On The 3-Cyclic Refined Neutrosophic Real Roots of Unity and Their Algebraic Classification

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Abstract

The objective of this paper is to find all formulas that describe the 3-cyclic refined neutrosophic real solutions of the equation $X^n = 1$ which are called 3-cyclic refined real roots of unity. Also, we classify the algebraic group represented by these solutions as a direct product of some familiar finite abelian groups. On the other hand, we illustrate many examples to clarify the validity of our work.

Keywords: 3-cyclic refined number; Root of unity; Abelian group; Direct product

1. Introduction

The concept of neutrosophic sets and neutrosophic structures is considered one of the most recent mathematical concepts with applications in many different scientific fields [1]. The n-cyclic refined neutrosophic structures were first defined in [9], where algebraic structures related to this definition were studied, such as rings and modules. Also, the groups and spaces generated by this mathematical systems were studied in [9-12, 14]. Then in [2-4], some famous algebraic problems were studied for the n-cyclic refined neutrosophic rings, where some conjectures were put forward that discuss the classification of the group of units such as generalized Von Shtawzen's conjecture [5].

The problem of the group of units for 3-cyclic and 4-cyclic refined neutrosophic rings of integers was studied by many researchers [7-8, 13], where it was classified in [6] as a proof of first and second Von Shtawzen's conjectures. In [15-16], the problem of finding integer n-cyclic refined neutrosophic solutions of the equation $X^n = 1$ was discussed, and many good formulas were proven.

In this work, we study the same problem for a generalized set, where we find the formulas for the solutions of the equation $X^n = 1$ in the 3-cyclic real ring, with a classification of its group structure.

2. Main Discussion

Definition:

Let $X = x_0 + \sum_{i=1}^{3} x_i I_i \in R_3(I)$, then X is called n-th root of unity in $R_3(I)$ if: $X^n = 1$.

Remark:

It is known that $\mathbb{R}_3(I) \cong \mathbb{R} \times \mathbb{R} \times \mathbb{C}$ with:

$\varphi: \mathbb{R}_3(I) \to \mathbb{R} \times \mathbb{R} \times \mathbb{C}$ such that:

$$\varphi(X) = \left(x_0, \sum_{i=0}^{3} x_i, x_0 + x_3 + x_1\,\alpha + x_2\alpha^2 \right) \ ; \ \alpha = e^{\frac{2\pi}{3}i}$$

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If $X^n = 1$, then: $\begin{cases} x_0^n = 1 \\ (\sum_{i=0}^{3} x_i)^n = 1 \\ (x_0 + x_3 + x_1\alpha + x_2\alpha^2)^n = 1 \end{cases}$

For odd values of n:

$$\begin{cases} x_0^n = 1 \Leftrightarrow x_0 = 1 \quad (1) \\ (\sum_{i=0}^{3} x_i)^n = 1 \Leftrightarrow \sum_{i=0}^{3} x_i = 1 \Leftrightarrow x_1 + x_2 + x_3 = 0 \quad (2) \end{cases}$$

On the other hand:

$$(x_0 + x_3 + x_1\alpha + x_2\alpha^2)^n = 1 \Leftrightarrow x_0 + x_3 + x_1\alpha + x_2\alpha^2 = e^{\frac{2\pi k}{n}i} \qquad (3) \qquad \text{where } 0 \le k \le n-1.$$

Thus, $-x_2 - x_1 + x_1\alpha + x_2\alpha^2 = e^{\frac{2\pi k}{n}i} - 1$

Hence: $x_1(\alpha - 1) + x_2(\alpha^2 - 1) = \beta_k - 1 \quad ;\beta_k = e^{\frac{2\pi k}{n}i}$

$\alpha = e^{\frac{2\pi}{3}i} = \frac{-1}{2} + \frac{\sqrt{3}}{2}i, \alpha^2 = \frac{-1}{2} - \frac{\sqrt{3}}{2}i$, there for:

$$x_1\left(\frac{-3}{2} + \frac{\sqrt{3}}{2}i\right) + x_2\left(\frac{-3}{2} - \frac{\sqrt{3}}{2}i\right) = \left(\cos\left(\frac{2\pi k}{n}\right) - 1\right) + i\ \sin\left(\frac{2\pi k}{n}\right),$$

