



## The Zariski topology on the graded second spectrum of a graded module

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### Abstract

Let  $R$  be a  $G$ -graded ring and  $M$  be a  $G$ -graded  $R$ -module. The graded second spectrum of  $M$ , denoted by  $Spec_G^s(M)$ , is the set of all graded second submodules of  $M$ . In this paper, we define a topology on  $Spec_G^s(M)$  which is analogous to that for  $Spec_G(R)$ , and investigate several topological properties of this topology.

**Keywords:** Graded second submodules; Graded second spectrum; Graded prime submodules; Zariski topology

### 1 Introduction and Preliminaries

Let  $G$  be a multiplicative group with identity  $e$  and  $R$  be a commutative ring with identity 1. Then  $R$  is called a  $G$ -graded ring if there exist additive subgroups  $R_g$  of  $R$  indexed by the elements  $g \in G$  such that  $R = \bigoplus_{g \in G} R_g$  and  $R_g R_h \subseteq R_{gh}$  for all  $g, h \in G$ . The elements of  $R_g$  are called homogeneous elements of degree  $g$ . The homogeneous elements of the ring  $R$  are denoted by  $h(R)$ , i.e.  $h(R) = \bigcup_{g \in G} R_g$ . In particular,  $R_e$  is a subring of  $R$  and  $1 \in R_e$ . If  $x \in R$ , then  $x$  can be written uniquely as  $\sum_{g \in G} x_g$ , where  $x_g$  is the component of  $x$  in  $R_g$ .

Let  $R$  be a  $G$ -graded ring and  $I$  be an ideal of  $R$ . Then  $I$  is called a  $G$ -graded ideal of  $R$ , denoted by  $I \triangleleft_G R$ , if  $I = \bigoplus_{g \in G} (I \cap R_g)$ . A subring  $H$  of  $R$  is said to be graded subring if  $H = \bigoplus_{g \in G} (H \cap R_g)$  (see<sup>9</sup>). A proper graded ideal  $J$  of  $R$  is said to be graded maximal if there is no graded ideal  $K$  of  $R$  such that  $J \subsetneq K \subsetneq R$ . We will denote the set of all graded maximal ideals of  $R$  by  $Max_G(R)$ . The graded prime spectrum of  $R$  is the set of all graded prime ideals of  $R$  and it is denoted by  $Spec_G(R)$ . It has been proved that  $Spec_G(R)$  is a topological space whose closed sets are  $V_G^R(I) = \{p \in Spec_G(R) \mid I \subseteq p\}$  for each graded ideal  $I$  of  $R$  (see, for example,<sup>10-12</sup>).

Let  $R$  be a  $G$ -graded ring and  $M$  be a left  $R$ -module. Then  $M$  is said to be a  $G$ -graded  $R$ -module if  $M = \bigoplus_{g \in G} M_g$  with  $R_g M_h \subseteq M_{gh}$  for all  $g, h \in G$ , where  $M_g$  is an additive subgroup of  $M$  for all  $g \in G$ . The elements of  $M_g$  are called homogeneous of degree  $g$ . The set of all homogeneous elements of  $M$  is denoted by  $h(M)$ , i.e.  $h(M) = \bigcup_{g \in G} M_g$ . If  $m \in M$ , then  $m$  can be written uniquely as  $\sum_{g \in G} m_g$ , where  $m_g$  is the component of  $m$  in  $M_g$ . Let  $M = \bigoplus_{g \in G} M_g$  be a  $G$ -graded  $R$ -module and  $N$  be a submodule of  $M$ . For  $g \in G$ , let  $N_g = N \cap M_g$ . Then  $N$  is called a  $G$ -graded submodule of  $M$ , denoted by  $N \leq_G M$ , if  $N = \bigoplus_{g \in G} N_g$ . In

this case,  $N_g$  is called the  $g$ -component of  $N$  for  $g \in G$ . Also,  $M/N$  becomes  $G$ -graded  $R$ -module with  $g$ -component  $(M/N)_g = (M_g + N)/N$  for  $g \in G$ . Moreover,  $M_g$  is an  $R_e$ -module and  $N_g$  is an  $R_e$ -submodule of  $M_g$  for  $g \in G$ , see.<sup>9</sup>

Let  $M$  be a  $G$ -graded  $R$ -module. Let  $I$  be a  $G$ -graded ideal of  $R$  and  $N$  be a  $G$ -graded submodule of  $M$ . Then  $\text{Ann}_R(N) = \{r \in R \mid rN = \{0\}\}$  is a  $G$ -graded ideal of  $R$ . In addition,  $\text{Ann}_M(I) = \{m \in M \mid Im = \{0\}\}$  and  $(N :_M I) = \{m \in M \mid Im \subseteq N\}$  are  $G$ -graded submodules of  $M$ . In particular,  $\text{Ann}_M(I)$  becomes a  $G$ -graded  $R/I$ -module by  $(\text{Ann}_M(I))_g = \text{Ann}_M(I) \cap M_g$  for all  $g \in G$ .

A  $G$ -graded  $R$ -module  $M$  is called a comultiplication graded  $R$ -module if every  $N \leq_G M$  has the form  $\text{Ann}_M(I)$  for some  $I \triangleleft_G R$  (see, for example,<sup>2,6</sup>). It is clear that if  $N$  is a graded submodule of a comultiplication graded module  $M$ , then  $N = \text{Ann}_M(\text{Ann}_R(N))$ . A  $G$ -graded  $R$ -module  $M$  is said to be graded Artinian if it satisfies the descending chain condition for graded submodules, see.<sup>9</sup>

Let  $M$  and  $M'$  be graded  $R$ -modules. An  $R$ -module homomorphism  $f : M \rightarrow M'$  is called graded homomorphism if  $f(M_g) \subseteq M'_g$  for all  $g \in G$  (see<sup>9</sup>). A non-zero graded submodule  $S$  of  $M$  is said to be graded second if for each  $r \in h(R)$ , the graded  $R$ -homomorphism  $f : S \rightarrow S$  defined by  $f(x) = rx$  is either surjective or zero. In other words,  $S$  is a graded second submodule of  $M$  if  $rS = S$  or  $rS = 0$  for every  $r \in h(R)$ . This implies that  $p = \text{Ann}_R(S)$  is a graded prime ideal of  $R$  and  $S$  is called a  $p$ -graded second submodule. The graded second spectrum of  $M$ , denoted by  $\text{Spec}_G^s(M)$ , is the set of all graded second submodules of  $M$ . For  $p \in \text{Spec}_G(R)$ ,  $p\text{-Spec}_G^s(M)$  denotes the collection of all  $p$ -graded second submodules, i.e.  $p\text{-Spec}_G^s(M) = \{S \in \text{Spec}_G^s(M) \mid \text{Ann}_R(S) = p\}$ . Graded second submodules of graded modules have been studied by various authors (see, for example,<sup>5,8,13</sup>).

A non-zero graded submodule  $N$  of a graded  $R$ -module  $M$  is said to be graded large if for every non-zero graded submodule  $L$  of  $M$ , we have  $N \cap L \neq 0$ . A graded  $R$ -module  $M$  is said to be graded simple if  $\{0\}$  and  $M$  are its only graded submodules. A non-zero graded submodule  $N$  of a graded  $R$ -module  $M$  is called graded minimal if it contains no other non-zero graded submodules of  $M$ . Equivalently,  $N$  is a non-zero graded submodule of  $M$  which is a graded simple module. The set of all graded minimal submodules of a  $G$ -graded  $R$ -module  $M$  will be denoted by  $\text{Min}_G(M)$ . The graded socle of a  $G$ -graded  $R$ -module  $M$ , denoted by  $\text{Soc}_G^R(M)$ , is defined to be the sum of all graded minimal submodules of  $M$ . If there are no graded minimal submodules of  $M$ , we put  $\text{Soc}_G^R(M) = \{0\}$ , see.<sup>9</sup> A graded  $R$ -module  $M$  is said to be graded cocyclic if  $\text{Soc}_G^R(M)$  is a graded large and a graded simple submodule of  $M$ , see.<sup>6</sup>

The Zariski-topology on some graded spectrums of graded rings and graded modules has been studied by several authors (see, for example,<sup>3,4,10,11,14</sup>).

In this paper, we introduce the notion of the  $G$ -cotop module which is analogous to the  $G$ -top module in<sup>10</sup> and the primary  $G$ -top module in.<sup>14</sup> For this, we define the variety of any graded submodule  $N$  of a  $G$ -graded  $R$ -module  $M$  by  $V_G^{s*}(N) = \{S \in \text{Spec}_G^s(M) \mid S \subseteq N\}$  and we set  $\Omega^{s*}(M) = \{V_G^{s*}(N) \mid N \leq_G M\}$ . Then  $M$  is called a  $G$ -cotop module if  $\Omega^{s*}(M)$  is closed under finite union. In this case,  $\Omega^{s*}(M)$  generates a topology. We call it the quasi-Zariski topology on  $\text{Spec}_G^s(M)$ . After this, we define another variety for any  $N \leq_G M$  by  $V_G^s(N) = \{S \in \text{Spec}_G^s(M) \mid \text{Ann}_R(N) \subseteq \text{Ann}_R(S)\}$ . Then the collection  $\Omega^s(M) = \{V_G^s(N) \mid N \leq_G M\}$  satisfies the axioms for closed sets of a topology on  $\text{Spec}_G^s(M)$  and we call this topology the Zariski topology on  $\text{Spec}_G^s(M)$ . By,<sup>13</sup> the map  $\varphi : \text{Spec}_G^s(M) \rightarrow \text{Spec}_G(R/\text{Ann}_R(M))$  defined by  $\varphi(S) = \text{Ann}_R(S)/\text{Ann}_R(M)$  is called the natural map of  $\text{Spec}_G^s(M)$ . We show that  $\varphi$  is continuous and we investigate some conditions under which  $\varphi$  is injective, open and closed. In addition, using this map, we give some relations between the properties of  $\text{Spec}_G^s(M)$  and  $\text{Spec}_G(R/\text{Ann}_R(M))$  such as the connectedness. Furthermore, we find a base for the Zariski-topology on  $\text{Spec}_G^s(M)$  and we investigate  $\text{Spec}_G^s(M)$  with the Zariski-topology from the viewpoint of being quasi-compact, irreducible,  $T_0$ -space and  $T_1$ -space.

Throughout this paper,  $G$  is a multiplicative group and every ring is commutative with identity. Let  $M$  be a  $G$ -graded  $R$ -module. For any graded ideal  $I$  of  $R$  containing  $\text{Ann}_R(M)$ ,  $\bar{I}$  and  $\bar{R}$  will denote  $I/\text{Ann}_R(M)$  and  $R/\text{Ann}_R(M)$ , respectively. In addition, we denote  $r + \text{Ann}_R(M)$  by  $\bar{r}$  for any  $r \in R$ .

## 2 Topologies on $Spec_G^s(M)$

In this section, we first introduce the notion of the  $G$ -cotop modules and discover some properties of such modules. Then, we define a variety of any graded submodule of a graded module and we use it to define the Zariski-topology on  $Spec_G^s(M)$ .

**Proposition 2.1.** *Let  $M$  be a  $G$ -graded  $R$ -module. For any  $G$ -graded submodule  $N$  of  $M$ , we define the variety of  $N$  by  $V_G^{s*}(N) = \{S \in Spec_G^s(M) \mid S \subseteq N\}$ . Then the following hold:*

1.  $V_G^{s*}(M) = Spec_G^s(M)$  and  $V_G^{s*}(0) = \emptyset$ .
2.  $\bigcap_{i \in I} V_G^{s*}(N_i) = V_G^{s*}(\bigcap_{i \in I} N_i)$  for any index set  $I$  and any family of graded submodules  $\{N_i\}_{i \in I}$ .
3.  $V_G^{s*}(N) \cup V_G^{s*}(N') \subseteq V_G^{s*}(N + N')$  for any  $N, N' \leq_G M$ .

