On the Nature of Solutions of Discrete Time Lyapunov Equations

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Abstract

This paper provides a method to solve the discrete time Lyapunov equation. Identified and discussed. If the equation takes the following form:

$D(\lambda y+\mu z) = \lambda Dy + \mu Dz,\ y, z \in y;\ \lambda, \mu \in F$.

If ∃ a constant e∈∞ ∋ ||Dy|| ≤ e ||y||, y ∀Y. and D is bounded, then D is called a linear operator equation. In particular, (Lyapunov and Sylvester operator equations) are very important in differential equations, integral equations and many other branches of mathematics. The study of solutions and of the above equestion We also discussed operator equations and special kinds of operators and studied some elementary operators. These operators are generalizations of operators

$\tau_{AD}: D(H) \to D(H)$

$\tau_{AD}: \tau_{AD}(y) = Ay - yD, \quad y \in D(H)$

Keywords: Time Lyapunov Equation; bounded linear operators; Sylvester operator equations; complex Hilbert space; Banach algebra.

1. Introduction

For communication problems using elasticity theory, see the examples. They cover mixed bounded problems and some math physics problems. Those problems become integral problems, including both linear and nonlinear types. (Abdou. (M, A: 2003) Purpose: This paper seeks solutions to the second kind Fredholm Volterra integral equation in the Banach space L2(Ω)×C[0, T]. Both the Volterra integral term in [0, T] and the terms in the Fredholm integral Li2(Ω) are linear. Furthermore, every integral term in Volterra or Fredholm is linear. There is a nonlinear integral term as well.

. [1]. (Kuffi. They studied a more general solution of the Lyapunov equations in continuous time. : A*XD + DXA = W [11]. (Wu. J, et al: 2023) some methods use neural networks to represent Lyapunov functions [16].

This research studies natural solutions to the discrete-time Lyapunov equation. We also examine solutions to discrete-time Lyapunov operators. Additionally, we discuss these operators, their equations, and special cases.

2. Basic Properties of some Types operators:

Definition 2.1, [14]:

The normed space is the binary (X**,** ll. ll) such that X vector space over field of real or complex numbers **and** ll. ll **is a norm over X.**

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function ll ll : X → R.

Definition 2.2, [14]:

If ∃ a multiplication (u,x) = ux: XxX → X is linear, then a Banach space X is referred to as a Banach algebra. *for each factor*

If ∃ the inner function <.,.>: X xX →lR or C is defined. Then X is an inner product space.

Definition 2.3, [9]:

assuming that B operates in c(H). then it gets proven that B*, the only bounded linear operator, exists on H.

∋ < Bw, n> = <w, B^*n> for all w and n in H, B^* is called the adjoint

Definition 2.4, [9]:

Let $D \epsilon B(H)$ If the linear operator C is bounded on H , ∋ DC = CD = I then D is said to be invertible

Proposition 2.1:

If bounded linear operators g and 𝒭 and a scalar λ are on H, then

1. It equals (G+ 𝒭) * it equals G^* + 𝒭 *
2. (λG) * it equals $\bar{\lambda}G^*$, $\bar{\lambda}$ Є C

3- < G^*m, Ë > it equals < m, GË >   for all Ë, m Є H

4- $(G^*)^*$ it equals G

5- (G𝒭)* it equals 𝒭* G^*

6- ll G^* ll it equals ll G ll

7- ll G^* G ll it equals ll G G^* ll it equals ll G ll ^{2}

8- G^*G it equals o   if and only if G it equals o

Proposition 2.2, [15]:

The follomming conditions on a bounded operator B on H are equivalent.

1- B is Hermitian.

2- < Bz, y > = <𝓑, By >   ∀𝓑, y in H

1- <Bz,𝓑 > is real   ∀𝓑 in H

Remark 2.1 [15]:

1. If ß and *f* are Hermitian, then ß plus Ƒ is also Hermitian

2. If ß is Hermitian and β is real then β multiplication ß is Hermitian.

3. If ß is any operator then ß* multiplication ß and ß plus ß* are adjoint.

Definition 2.5, [15]:

A matrix K is called self adjoint. **if k=k***

Remark 2.2

1- If *f* is a Hermitian operator, then Ƒ equals its adjoint. Then Ƒ is a normal operator.