This implies:

$$\begin{cases} \frac{-3}{2}x_1 - \frac{3}{2}x_2 = \cos\left(\frac{2\pi k}{n}\right) - 1 \\ i\left(\frac{\sqrt{3}}{2}x_1 - \frac{\sqrt{3}}{2}x_2\right) = i\ \sin\left(\frac{2\pi k}{n}\right) \end{cases}$$

Hence: $\begin{cases} x_1 + x_2 = \frac{-2}{3}\cos\left(\frac{2\pi k}{n}\right) + \frac{2}{3} \\ x_1 - x_2 = \frac{2}{\sqrt{3}}\sin\left(\frac{2\pi k}{n}\right) \end{cases}$

Hence: $\begin{cases} x_1 = \frac{-1}{3}\cos\left(\frac{2\pi k}{n}\right) + \frac{1}{3} + \frac{1}{\sqrt{3}}\sin\left(\frac{2\pi k}{n}\right) \\ x_2 = \frac{-1}{3}\cos\left(\frac{2\pi k}{n}\right) + \frac{1}{3} - \frac{1}{\sqrt{3}}\sin\left(\frac{2\pi k}{n}\right) \end{cases}$

$$x_3 = -x_1 - x_2 = \frac{2}{3}\cos\left(\frac{2\pi k}{n}\right) - \frac{2}{3}$$

And $X = 1 + (-\frac{1}{3}\cos\left(\frac{2\pi k}{n}\right) + \frac{1}{3} + \frac{1}{\sqrt{3}}\sin\left(\frac{2\pi k}{n}\right)) I_1 + \left(-\frac{1}{3}\cos\left(\frac{2\pi k}{n}\right) + \frac{1}{3} - \frac{1}{\sqrt{3}}\sin\left(\frac{2\pi k}{n}\right)\right) I_2 + \left(\frac{2}{3}\cos\left(\frac{2\pi k}{n}\right) - \frac{2}{3}\right) I_3 \ ; 0 \le k \le n-1.$

For even values of n:

$$\begin{cases} x_0^n = 1 \Leftrightarrow x_0 \in \{1, -1\} \\ (\sum_{i=0}^{3} x_i)^n = 1 \Leftrightarrow \sum_{i=0}^{3} x_i \in \{1, -1\} \\ (x_0 + x_3 + x_1\alpha + x_2\alpha^2)^n = 1 \Leftrightarrow x_0 + x_3 + x_1\alpha + x_2\alpha^2 = \beta_k \quad ; 0 \le k \le n-1 \end{cases}$$

We discuss the possible cases:

Case (1):

if $x_0 = 1$, $\sum_{i=0}^{3} x_i = 1$, then:

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$$\begin{cases} x_1 = \dfrac{-1}{3}\cos\left(\dfrac{2\pi k}{n}\right) + \dfrac{1}{3} + \dfrac{1}{\sqrt{3}}\sin\left(\dfrac{2\pi k}{n}\right) \\ x_2 = \dfrac{-1}{3}\cos\left(\dfrac{2\pi k}{n}\right) + \dfrac{1}{3} - \dfrac{1}{\sqrt{3}}\sin\left(\dfrac{2\pi k}{n}\right) \end{cases}$$

$x_3 = \frac{2}{3}\cos\left(\frac{2\pi k}{n}\right) - \frac{2}{3}$,hence:

$X = 1 + \left(\frac{-1}{3}\cos\left(\frac{2\pi k}{n}\right) + \frac{1}{3} + \frac{1}{\sqrt{3}}\sin\left(\frac{2\pi k}{n}\right)\right) I_1 + \left(\frac{-1}{3}\cos\left(\frac{2\pi k}{n}\right) + \frac{1}{3} - \frac{1}{\sqrt{3}}\sin\left(\frac{2\pi k}{n}\right)\right) I_2 + \left(\frac{2}{3}\cos\left(\frac{2\pi k}{n}\right) - \frac{2}{3}\right) I_3.$