*Proof.* The proof is straightforward. □

By Proposition 2.1, the collection  $\Omega^{s*}(M) = \{V_G^{s*}(N) \mid N \leq_G M\}$  satisfies the axioms for closed sets of a topology on  $Spec_G^s(M)$  if and only if  $\Omega^{s*}(M)$  is closed under finite union. In this case, we call  $M$  a  $G$ -cotop module and the generated topology,  $\tau_G^{s*}$ , the quasi-Zariski topology on  $Spec_G^s(M)$ .

**Theorem 2.2.** *Every comultiplication  $G$ -graded  $R$ -module  $M$  is a  $G$ -cotop module.*

*Proof.* Let  $N, N' \leq_G M$  and  $S \in V_G^{s*}(N + N')$ . It is enough to prove that  $S \in V_G^{s*}(N) \cup V_G^{s*}(N')$ . Assume that  $S \not\subseteq N$  and let  $s \in S$ . Since  $S \not\subseteq N$  and  $M$  is a comultiplication graded module, then  $\exists a \in S$  such that  $a \notin N = Ann_M(Ann_R(N))$  and thus  $\exists r \in h(R) \cap Ann_R(N)$  such that  $ra \neq 0$ . Since  $S \in Spec_G^s(M)$  and  $rS \neq 0$ , then  $rS = S$  and hence  $s = rm$  for some  $m \in S$ . But  $S \subseteq N + N'$ . This follows that  $m = n + n'$  for some  $n \in N$  and  $n' \in N'$ . So  $s = r(n + n') = rn'$  as  $r \in Ann_R(N)$  and so  $s \in N'$ . This implies that  $S \subseteq N$  or  $S \subseteq N'$ . Therefore  $S \in V_G^{s*}(N) \cup V_G^{s*}(N')$ . □

Recall that an  $R$ -module  $M$  is called secondless if it has no second submodules. On the other hand, a  $G$ -graded  $R$ -module  $M$  is said to be graded secondless if it has no graded second submodules ( i.e  $Spec_G^s(M) = \emptyset$  ). For a graded submodule  $N$  of a  $G$ -graded  $R$ -module  $M$ , the graded second socle (or graded second radical) of  $N$  is defined as the sum of all graded second submodules of  $M$  contained in  $N$  and it is denoted by  $soc_G(N)$  (or  $sec_G(N)$ ). If  $N$  does not contain any graded second submodule, we put  $soc_G(N) = \{0\}$ . Moreover,  $N$  is called a graded socle submodule if  $N \neq 0$  and  $soc_G(N) = N$  (see<sup>13</sup>).

**Remark 2.3.** If a  $G$ -graded  $R$ -module  $M$  is graded secondless, then  $M$  is trivially a  $G$ -cotop module.

**Lemma 2.4.** *Let  $M$  be a  $G$ -graded  $R$ -module. Let  $N$  be a  $G$ -graded  $R$ -submodule of  $M$  and  $p$  be a  $G$ -graded ideal of  $R$ . Then the following hold:*

1.  $N$  is a graded second  $R$ -submodule of  $M$  if and only if  $N$  is a graded second  $\bar{R}$ -submodule of  $M$ .
2. If  $S$  is a  $G$ -graded second  $R/p$ -submodule of  $Ann_M(p)$ , then  $S$  is a  $G$ -graded second  $\bar{R}$ -submodule of  $Ann_M(p)$ .
3.  $V_G^{s*}(N) = V_G^{s*}(soc_G(N))$ .
4. If  $p \in Spec_G(R)$  and  $N' \in p\text{-}Spec_G^s(M)$ , then  $N' \subseteq N$  or  $(N + N')/N \in p\text{-}Spec_G^s(M/N)$ .

*Proof.* (1) follows from [8, Proposition 2.4] and (2) is obvious.

(3) Since  $soc_G(N) \subseteq N$ , then  $V_G^{s*}(soc_G(N)) \subseteq V_G^{s*}(N)$ . Let  $S \in V_G^{s*}(N)$ . Then  $S \in Spec_G^s(M)$  and  $S \subseteq N$ . So  $S = soc_G(S) \subseteq soc_G(N)$  and thus  $S \in V_G^{s*}(soc_G(N))$ .

(4) Assume that  $N' \not\subseteq N$ . Then  $(N + N')/N \in Spec_G^s(M/N)$ . It is clear that  $p \subseteq Ann_R((N + N')/N)$ . Let  $r \in Ann_R((N + N')/N) \cap h(R)$ . By hypothesis,  $rN' = 0$  or  $rN' = N'$ . If  $rN' = N'$ , then  $N' \subseteq N$  which is a contradiction. Thus  $r \in Ann_R(N') = p$ . Therefore  $Ann_R((N + N')/N) = p$ . □

**Proposition 2.5.** *Let  $M$  be a  $G$ -graded  $R$ -module. Then the following hold:*

1. *If  $M_g$  is a secondless  $R_e$ -module for every  $g \in G$ , then  $M$  is a graded secondless  $R$ -module. The converse is true if  $R$  is trivially graded by  $G$ .*
2. *If  $N \leq_G M$  such that both  $N$  and  $M/N$  are graded secondless  $R$ -modules, then  $M$  is a graded secondless  $R$ -module.*

*Proof.* (1) Suppose that  $M$  is not a graded secondless  $R$ -module. Then  $\exists S \in \text{Spec}_G^s(M)$ . Since  $S \neq 0$ , then  $\exists g \in G$  such that  $S_g \neq 0$ . By [8, Theorem 2.3],  $S_g$  is a second  $R_e$ -submodule of  $M_g$ , a contradiction. Conversely, assume by way of contradiction that  $\exists t \in G$  such that  $M_t$  is not a secondless  $R_e$ -module which implies that there is a second  $R_e$ -submodule  $H$  of  $M_g$ . Since  $R$  is trivially graded by  $G$ , then it is easy to see that  $H \in \text{Spec}_G^s(M)$  which is also a contradiction.

(2) Suppose that  $M$  is not graded secondless. Then  $\exists S \in \text{Spec}_G^s(M)$ . Let  $p = \text{Ann}_R(S)$ . So we get  $S \in p\text{-Spec}_G^s(M)$ . Since  $M/N$  is graded secondless, then  $(N + S)/N \notin p\text{-Spec}_G^s(M)$  and hence  $S \subseteq N$  by Lemma 2.4(4). Therefore  $S \in \text{Spec}_G^s(N) = \emptyset$ , a contradiction.  $\square$

If we take the ring of integers  $R = \mathbb{Z}$  as a  $\mathbb{Z}_2$ -graded  $\mathbb{Z}$ -module by  $R_0 = R$  and  $R_1 = \{0\}$ , then  $R_0$  and  $R_1$  are secondless  $R_0$ -module which follows that  $\mathbb{Z}$  is a  $\mathbb{Z}_2$ -graded secondless  $\mathbb{Z}$ -module by Proposition 2.5.

A graded second submodule  $S$  of a graded  $R$ -module  $M$  is said to be graded extraordinary if whenever  $N_1$  and  $N_2$  are graded socle submodules of  $M$  with  $S \subseteq N_1 + N_2$ , then  $S \subseteq N_1$  or  $S \subseteq N_2$ .

**Theorem 2.6.** *Let  $M$  be a  $G$ -graded  $R$ -module. Then the following statements are equivalent:*

1.  *$M$  is a  $G$ -cotop module.*
2. *Every graded second submodule of  $M$  is graded extraordinary.*
3.  *$V_G^{s*}(N) \cup V_G^{s*}(N') = V_G^{s*}(N + N')$  for any graded socle submodules  $N, N'$  of  $M$ .*

*Proof.* If  $M$  is graded secondless, then the result is trivially true. Suppose that  $M$  is not graded secondless.

(1) $\Rightarrow$ (2): Let  $S \in \text{Spec}_G^s(M)$  and let  $N$  and  $L$  be graded socle submodules of  $M$  such that  $S \subseteq N + L$ . By hypothesis,  $V_G^{s*}(N) \cup V_G^{s*}(L) = V_G^{s*}(H)$  for some  $H \leq_G M$ . Now there is an index set  $I$  such that  $N = \sum_{i \in I} S_i$ , where  $S_i \in \text{Spec}_G^s(M)$  and  $S_i \subseteq N$  for every  $i \in I$ . For each  $i \in I$ ,  $S_i \in V_G^{s*}(N) \subseteq V_G^{s*}(H)$  and hence  $S_i \subseteq H$  which implies that  $N \subseteq H$ . Similarly  $L \subseteq H$ . Thus  $N + L \subseteq H$ . So  $V_G^{s*}(N) \cup V_G^{s*}(L) \subseteq V_G^{s*}(N + L) \subseteq V_G^{s*}(H) = V_G^{s*}(N) \cup V_G^{s*}(L)$  and so  $V_G^{s*}(N + L) = V_G^{s*}(N) \cup V_G^{s*}(L)$ . But  $S \in V_G^{s*}(N + L)$ . Therefore  $S \subseteq N$  or  $S \subseteq L$ , as desired.

(2) $\Rightarrow$ (3): It is clear.

(3) $\Rightarrow$ (1): Let  $W, W' \leq_G M$ . By Lemma 2.4(3), we have  $V_G^{s*}(W) \cup V_G^{s*}(W') = V_G^{s*}(\text{soc}_G(W)) \cup V_G^{s*}(\text{soc}_G(W'))$ . If  $\text{soc}_G(W) = 0$  or  $\text{soc}_G(W') = 0$ , then the result is trivially true. If not, then  $\text{soc}_G(W)$  and  $\text{soc}_G(W')$  are graded socle submodules of  $M$ . By hypothesis,  $V_G^{s*}(W) \cup V_G^{s*}(W') = V_G^{s*}(\text{soc}_G(W) + \text{soc}_G(W'))$ . Therefore  $M$  is a  $G$ -cotop module.  $\square$

**Corollary 2.7.** *1. Every graded submodule  $N$  of a  $G$ -cotop module  $M$  is a  $G$ -cotop module.*

2.  *$M$  is a  $G$ -cotop  $R$ -module if and only if  $M$  is a  $G$ -cotop  $\overline{R}$ -module.*
3. *Let  $M$  be a  $G$ -graded  $R$ -module and  $p \triangleleft_G R$ . If  $M$  is a  $G$ -cotop  $R$ -module, then  $\text{Ann}_M(p)$  is a  $G$ -cotop  $R/p$ -module.*
4. *Let  $H$  be a graded subring of a graded ring  $R$  and let  $M$  be a  $G$ -graded  $R$ -module such that it is a  $G$ -cotop as graded  $H$ -module. Then  $M$  is a  $G$ -cotop  $R$ -module.*
5. *Let  $f : M \rightarrow M'$  be a  $G$ -graded module isomorphism of  $G$ -graded  $R$ -modules. If  $M$  is a  $G$ -cotop module, then  $M'$  is a  $G$ -cotop module.*

*Proof.* (1) Let  $S \in \text{Spec}_G^s(N)$  and  $L_1, L_2$  be graded socle submodules of  $N$  such that  $S \subseteq L_1 + L_2$ . Clearly,  $\text{Spec}_G^s(N) \subseteq \text{Spec}_G^s(M)$ . So  $S \in \text{Spec}_G^s(M)$ . Now it is easy to see that  $L_1, L_2$  are graded socle submodules of  $M$ . Since  $M$  is a  $G$ -cotop module, then  $S \subseteq L_1$  or  $S \subseteq L_2$  by Theorem 2.6. Therefore  $N$  is a  $G$ -cotop module by Theorem 2.6 again.