2- £ is a normal if S is a unitary operator, that is, S*S= S S* =l.

3 - · **Ƒ** is a normal operator if it is a skew Hermitian function, that is, **-Ƒ = Ƒ*,**

Remark 2.3 [15]:

1- iK is Hermitian if and only if K is skew – Hermitian.

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2. K- K* is skew and Hermitian for every operator K.

Lemma 2.1:

If Z is a linear operator and bounded on complex Hilbert space H.

, and also ⟨ Z 𝒷 𝒷 ⟩ = 0 ∀𝒷 in H, then Z = 0.

Proof:

Suppose 𝒷, y ϵ H and V = a multiplication 𝒷 plus Ëy where a and Ë ϵ C Then Ëy assumption < Z multiplication V, V> = 0 Thus $|a|^2$ <Z multiplication 𝒷,𝒷> plus a$\bar{\text{Ë}}$ < Z multiplication 𝒷, y> + Ë$\bar{a}$ <Z y,𝒷> + $|\text{Ë}|^2$ <Z y, y> = 0 Since <Z multiplication 𝒷, 𝒷 > = <Z multiplication y , y> = 0

then a multiplication $\bar{\text{Ë}}$ <Z multiplication 𝒷, y> + Ë$\bar{a}$ <Z multiplication y,𝒷> = 0

Now, if we take a it equals Ë it equals l, then <Z multiplication 𝒷, y> + <Z multiplication y,𝒷> it equals 0. So, if we put a it equals i and Ë it equals 1, then ii < Z multiplication 𝒷, y> - I <Z multiplication y,𝒷> it equals 0. Thus <Z multiplication 𝒷, y> it equals 0 ∀ y in H; this leads Z multiplication 𝒷 it equals 0 for all𝒷 in H and Z it equals 0.

Proposition 2.3, [15]:

The following conditions are the same on operator A

1- A is unitary

2- A^* is unitary

3- A is bijective and $A^{-1} = A^*$

Proposition 2.4, [15]:

The following conditions **are the same** on **operator** B

1- Ë is normal.

2- Ë^* is normal.

3- ll Ë^*zll = ll Ëzll for allË in H

Proof:

1- implies (2): Follows from $(\text{Ë}^*)^* =\text{Ë}$

2- implies (3):

< Ë^* multiplication Ëz,Ë > it equals <Ëz, Ëz> it equals ll Ëz ll^2 ∀ Ë in H,

Hence, < Ë^* multiplication Ëz,Ë> it equals ll Ë^*z ll ^{2} it equals < Ë^*z, Ë^*z> it equals < Ë multiplication Ë^*z, y> This implies < (Ë^* multiplication Ë-Ë Ë^*)Ë ,Ë > it equals 0 ∀z Є H

Thus by leman (1), (Ë^* multiplication Ë-ËË^*) it equals o and Hence Ë^* multiplication Ë it equals Ë multiplication Ë^*

Proposition 2.5:

If f is hyponormal, then $\| f \|^n$ it equals $\| f \|^n$ for every positive integer n.

Proof:

See [8]

Another type of operators that we will face in our work is the quasinipotent operators. But first note that an operator Ƒ is referred to as Nilpotent if ∃ a positive integer

m ∋ $Ƒ^m$ it equals 0 The Index of Nilpotency refers to the lowest such m. An operator

Ƒ is said to be quash nilpotent operator if $\| Ƒ^n \|^{1/n} \rightarrow$ o

As n→ ∞ it is obvious that Nilpotence implies quasinilpotence

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Proposition 2.6:

The equivalent is as follows: When ß operates on H, then.

1- ß is hyponormal.

2- ll ß*nll ≤ ll ßn ll ∀n Є H

Proof:

Since ß be a hyponormal operator, therefore ß * ß – ß ß* ≥ 0

Now ∀ n in H

ll ß*nll 2 = < (ß*n, ß*Type equation here.n) > it equals < (ß multiplication ß*n, n) > least than or equals < (ß* multiplication ßn, n)>

it equals < ßn, ßn> it equals ll ßn ll2 ∀ n in H

Conversely ∀ n is H

< ß* ßn , ßn > = < ß n , ßn > = ll ßn ll2 ≥ ll ß*ll 2

= < ß*n, ß*n) > it equals < (ß multiplication ß*n, y) >.

therefore < (ß* multiplication ß - ß multiplication ß*) n, n > ≥ 0 which implies that ß multiplication ß* - ß multiplication ß* ≥ 0 and ß is hyponormal operator

Definition 2.6, [8]:

If K goes with K Multiplication K*, then an operator K in B(H) is quasinormal.