Case (2):

if $x_0 = \sum_{i=0}^{3} x_i = -1$, then $x_1 + x_2 + x_3 = 0\ , x_3 = -x_1 - x_2$

Also: $-1 - x_2 - x_1 + x_1\alpha + x_2\alpha^2 = \beta_k$,

$$\begin{cases} x_1(\alpha - 1) + x_2(\alpha^2 - 1) = \beta_k + 1, \\ x_1\left(-\dfrac{3}{2} + \dfrac{\sqrt{3}}{2}i\right) + x_2\left(-\dfrac{3}{2} - \dfrac{\sqrt{3}}{2}i\right) = \left[\cos\left(\dfrac{2\pi k}{n}\right) + 1\right] + i\sin\left(\dfrac{2\pi k}{n}\right), \\ -\dfrac{3}{2}(x_1 + x_2) = \cos\left(\dfrac{2\pi k}{n}\right) + 1 \\ \dfrac{\sqrt{3}}{2}(x_1 - x_2) = \sin\left(\dfrac{2\pi k}{n}\right) \end{cases}$$

Hence: $\begin{cases} x_1 + x_2 = -\frac{2}{3}\cos\left(\frac{2\pi k}{n}\right) - \frac{2}{3} \\ x_1 - x_2 = \frac{2}{\sqrt{3}}\sin\left(\frac{2\pi k}{n}\right) \end{cases}$

$$\begin{cases} x_1 = \dfrac{-1}{3}\cos\left(\dfrac{2\pi k}{n}\right) - \dfrac{1}{3} + \dfrac{1}{\sqrt{3}}\sin\left(\dfrac{2\pi k}{n}\right) \\ x_2 = \dfrac{-1}{3}\cos\left(\dfrac{2\pi k}{n}\right) - \dfrac{1}{3} - \dfrac{1}{\sqrt{3}}\sin\left(\dfrac{2\pi k}{n}\right) \\ x_3 = \dfrac{2}{3}\cos\left(\dfrac{2\pi k}{n}\right) + \dfrac{2}{3} \end{cases}$$

$X = -1 + \left(\frac{-1}{3}\cos\left(\frac{2\pi k}{n}\right) - \frac{1}{3} + \frac{1}{\sqrt{3}}\sin\left(\frac{2\pi k}{n}\right)\right) I_1 + \left(\frac{-1}{3}\cos\left(\frac{2\pi k}{n}\right) - \frac{1}{3} - \frac{1}{\sqrt{3}}\sin\left(\frac{2\pi k}{n}\right)\right) I_2 + \left(\frac{2}{3}\cos\left(\frac{2\pi k}{n}\right) + \frac{2}{3}\right) I_3.$

Case (3):

if $x_0 = 1, \sum_{i=0}^{3} x_i = -1, x_1 + x_2 + x_3 = -2\ , x_3 = -2 - x_1 - x_2$,

Also: $x_0 + x_3 + x_1\alpha + x_2\alpha^2 = \beta_k,\ 1 - 2 - x_1 - x_2 + x_1\alpha + x_2\alpha^2 = \beta_k$,

$x_1(\alpha - 1) + x_2(\alpha^2 - 1) = \beta_k + 1$, hence:

$$\begin{cases} x_1 = \dfrac{-1}{3}\cos\left(\dfrac{2\pi k}{n}\right) - \dfrac{1}{3} + \dfrac{1}{\sqrt{3}}\sin\left(\dfrac{2\pi k}{n}\right) \\ x_2 = \dfrac{-1}{3}\cos\left(\dfrac{2\pi k}{n}\right) - \dfrac{1}{3} - \dfrac{1}{\sqrt{3}}\sin\left(\dfrac{2\pi k}{n}\right) \\ x_3 = \dfrac{2}{3}\cos\left(\dfrac{2\pi k}{n}\right) - \dfrac{4}{3} \end{cases}$$