(2) If  $M$  is a graded secondless  $R$ -module, then  $M$  is a graded secondless  $\bar{R}$ -module by Lemma 2.4(1). Thus the result is clearly true by Remark 2.3. Suppose that  $M$  is not a graded secondless  $R$ -module. Then by Lemma 2.4(1) and Theorem 2.6, we have  $M$  is a  $G$ -cotop module  $\Leftrightarrow$  Every graded second  $R$ -submodule of  $M$  is graded extraordinary  $\Leftrightarrow$  Every graded second  $\bar{R}$ -submodule of  $M$  is graded extraordinary  $\Leftrightarrow M$  is a  $G$ -cotop  $\bar{R}$ -module.

(3) Suppose that  $\text{Ann}_M(p)$  is not a graded secondless  $R/p$ -module. Let  $S$  be a graded second  $R/p$ -submodule of  $\text{Ann}_M(p)$  and  $N, K$  be graded socle  $R/p$ -submodules of  $\text{Ann}_M(p)$  with  $S \subseteq N + K$ . By Lemma 2.4(2),  $S$  is a graded second  $\bar{R}$ -submodule of  $\text{Ann}_M(p)$ . Also, it is easy to see that  $N, K$  are graded socle  $\bar{R}$ -submodules of  $\text{Ann}_M(p)$ . But  $M$  is a  $G$ -cotop  $R$ -module. So  $\text{Ann}_M(p)$  is a  $G$ -cotop  $R$ -module by (1) and hence it is a  $G$ -cotop  $\bar{R}$ -module by (2). This implies that  $S \subseteq N$  or  $S \subseteq K$  by Theorem 2.6. Therefore  $\text{Ann}_M(p)$  is a  $G$ -cotop  $R/p$ -module by Theorem 2.6 again.

(4) Suppose that  $M$  is not a graded secondless  $R$ -module. Let  $S$  be a graded second  $R$ -submodule of  $M$  and  $W_1, W_2$  be graded socle  $R$ -submodules of  $M$  such that  $S \subseteq W_1 + W_2$ . It is straightforward to see that  $S$  is a graded second  $H$ -submodule of  $M$  and  $W_1, W_2$  are graded socle  $H$ -submodules of  $M$ . But  $M$  is a  $G$ -cotop  $H$ -module. By Theorem 2.6,  $S \subseteq W_1$  or  $S \subseteq W_2$ . Therefore  $M$  is a  $G$ -cotop  $R$ -module.

(5) Suppose that  $M'$  is not graded secondless. Let  $S' \in \text{Spec}_G^s(M')$  and  $N'_1, N'_2$  be graded socle submodules of  $M'$  such that  $S' \subseteq N'_1 + N'_2$ . Since  $f$  is a graded module isomorphism, then  $f^{-1}(S') \in \text{Spec}_G^s(M)$  and  $f^{-1}(N'_1), f^{-1}(N'_2)$  are graded socle submodules of  $M$ . Now  $f^{-1}(S') \subseteq f^{-1}(N'_1 + N'_2) = f^{-1}(N'_1) + f^{-1}(N'_2)$ . But  $M$  is a  $G$ -cotop module. Hence  $f^{-1}(S') \subseteq f^{-1}(N'_1)$  or  $f^{-1}(S') \subseteq f^{-1}(N'_2)$ . By the surjectivity of  $f$ , we obtain  $S' \subseteq N'_1$  or  $S' \subseteq N'_2$ . Therefore  $M'$  is a  $G$ -cotop module.  $\square$

A graded ring  $F$  is called a graded field if every non-zero homogeneous element of  $F$  is unit. As usual, the set of all units of a graded ring  $R$  will be denoted by  $U(R)$ . A graded  $R$ -module  $M$  is said to be graded cyclic if there exists  $m \in h(M)$  such that  $M = Rm$ , see.<sup>9</sup>

**Lemma 2.8.** *Let  $M$  be a  $G$ -graded  $R$ -module. Then the following hold:*

1. *If  $R$  is a graded field, then every non-zero graded submodule of  $M$  is graded second.*
2. *If  $A \subseteq h(M)$  is a base for  $M$  such that  $|A| > 1$ , then there exist non-zero graded submodules  $N_1$  and  $N_2$  of  $M$  such that  $N_1 \not\subseteq N_2$  and  $N_2 \not\subseteq N_1$ .*
3. *If  $R$  is a graded field and  $M$  is a non-zero graded cyclic  $R$ -module, then  $M$  is a graded simple  $R$ -module.*

*Proof.* (1) Let  $N$  be a non-zero graded submodule of  $M$ . Let  $r \in h(R)$ . If  $r = 0$ , then  $rN = 0$ . If  $r \neq 0$ , then  $r \in U(R)$ . So for any  $n \in N$ , we have  $n = r(r^{-1}n) \in rN$  and thus  $rN = 0$  or  $rN = N$ . Therefore  $N \in \text{Spec}_G^s(M)$ . (2) Assume by way of contradiction that for any non-zero graded submodules  $N_1$  and  $N_2$  of  $M$ , we have  $N_1 \subseteq N_2$  or  $N_2 \subseteq N_1$ . By hypothesis, there exist  $m_1, m_2 \in A$  such that  $m_1 \neq m_2$ . It is clear that  $m_1, m_2 \neq 0$  as  $A$  is a base for  $M$ . Since  $A \subseteq h(M)$ , then  $Rm_1$  and  $Rm_2$  are non-zero graded submodules of  $M$ . Hence  $Rm_1 \subseteq Rm_2$  or  $Rm_2 \subseteq Rm_1$ . If  $Rm_1 \subseteq Rm_2$ , then  $m_1 = rm_2$  for some  $r \in R$  and thus  $m_1 + (-r)m_2 = 0$ . Since  $A$  is a linearly independent set, then  $r = 0$  and thus  $m_1 = 0$ , a contradiction. Similarly, if  $Rm_2 \subseteq Rm_1$ , we also have a contradiction.

(3) By hypothesis,  $M = Rm$  for some  $m \in h(M)$ . Let  $x \in M$  and  $m' \in h(M) - \{0\}$ . Then  $x = rm$  and  $m' = cm$  for some  $r, c \in R$ . Now  $m' \in M_g$  and  $m \in M_h$  for some  $g, h \in G$ . So  $m' = c_{gh^{-1}}m$ . Let  $a = c_{gh^{-1}}$ . Since  $m' \neq 0$ , then  $a \neq 0$  and thus  $a \in U(R)$ . This implies that  $m = a^{-1}m'$  and hence  $x = r(a^{-1}m') \in Rm'$ . So we obtain  $M = Rm'$ . This means that  $M$  is a non-zero graded cyclic module generated by any non-zero homogeneous element. Therefore  $M$  is a graded simple  $R$ -module.  $\square$

**Corollary 2.9.** *Let  $M$  be a  $G$ -graded module over a graded field  $F$ . If  $A \subseteq h(M)$  is a base for  $M$ , then  $M$  is a  $G$ -cotop module if and only if  $|A| \leq 1$ .*

*Proof.*  $\Rightarrow$ : Suppose that  $|A| > 1$ . By Lemma 2.8(2), there exist non-zero graded submodules  $N_1$  and  $N_2$  of  $M$  such that  $N_1 \not\subseteq N_2$  and  $N_2 \not\subseteq N_1$ . Also  $N_1, N_2, N_1 + N_2 \in \text{Spec}_G^s(M)$  by Lemma 2.8(1). Thus

$N_1 = soc_G(N_1)$  and  $N_2 = soc_G(N_2)$ . Since  $M$  is a  $G$ -cotop module and  $N_1 + N_2 \subseteq N_1 + N_2$ , then  $N_1 \subseteq N_2$  or  $N_2 \subseteq N_1$  by Theorem 2.6, a contradiction.

⇐: If  $M = 0$ , then  $M$  is a graded secondless module and hence it a  $G$ -cotop module by Remark 2.3. If  $M \neq 0$ , then  $|A| = 1$  and thus  $M$  is a graded simple  $F$ -module by Lemma 2.8(3). Therefore  $M$  is a  $G$ -cotop module. □

The following is an example on a non  $G$ -cotop module.

**Example 2.10.** Let  $F$  be any field. Take  $G = Z_2$  and  $M = F \times F$ . Then  $F$  is a  $G$ -graded field by  $F_0 = F$  and  $F_1 = \{0\}$ . Also  $M$  is a  $G$ -graded  $F$ -module by  $M_0 = F \times \{0\}$  and  $M_1 = \{0\} \times F$ . Moreover  $M_0, M_1 \leq_G M$ . Since  $F$  is a  $G$ -graded field, then  $M \in Spec_G^s(M)$  and  $M_0, M_1$  are graded socle submodules of  $M$  by Lemma 2.8(1). Now  $M \subseteq M_0 + M_1$ . But  $M \not\subseteq M_0$  and  $M \not\subseteq M_1$ . Therefore  $M$  is not a  $G$ -cotop module by Theorem 2.6.

Let  $M$  be a  $G$ -cotop module and  $Y \subseteq Spec_G^s(M)$ . The closure of  $Y$  in  $Spec_G^s(M)$  for the quasi-Zariski topology will be expressed by  $Cl(Y)$ .

**Lemma 2.11.** Let  $M$  be a  $G$ -cotop module and  $Y \subseteq Spec_G^s(M)$ . Let  $T(Y) = \sum_{S \in Y} S$ . Then  $Cl(Y) = V_G^{s*}(T(Y))$ . In particular, if  $M \in Y$ , then  $Y$  is dense in  $Spec_G^s(M)$ , that is,  $Cl(Y) = Spec_G^s(M)$ .

*Proof.* It is clear that  $Y \subseteq V_G^{s*}(T(Y))$ . Let  $V_G^{s*}(N)$  be any closed set containing  $Y$ , where  $N \leq_G M$ . It is enough to show that  $V_G^{s*}(T(Y)) \subseteq V_G^{s*}(N)$ . Let  $S \in V_G^{s*}(T(Y))$ . Then  $S \subseteq T(Y)$ . Note that for any  $S' \in Y$ , we have  $S' \subseteq N$  and hence  $T(Y) \subseteq N$ . This implies that  $S \subseteq N$  and thus  $S \in V_G^{s*}(N)$ . Therefore  $Cl(Y) = V_G^{s*}(T(Y))$ . Now if  $M \in Y$ , then  $Cl(Y) = V_G^{s*}(T(Y)) = V_G^{s*}(M) = Spec_G^s(M)$ . □

**Corollary 2.12.** Let  $M$  be a  $G$ -cotop module. Then  $Spec_G^s(M)$  is a  $T_0$ -space.

*Proof.* Let  $S_1, S_2 \in Spec_G^s(M)$  with  $S_1 \neq S_2$ . If  $Cl(\{S_1\}) = Cl(\{S_2\})$ , then  $V_G^{s*}(S_1) = V_G^{s*}(S_2)$  by Lemma 2.11. This implies that  $S_1 = S_2$ , a contradiction. Thus  $Cl(\{S_1\}) \neq Cl(\{S_2\})$ . This means that the closures of any two distinct points of  $Spec_G^s(M)$  are distinct. Therefore  $Spec_G^s(M)$  is a  $T_0$ -space. □

**Lemma 2.13.** Let  $M$  be a  $G$ -graded  $R$ -module. If  $N \leq_G M$  and  $I \triangleleft_G R$ , then  $V_G^{s*}(N) \cup V_G^{s*}(Ann_M(I)) = V_G^{s*}((N :_M I))$ .