Definition 2.7, [5]:

On a Hilbert space H, an operator M is binormal. if M and its adjoint M* commute. This means M*M = MM*.

<u>Remarks 2.3</u>

1. All normal operators are binormal.

2- Every normal operator that is quashable is binormal

Proof:

1- Let £ be a normal operator then (£ * £) multiplication (£ £ *) it equals (£ £ *) multiplication (£ * £), hence £ is a binormal operator.

2- Let £ be aquas normal operator, then £ multiplication (£ * £) it equals (£ * £) lead to (£ * £) multiplication £ * it equals £ * multiplication (£ * £) Now (£ * £) multiplication (£ £ *) it equals £ multiplication (£ * £) multiplication £ * it equals (£ £ *) multiplication (£ * £) and £ is binormal operator.

Proposition 2.7, [4]:

Let Đ be a hyponormal and binormal operator, then Đ Ž is hyponormal for Ž ≥ 1

Proof:

Since Đ is hyponormal, then Đ* multiplication Đ great than or equal to Đ Đ* great than or equal 0

Thus, one can easily check that Đ* multiplication (Đ* multiplication Đ) multiplication Đ That is, Đ*2 multiplication Đ2 great than or equal to (Đ* multiplication Đ)2 and (Đ multiplication Đ*) and (Đ multiplication Đ*)2 great than or equal Đ2 multiplication Đ*2. But Đ* multiplication Đ great than or equal Đ multiplication Đ* and [Đ*Đ, ĐĐ*] = 0 then by using [6] , we get (Đ*Đ)2 great than or equal (Đ multiplication Đ*)2 Therefore Đ*2 multiplication Đ2 great than or equal (Đ* multiplication Đ)2 great than or equal (Đ multiplication Đ*)2 great than or equal Đ2 multiplication Đ*2 and Đ2 is hyponormal .

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Since then Đ*ŽTŽ great than or equal (Đ* multiplication Đ) multiplication Ž great than or equal (Đ multiplication Đ*) multiplication Ž great than or equal ĐŽĐ*Ž to some Integer Ž great than or equal 2 . then Đ* multiplication Ž ĐŽ great than or equal (Đ*Đ)Ž great than or equal (Đ multiplication Đ*)Ž lead to Đ* Ž+1 multiplication ĐŽ+1 great than or equal (Đ*Đ)Ž+1 and (Đ*Đ)Ž great than or equal (ĐĐ*)Ž great than or equal ĐŽĐ*Ž Implies that (ĐĐ*) Ž+1 multiplication ĐŽ+1 Đ* Ž+1 But (Đ*Đ)Ž great than or equal (ĐĐ*) Implies that (Đ*Đ)Ž+1 great than or equal (ĐĐ*)Ž+1 Hence Đ*Ž+1 multiplication ĐŽ+1 great than or equal (Đ* multiplication Đ)Ž+1 great than or equal (ĐĐ*)Ž+1 great than or equal ĐŽ+1 multiplication Đ*Ž+1 Thus Đ Ž+1 is hyponormal .

Definition 2.8, [6]:

Consider the operator T in Hilbert space. If T* multiplication T commutes T plusT*, then H we say that T is θ operator.

Proposition 2.8:

The following statements hold if $\mathscr{R}$ is an operator on Hilbert space H:
1. The operator $\mathscr{R}$ is θ operator
2- * $\mathscr{R}$ * multiplication [$\mathscr{R}$ *, $\mathscr{R}$] it equals [$\mathscr{R}$ *, $\mathscr{R}$] multiplication $\mathscr{R}$

Proof:

Let $\mathscr{R}$ be a Θ -operator, then

$\mathcal{R}^*$ multiplication$[\mathcal{R}^*\mathcal{R}]$it equals $\mathcal{R}^*$multiplication $[\mathcal{R}^*\mathcal{R} - \mathcal{R}\mathcal{R}^*]$it equals $\mathcal{R}^*$ multiplication $\mathcal{R}^*\mathcal{R} -$ $\mathcal{R}^*$ multiplication$\mathcal{R}\mathcal{R}^*$ By Θ operator we get .