$X = 1 + \left(\frac{-1}{3}\cos\left(\frac{2\pi k}{n}\right) - \frac{1}{3} + \frac{1}{\sqrt{3}}\sin\left(\frac{2\pi k}{n}\right)\right) I_1 + \left(\frac{-1}{3}\cos\left(\frac{2\pi k}{n}\right) - \frac{1}{3} - \frac{1}{\sqrt{3}}\sin\left(\frac{2\pi k}{n}\right)\right) I_2 + \left(\frac{2}{3}\cos\left(\frac{2\pi k}{n}\right) - \frac{4}{3}\right) I_3.$

Case (4):

if $x_0 = -1, \sum_{i=0}^{3} x_i = 1, x_1 + x_2 + x_3 = 2\ , x_3 = 2 - x_1 - x_2$,

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Also: $x_0 + x_3 + x_1\alpha + x_2\alpha^2 = \beta_k, -1 + 2 - x_1 - x_2 + x_1\alpha + x_2\alpha^2 = \beta_k,$

$x_1(\alpha - 1) + x_2(\alpha^2 - 1) = \beta_k - 1$, hence:

$$\begin{cases} x_1 = \dfrac{-1}{3}\cos\left(\dfrac{2\pi k}{n}\right) + \dfrac{1}{3} + \dfrac{1}{\sqrt{3}}\sin\left(\dfrac{2\pi k}{n}\right) \\ x_2 = \dfrac{-1}{3}\cos\left(\dfrac{2\pi k}{n}\right) + \dfrac{1}{3} - \dfrac{1}{\sqrt{3}}\sin\left(\dfrac{2\pi k}{n}\right) \\ \qquad x_3 = \dfrac{2}{3}\cos\left(\dfrac{2\pi k}{n}\right) + \dfrac{4}{3} \end{cases}$$

$X = -1 + \left(\frac{-1}{3}\cos\left(\frac{2\pi k}{n}\right) + \frac{1}{3} + \frac{1}{\sqrt{3}}\sin\left(\frac{2\pi k}{n}\right)\right) I_1 + \left(\frac{-1}{3}\cos\left(\frac{2\pi k}{n}\right) + \frac{1}{3} - \frac{1}{\sqrt{3}}\sin\left(\frac{2\pi k}{n}\right)\right) I_2 + \left(\frac{2}{3}\cos\left(\frac{2\pi k}{n}\right) + \frac{4}{3}\right) I_3.$

In all cases $0 \le k \le n - 1$.

Example:

Find all 3-cyclic refined 5-th roots of unity.

Solution:

Since n=5 is odd, then:

$X_k = 1 + \left(\frac{-1}{3}\cos\left(\frac{2\pi k}{5}\right) + \frac{1}{3} + \frac{1}{\sqrt{3}}\sin\left(\frac{2\pi k}{5}\right)\right) I_1 + \left(\frac{-1}{3}\cos\left(\frac{2\pi k}{5}\right) + \frac{1}{3} - \frac{1}{\sqrt{3}}\sin\left(\frac{2\pi k}{5}\right)\right) I_2 + \left(\frac{2}{3}\cos\left(\frac{2\pi k}{5}\right) - \frac{2}{3}\right) I_3$; $0 \le k \le 4$.

$X_1 = 1 + \left(\frac{-1}{3}\cos\left(\frac{2\pi}{5}\right) + \frac{1}{3} + \frac{1}{\sqrt{3}}\sin\left(\frac{2\pi}{5}\right)\right) I_1 + \left(\frac{-1}{3}\cos\left(\frac{2\pi}{5}\right) + \frac{1}{3} - \frac{1}{\sqrt{3}}\sin\left(\frac{2\pi}{5}\right)\right) I_2 + \left(\frac{2}{3}\cos\left(\frac{2\pi}{5}\right) - \frac{2}{3}\right) I_3$.