*Proof.* It is clear that  $V_G^{s*}(N) \cup V_G^{s*}(Ann_M(I)) \subseteq V_G^{s*}((N :_M I))$ . Let  $S \in V_G^{s*}((N :_M I))$ . Suppose that  $S \not\subseteq N$  and it is enough to show that  $S \subseteq Ann_M(I)$ . Let  $s \in S$  and  $i \in I$ . Now  $S \subseteq (N :_M I)$ . So  $iS \subseteq N$ . Since  $S \in Spec_G^s(M)$ , then  $iS = \{0\}$  and hence  $is = 0$ , as desired. □

**Corollary 2.14.** Let  $M$  be a  $G$ -graded  $R$ -module. If  $I, J \triangleleft_G R$ , then  $V_G^{s*}(Ann_M(I)) \cup V_G^{s*}(Ann_M(J)) = V_G^{s*}(Ann_M(IJ))$ .

*Proof.* Clearly,  $(Ann_M(I) :_M J) = Ann_M(IJ)$ . By Lemma 2.13,  $V_G^{s*}(Ann_M(I)) \cup V_G^{s*}(Ann_M(J)) = V_G^{s*}((Ann_M(I) :_M J)) = V_G^{s*}(Ann_M(IJ))$ . □

Let  $M$  be a  $G$ -graded  $R$ -module. By Corollary 2.14, the family  $\Omega^{s'} = \{V_G^{s*}(Ann_M(I)) \mid I \triangleleft_G R\}$  always induce a topology on  $Spec_G^s(M)$  and it is denoted by  $\tau_G^{s'}$ . Now, the proof of the following proposition is obvious.

**Proposition 2.15.** Let  $M$  be a  $G$ -graded  $R$ -module. If  $M$  is a  $G$ -cotop module, then the quasi-Zariski topology  $\tau_G^{s*}$  on  $Spec_G^s(M)$  is finer than  $\tau_G^{s'}$ .

Now, we define another variety for a graded submodule  $N$  of a  $G$ -graded  $R$ -module  $M$ . We set  $V_G^s(N) = \{S \in Spec_G^s(M) \mid Ann_R(N) \subseteq Ann_R(S)\}$ . We state some properties of this variety in the following theorem to construct the Zariski-topology on  $Spec_G^s(M)$ .

**Theorem 2.16.** *Let  $M$  be a  $G$ -graded  $R$ -module. Then the following hold:*

1.  $V_G^s(M) = \text{Spec}_G^s(M)$  and  $V_G^s(0) = \emptyset$ .
2.  $\bigcap_{i \in I} V_G^s(N_i) = V_G^s(\bigcap_{i \in I} \text{Ann}_M(\text{Ann}_R(N_i)))$ , for any family  $\{N_i\}_{i \in I}$  of graded submodules.
3.  $V_G^s(N) \cup V_G^s(N') = V_G^s(N + N')$  for any graded submodules  $N, N'$  of  $M$ .

*Proof.* (1) It is clear.

(2) Let  $S \in \bigcap_{i \in I} V_G^s(N_i)$ . For all  $i \in I$ , we have  $\text{Ann}_R(N_i) \subseteq \text{Ann}_R(S)$  and thus  $\text{Ann}_M(\text{Ann}_R(S)) \subseteq \text{Ann}_M(\text{Ann}_R(N_i))$ . Hence  $\text{Ann}_M(\text{Ann}_R(S)) \subseteq \bigcap_{i \in I} \text{Ann}_M(\text{Ann}_R(N_i))$ . It follows that  $\text{Ann}_R(\bigcap_{i \in I} \text{Ann}_M(\text{Ann}_R(N_i))) \subseteq \text{Ann}_R(\text{Ann}_M(\text{Ann}_R(S))) = \text{Ann}_R(S)$ . So  $S \in V_G^s(\bigcap_{i \in I} \text{Ann}_M(\text{Ann}_R(N_i)))$  and so  $\bigcap_{i \in I} V_G^s(N_i) \subseteq V_G^s(\bigcap_{i \in I} \text{Ann}_M(\text{Ann}_R(N_i)))$ . Conversely, let  $S \in V_G^s(\bigcap_{i \in I} \text{Ann}_M(\text{Ann}_R(N_i)))$ . Then  $\text{Ann}_R(\bigcap_{i \in I} \text{Ann}_M(\text{Ann}_R(N_i))) \subseteq \text{Ann}_R(S)$ . For all  $i \in I$ , we have  $\bigcap_{i \in I} \text{Ann}_M(\text{Ann}_R(N_i)) \subseteq \text{Ann}_M(\text{Ann}_R(N_i))$  and hence  $\text{Ann}_R(N_i) = \text{Ann}_R(\text{Ann}_M(\text{Ann}_R(N_i))) \subseteq \text{Ann}_R(\bigcap_{i \in I} \text{Ann}_M(\text{Ann}_R(N_i))) \subseteq \text{Ann}_R(S)$ . Therefore  $S \in \bigcap_{i \in I} V_G^s(N_i)$ .

(3) Since  $N, N' \subseteq N + N'$ , then  $V_G^s(N) \subseteq V_G^s(N + N')$  and  $V_G^s(N') \subseteq V_G^s(N + N')$  and thus  $V_G^s(N) \cup V_G^s(N') \subseteq V_G^s(N + N')$ . For the reverse inclusion, let  $S \in V_G^s(N + N')$ . Then  $\text{Ann}_R(N + N') \subseteq \text{Ann}_R(S)$ . So we get  $\text{Ann}_R(N)\text{Ann}_R(N') \subseteq \text{Ann}_R(S) \in \text{Spec}_G(R)$ . By [11, Proposition 1.2], we have  $\text{Ann}_R(N) \subseteq \text{Ann}_R(S)$  or  $\text{Ann}_R(N') \subseteq \text{Ann}_R(S)$ . Therefore  $S \in V_G^s(N) \cup V_G^s(N')$ .  $\square$

In view of Theorem 2.16, the collection  $\Omega^s(M) = \{V_G^s(N) \mid N \leq_G M\}$  satisfies the axioms for closed sets of a topology on  $\text{Spec}_G^s(M)$ . This topology is called the Zariski-topology on  $\text{Spec}_G^s(M)$  and it is denoted by  $\tau_G^s$ .

In the following lemma, we state some relations between the varieties  $V_G^{s*}(N)$  and  $V_G^s(N)$  for any graded submodule  $N$  of a  $G$ -graded  $R$ -module  $M$ .

**Lemma 2.17.** *Suppose that  $N$  and  $N'$  are graded submodules of a  $G$ -graded  $R$ -module  $M$  and that  $I$  is a  $G$ -graded ideal of  $R$ . Then the following hold:*

1. If  $\text{Ann}_R(N) = \text{Ann}_R(N')$ , then  $V_G^s(N) = V_G^s(N')$ . The converse is also true if  $N, N' \in \text{Spec}_G^s(M)$ .
2.  $V_G^s(N) = \bigcup_{p \in V_G^R(\text{Ann}_R(N))} p\text{-Spec}_G^s(M)$
3.  $V_G^s(N) = V_G^s(\text{Ann}_M(\text{Ann}_R(N))) = V_G^{s*}(\text{Ann}_M(\text{Ann}_R(N)))$ . In particular,  $V_G^s(\text{Ann}_M(I)) = V_G^{s*}(\text{Ann}_M(I))$ .

*Proof.* The proof is straightforward.  $\square$

**Theorem 2.18.** *For any  $G$ -graded  $R$ -module  $M$ , the Zariski-topology  $\tau_G^s$  on  $\text{Spec}_G^s(M)$  is identical with  $\tau_G^{s*}$ . Moreover, if  $M$  is a  $G$ -cotop module, then the quasi-Zariski topology  $\tau_G^{s*}$  on  $\text{Spec}_G^s(M)$  is finer than the Zariski-topology  $\tau_G^s$ .*

*Proof.* It is clear by Lemma 2.17(3).  $\square$

**Proposition 2.19.** *Let  $M$  be a  $G$ -graded module over a graded field  $F$ . If  $A \subseteq h(M)$  is a base for  $M$  such that  $|A| \leq 1$ , then  $M$  is a  $G$ -cotop module and  $\tau_G^s = \tau_G^{s*} =$  the trivial topology on  $\text{Spec}_G^s(M)$ .*

*Proof.* Since  $|A| \leq 1$ , then  $M$  is a  $G$ -cotop module by Corollary 2.9. It is clear that if  $|A| = 0$ , then  $\tau_G^s = \tau_G^{s*} =$  the trivial topology on  $\text{Spec}_G^s(M)$ . If  $|A| = 1$ , then  $M = Fm$  for some  $m \in A$ . If  $m = 0$ , then  $M = 0$  and the result is trivially true. Suppose that  $m \neq 0$ . By Lemma 2.8(3),  $M$  is a graded simple  $F$ -module. Now let  $U \in \tau_G^{s*}$ . Then  $U = \text{Spec}_G^s(M) - V_G^{s*}(N)$  for some  $N \leq_G M$ . But  $N = 0$  or  $N = M$ . This implies that  $\tau_G^{s*} = \{\emptyset, \text{Spec}_G^s(M)\}$ . By Theorem 2.18,  $\tau_G^s \subseteq \tau_G^{s*} = \{\emptyset, \text{Spec}_G^s(M)\} \subseteq \tau_G^s$ . Therefore  $\tau_G^s = \tau_G^{s*} =$  the trivial topology on  $\text{Spec}_G^s(M)$ .  $\square$

### 3 Relating $Spec_G^s(M)$ and $Spec_G(\overline{R})$

Let  $M$  be a  $G$ -graded  $R$ -module. From now on, we assume that  $Spec_G^s(M)$  is a non-empty set and is equipped the Zariski-topology. We also consider the natural map  $\varphi$  of  $Spec_G^s(M)$  as described in the introduction. In this section, we provide some relations between  $Spec_G^s(M)$  and  $Spec_G(\overline{R})$  for any  $G$ -graded  $R$ -module  $M$ .