$\mathcal{R}^*$ multiplication$[\mathcal{R}^*, \mathcal{R}]$it equals$\mathcal{R}^*$ multiplication$\mathcal{R}\,\mathcal{R} - \mathcal{R}$ multiplication $\mathcal{R}^*\mathcal{R}$it equals$(\mathcal{R}^*\mathcal{R} -$ $\mathcal{R}\mathcal{R}^*)$ multiplication$\mathcal{R}$it equals$[\mathcal{R}^*, \mathcal{R}]$ multiplication$\mathcal{R}$

Conversely, since $\mathscr{R}^*$ multiplication ($\mathscr{R}^*$ $\mathscr{R}$) it equals [$\mathscr{R}^*$ $\mathscr{R}$] multiplication $\mathscr{R}$ then

$\mathcal{R}^*\mathcal{R}$ multiplication$[\mathcal{R} + \mathcal{R}^*]$ it equals $\mathcal{R}^*$multiplication $\mathcal{R}\,\mathcal{R}$
$+ \mathcal{R}^*$ multiplication $\mathcal{R}\,\mathcal{R}^*$ it equals $\mathcal{R}^*\,\mathcal{R}^*$ multiplication $\mathcal{R}$
$+ \mathcal{R}\,\mathcal{R}^*$multiplication $\mathcal{R}$ it equals
$= (\mathcal{R}^* + \mathcal{R})$ multiplication$(\mathcal{R}^*\mathcal{R})$

Hence $\mathscr{R}$ is a Θ- operator.

Proposition 2.9:

$\mathscr{B}$ -1 is a ، Θ -operator if $\mathscr{B}$ is a Θ -operator and $\mathscr{B}$-1 exists.

Proof:

since $\mathscr{B}^{-1}$ exists, then $\mathscr{B}^{*-1}$ exists suppose that $\mathscr{B}$ ЄΘ, then ($\mathscr{B}^*$ multiplication $\mathscr{B}$) ($\mathscr{B}$ +$\mathscr{B}^*$) it equals ($\mathscr{B}$ + $\mathscr{B}^*$) ($\mathscr{B}$ * multiplication $\mathscr{B}$). Multiply the equality on the right by$\mathscr{B}$-1 and also on the left by$\mathscr{B}$*-1, we obtain $\mathscr{B}$+$\mathscr{B}\mathscr{B}^*\mathscr{B}^{-1}$=$\mathscr{B}^{*-1}\mathscr{B}\mathscr{B}^*$+$\mathscr{B}^*$ Multiply the equality on the right by and also on the left by$\mathscr{B}$*-1$\mathscr{B}$-1

we obtain$\mathscr{B}^{*-1}\mathscr{B}^{*-1}\mathscr{B}^{-1}$ +$\mathscr{B}^{-1}\mathscr{B}^{*-1}\mathscr{B}^{-1}$ it equals $\mathscr{B}^{*-1}$ multiplication $\mathscr{B}^{-1}\mathscr{B}^{*-1}$ +$\mathscr{B}^{*-1}$ multiplication $\mathscr{B}^{-1\mathscr{B}-1}$

Thus ($\mathscr{B}^{*-1}$ +$\mathscr{B}^{-1}$) multiplication $\mathscr{B}^{*-1}$ $\mathscr{B}^{-1}$ it equals $\mathscr{B}^{*-1\mathscr{B}-1}$ multiplication ($\mathscr{B}^{*-1}$+$\mathscr{B}^{-1}$) and$\mathscr{B}^{-1}$ is a Θ-operator

Remarks 2.4:

1. Every normal operator may be Θ - operator.

1. Each of the questionably operators is Θ - operator.

Proof:

1. allow Ú be an anormal operator. then

(Ú* multiplication Ú) (Ú + Ú*) it equals Ú* multiplication Ú multiplication Ú + Ú* Ú multiplication Ú* it equals Ú Ú* Ú + Ú*Ú multiplication Ú

it equals (Ú + Ú*) (Ú multiplication Ú*) Thus Ú is Θ – operator

2-Let Ú be aquas normal operator then Ú multiplication (Ú*Ú) it equals (Ú*Ú) T therefore (Ú*Ú) multiplication Ú* it equals Ú* multiplication (Ú*Ú) thus (Ú*Ú) multiplication (Ú+Ú *) it equals (Ú*Ú) multiplication Ú + (Ú*Ú) multiplication Ú*

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It equals Ú multiplication ($Ú^*Ú$) +$Ú^*$($Ú^*Ú$) it equals (Ú+ $Ú^*$) multiplication $Ú^*$ Hence Ú is ϴ- operator

3. Some types of operator Equations

Definition 3.1 [7]:

The equation that has the following form

$$\hat{U}(y) = C \quad (1)$$

are called operator equations. In the Hilbert space H, we know the operators F and C. We need to determine the operator Y, which is currently unknown.

3.1 1. The Lyapunov equation for discrete time:

Note that The operator equation

$$y - K^*yK = Ú \quad \ldots\ldots\ldots\ldots (2)$$

1- Is called the Lyapunov equation for discrete time such that. Operators K and Ú operate in B(H). We must determine a non-known operator, y. [3]

In most cases, there may be only one solution, infinite solutions, or no solution.

Example (3.1.1):

Suppose $\mathcal{R} = L_2(M)$ the unilateral shift operator N on $L_2(M)$ is defined as $N(y_1, y_2 \ldots) = (o, y_1, y_2, \ldots)$

Consider eÚuestion (2) where K= n and Ú =o Therefore eq (2) reduces to the equation of the operator.

$$y - DyN = o$$

Where D is the bilateral shift operator which defined as $D(y_1, y_2, \ldots (y_2, y_3, y_4, \ldots)$

2- and n^*=D the above equation has infinite solutions. It is a discrete-time Lyapunov equation.

3- Namely y = o, y = N, and y = MI, where M is a constant.

Proposition (3.1.1):

Proof:

Assume self-adjoint operators Þ and $\mathscr{R}$. Equation (2) possesses a unique solution, Y, which makes it self-adjoint.

Think about question (2). So, Y* - $\mathscr{R}$ Y $\mathscr{R}$ = Þ. *Since $\mathscr{R}$ and Þ are self-adjoint operators, Y - $\mathscr{R}$ Y*

$\mathscr{R}$ = Þ, and Y - K*YK = Þ, but question (2) has a unique solution; thus, Y = Y*, Y is a self-adjoint operator as a result.

We will now look at equation (2)'s answer. we concentrate on normal operators, covering (bi-normal, quasi-normal, and ϴ-operator types.

Examples show that equation (2) can yield non-standard solutions. These solutions defy the usual rules for operators. Yet, for known operators $\mathscr{R}$ and Þ they remain normal. Operators categorize them as binormal, quasi-normal, and ϴ operator.

Example (3.1.2):

4- Consider equestion. (2). take $\mathscr{R}$ =il, Þ = 0. This equation has these solutions: Y=0, Y=M I, Y=n. M is arbitrary, and n is non-normal (not abnormal, quasi-normal, or ϴ-operator

3.2 Basic Jordan *- Derivation

Definition 3.2.1, [10]:

Let k be aring, A derivation h on k is a mapping of k into itself that hold the following

1- g (c+d) = g (c) + g (d) for all c, d ϵ k

2- g (cd) = c g (b) + g (c)d for all c, d ϵ k

Definition 3.2.2:

Assuming k be aring A Jordan derivation h for k is a mapping of k into itself that hold the following:

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1- g (c+d) = g (c)+ g (d) for all c, d in k

2- g (c^2) = c g (c) + g (c)c for all c in k

Definition (3.2.3), [8]:

Let R be aring R is said to be *- ring if there exists a map * of R into itself that satisfies the bellow

1- $*^{\wedge 2}$ (g) it equals g $\forall$ g Є R

2- * (g +d) it equals * (g) +*(d) $\forall$ g, d Є R

3- * (g d) it equals * (d) * (g) $\forall$ g,d ϵ R

Definition (3.2.4), [13]:

Assuming k be *-ring, Derivation J *- of Jordan on k is amapping from k into itself that satisfies the following:

1- J (g +d) = J(g) + J(d) $\forall$ g.d in k

2- J (g^2) = g J(\g) + J(g) g^* $\forall$ g in k

As an illustration consider the next example.

