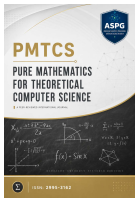
Pure Mathematics for Theoretical Computer Science

♦ ISSN: 2995-3162 Vol: 04, No. 02, 2024 ♦ PP. 19–25

Article DOI: https://doi.org/10.54216/PMTCS.040203



FULL LENGTH ARTICLE

On the Nature of Solutions of Discrete Time Lyapunov Equations

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Received: January 27, 2024 Revised: May 20, 2024 Accepted: August 16, 2024 ⋆ Corresponding author

ABSTRACT

This paper provides a method to solve the discrete-time Lyapunov equation. Linear operator equations are identified and discussed. If an operator D satisfies

$$D(\lambda y+\mu z)=\lambda Dy+\mu Dz, \qquad y,z\in Y, \quad \lambda,\mu\in F,$$

and if there exists a constant $c\in\mathbb{R}^+$ such that $\|Dy\|\leq c\|y\|$ for every $y\in Y$, then D is a bounded linear operator. Lyapunov and Sylvester operator equations are particularly important in differential equations, integral equations, and many other branches of mathematics. We study solutions of the above equation, discuss operator equations and special kinds of operators, and examine some elementary operators. These operators generalize

$$\tau_{AD}:\mathscr{B}(H)\longrightarrow\mathscr{B}(H), \qquad \tau_{AD}(y)=Ay-yD, \quad y\in\mathscr{B}(H).$$

Keywords: Time Lyapunov equation ▪ Bounded linear operators ▪ Sylvester operator equations ▪ Complex Hilbert space ▪ Banach algebra

1. INTRODUCTION

For communication problems using elasticity theory, see the examples. They cover mixed bounded problems and some mathematical-physics problems. Those problems become integral problems, including both linear and nonlinear types. Abdou [1] sought solutions to the second-kind Fredholm–Volterra integral equation in the Banach space $L^2(\Omega)\times C[0,T]$. Both the Volterra integral term on $[0,T]$ and the terms in the Fredholm integral in $L^2(\Omega)$ are linear. Furthermore, every Volterra or Fredholm integral term is linear, while there is a nonlinear integral term as well.

Kuffi and Satar studied a more general solution of the continuous-time Lyapunov equation

$$A^*XD+DXA=W$$

[11]. Some methods use neural networks to represent Lyapunov functions [16].

This research studies natural solutions to the discrete-time Lyapunov equation. We also examine solutions to discrete-time Lyapunov operators and discuss these operators, their equations, and special cases.

2. BASIC PROPERTIES OF SOME TYPES OF OPERATORS

Definition 2.1 [14].

The pair $(X,\|\cdot\|)$ is a normed space if X is a vector space over the field of real or complex numbers and

$$\|\cdot\|:X\longrightarrow\mathbb{R}$$

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is a norm on X.

Definition 2.2 [14].

If there is a multiplication $(u,x) \mapsto ux : X \times X \to X$ that is linear in each factor, then a Banach space X is referred to as a Banach algebra. If an inner product

$$\langle \cdot, \cdot \rangle : X \times X \longrightarrow \mathbb{R} \quad \text{or} \quad \mathbb{C}$$

is defined, then X is an inner-product space.

Definition 2.3 [9].

Assume that $B \in \mathscr{B}(H)$. Then there exists a unique bounded linear operator B^* on H such that

$$\langle Bw, n \rangle = \langle w, B^* n \rangle, \qquad w, n \in H.$$

The operator B^* is called the adjoint of B.

Definition 2.4 [9].

Let $D \in \mathscr{B}(H)$. If there is a bounded linear operator C on H such that

$$DC = CD = I,$$

then D is said to be invertible.

Proposition 2.1.

If G and R are bounded linear operators on H and λ is a scalar, then

1. $(G+R)^* = G^* + R^*$;
2. $(\lambda G)^* = \overline{\lambda} G^*$, $\lambda \in \mathbb{C}$;
3. $\langle G^* m, z \rangle = \langle m, Gz \rangle$ for all $z, m \in H$;
4. $(G^*)^* = G$;
5. $(GR)^* = R^* G^*$;
6. $\|G^*\| = \|G\|$;
7. $\|G^* G\| = \|GG^*\| = \|G\|^2$;
8. $G^* G = 0$ if and only if $G = 0$.