**Proposition 3.1.** *Let  $M$  be a  $G$ -graded  $R$ -module. Then  $\varphi^{-1}(V_G^s(\overline{I})) = V_G^s(Ann_M(I))$ , for every graded ideal  $I$  of  $R$  containing  $Ann_R(M)$ , and therefore the natural map  $\varphi$  of  $Spec_G^s(M)$  is continuous.*

*Proof.* For any  $S \in Spec_G^s(M)$ , we have  $S \in \varphi^{-1}(V_G^s(\overline{I})) \Leftrightarrow \varphi(S) = \overline{Ann_R(S)} \in V_G^s(\overline{I}) \Leftrightarrow \overline{I} \subseteq \overline{Ann_R(S)} \Leftrightarrow I \subseteq Ann_R(S) \Leftrightarrow S \subseteq Ann_M(I) \Leftrightarrow Ann_R(Ann_M(I)) \subseteq Ann_R(S) \Leftrightarrow S \in V_G^s(Ann_M(I))$ . Therefore  $\varphi^{-1}(V_G^s(\overline{I})) = V_G^s(Ann_M(I))$ .  $\square$

**Proposition 3.2.** *The following are equivalent for any  $G$ -graded  $R$ -module  $M$ :*

1. *If whenever  $S_1, S_2 \in Spec_G^s(M)$  with  $V_G^s(S_1) = V_G^s(S_2)$ , then  $S_1 = S_2$ .*
2.  *$|p\text{-}Spec_G^s(M)| \leq 1$  for every  $p \in Spec_G(R)$ .*
3. *The natural map  $\varphi$  is injective.*

*Proof.* (1) $\Rightarrow$ (2): Let  $p \in Spec_G(R)$  and  $S_1, S_2 \in p\text{-}Spec_G^s(M)$ . Then  $S_1, S_2 \in Spec_G^s(M)$  and  $Ann_R(S_1) = Ann_R(S_2) = p$ . By Lemma 2.17(1), we have  $V_G^s(S_1) = V_G^s(S_2)$  and hence  $S_1 = S_2$  by the assumption (1). (2) $\Rightarrow$ (3): Assume that  $\varphi(S_1) = \varphi(S_2)$ , where  $S_1, S_2 \in Spec_G^s(M)$ . Then  $Ann_R(S_1) = Ann_R(S_2)$ . Let  $p = Ann_R(S_1) \in Spec_G(R)$ . Then we get  $S_1, S_2 \in p\text{-}Spec_G^s(M)$  and by hypothesis, we obtain  $S_1 = S_2$ . (3) $\Rightarrow$ (1): Let  $S_1, S_2 \in Spec_G^s(M)$  with  $V_G^s(S_1) = V_G^s(S_2)$ . By Lemma 2.17(1), we have  $Ann_R(S_1) = Ann_R(S_2)$ . So  $\varphi(S_1) = \varphi(S_2)$ . Since  $\varphi$  is injective, then  $S_1 = S_2$ .  $\square$

**Proposition 3.3.** *Let  $M$  be a  $G$ -graded  $R$ -module. If  $M$  is a comultiplication graded module, then the natural map  $\varphi$  of  $Spec_G^s(M)$  is injective.*

*Proof.* Assume that  $\varphi(S_1) = \varphi(S_2)$ , where  $S_1, S_2 \in Spec_G^s(M)$ . Then  $Ann_R(S_1) = Ann_R(S_2)$ . Since  $M$  is a comultiplication graded module, then  $S_1 = Ann_M(Ann_R(S_1)) = Ann_M(Ann_R(S_2)) = S_2$ .  $\square$

Let  $M$  be a  $G$ -graded  $R$ -module. Recall that a proper graded submodule  $N$  of  $M$  is said to be graded completely irreducible if  $N = \bigcap_{i \in I} N_i$ , where  $\{N_i\}_{i \in I}$  is a family of graded submodules of  $M$ , implies that  $N = N_i$  for some  $i \in I$ . By [1, Theorem 2.1], every proper graded submodule of  $M$  is an intersection of graded completely irreducible submodules of  $M$ . Therefore, if  $M \neq 0$ , then the intersection of all graded completely irreducible submodules of  $M$  is zero.

Let  $M$  and  $M'$  be  $G$ -graded  $R$ -modules. By  $M \cong_G M'$ , we mean that there exists a  $G$ -graded module isomorphism  $f : M \rightarrow M'$ .

**Lemma 3.4.** *Let  $M, M'$  be  $G$ -graded  $R$ -modules and  $p \triangleleft_G R$ . Then the following hold:*

1. *If  $S$  is a graded second  $R/p$ -submodule of  $Ann_M(p)$ , then  $S$  is a graded second  $R$ -submodule of  $M$ .*
2. *If  $M$  is graded cocyclic and  $M \cong_G M'$ , then  $M'$  is graded cocyclic.*
3. *If  $M$  is a graded cocyclic  $R$ -module and  $N$  is a non-zero graded submodule of  $M$ , then  $N$  is a graded cocyclic  $R$ -module.*

*Proof.* (1) By Lemma 2.4(2),  $S$  is a graded second  $\bar{R}$ -submodule of  $Ann_M(p)$ . So  $S$  is a graded second  $R$ -submodule of  $Ann_M(p)$  by Lemma 2.4(1). Therefore  $S$  is a graded second  $R$ -submodule of  $M$ .  
 (2) Since  $M \cong_G M'$ , then there exists a  $G$ -graded module isomorphism  $f : M \rightarrow M'$ . By hypothesis,  $Soc_G^R(M)$  is a graded large and a graded simple submodule of  $M$ . Thus  $Soc_G^R(M') = f(Soc_G^R(M))$  is a graded large and a graded simple submodule of  $M'$ . Therefore  $M'$  is a graded cocyclic  $R$ -module.  
 (3) Since  $N \subseteq M$ , then  $Soc_G^R(N) \subseteq Soc_G^R(M)$ . By hypothesis,  $Soc_G^R(M)$  is a graded simple submodule of  $M$ . So  $Soc_G^R(N) = 0$  or  $Soc_G^R(N) = Soc_G^R(M)$ . Since  $N \neq 0$  and  $Soc_G^R(M)$  is a graded large submodule of  $M$ , then  $Soc_G^R(M) \cap N \neq 0$ . Now, it is easy to see that  $Soc_G^R(M) \cap N$  is a graded simple submodule of  $N$ . This follows that  $Soc_G^R(N) \neq 0$  and thus  $Soc_G^R(N) = Soc_G^R(M)$ . But  $Soc_G^R(N) \subseteq N$ . So  $Soc_G^R(N)$  is a graded large and a graded simple submodule of  $N$ , as needed.  $\square$

The following is a result of Proposition 3.2 and Lemma 3.4.

**Corollary 3.5.** *Let  $M$  be a  $G$ -graded  $R$ -module. If the natural map  $\varphi$  is injective, then  $Ann_M(p)$  is a graded cocyclic module for every  $p \in Max_G(R)$  with  $Ann_M(p) \neq 0$ .*

*Proof.* Let  $p \in Max_G(R)$  such that  $Ann_M(p) \neq 0$ . Then  $Ann_M(p)$  is a graded module over the graded field  $R/p$  which follows that  $Ann_M(p)$  is a graded second  $R/p$ -submodule of  $Ann_M(p)$  by Lemma 2.8(1). Thus  $Ann_M(p)$  is a graded second  $R$ -submodule of  $M$  by Lemma 3.4(1). Since  $p \subseteq Ann_R(Ann_M(p))$  and  $Ann_M(p) \neq 0$ , then  $Ann_R(Ann_M(p)) = p$ . Note that there exists a graded completely irreducible submodule  $C$  of  $M$  such that  $Ann_M(p) \not\subseteq C$ . If not, then  $Ann_M(p) \subseteq \bigcap H = 0$ , where the intersection runs over all graded completely irreducible submodules of  $M$ , a contradiction. Now, if  $Ann_M(p) \cap C \neq 0$ , then  $Ann_R(Ann_M(p) \cap C) \neq R$ . But  $p = Ann_R(Ann_M(p)) \subseteq Ann_R(Ann_M(p) \cap C)$ . So  $Ann_R(Ann_M(p) \cap C) = p = Ann_R(Ann_M(p))$ . Also, it is easy to see that  $Ann_M(p) \cap C \in Spec_G^s(M)$ . Since  $\varphi$  is injective, then  $|p-Spec_G^s(M)| \leq 1$  by Proposition 3.2 and thus  $Ann_M(p) = Ann_M(p) \cap C$ . This implies that  $Ann_M(p) \subseteq C$ , a contradiction. So we get  $Ann_M(p) \cap C = 0$  and hence  $Ann_M(p) \cong_G (C + Ann_M(p))/C \subseteq M/C$ . By [13, Lemma 2.10],  $M/C$  is a graded cocyclic module and thus  $Ann_M(p)$  is graded cocyclic by Lemma 3.4(2), (3).  $\square$

**Corollary 3.6.** *If  $M$  is a comultiplication graded  $R$ -module, then  $Ann_M(p)$  is a graded cocyclic  $R$ -module for every  $p \in Max_G(R)$  with  $Ann_M(p) \neq 0$ .*

*Proof.* By Corollary 3.5 and Proposition 3.3.  $\square$

**Theorem 3.7.** *Let  $M$  be a  $G$ -graded  $R$ -module. If the natural map  $\varphi$  of  $Spec_G^s(M)$  is surjective, then  $\varphi$  is both open and closed; more precisely, for every  $N \leq_G M$ ,  $\varphi(V_G^s(N)) = V_G^R(Ann_R(N))$  and  $\varphi(Spec_G^s(M) - V_G^s(N)) = Spec_G(\bar{R}) - V_G^R(Ann_R(N))$ .*

*Proof.* By Proposition 3.1, we have  $\varphi^{-1}(V_G^R(\bar{I})) = V_G^s(Ann_M(I))$  for every  $I \triangleleft_G R$  containing  $Ann_R(M)$  and thus  $\varphi^{-1}(V_G^R(Ann_R(N))) = V_G^s(Ann_M(Ann_R(N))) = V_G^s(N)$ . So  $\varphi(V_G^s(N)) = V_G^R(Ann_R(N))$  as  $\varphi$  is surjective. For the second part, note that  $Spec_G^s(M) - V_G^s(N) = Spec_G^s(M) - \varphi^{-1}(V_G^R(Ann_R(N))) = \varphi^{-1}(Spec_G(\bar{R}) - V_G^R(Ann_R(N)))$ . This implies that  $\varphi(Spec_G^s(M) - V_G^s(N)) = Spec_G(\bar{R}) - V_G^R(Ann_R(N))$ .  $\square$

**Corollary 3.8.** *Let  $M$  be a  $G$ -graded  $R$ -module. Then the natural map  $\varphi$  of  $Spec_G^s(M)$  is bijective if and only if  $\varphi$  is homeomorphic.*

Let  $R$  be a  $G$ -graded ring and  $I \triangleleft_G R$ . The set  $D_r = Spec_G(R) - V_G^R(rR) = Spec_G(R) - V_G^R(r)$  is open in  $Spec_G(R)$  for each  $r \in h(R)$  and the family  $\{D_r \mid r \in h(R)\}$  forms a base for the Zariski topology on  $Spec_G(R)$  (see [10, Theorem 2.3]). The radical of  $I$ , the graded radical of  $I$ , the nilradical of  $I$  and the graded nilradical of  $I$  will be denoted by  $\sqrt{I}$ ,  $Gr(I)$ ,  $N(R)$  and  $N_G(R)$ , respectively. For details concerning the graded radical and the graded nilradical of a graded ideal of a graded ring, see.<sup>11</sup>

Now, we need the following lemma to give a simple characterization for the connectedness of  $Spec_G^s(M)$ .