Example (3.2.1):

Let R=Þ we noted in Example (3.1.1), that Þ is *- relative to the conjugation-defined map, ring. Assuming that £: Þ ⟶ Þ is defined as £ (Ü)=Ü-n¯for every Ü in Þ, f is a Jordan *- derivation

Proof:

1- $£(Ü_1 + Ü_2)$ it equals $(Ü_1 + Ü_2) - \overline{(Ü_1 + Ü_2)}$

it equals $(Ü_1 + \overline{Ü_1}) + (Ü_2 + \overline{Ü_2})$ $\forall Ü_1, Ü_2 \in C$

Hence £ ($Ü_1$+$Ü_2$) it equals £($Ü_1$) + £($Ü_2$)

2- $Ü£(Ü) + £(Ü)Ü^*$ it equals $Ü(Ü - \overline{Ü}) + (Ü - \bar{Ü})\bar{Ü} = Ü^2 - Ü\bar{Ü} + Ü\bar{Ü} - \bar{Ü}^2$

it equals $Ü^2 - \overline{Ü^2}$

Thus Ü £ (Ü)+ £ (Ü)$Ü^*$ it equals £ ($Ü^2$)

3.3 The Map μ_K:

The bounded operators on an infinite, separable Hilbert space are denoted by C(H).

., complex Hilbert space H. for K in C(H), define μK : C(H) ⟶ C(H) by μK(y) = y - k*yk for y in C(H). Then, Rang(μK) = {y - k*yk : y ∈ C(H)}.

This section will examine a few characteristics of uF. In fact, this is obviously line graph U $(\mathcal{B}y_1 + \mathcal{R}y_2) = (\mathcal{B}y_1 + \mathcal{R}y_2) - k^*(\mathcal{B}y_1 + \mathcal{R}y_2)K$.

uF. In fact, this is obviously line graph U $(\mathcal{B}y_1 + \mathcal{R}y_2) = (\mathcal{B}y_1 + \mathcal{R}y_2) - k^*(\mathcal{B}y_1 + \mathcal{R}y_2)K$.

$$= \mathcal{B}y_1 + \mathcal{R}y_2 - \mathcal{B}K^*y_1K - \mathcal{R}K^*y_2K = \mathcal{B}u(y_1) + \mathcal{R}u(y_2).$$

We remember that μ_K (y) is one-to-one if Ker (u_K)= {0}.

Assuming Y$_\epsilon$ Ker (μ_K) so u_K(y)=0, then y=K^*yK. If K is an isometric operator, then IЄ Ker (μ_K) and Ker (μ_K) ≠ {0}. Therefore, u_K is not one-to-one and thus it is non invertible.

Remark (3.3.1):

If $\mathcal{R}$ is an isometric operator, then U $\mathcal{R}$ is not one-to-one.

For example,

Example (3.3.1):

The eq. (2.1) so μ_K it equals I-$R_{K*}L_K$, where R_{K*} it equals K^*y and L_k= yK μ_k(y) it equals 0 therefore y it equals 0.

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y-K^*yK it equals 0 so y it equals K^*yK. assume K it equals u so I it equals Ciu it equals I and 0 it equals Cou it equals 0. K^*K it equals CU it equals I. Because uk is not one-to-one and because then is an isometric operator, K is also an isometric operator, making it non-invertible.

Operator M is a projection operator, as we recall.

in case $M^2 = M$ $M^* = M$ **[2]** and M on a Hilbert space H

Proposition (3.3.1):

The uK is said to be not one to one and so is not invertible. If K is a non-zero projection operator,

Proof:

Since K is a projection operator then $\mu_K(y)$ it equals y-K^*yK.

Validating that is simple. K∊ Ker (μ_K). Since K≠0 thus Ker (μ_K) ≠ 0. Therefore, u_K is not one to one.