$X_0 = 1$, $X_2 = 1 + \left(\frac{-1}{3}\cos\left(\frac{4\pi}{5}\right) + \frac{1}{3} + \frac{1}{\sqrt{3}}\sin\left(\frac{4\pi}{5}\right)\right) I_1 + \left(\frac{-1}{3}\cos\left(\frac{4\pi}{5}\right) + \frac{1}{3} - \frac{1}{\sqrt{3}}\sin\left(\frac{4\pi}{5}\right)\right) I_2 + \left(\frac{2}{3}\cos\left(\frac{4\pi}{5}\right) - \frac{2}{3}\right) I_3,$
$X_3 = 1 + \left(\frac{-1}{3}\cos\left(\frac{6\pi}{5}\right) + \frac{1}{3} + \frac{1}{\sqrt{3}}\sin\left(\frac{6\pi}{5}\right)\right) I_1 + \left(\frac{-1}{3}\cos\left(\frac{6\pi}{5}\right) + \frac{1}{3} - \frac{1}{\sqrt{3}}\sin\left(\frac{6\pi}{5}\right)\right) I_2 + \left(\frac{2}{3}\cos\left(\frac{6\pi}{5}\right) - \frac{2}{3}\right) I_3,$

$X_4 =$

Example:

Find all 3-cyclic refined 4-th roots of unity.

Solution:

Since n=4 is even, we have:

Case (1):

$X_0 = 1$

$X_1 = 1 + \left(\frac{1}{3} + \frac{1}{\sqrt{3}}\right) I_1 + \left(\frac{1}{3} - \frac{1}{\sqrt{3}}\right) I_2 - \frac{2}{3} I_3$.

$X_2 = 1 + \dfrac{2}{3} I_1 + \dfrac{2}{3} I_2 - \dfrac{4}{3} I_3$

$X_3 = 1 + \left(\frac{1}{3} - \frac{1}{\sqrt{3}}\right) I_1 + \left(\frac{1}{3} + \frac{1}{\sqrt{3}}\right) I_2 - \frac{2}{3} I_3$.

Case (2):

$X_0{}' = -1 - \frac{2}{3} I_1 - \frac{2}{3} I_2 + \frac{4}{3} I_3 = -X_2.$

$X_1{}' = -1 + \left(-\frac{1}{3} + \frac{1}{\sqrt{3}}\right) I_1 + \left(-\frac{1}{3} - \frac{1}{\sqrt{3}}\right) I_2 + \frac{2}{3} I_3 = -X_3$.

$X_2{}' = -1 = -X_0.$

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$$X_3{}' = -1 + \left(-\tfrac{1}{3} - \tfrac{1}{\sqrt{3}}\right) I_1 + \left(-\tfrac{1}{3} + \tfrac{1}{\sqrt{3}}\right) I_2 + \tfrac{2}{3} I_3 \;=\; -X_1\,.$$

Case (3):

$$X_0{}'' = 1 - \tfrac{2}{3} I_1 - \tfrac{2}{3} I_2 - \tfrac{2}{3} I_3.$$

$$X_1{}'' = 1 + \left(-\tfrac{1}{3} + \tfrac{1}{\sqrt{3}}\right) I_1 + \left(-\tfrac{1}{3} - \tfrac{1}{\sqrt{3}}\right) I_2 - \tfrac{4}{3} I_3.$$

$$X_2{}'' = 1 - 2I_3.$$

$$X_3{}'' = 1 + \left(-\tfrac{1}{3} - \tfrac{1}{\sqrt{3}}\right) I_1 + \left(-\tfrac{1}{3} + \tfrac{1}{\sqrt{3}}\right) I_2 - \tfrac{4}{3} I_3.$$

Case (4):

$$X_0{}''' = -1 + 2I_3 = -X_2{}''.$$

$$X_1{}''' = -1 + \left(\tfrac{1}{3} + \tfrac{1}{\sqrt{3}}\right) I_1 + \left(\tfrac{1}{3} - \tfrac{1}{\sqrt{3}}\right) I_2 + \tfrac{4}{3} I_3 = -X_3{}''.$$

$$X_2{}''' = -1 + \tfrac{2}{3} I_1 + \tfrac{2}{3} I_2 + \tfrac{2}{3} I_3 = -X_0{}''.$$

$$X_3{}''' = -1 + \left(\tfrac{1}{3} - \tfrac{1}{\sqrt{3}}\right) I_1 + \left(\tfrac{1}{3} + \tfrac{1}{\sqrt{3}}\right) I_2 + \tfrac{4}{3} I_3 = -X_1{}''.$$

Example:

Find all 3-cyclic refined 6-th roots of unity.