Proposition 2.2 [15].

The following conditions on a bounded operator B on H are equivalent:

1. B is Hermitian;
2. $\langle Bz, y \rangle = \langle z, By \rangle$ for all $z, y \in H$;
3. $\langle Bz, z \rangle$ is real for every $z \in H$.

Remark 2.1 [15].

1. If B and F are Hermitian, then $B+F$ is Hermitian.
2. If B is Hermitian and β is real, then βB is Hermitian.
3. If B is any operator, then $B^* B$ and $B + B^*$ are self-adjoint.

Definition 2.5 [15].

A matrix K is called self-adjoint if $K = K^*$.

Remark 2.2.

1. If F is Hermitian, then $F = F^*$; hence, F is a normal operator.
2. An operator S is normal if it is unitary; that is, $S^* S = SS^* = I$.
3. An operator F is normal if it is skew-Hermitian; that is, $-F = F^*$.

Remark 2.3 [15].

1. iK is Hermitian if and only if K is skew-Hermitian.
2. $K - K^*$ is skew-Hermitian for every operator K.

Lemma 2.1.

If Z is a bounded linear operator on a complex Hilbert space H and

$$\langle Zx, x \rangle = 0 \qquad \text{for every } x \in H,$$

then $Z = 0$.

Proof.

Suppose $x, y \in H$ and $v = ax + by$, where $a, b \in \mathbb{C}$. By assumption,

$$0 = \langle Zv, v \rangle = |a|^2 \langle Zx, x \rangle + a\overline{b} \langle Zx, y \rangle + b\overline{a} \langle Zy, x \rangle + |b|^2 \langle Zy, y \rangle.$$

Since the first and last terms vanish,

$$a\overline{b} \langle Zx, y \rangle + b\overline{a} \langle Zy, x \rangle = 0.$$

Taking $a = b = 1$ and then $a = i$, $b = 1$ gives $\langle Zx, y \rangle = 0$ for all $y \in H$. Thus $Zx = 0$ for every $x \in H$, and therefore $Z = 0$.

Proposition 2.3 [15].

The following conditions on an operator A are equivalent:

1. A is unitary;
2. A^* is unitary;
3. A is bijective and $A^{-1} = A^*$.

Proposition 2.4 [15].

The following conditions on an operator B are equivalent:

1. B is normal;
2. B^* is normal;
3. $\|B^* z\| = \|Bz\|$ for every $z \in H$.

Proof.

The implication (1)⇒(2) follows from $(B^*)^* = B$. For (2)⇒(3), normality gives

$$\|Bz\|^2 = \langle B^* Bz, z \rangle = \langle BB^* z, z \rangle = \|B^* z\|^2.$$

Conversely, if the norm equality holds, then

$$\langle (B^* B - BB^*) z, z \rangle = 0, \qquad z \in H.$$

Lemma 2.1 yields $B^* B = BB^*$, and hence B is normal.

Proposition 2.5.

If F is hyponormal, then

$$\|F^n\| = \|F\|^n$$

for every positive integer n.

Proof.

See [8].

Another type of operator encountered in this work is the quasinilpotent operator. An operator F is nilpotent if there is a positive integer m such that $F^m = 0$; the least such m is the index of nilpotency. An operator F is quasinilpotent if

$$\|F^n\|^{1/n} \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

It is clear that nilpotence implies quasinilpotence.

Proposition 2.6.

For an operator B on H, the following are equivalent:

1. B is hyponormal;
2. $\|B^*x\| \leq \|Bx\|$ for every $x \in H$.

Proof.

If B is hyponormal, then $B^*B - BB^* \geq 0$. Hence, for every $x \in H$,

$$\|B^*x\|^2 = \langle BB^*x, x\rangle \leq \langle B^*Bx, x\rangle = \|Bx\|^2.$$

Conversely, the norm inequality implies

$$\langle (B^*B - BB^*)x, x\rangle \geq 0,$$

so $B^*B - BB^* \geq 0$ and B is hyponormal.