Example (3.3.2):

IF $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$and $A^*A \neq I$. Then A is not isometric.

Let y =A^*yA,

Let $y = \begin{bmatrix} \propto & m \\ \ddot{E} & \mathcal{R} \end{bmatrix}$. After that $\begin{bmatrix} \propto & m \\ \ddot{E} & \mathcal{R} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} \propto & m \\ \ddot{E} & \mathcal{R} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ After simple computation one gets $y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ϵ Ker

(μ_k). so it should, μ_k is not onto and is not one-to-one

Proposition (3.3.2):

1- Rang $(\mu_A)^*$ = Rang (μ_A)

2- ∝ Rang (μ_A) = Rang (μ_A) $\forall \propto \epsilon C$

3- The μ_A is bounded operator

Proof:

1- $\text{Reng}\ (\mu A)^* = \{(y - A^* \times y \times A)^*, y \epsilon C(H)\} = \{y^* - A^* \times y^* \times A, y \epsilon C(H)\}$.

$\text{let}\ y_1 = y^*, \text{then}\ (\text{Rang}\ (\mu_A))^* = \{y_1 - A^* y_1 A, y_1 \epsilon\ C(H) = \text{Rang}(\mu\_A)\ \}$

.

2- $\propto \text{Rang}(\mu_A) = \{\propto (y - A^* \times y \times A), y \epsilon C(H)\}$

$= \{\propto y - A^* \propto yA, y \epsilon C(H)\}$.

$\text{let}\ y_1 = \propto y, \text{then}\ \propto \text{Rang}\ (\mu_A) = \{y_1 - AA^* y_1 A, y_1 \epsilon C(H)\}$

$= \text{Rang}(\mu_A)$.

3- $\|\mu_A(y)\|$ it equals $\|y - A_* yA\| \leq \|y\|$ plus $\|A^* yA\| \leq \|y\| \times (1$ plus $\|A\|^2)$

But $A \epsilon C(H)$, thus $\|u_A(y)\| \leq M\ \|y\|$, where

$M = (1 + \|A\|^2)$, so μ_A is bounded .

Proposition (3.3.3):

A linear manifold of operator in C(H) is Rang (μP).

Proof:

Rang (μP(ÿ)) = {Đ/ uP (ÿ)=Đ, x˙C(H)} is understood by us.

assuming that ÿ = 0 ϵ C(H) and μP (0) = 0, 0 ϵ Rang (μP).

1- Let $Đ_1$, $Đ_2$ ϵ Rang (μ_P) we must prove $Đ_1$ - $Đ_2$ ϵ Rang (μ_P)

Therefore,$\exists \ddot{y}_1 \in C(H)$ such that $\mu_P \ddot{y}_1 = Đ_1$ and $\exists\ \ddot{y}_2 \in C(H)$

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$$\text{Thus } \mu_P(\ddot{y}_2) = Đ_2. \text{Thus }, u_P = (\ddot{y}_1 - \ddot{y}_2) - P^* \times (\ddot{y}_1 - \ddot{y}_2)P \text{ But },$$

$= \ddot{y}_1 - \ddot{y}_2 - P^* \times \ddot{y}_1 P + P^* \ddot{y}_2 P$

$= (\ddot{y}_1 - P^* \times \ddot{y}_1 P) - (\ddot{y}_2 - P^* \times \ddot{y}_2 P)$

$= Đ_1 - Đ_2.$

Then$(\ddot{y}_1 - \ddot{y}_2) \epsilon C(H)$such that $u_P(\ddot{y}_1 - \ddot{y}_2) = Đ_1 - Đ_2$.

so $Đ_1 - Đ_2 \epsilon \text{ Rang } (\mu_P)$.

Thus the operator range (μP) forms a linear manifold.

Proposition (3.3.4):

If Mϵ Rang (μ_A), then M* does as well.

Proof:

Assuming Mϵ Rang (μ_A), there exists yϵC(H) ∋ μ_A (y) = M.

To show that M^* ϵ Rang (μ_A), there exists y∈C(H) ∋ μ_A (y)= M^1.