**Lemma 3.9.** *Let  $R$  be a  $G$ -graded ring. Then  $Spec_G(R)$  is disconnected if and only if  $R$  contains idempotent element other than 0 and 1.*

*Proof.*  $\Rightarrow$ : Suppose that  $Spec_G(R)$  is disconnected. Then  $Spec_G(R) = V_G^R(I) \cup V_G^R(J)$  for some  $I, J \triangleleft_G R$  such that  $V_G^R(I) \cap V_G^R(J) = \emptyset$  and  $V_G^R(I), V_G^R(J) \neq \emptyset$ . By [10, Proposition 2.1(2)],  $V_G^R(I + J) = V_G^R(I) \cap V_G^R(J) = \emptyset$  and hence  $I + J = R$ . So there exist  $a \in I$  and  $b \in J$  such that  $a + b = 1$  which follows that  $1 = 1_e = (a + b)_e = a_e + b_e$  with  $a_e \in I$  and  $b_e \in J$ . Also by [10, Proposition 2.1(3)], we have  $Spec_G(R) = V_G^R(IJ)$ . This implies that for every  $p \in Spec_G(R)$ , we have  $IJ \subseteq p$  and thus  $a_e b_e \in N_G(R)$  by [11, Proposition 2.5]. Since  $a_e b_e \in h(R)$ , then  $a_e b_e \in N(R)$  and hence there exists a positive integer  $m$  such that  $(a_e b_e)^m = 0$ . Note that  $V_G^R(Ra_e^m + Rb_e^m) = \emptyset$ . If not, then  $\exists p \in Spec_G(R)$  such that  $a_e^m, b_e^m \in p$  which follows that  $a_e, b_e \in \sqrt{p}$ . But  $a_e, b_e \in h(R)$ . So we get  $a_e, b_e \in Gr(p) = p$  by [11, Proposition 2.4(5)]. Thus  $1 = a_e + b_e \in p$ , a contradiction. Therefore  $Ra_e^m + Rb_e^m = R$  which implies that  $1 = ra_e^m + r'b_e^m$  for some  $r, r' \in R$ . Note that  $(ra_e^m)^2 = (ra_e^m)(1 - r'b_e^m) = ra_e^m - rr'(a_e b_e)^m = ra_e^m$ . If  $ra_e^m = 0$ , then  $b_e^m \in U(R)$  and thus  $D_{b_e^m} = Spec_G(R)$  by [11, Proposition 3.6]. So we obtain  $V_G^R(Rb_e^m) = \emptyset$  and so  $V_G^R(J) \subseteq V_G^R(Rb_e^m) = \emptyset$ , a contradiction. Similarly, if  $ra_e^m = 1$ , then we have  $V_G^R(I) = \emptyset$ , a contradiction. Therefore,  $ra_e^m$  is an idempotent element other than 0 and 1.

$\Leftarrow$ : Suppose that  $x$  is an idempotent element of  $R$  other than 0 and 1. By [15, Lemma 4.4],  $x \in R_e$  and thus  $1 - x \in R_e$  as  $1 \in R_e$ . By [10, Theorem 2.3(2)],  $D_x \cap D_{1-x} = D_{x(1-x)} = D_0 = \emptyset$ . Now for any  $p \in Spec_G(R)$ , we have  $x \notin p$  or  $1 - x \notin p$  which follows that  $p \in D_x \cup D_{1-x}$  and thus  $Spec_G(R) = D_x \cup D_{1-x}$ . Note that if  $D_x = \emptyset$ , then  $x \in N(R)$  by [11, Proposition 3.6(2)]. So  $x^n = 0$  for some positive integer  $n$ . But  $x$  is idempotent which follows that  $x = 0$ , a contradiction. Thus  $D_x \neq \emptyset$ . Similarly, if  $D_{1-x} = \emptyset$ , then we obtain  $x = 1$ , a contradiction. Hence  $D_{1-x} \neq \emptyset$ . Now  $Spec_G(R)$  can be written as a union two non-empty disjoint open sets which means that  $Spec_G(R)$  is disconnected.  $\square$

The following is a result for Proposition 3.1, Theorem 3.7 and Lemma 3.9.

**Corollary 3.10.** *Let  $M$  be a  $G$ -graded  $R$ -module such that the natural map  $\varphi$  of  $Spec_G^s(M)$  is surjective. Then the following statements are equivalent:*

1.  $Spec_G^s(M)$  is connected.
2.  $Spec_G(\overline{R})$  is connected.
3. The only idempotent elements of  $\overline{R}$  are  $\overline{0}$  and  $\overline{1}$ .

*Proof.* (1) $\Rightarrow$ (2): Since  $\varphi$  is continuous surjective map and  $Spec_G^s(M)$  is connected, then  $Spec_G(\overline{R})$  is connected.

(2) $\Rightarrow$ (1): Assume by way of contradiction that  $Spec_G^s(M)$  is disconnected. Then there exists a clopen set  $U$  in  $Spec_G^s(M)$  such that  $U \neq \emptyset$  and  $U \neq Spec_G^s(M)$ . Since  $\varphi$  is surjective, then  $\varphi(U)$  is clopen in  $Spec_G(\overline{R})$  by Theorem 3.7. This follows that  $\varphi(U) = \emptyset$  or  $\varphi(U) = Spec_G(\overline{R})$ . Since  $U$  is open in  $Spec_G^s(M)$ , then  $U = Spec_G^s(M) - V_G^s(N)$  for some  $N \leq_G M$ . Now if  $\varphi(U) = Spec_G(\overline{R})$ , then  $Spec_G(\overline{R}) = \varphi(U) = Spec_G(\overline{R}) - V_G^R(Ann_R(N))$  by Theorem 3.7, again. Thus  $V_G^R(Ann_R(N)) = \emptyset$  which implies that  $\emptyset = \varphi^{-1}(V_G^R(Ann_R(N))) = V_G^s(Ann_M(Ann_R(N))) = V_G^s(N)$  by Proposition 3.1 and Lemma 2.17(3). So  $U = Spec_G^s(M)$ , a contradiction. If  $\varphi(U) = \emptyset$ , then  $U \subseteq \varphi^{-1}(\varphi(U)) = \emptyset$ . It follows that  $U = \emptyset$  which is also a contradiction. Therefore  $Spec_G^s(M)$  is connected.

The equivalence of (2) and (3) follows from Lemma 3.9.  $\square$

**Corollary 3.11.** *Let  $M$  be a  $G$ -graded  $R$ -module such that the natural map  $\varphi$  of  $Spec_G^s(M)$  is surjective. If either  $R$  is a graded local ring or  $Ann_R(M) \in Spec_G(R)$ , then both  $Spec_G^s(M)$  and  $Spec_G(\overline{R})$  are connected.*

*Proof.* First, assume that  $R$  is a graded local ring. Then the only idempotent elements of  $\overline{R}$  are  $\overline{0}$  and  $\overline{1}$  (see the proof of [15, Lemma 4.4]). Now the result follows from Corollary 3.10. Secondly, assume that  $Ann_R(M) \in Spec_G(R)$  and let  $r$  be idempotent element of  $R$ . Then  $\overline{r} = \overline{r^2}$  and hence  $r(r-1) \in Ann_R(M)$ . Again, by [15, Lemma 4.4], we have  $r \in R_e$  and thus  $r - 1 \in R_e$ . This follows that  $r \in Ann_R(M)$  or  $r - 1 \in Ann_R(M)$ . Therefore  $\overline{r} = \overline{0}$  or  $\overline{r} = \overline{1}$ . By Corollary 3.10,  $Spec_G^s(M)$  and  $Spec_G(\overline{R})$  are connected.  $\square$

**Proposition 3.12.** *Let  $M$  and  $M'$  be  $G$ -graded  $R$ -modules. If  $f : M \rightarrow M'$  is a graded module monomorphism, then the mapping  $\pi : \text{Spec}_G^s(M) \rightarrow \text{Spec}_G^s(M')$  by  $\pi(S) = f(S)$  is continuous.*

*Proof.* It is easy to see that  $\pi$  is well-defined. Now for any  $S \in \text{Spec}_G^s(M)$  and any closed set  $V_G^s(N')$  in  $\text{Spec}_G^s(M')$ , where  $N' \leq_G M'$ , we have  $S \in \pi^{-1}(V_G^s(N')) = \pi^{-1}(V_G^{s*}(\text{Ann}_{M'}(\text{Ann}_R(N')))) \Leftrightarrow \pi(S) = f(S) \subseteq \text{Ann}_{M'}(\text{Ann}_R(N')) \Leftrightarrow S = f^{-1}(f(S)) \subseteq f^{-1}(\text{Ann}_{M'}(\text{Ann}_R(N'))) = \text{Ann}_M(\text{Ann}_R(N')) \Leftrightarrow S \in V_G^{s*}(\text{Ann}_M(\text{Ann}_R(N'))) = V_G^s(\text{Ann}_M(\text{Ann}_R(N')))$ . So  $\pi^{-1}(V_G^s(N')) = V_G^s(\text{Ann}_M(\text{Ann}_R(N')))$  and so  $\pi$  is continuous.  $\square$

Let  $M$  be a  $G$ -graded  $R$ -module. Now, we write  $X_r^s = \text{Spec}_G^s(M) - V_G^s(\text{Ann}_M(r))$  for each  $r \in h(R)$  and prove that  $B = \{X_r^s \mid r \in h(R)\}$  is a base for the Zariski topology on  $\text{Spec}_G^s(M)$ , which is similar to that on  $\text{Spec}_G(R)$ .

**Proposition 3.13.** *For any  $G$ -graded  $R$ -module  $M$ , the set  $B = \{X_r^s \mid r \in h(R)\}$  forms a base for the Zariski topology on  $\text{Spec}_G^s(M)$ .*

*Proof.* Let  $U = \text{Spec}_G^s(M) - V_G^s(N)$  be an open set in  $\text{Spec}_G^s(M)$ , where  $N \leq_G M$ . Let  $S \in U$  and it is enough to find an element  $r \in h(R)$  such that  $S \in X_r^s \subseteq U$ . Since  $S \in U$ , then  $\text{Ann}_R(N) \not\subseteq \text{Ann}_R(S)$  and hence there exists  $t \in R$  and  $g \in G$  such that  $t_g \in \text{Ann}_R(N) - \text{Ann}_R(S)$ . Take  $r = t_g \in h(R)$ . If  $S \notin X_r^s$ , then  $rR \subseteq \text{Ann}_R(\text{Ann}_M(r)) \subseteq \text{Ann}_R(S)$  and thus  $r \in \text{Ann}_R(S)$ , a contradiction. This implies that  $S \in X_r^s$ . Since  $r \in \text{Ann}_R(N)$ , then  $\text{Ann}_M(\text{Ann}_R(N)) \subseteq \text{Ann}_M(r)$ . Now let  $S' \in X_r^s$ . Then we have  $\text{Ann}_R(\text{Ann}_M(r)) \not\subseteq \text{Ann}_R(S')$ . If  $S' \notin U$ , then  $\text{Ann}_R(N) \subseteq \text{Ann}_R(S')$  which follows that  $\text{Ann}_M(\text{Ann}_R(S')) \subseteq \text{Ann}_M(\text{Ann}_R(N)) \subseteq \text{Ann}_M(r)$ . So  $\text{Ann}_R(\text{Ann}_M(r)) \subseteq \text{Ann}_R(\text{Ann}_M(\text{Ann}_R(S')))$  =  $\text{Ann}_R(S')$ , a contradiction. This means that  $S' \in U$  and hence  $S \in X_r^s \subseteq U$ , as desired.  $\square$

**Proposition 3.14.** *Let  $M$  be a  $G$ -graded  $R$ -module and  $r \in h(R)$ . Then:*

1.  $\varphi^{-1}(D_{\bar{r}}) = X_r^s$ .
2.  $\varphi(X_r^s) \subseteq D_{\bar{r}}$ . If the natural map  $\varphi$  is surjective, then the equality holds.
3.  $X_a^s \cap X_b^s = X_{ab}^s$ , for any  $a, b \in h(R)$ .
4. If  $r \in N(R)$ , then  $X_r^s = \emptyset$ .
5. If  $r \in U(R)$ , then  $X_r^s = X_{r^{-1}}^s = \text{Spec}_G^s(M)$ .

*Proof.* (1)  $\varphi^{-1}(D_{\bar{r}}) = \varphi^{-1}(\text{Spec}_G(\bar{R}) - V_{\bar{G}}^{\bar{R}}(\bar{r}\bar{R})) = \text{Spec}_G^s(M) - \varphi^{-1}(V_{\bar{G}}^{\bar{R}}(\bar{r}\bar{R})) = \text{Spec}_G^s(M) - V_G^s(\text{Ann}_M(r)) = X_r^s$  by Proposition 3.1.