Definition 2.6 [8].

An operator $K \in \mathscr{B}(H)$ is quasinormal if K commutes with K^*K; equivalently,

$$K(K^*K) = (K^*K)K.$$

Definition 2.7 [5].

An operator M on a Hilbert space H is binormal if M^*M and MM^* commute.

Remark 2.4.

1. Every normal operator is binormal.
2. Every quasinormal operator is binormal.

Proof.

If F is normal, then $F^*F = FF^*$, and hence these two operators commute. If F is quasinormal, then

$$F(F^*F) = (F^*F)F.$$

Taking adjoints gives $(F^*F)F^* = F^*(F^*F)$, and therefore

$$\begin{aligned}(F^*F)(FF^*) &= F(F^*F)F^* \\ &= (FF^*)(F^*F).\end{aligned}$$

Thus F is binormal.

Proposition 2.7 [4].

Let D be a hyponormal and binormal operator. Then D^n is hyponormal for every integer $n \geq 1$.

Proof.

Since D is hyponormal,

$$D^*D \geq DD^* \geq 0.$$

The binormality assumption gives $[D^*D, DD^*] = 0$. Consequently,

$$D^{*2}D^2 \geq (D^*D)^2 geq (DD^*)^2 geq D^2D^{*2},$$

so D^2 is hyponormal. Continuing inductively, assume the corresponding inequalities hold to an integer $n \geq 2$. Then

$$D^{*(n+1)}D^{n+1} \geq (D^*D)^{n+1} \geq (DD^*)^{n+1} \geq D^{n+1}D^{*(n+1)}.$$

Thus D^{n+1} is hyponormal.

Definition 2.8 [6].

An operator T on a Hilbert space H is called a θ-operator if T^*T commutes with $T + T^*$.

Proposition 2.8.

For an operator R on a Hilbert space H, the following are equivalent:

1. R is a θ-operator;
2. $R^*[R^*, R] = [R^*, R]R$.

Proof.

If R is a θ-operator, then expansion of

$$(R^*R)(R + R^*) = (R + R^*)(R^*R)$$

and rearrangement gives

$$R^*(R^*R - RR^*) = (R^*R - RR^*)R,$$

which is the second assertion. Conversely, expanding the second assertion and rearranging recovers the commutation relation above; hence R is a θ-operator.

Proposition 2.9.

If B is an invertible θ-operator, then B^{-1} is a θ-operator.

Proof.

Since B^{-1} exists, B^{*-1} exists. Starting from

$$(B^*B)(B + B^*) = (B + B^*)(B^*B)$$

and multiplying by the appropriate inverses on the left and right gives

$$(B^{*-1} + B^{-1})B^{*-1}B^{-1} = B^{*-1}B^{-1}(B^{*-1} + B^{-1}).$$

Thus B^{-1} is a θ-operator.

Remark 2.5.

1. Every normal operator is a θ-operator.
2. Every quasinormal operator is a θ-operator.

Proof.

For a normal operator U, the equality $U^*U = UU^*$ implies

$$(U^*U)(U+U^*) = (U+U^*)(U^*U).$$

For a quasinormal operator, $U(U^*U) = (U^*U)U$, and taking adjoints gives $(U^*U)U^* = U^*(U^*U)$. Adding the two equalities again proves that U^*U commutes with $U+U^*$.

3. SOME TYPES OF OPERATOR EQUATIONS

Definition 3.1 [7].

An equation of the form

$$\mathscr{F}(Y) = C \tag{1}$$

is called an operator equation. In the Hilbert space H, the operators $\mathscr{F}$ and C are known, and the unknown operator Y is to be determined.

3.1 The Lyapunov Equation for Discrete Time

Consider the operator equation

$$Y - K^*YK = U. \tag{2}$$

This is called the discrete-time Lyapunov equation, where $K, U \in \mathscr{B}(H)$ and the unknown operator is Y [3]. In general, the equation can have a unique solution, infinitely many solutions, or no solution.