However, since μ_A (y) = M, (y-A*× yA) *=Q*. Therefore, y* = ⟦A⟧ * × y* × A=M*. since $y \in C(H)$ then $y^* \in C(H)$so $M^* \in$ Rang

4. Elementary operators:

We look into elementary operators in this part. The following operators are generalization of the operator TAB: B(H) Leads to B(H) where

T_{AB} it equals T_{AB} (x) it equals AX-XB, X ЄB(H).

We start with the definition that follows. [2]

Definition 4.1:

An elementary operator (∆) on B(H) takes a shape.

$$\Delta(Y) = \sum_{j=0}^{m} \mathcal{R}_i \times Đ_j$$

The operators $\{\mathcal{R}_j\}_{j=1}^{m}$ and $\{Đ_j\}_{j=1}^{m}$ are defined in B(H).

∆ is a map that is linear.

going from B(H) to B(H). Now, we try to study the properties of ∆ by virtue of those of $K_i'^s$ and $D_i'^s$, or inversely, characterization of properties of K_i and D_i by using the properties of ∆.

try to study the properties of ∆ by virtue of those of $K_i'^s$ and $D_i'^s$, or inversely, characterization of properties of K_i and D_i by using the properties of ∆.

Definition 4.2, [2]:

Consider Y to be a Banach space over 1. Also assume K∈B(H) Next, define by

$\sigma_\pi(K) = \{\lambda Є ¢ : K - \lambda I \text{ is not bounded below}\}$

$\sigma_\pi(K)$ is called the approrimate point spectrum of K

Also, Define $\sigma_\delta(K) = \{\lambda Є ¢ : K - \lambda I \text{ is not surjective}\}$

$\sigma_\pi(K)$ is said the defect spectrum of K.

Theorem 4.1, [12]:

Let each of $\{k_i\}_{i=1}^{n}$ and $\{D\}_{i=1}^{n}$ be a commutative set of operators then $\sigma(\Delta) = \sum_{i=1}^{n} \sigma(K_i)\, \sigma(D_i)$.

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Theorem 4.2., [12]:

If M ,N $\in B(H)$ and then MN= NM

1- $\sigma_\pi(M+N) \subseteq \sigma_\pi(M) + \sigma_\pi(N)$

2- $\sigma_\pi(MN) \subseteq \sigma_\pi(M)\sigma_\pi(N)$

Theorem 4.3, [12]:

1- $\sigma_\delta \times (M+N) \subseteq \sigma_\delta \times (M) + \sigma_\delta \times (N)$

2- $\sigma_\delta \times (MN) \subseteq \sigma_\delta \times (M) \times \sigma_\delta(N)$

Theorem 4.4:

Let each of $\{M_i\}_{i=1}^n$ and $\{N_i\}_{i=1}^n$ be a commutative family of operators on H, then

$$\sigma_\pi(\Delta) \subseteq \left\{\sum_{i=1}^{n} \alpha_i N_i : \alpha_i \in \sigma_\pi(M_i), N_i \in \sigma_\delta(N_i)\right\}$$

And $\sigma_\delta(\Delta) \subseteq \{\sum_{i=1}^{n} \alpha_i N_i : \alpha_i \in \sigma_\delta(M_i), N_i \in \sigma_\pi(N_i)\}$

Proof

Define operators L , R on B(H) by

$L_A(x) = Ax, R_A(x) = xA$.

So, the elementary operator Δ can be written as

$\Delta(x) = \sum_{i=1}^{n} R_{Bi}\,(L_{Ai}(x))$.

So, by theorem (4.2) $\sigma_\pi(\Delta) \subseteq \sum_{i=1}^{n} \sigma_\pi\,(R_{Ai}(L_{Bi}(x))$.

and by theorem (4.3) we get

$\sigma_\pi(\Delta) \subseteq \sum_{i=1}^{n} \sigma_\pi\,(A_i)\sigma_\delta(A_i)$.

Also, by theorem (4.3) we have

$$\sigma_\sigma(\Delta) \subseteq \sum_{i=1}^{n} \sigma_\sigma\,(R_{Ai}(L_{Bi}(x))$$

We get $\sigma_\delta(\Delta) \subseteq \sum_{i=1}^{n} \sigma_\delta\,(A_i)\sigma_\pi(B_i)$

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