(2) Trivial.

(3) By [10, Theorem 2.3(2)], we have  $D_{\bar{a}} \cap D_{\bar{b}} = D_{\bar{ab}}$  and thus  $X_a^s \cap X_b^s = \varphi^{-1}(D_{\bar{a}}) \cap \varphi^{-1}(D_{\bar{b}}) = \varphi^{-1}(D_{\bar{ab}}) = X_{ab}^s$ .

(4) Suppose that  $r \in N(R)$ . By [11, Proposition 3.6(2)], we have  $D_r = \emptyset$  and hence  $D_{\bar{r}} = \emptyset$ . This follows that  $X_r^s = \varphi^{-1}(D_{\bar{r}}) = \emptyset$  by (1).

(5) Assume that  $r \in U(R)$ . Then  $D_r = \text{Spec}_G(R)$  by [11, Proposition 3.6(3)]. This implies that  $D_{\bar{r}} = \text{Spec}_G(\bar{R})$  and thus  $X_r^s = \varphi^{-1}(D_{\bar{r}}) = \varphi^{-1}(\text{Spec}_G(\bar{R})) = \text{Spec}_G^s(M)$  by (1). Now, since  $r \in U(R) \cap h(R)$ , then  $r^{-1} \in U(R) \cap h(R)$ . Therefore  $X_{r^{-1}}^s = \text{Spec}_G^s(M) = X_r^s$ .  $\square$

Let  $M$  be a  $G$ -graded  $R$ -module. By [10, Theorem 2.3(4)], each  $D_r$  is quasi-compact and hence  $D_1 = \text{Spec}_G(R)$  is quasi-compact. In the following theorem, we give a similar result for  $\text{Spec}_G^s(M)$  and its basic open sets.

**Theorem 3.15.** *Let  $M$  be a  $G$ -graded  $R$ -module. If the natural map  $\varphi$  is surjective, then the open set  $X_r^s$  is quasi compact for every  $r \in h(R)$ . Specifically,  $\text{Spec}_G^s(M)$  is quasi compact.*

*Proof.* Let  $r \in h(R)$  and  $\zeta = \{X_t^s \mid t \in \Delta\}$  be a basic open cover for  $X_r^s$ , where  $\Delta$  is an index set subset of  $h(R)$ . Then  $X_r^s \subseteq \bigcup_{t \in \Delta} X_t^s$  and hence  $D_{\bar{r}} = \varphi(X_r^s) \subseteq \bigcup_{t \in \Delta} \varphi(X_t^s) = \bigcup_{t \in \Delta} D_{\bar{t}}$  by Proposition 3.14(1).

So  $\bar{\zeta} = \{D_{\bar{t}} \mid t \in \Delta\}$  is a basic open cover for the quasi compact set  $D_{\bar{r}}$ . Thus it has a finite subcover  $\bar{\bar{\zeta}} = \{D_{\bar{t}_i} \mid i = 1, \dots, n\}$ , where  $t_i \in \Delta$  for any  $i = 1, \dots, n$ . Now  $X_r^s = \varphi^{-1}(D_{\bar{r}}) \subseteq \bigcup_{i=1}^n \varphi^{-1}(D_{\bar{t}_i}) = \bigcup_{i=1}^n X_{t_i}^s$  and this follows that  $\zeta = \{X_{t_i}^s \mid i = 1, \dots, n\} \subseteq \zeta$  is a finite subcover for  $X_r^s$ . Therefore  $X_r^s$  is quasi compact. Now  $Spec_G^s(M) = X_1^s$ . So  $Spec_G^s(M)$  is quasi compact.  $\square$

#### 4 Irreducibility in $Spec_G^s(M)$

Let  $M$  be a  $G$ -graded  $R$ -module and  $Y$  be a subset of  $Spec_G^s(M)$ . We will denote the closure of  $Y$  in  $Spec_G^s(M)$  by  $Cl(Y)$  and the summation  $\sum_{S \in Y} S$  by  $T(Y)$ . If  $Z$  is a subset of  $Spec_G(R)$ , then the intersection of all members of  $Z$  will be expressed by  $\xi(Y)$ .

**Proposition 4.1.** *Let  $M$  be a  $G$ -graded  $R$ -module and  $Y \subseteq Spec_G^s(M)$ . Then  $Cl(Y) = V_G^s(T(Y))$ . In particular,  $Y$  is closed if and only if  $V_G^s(T(Y)) = Y$ .*

*Proof.* It is easy to see that  $Y \subseteq V_G^s(T(Y))$ . Let  $V_G^s(N)$  be any closed set in  $Spec_G^s(M)$  such that  $Y \subseteq V_G^s(N)$  and it is enough to show that  $V_G^s(T(Y)) \subseteq V_G^s(N)$ . So let  $S' \in V_G^s(T(Y))$ . Then  $Ann_R(T(Y)) \subseteq Ann_R(S')$ . Note that for any  $Q \in Y$ , we have  $Ann_R(N) \subseteq Ann_R(Q)$  and thus  $Ann_R(N) \subseteq \bigcap_{Q \in Y} Ann_R(Q) = Ann_R(T(Y)) \subseteq Ann_R(S')$ . Hence  $S' \in V_G^s(N)$ . Therefore  $V_G^s(T(Y))$  is the smallest closed set containing  $Y$  and this means that  $Cl(Y) = V_G^s(T(Y))$ .  $\square$

Recall that a topological space  $X$  is irreducible if for any decomposition  $X = F_1 \cup F_2$  with closed subsets  $F_i$  of  $X$  with  $i = 1, 2$ , we have  $F_1 = X$  or  $F_2 = X$ . A subset  $X'$  of  $X$  is irreducible if it is an irreducible topological space with the induced topology. An irreducible component of a topological space is a maximal irreducible subset of it. Let  $X$  be a topological space. It is easy to see that every singleton subset of  $X$  is irreducible. Furthermore, a subset  $Y$  of  $X$  is irreducible if and only if its closure is irreducible (see<sup>7</sup>).

**Theorem 4.2.** *Let  $M$  be a  $G$ -graded  $R$ -module. If  $S \in Spec_G^s(M)$ , then  $Cl(\{S\}) = V_G^s(S)$  and  $V_G^s(S)$  is an irreducible closed subset of  $Spec_G^s(M)$ .*

*Proof.* By Proposition 4.1,  $Cl(\{S\}) = V_G^s(T(\{S\})) = V_G^s(S)$ . Since  $\{S\}$  is irreducible, then  $Cl(\{S\}) = V_G^s(S)$  is irreducible, as desired.  $\square$

**Proposition 4.3.** *Let  $M$  be a  $G$ -graded  $R$ -module and  $Y \subseteq Spec_G^s(M)$ . Then :*

1. *If  $T(Y) \in Spec_G^s(M)$ , then  $Y$  is irreducible.*
2. *If  $Y$  is irreducible, then  $\Upsilon = \{Ann_R(S) \mid S \in Y\}$  is an irreducible subset of  $Spec_G(R)$ , i.e.,  $\xi(\Upsilon) = Ann_R(T(Y))$  is a graded prime ideal of  $R$ .*

*Proof.* (1) Assume that  $T(Y) \in Spec_G^s(M)$ . By Proposition 4.1 and Theorem 4.2,  $V_G^s(T(Y)) = Cl(Y)$  is irreducible. Therefore  $Y$  is irreducible.

(2) Assume that  $Y$  is irreducible. Then  $\varphi(Y) = Y'$  is an irreducible subset of  $Spec_G(\bar{R})$  since  $\varphi$  is continuous by Proposition 3.1. Note that  $\xi(Y') = \xi(\varphi(Y)) = \bigcap_{S \in Y} Ann_R(S) = \overline{\bigcap_{S \in Y} Ann_R(S)} = \overline{Ann_R(T(Y))}$  and

thus  $\xi(Y') = \overline{Ann_R(T(Y))} \in Spec_G(\bar{R})$  by [14, Lemma 4.3]. This implies that  $\xi(\Upsilon) = Ann_R(T(Y)) \in Spec_G(R)$  and thus  $\Upsilon$  is an irreducible subset of  $Spec_G(R)$  by [14, Lemma 4.3] again.  $\square$

**Corollary 4.4.** *Let  $M$  be a  $G$ -graded  $R$ -module. Assume that  $p\text{-}Spec_G^s(M) \neq \emptyset$  for some  $p \in Spec_G(R)$ . Then the following hold:*

1.  $p\text{-Spec}_G^s(M)$  is irreducible.
2. If  $p \in \text{Max}_G(R)$ , then  $p\text{-Spec}_G^s(M)$  is an irreducible closed subset of  $\text{Spec}_G^s(M)$ .

*Proof.* By [5, Proposition 2.4(a)], we have  $T(p\text{-Spec}_G^s(M)) \in p\text{-Spec}_G^s(M)$ . So  $p\text{-Spec}_G^s(M)$  is irreducible subset of  $\text{Spec}_G^s(M)$  by Proposition 4.3(1). Now it is easy to see that  $\text{Ann}_M(p) \neq 0$ . So if  $p \in \text{Max}_G(R)$ , then  $\text{Ann}_R(\text{Ann}_M(p)) = p$ . By Lemma 2.17(2), we obtain  $V_G^s(\text{Ann}_M(p)) = \bigcup_{q \in V_G^R(\text{Ann}_R(\text{Ann}_M(p)))} q\text{-Spec}_G^s(M) = \bigcup_{q \in \{p\}} q\text{-Spec}_G^s(M) = p\text{-Spec}_G^s(M)$ . Therefore  $p\text{-Spec}_G^s(M) = V_G^s(\text{Ann}_M(p))$  and this follows that  $p\text{-Spec}_G^s(M)$  is an irreducible closed subset of  $\text{Spec}_G^s(M)$ .  $\square$

**Proposition 4.5.** *Let  $M$  be a  $G$ -graded  $R$ -module and  $Y \subseteq \text{Spec}_G^s(M)$  such that  $\text{Ann}_R(T(Y)) = p$  is a graded prime ideal of  $R$ . If  $p\text{-Spec}_G^s(M) \neq \emptyset$ , then  $Y$  is irreducible.*

*Proof.* Let  $S \in p\text{-Spec}_G^s(M)$ . By Lemma 2.17(1) and Proposition 4.1, we have  $V_G^s(S) = V_G^s(T(Y)) = Cl(Y)$ . So  $Cl(Y)$  is irreducible by Theorem 4.2 and thus  $Y$  is irreducible.  $\square$

**Theorem 4.6.** *Let  $M$  be a  $G$ -graded  $R$ -module and  $Y \subseteq \text{Spec}_G^s(M)$ . If the natural map  $\varphi$  is surjective, then  $Y$  is an irreducible closed subset of  $\text{Spec}_G^s(M)$  if and only if  $Y = V_G^s(S)$  for some  $S \in \text{Spec}_G^s(M)$ .*

*Proof.* Assume that  $Y$  is an irreducible closed subset of  $\text{Spec}_G^s(M)$ . Then  $\text{Ann}_R(T(Y)) \in \text{Spec}_G(R)$  by Proposition 4.3(2). Thus  $\overline{\text{Ann}_R(T(Y))} \in \text{Spec}_G(\overline{R})$ . Since  $\varphi$  is surjective, then  $\text{Ann}_R(S) = \text{Ann}_R(T(Y))$  for some  $S \in \text{Spec}_G^s(M)$ . By Lemma 2.17(1) and Proposition 4.1,  $V_G^s(S) = V_G^s(T(Y)) = Cl(Y) = Y$ . Conversely, if  $Y = V_G^s(S)$  for some  $S \in \text{Spec}_G^s(M)$ , then  $Y$  is closed and it is irreducible by Theorem 4.2.  $\square$