Example 3.1.1.

Suppose $H = \ell^2(\mathbb{N})$. The unilateral shift operator N on $\ell^2(\mathbb{N})$ is defined by

$$N(y_1, y_2, \ldots) = (0, y_1, y_2, \ldots).$$

Consider (2) with $K = N$ and $U = 0$. If $D = N^*$ is the backward shift,

$$D(y_1, y_2, \ldots) = (y_2, y_3, y_4, \ldots),$$

then the equation becomes

$$Y - DYN = 0.$$

It has infinitely many solutions, including $Y = 0$, $Y = N$, and $Y = MI$, where M is an arbitrary constant.

Proposition 3.1.1.

Assume that K and U are self-adjoint operators. If (2) has a unique solution Y, then Y is self-adjoint.

Proof.

Taking adjoints in (2) gives

$$Y^* - KY^*K = U.$$

Since $K = K^*$ and $U = U^*$, both Y and Y^* solve the same equation. Uniqueness therefore gives $Y = Y^*$.

We now examine solutions of (2), concentrating on normal operators, including binormal, quasinormal, and θ-operator types. The equation can yield non-normal solutions even when the known operators K and U are normal.

Example 3.1.2.

In (2), take $K = iI$ and $U = 0$. The equation has the solutions $Y = 0$, $Y = MI$, and $Y = N$, where M is arbitrary and N is non-normal (and is not binormal, quasinormal, or a θ-operator).

3.2 Basic Jordan $*$-Derivation

Definition 3.2.1 [10].

Let R be a ring. A derivation g on R is a mapping of R into itself that satisfies

$$g(c+d) = g(c) + g(d),$$
$$g(cd) = c\,g(d) + g(c)d,$$

for all $c, d \in R$.

Definition 3.2.2.

Let R be a ring. A Jordan derivation g on R is a mapping of R into itself such that

$$g(c+d) = g(c) + g(d),$$
$$g(c^2) = c\,g(c) + g(c)c,$$

for all $c, d \in R$.

Definition 3.2.3 [8].

A ring R is a $*$-ring if there is a map $* : R \to R$ satisfying

$$(g^*)^* = g,$$
$$(g+d)^* = g^* + d^*,$$
$$(gd)^* = d^*g^*,$$

for all $g, d \in R$.

Definition 3.2.4 [13].

Let R be a $*$-ring. A Jordan $*$-derivation J on R is a mapping from R into itself satisfying

$$J(g+d) = J(g) + J(d),$$
$$J(g^2) = gJ(g) + J(g)g^*,$$

for all $g, d \in R$.

Example 3.2.1.

Let $R = \mathbb{C}$, which is a $*$-ring under complex conjugation. Define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = z - \bar{z}.$$

Then f is a Jordan $*$-derivation.

Proof.

For $z_1, z_2 \in \mathbb{C}$,

$$\begin{aligned} f(z_1+z_2) &= (z_1+z_2) - \overline{(z_1+z_2)} \\ &= (z_1 - \bar{z}_1) + (z_2 - \bar{z}_2) \\ &= f(z_1) + f(z_2). \end{aligned}$$

Moreover,

$$\begin{aligned} zf(z)+f(z)z^* &= z(z-\bar{z})+(z-\bar{z})\bar{z} \\ &= z^2-\bar{z}^2 = f(z^2). \end{aligned}$$

3.3 The Map μ_K

The bounded operators on an infinite, separable complex Hilbert space H are denoted by $\mathscr{B}(H)$. For $K \in \mathscr{B}(H)$, define

$$\mu_K : \mathscr{B}(H) \longrightarrow \mathscr{B}(H), \qquad \mu_K(Y) = Y - K^*YK.$$

Then

$$\mathrm{Ran}(\mu_K) = \{Y - K^*YK : Y \in \mathscr{B}(H)\}.$$

The map μ_K is linear because, for scalars a, b,

$$\begin{aligned} \mu_K(aY_1+bY_2) &= aY_1+bY_2-K^*(aY_1+bY_2)K \\ &= a\mu_K(Y_1)+b\mu_K(Y_2). \end{aligned}$$

Recall that μ_K is one-to-one if $\mathrm{Ker}(\mu_K) = \{0\}$. If $Y \in \mathrm{Ker}(\mu_K)$, then $Y = K^*YK$. If K is an isometry, then $I \in \mathrm{Ker}(\mu_K)$, so μ_K is not one-to-one and hence is not invertible.

