] Nikolis A, Seimenis I. Solving dynamical systems with cubic trigonometric splines. Appl Math E-Notes [Electronic Only] 2005;5:116–23.

[5] Vijayakumar V, Udhayakumar R, Kavitha K. On the approximate controllability of neutral integro-differential inclusions of Sobolev-type with infinite delay. Evol Equations & Control Theory 2021;10:271–396. https://doi.org/10.3934/eect.2020066.

[6] Zahra WK. Trigonometric B-Spline Collocation Method for Solving PHI-Four and Allen–Cahn Equations. Mediterr J Math 2017;14. https://doi.org/10.1007/s00009-017-0916-8.

[7] Chandru M, Shanthi V. Fitted Mesh Method for Singularly Perturbed Robin Type Boundary Value Problem with Discontinuous Source Term. Int J Appl Comput Math 2015;1:491–501. https://doi.org/10.1007/s40819-015-0030-1.

[8] Chandru M, Shanthi V. An asymptotic numerical method for singularly perturbed fourth order ODE of convection-diffusion type turning point problem. Neural, Parallel, Sci Comput 2016;24:473–88.

[9] Debela HG, Duressa GF. Uniformly convergent numerical method for singularly perturbed <scp>convection-diffusion</scp> type problems with nonlocal boundary condition. Int J Numer Methods Fluids 2020;92:1914–26. https://doi.org/10.1002/fld.4854.

[10] Debela HG, Duressa GF. Accelerated Exponentially Fitted Operator Method for Singularly Perturbed Problems with Integral Boundary Condition. Int J Differ Equations 2020;2020:1–8. https://doi.org/10.1155/2020/9268181.

[11] Debela HG, Duressa GF. Accelerated fitted operator finite difference method for singularly perturbed delay differential equations with non-local boundary condition. J Egypt Math Soc 2020;28:1–16.

[12] Jiwari R, Singh S, Singh P. Local RBF-FD-Based Mesh-free Scheme for Singularly Perturbed Convection-Diffusion-Reaction Models with Variable Coefficients. J Math 2022;2022. https://doi.org/10.1155/2022/3119482.

[13] Kadalbajoo MK, Patidar KC. Singularly perturbed problems in partial differential equations: a survey. Appl Math Comput 2003;134:371–429. https://doi.org/10.1016/s0096-3003(01)00291-0.

[14] Kumar N, Toprakseven Ş, Jiwari R. A numerical method for singularly perturbed convection–diffusion–reaction equations on polygonal meshes. Comput Appl Math 2024;43. https://doi.org/10.1007/s40314-023-02553-x.

[15] Hepson OE, Dag I. Numerical Solution of Singularly Perturbed Problems via both Galerkin and Subdomain Galerkin methods. Appl Math & Inf Sci 2018;12:495–500. https://doi.org/10.18576/amis/120303.

[16] Khan MA, Tirmizi IA, Twizell EH. Non polynomial spline approach to the solution of a system of third-order boundary-value problems. Appl Math Comput 2005;168:152–63.

[17] Rashidinia J, Mohammadi R. Non-Polynomial Spline approximations for the solution of singularlyperturbed boundary value problems. TWMS J Pure Appl Math 2010;1:236–51.

DOI: https://doi.org/10.54216/PMTCS.040204
Received: January 28, 2024 Revised: May 20, 2024 Accepted: August 17, 2024