**Corollary 4.7.** *Let  $M$  be a  $G$ -graded  $R$ -module and  $S \in \text{Spec}_G^s(M)$ . If  $\overline{\text{Ann}_R(S)}$  is a minimal graded prime ideal of  $\overline{R}$ , then  $V_G^s(S)$  is an irreducible component of  $\text{Spec}_G^s(M)$ . The converse is true if the natural map  $\varphi$  is surjective.*

*Proof.* By Theorem 4.2,  $V_G^s(S)$  is irreducible and it is enough to show that it is a maximal irreducible. Let  $Y$  be irreducible subset of  $\text{Spec}_G^s(M)$  with  $V_G^s(S) \subseteq Y$  and it remains to prove  $Y = V_G^s(S)$ . Since  $S \in V_G^s(S) \subseteq Y$ , then  $S \in Y$  and thus  $\overline{\text{Ann}_R(T(Y))} \subseteq \overline{\text{Ann}_R(S)}$ . By Proposition 4.3(2),  $\overline{\text{Ann}_R(T(Y))} \in \text{Spec}_G(\overline{R})$ . But  $\overline{\text{Ann}_R(S)}$  is a minimal graded prime ideal of  $\overline{R}$ . So  $\overline{\text{Ann}_R(T(Y))} = \overline{\text{Ann}_R(S)}$  and this follows that  $V_G^R(\overline{\text{Ann}_R(T(Y))}) = V_G^R(\overline{\text{Ann}_R(S)})$ . Thus  $V_G^s(T(Y)) = \varphi^{-1}(V_G^R(\overline{\text{Ann}_R(T(Y))})) = \varphi^{-1}(V_G^R(\overline{\text{Ann}_R(S)})) = V_G^s(S)$  by Lemma 2.17(3) and Proposition 3.1. Since  $Y \subseteq V_G^s(T(Y))$ , then  $Y \subseteq V_G^s(S)$  and hence  $Y = V_G^s(S)$ . Conversely, assume that  $\varphi$  is surjective. Let  $\overline{p} \in \text{Spec}_G(\overline{R})$  with  $\overline{p} \subseteq \overline{\text{Ann}_R(S)}$  and it is enough to show that  $\overline{p} = \overline{\text{Ann}_R(S)}$ . By the surjectivity of  $\varphi$ , there exists  $S' \in \text{Spec}_G^s(M)$  such that  $\text{Ann}_R(S') = p$ . Then  $\text{Ann}_R(S') \subseteq \text{Ann}_R(S)$ . So we get  $V_G^s(S) \subseteq V_G^s(S')$ . Since  $V_G^s(S)$  is irreducible component and  $V_G^s(S')$  is irreducible by Theorem 4.2, then  $V_G^s(S) = V_G^s(S')$ . By Lemma 2.17(1), we have  $\text{Ann}_R(S) = \text{Ann}_R(S') = p$ . Therefore  $\overline{p} = \overline{\text{Ann}_R(S)}$ , as needed.  $\square$

**Theorem 4.8.** *Let  $M$  be a  $G$ -graded  $R$ -module. Then the following statements are equivalent for any  $S_1, S_2 \in \text{Spec}_G^s(M)$ :*

1.  $\text{Spec}_G^s(M)$  is a  $T_0$ -space.
2. If  $V_G^s(S_1) = V_G^s(S_2)$ , then  $S_1 = S_2$ .
3.  $|p\text{-Spec}_G^s(M)| \leq 1$ , for every  $p \in \text{Spec}_G(R)$ .
4. The natural map  $\varphi$  is injective.

*Proof.* The equivalence of (2), (3) and (4) follows from Proposition 3.2. Also (1) $\Leftrightarrow$ (2) follows from Theorem 4.2 and the fact that a topological space is a  $T_0$ -space if the closures of distinct points are distinct.  $\square$

**Proposition 4.9.** *Let  $M$  be a  $G$ -graded module over a graded field  $F$ . If  $\text{Spec}_G^s(M)$  is a  $T_0$ -space, then  $|A| \leq 1$  for any base  $A$  of  $M$  with  $A \subseteq h(M)$ .*

*Proof.* Suppose that there exists a base  $A$  of  $M$  such that  $A \subseteq h(M)$  and  $|A| > 1$ . By Lemma 2.8(1),(2), there exist  $N_1, N_2 \in \text{Spec}_G^s(M)$  with  $N_1 \neq N_2$ . Since  $F$  is a  $G$ -graded field, then  $r \in U(F)$  for every  $r \in h(F) - \{0\}$  and hence  $B = \{X_r \mid r \in h(F)\} = \{\text{Spec}_G^s(M), \emptyset\}$  by Proposition 3.14(5). This follows that  $\tau_G^s =$  the trivial topology on  $\text{Spec}_G^s(M)$ . Now it is easy to see that  $\text{Spec}_G^s(M)$  is not a  $T_0$ -space, a contradiction.  $\square$

Recall that a topological space  $X$  is called a  $T_1$ -space if every singleton subset of  $X$  is closed. Now, we need the following lemma to investigate  $\text{Spec}_G^s(M)$  with the Zariski topology from the viewpoint of being a  $T_1$ -space.

**Lemma 4.10.** *Let  $M$  be a  $G$ -graded  $R$ -module. If  $Q$  is a graded minimal submodule of  $M$ , then  $\text{Ann}_R(Q)$  is a graded maximal ideal of  $R$ .*

*Proof.* Since  $Q \neq 0$ , then  $\text{Ann}_R(Q) \neq R$ . Let  $J \triangleleft_G R$  such that  $\text{Ann}_R(Q) \subseteq J \subseteq R$  and  $\text{Ann}_R(Q) \neq J$ . Then  $\exists t \in J - \text{Ann}_R(Q)$ . So  $\exists m \in h(M) \cap Q$  such that  $tm \neq 0$ . Note that  $Jm$  is a non-zero graded submodule of  $M$ . But  $Jm \subseteq Q$ . So  $Jm = Q$  as  $Q$  is a graded minimal submodule. Thus  $m = jm$  for some  $j \in J$ . This implies that  $1 - j \in \text{Ann}_R(m) \subseteq \text{Ann}_R(Jm) = \text{Ann}_R(Q) \subseteq J$ . Hence  $1 \in J$ . Therefore  $J = R$ .  $\square$

**Theorem 4.11.** *Let  $M$  be a  $G$ -graded  $R$ -module such that every graded second submodule of  $M$  contains a graded minimal submodule and let  $S \in \text{Spec}_G^s(M)$ . Then the set  $\{S\}$  is closed in  $\text{Spec}_G^s(M)$  if and only if  $S$  is a graded minimal submodule of  $M$  and  $p\text{-Spec}_G^s(M) = \{S\}$ , where  $p = \text{Ann}_R(S)$ .*

*Proof.* Suppose that  $\{S\}$  is closed in  $\text{Spec}_G^s(M)$ . Then  $Cl(\{S\}) = \{S\}$  and hence  $V_G^s(S) = \{S\}$  by Theorem 4.2. Now  $\exists W \in \text{Min}_G(M)$  such that  $W \subseteq S$ . So we get  $\text{Ann}_R(S) \subseteq \text{Ann}_R(W)$ , which implies that  $W \in V_G^s(S) = \{S\}$ . Hence  $W = S$  and this means that  $S \in \text{Min}_G(M)$ . Also it is clear that  $p\text{-Spec}_G^s(M) = \{S\}$ . Conversely, let  $H \in Cl(\{S\})$ . Then  $H \in V_G^s(S)$  and hence  $\text{Ann}_R(S) \subseteq \text{Ann}_R(H)$ . By Lemma 4.10, we have  $\text{Ann}_R(S) \in \text{Max}_G(R)$ . This follows that  $\text{Ann}_R(S) = \text{Ann}_R(H)$ . Since  $p\text{-Spec}_G^s(M) = \{S\}$ , then  $H = S$ . Therefore  $Cl(\{S\}) = \{S\}$ , as desired.  $\square$

**Corollary 4.12.** *Let  $M$  be a  $G$ -graded  $R$ -module such that every graded second submodule contains a graded minimal submodule. Then  $\text{Spec}_G^s(M)$  is a  $T_1$ -space if and only if  $\text{Min}_G(M) = \text{Spec}_G^s(M)$ .*

*Proof.* Assume that  $\text{Spec}_G(M)$  is a  $T_1$ -space. By [5, Proposition 2.3(c)],  $\text{Min}_G(M) \subseteq \text{Spec}_G^s(M)$ . Let  $S \in \text{Spec}_G^s(M)$ . Since  $\text{Spec}_G^s(M)$  is a  $T_1$ -space, then  $\{S\}$  is closed in  $\text{Spec}_G^s(M)$ . By Theorem 4.11, we have  $S \in \text{Min}_G(M)$ . For the converse, assume that  $\text{Min}_G(M) = \text{Spec}_G^s(M)$ . Let  $W \in \text{Spec}_G^s(M)$  and it is enough to show that  $\{W\}$  is closed in  $\text{Spec}_G^s(M)$ . Let  $p = \text{Ann}_R(W)$  and  $H \in p\text{-Spec}_G^s(M)$ . By [5, Proposition 2.4(a)], we have  $H + W \in p\text{-Spec}_G^s(M)$ . This follows that  $H + W \in \text{Min}_G(M)$ . But  $H, W \subseteq H + W$ . So  $H = W$  and so  $p\text{-Spec}_G^s(M) = \{W\}$ . By Theorem 4.11,  $\{W\}$  is closed in  $\text{Spec}_G^s(M)$ .  $\square$

Let  $M$  be a  $G$ -graded  $R$ -module and  $S \in \text{Spec}_G^s(M)$ . We say that  $S$  is a maximal graded second submodule if it is not properly contained in another graded second submodule of  $M$  (see<sup>8</sup>).

**Proposition 4.13.** *Let  $M$  be a  $G$ -graded  $R$ -module. Then the following hold:*

1. *If  $\text{Spec}_G^s(M)$  is a  $T_1$ -space, then every graded second submodule of  $M$  is a maximal graded second.*
2. *If  $M$  is a graded Artinian module, then  $\text{Spec}_G^s(M)$  is a  $T_1$ -space if and only if  $\text{Spec}_G^s(M)$  is discrete.*

*Proof.* (1) Let  $S \in \text{Spec}_G^s(M)$  and we show that it is maximal graded second. So let  $S' \in \text{Spec}_G^s(M)$  with  $S \subseteq S'$ . By hypothesis,  $\{S\} = \text{Cl}(\{S\}) \subseteq \text{Cl}(\{S'\}) = \{S'\}$  and thus  $S = S'$ .

(2) If  $M = 0$ , then the result is trivially true. Suppose that  $M \neq 0$ . Clearly, if  $\text{Spec}_G^s(M)$  is discrete, then it is a  $T_1$ -space. Conversely, assume that  $\text{Spec}_G^s(M)$  is a  $T_1$ -space. Let  $\text{Max}_G^s(M)$  be the set of all maximal graded second submodules of  $M$ . By (1),  $\text{Spec}_G^s(M) \subseteq \text{Max}_G^s(M)$ . But by [8, Theorem 2.11],  $\text{Max}_G^s(M)$  is a finite set and hence  $\text{Spec}_G^s(M)$  is finite. Now, since  $\text{Spec}_G^s(M)$  is a  $T_1$ -space, then it is discrete with respect to the Zariski topology.  $\square$

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