Remark 3.3.1.

If K is an isometric operator, then μ_K is not one-to-one.

Example 3.3.1.

The map in (2) can be written as

$$\mu_K = I - R_{K^*}L_K,$$

where $R_{K^*}(Y) = K^*Y$ and $L_K(Y) = YK$. If K is an isometry, then $K^*K = I$ and

$$\mu_K(I) = I - K^*K = 0.$$

Thus μ_K is not one-to-one.

Recall that an operator P on a Hilbert space H is a projection if

$$P^2 = P = P^*$$

[2].

Proposition 3.3.1.

If K is a nonzero projection, then μ_K is not one-to-one and therefore is not invertible.

Proof.

Since K is a projection,

$$\mu_K(K) = K - K^*KK = K - K = 0.$$

Thus $K \in \mathrm{Ker}(\mu_K)$. Since $K \neq 0$, the kernel is nontrivial.

Example 3.3.2.

Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then $A^*A \neq I$, so A is not isometric. Let

$$Y = A^*YA, \qquad Y = \begin{bmatrix} \alpha & m \\ b & r \end{bmatrix}.$$

Thus

$$\begin{bmatrix} \alpha & m \\ b & r \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \alpha & m \\ b & r \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

A direct computation gives

$$Y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathrm{Ker}(\mu_A),$$

so μ_A is not one-to-one (and is not onto).

Proposition 3.3.2.

For $A \in \mathscr{B}(H)$:

1. $\mathrm{Ran}(\mu_A)^* = \mathrm{Ran}(\mu_A)$;
2. $\alpha\,\mathrm{Ran}(\mu_A) = \mathrm{Ran}(\mu_A)$ for every $\alpha \in \mathbb{C}$, $\alpha \neq 0$;
3. μ_A is a bounded operator.

Proof.

First,

$$\begin{aligned} \mathrm{Ran}(\mu_A)^* &= \{(Y - A^*YA)^* : Y \in \mathscr{B}(H)\} \\ &= \{Y^* - A^*Y^*A : Y \in \mathscr{B}(H)\} \\ &= \mathrm{Ran}(\mu_A). \end{aligned}$$

For a nonzero scalar α,

$$\begin{aligned} \alpha\,\mathrm{Ran}(\mu_A) &= \{\alpha Y - A^*(\alpha Y)A : Y \in \mathscr{B}(H)\} \\ &= \mathrm{Ran}(\mu_A). \end{aligned}$$

Finally,

$$\begin{aligned} \|\mu_A(Y)\| &\leq \|Y\| + \|A^*YA\| \\ &\leq (1+\|A\|^2)\|Y\|. \end{aligned}$$

Thus μ_A is bounded.

Proposition 3.3.3.

The range $\mathrm{Ran}(\mu_P)$ is a linear manifold in $\mathscr{B}(H)$.

Proof.

Clearly, $0 = \mu_P(0) \in \mathrm{Ran}(\mu_P)$. Let $D_1, D_2 \in \mathrm{Ran}(\mu_P)$. There exist $Y_1, Y_2 \in \mathscr{B}(H)$ such that $\mu_P(Y_1) = D_1$ and $\mu_P(Y_2) = D_2$. Then

$$\begin{aligned} \mu_P(Y_1 - Y_2) &= (Y_1 - Y_2) - P^*(Y_1 - Y_2)P \\ &= (Y_1 - P^*Y_1P) - (Y_2 - P^*Y_2P) \\ &= D_1 - D_2. \end{aligned}$$

Therefore $D_1 - D_2 \in \mathrm{Ran}(\mu_P)$, proving the assertion.

Proposition 3.3.4.

If $M \in \mathrm{Ran}(\mu_A)$, then $M^* \in \mathrm{Ran}(\mu_A)$ as well.

Proof.

If $M \in \mathrm{Ran}(\mu_A)$, there exists $Y \in \mathscr{B}(H)$ such that

$$\mu_A(Y) = Y - A^*YA = M.$$

Taking adjoints gives

$$Y^* - A^*Y^*A = M^*.$$

Since $Y^* \in \mathscr{B}(H)$, it follows that $M^* \in \mathrm{Ran}(\mu_A)$.

4. ELEMENTARY OPERATORS

We investigate elementary operators in this section. They generalize the operator

$$T_{AB} : \mathscr{B}(H) \longrightarrow \mathscr{B}(H), \qquad T_{AB}(X) = AX - XB.$$

Definition 4.1 [2].
An elementary operator Δ on $\mathscr{B}(H)$ has the form

$$\Delta(Y) = \sum_{j=1}^{m} R_j Y D_j,$$

where $\{R_j\}_{j=1}^m$ and $\{D_j\}_{j=1}^m$ are contained in $\mathscr{B}(H)$. Thus Δ is a linear map from $\mathscr{B}(H)$ to $\mathscr{B}(H)$. We study the properties of Δ through those of R_j and D_j and, conversely, characterize properties of the coefficients through those of Δ.

Definition 4.2 [2].
Let X be a Banach space and $K \in \mathscr{B}(X)$. Define

$$\sigma_\pi(K) = \{\lambda \in \mathbb{C} : K - \lambda I \text{ is not bounded below}\}.$$

The set $\sigma_\pi(K)$ is called the approximate point spectrum of K. Also define

$$\sigma_\delta(K) = \{\lambda \in \mathbb{C} : K - \lambda I \text{ is not surjective}\}.$$

The set $\sigma_\delta(K)$ is called the defect spectrum of K.

Theorem 4.1 [12].
Let $\{K_i\}_{i=1}^n$ and $\{D_i\}_{i=1}^n$ be commuting families of operators. Then

$$\sigma(\Delta) = \sum_{i=1}^{n} \sigma(K_i)\sigma(D_i).$$

Theorem 4.2 [12].
If $M, N \in \mathscr{B}(H)$ and $MN = NM$, then

$$\sigma_\pi(M+N) \subseteq \sigma_\pi(M) + \sigma_\pi(N),$$
$$\sigma_\pi(MN) \subseteq \sigma_\pi(M)\sigma_\pi(N).$$

Theorem 4.3 [12].
Under the same assumptions,

$$\sigma_\delta(M+N) \subseteq \sigma_\delta(M) + \sigma_\delta(N),$$
$$\sigma_\delta(MN) \subseteq \sigma_\delta(M)\sigma_\delta(N).$$

Theorem 4.4.
Let $\{M_i\}_{i=1}^n$ and $\{N_i\}_{i=1}^n$ be commuting families of operators on H. Then

$$\sigma_\pi(\Delta) \subseteq \left\{ \sum_{i=1}^{n} \alpha_i\beta_i : \begin{array}{l} \alpha_i \in \sigma_\pi(M_i), \\ \beta_i \in \sigma_\delta(N_i) \end{array} \right\},$$
$$\sigma_\delta(\Delta) \subseteq \left\{ \sum_{i=1}^{n} \alpha_i\beta_i : \begin{array}{l} \alpha_i \in \sigma_\delta(M_i), \\ \beta_i \in \sigma_\pi(N_i) \end{array} \right\}.$$

Proof.
Define the left- and right-multiplication operators on $\mathscr{B}(H)$ by

$$L_A(X) = AX, \qquad R_A(X) = XA.$$

The elementary operator can be written as

$$\Delta(X) = \sum_{i=1}^{n} R_{N_i}\big(L_{M_i}(X)\big).$$

By Theorems 4.2 and 4.3,

$$\sigma_\pi(\Delta) \subseteq \sum_{i=1}^{n} \sigma_\pi(M_i)\sigma_\delta(N_i),$$
$$\sigma_\delta(\Delta) \subseteq \sum_{i=1}^{n} \sigma_\delta(M_i)\sigma_\pi(N_i),$$

which proves the stated inclusions.

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