Solution:

Case (1):

$$X_0{}^{(1)} = 1\left(-\tfrac{1}{3}(1) + \tfrac{1}{3}\right) I_1 + \left(-\tfrac{1}{3} + \tfrac{1}{3}\right) I_2 = \;1.$$

$$X_1{}^{(1)} = 1 + \left(-\tfrac{1}{3}(\tfrac{1}{2}) + \tfrac{1}{3} + \tfrac{1}{\sqrt{3}}(\tfrac{\sqrt{3}}{2})\right) I_1 + \left(-\tfrac{1}{3}\left(\tfrac{1}{2}\right) + \tfrac{1}{3} - \tfrac{1}{\sqrt{3}}(\tfrac{\sqrt{3}}{2})\right) I_2 + \left(\tfrac{2}{3}\left(\tfrac{1}{2}\right) - \tfrac{2}{3}\right) I_3 = \;1 + \tfrac{2}{3} I_1 - \tfrac{1}{3} I_2 - \tfrac{1}{3} I_3\;\;.$$

$$X_2{}^{(1)} = 1 + \left[-\tfrac{1}{3}(-\tfrac{1}{2}) + \tfrac{1}{3} + \tfrac{1}{\sqrt{3}}(\tfrac{\sqrt{3}}{2})\right] I_1 + \left[-\tfrac{1}{3}\left(-\tfrac{1}{2}\right) + \tfrac{1}{3} - \tfrac{1}{\sqrt{3}}(\tfrac{\sqrt{3}}{2})\right] I_2 + \left[\tfrac{2}{3}\left(-\tfrac{1}{2}\right) - \tfrac{2}{3}\right] I_3 = \;1 + I_1 - I_3.$$

$$X_3{}^{(1)} = 1 + \left[-\tfrac{1}{3}(-1) + \tfrac{1}{3}\right] I_1 + \left[-\tfrac{1}{3}(-1) + \tfrac{1}{3}\right] I_2 + \left[\tfrac{2}{3}(-1) - \tfrac{2}{3}\right] I_3 = \;1 + \tfrac{2}{3} I_1 + \tfrac{2}{3} I_2 - \tfrac{4}{3} I_3.$$

$$X_4{}^{(1)} = 1 + \left[-\tfrac{1}{3}(-\tfrac{1}{2}) + \tfrac{1}{3} + \tfrac{1}{\sqrt{3}}(\tfrac{-\sqrt{3}}{2})\right] I_1 + \left[-\tfrac{1}{3}\left(-\tfrac{1}{2}\right) + \tfrac{1}{3} - \tfrac{1}{\sqrt{3}}(\tfrac{-\sqrt{3}}{2})\right] I_2 + \left[\tfrac{2}{3}\left(-\tfrac{1}{2}\right) - \tfrac{2}{3}\right] I_3 = \;1 + I_2 - I_3.$$

$$X_5{}^{(1)} = 1 + \left[-\tfrac{1}{3}(\tfrac{1}{2}) + \tfrac{1}{3} + \tfrac{1}{\sqrt{3}}(\tfrac{-\sqrt{3}}{2})\right] I_1 + \left[-\tfrac{1}{3}\left(\tfrac{1}{2}\right) + \tfrac{1}{3} - \tfrac{1}{\sqrt{3}}(\tfrac{-\sqrt{3}}{2})\right] I_2 + \left[\tfrac{2}{3}\left(\tfrac{1}{2}\right) - \tfrac{2}{3}\right] I_3 = \;1 - \tfrac{1}{3} I_1 + \tfrac{2}{3} I_2 - \tfrac{1}{3} I_3.$$

Case (2):

$$-X_0{}^{(1)} = -1.$$

$$-X_1{}^{(1)} = -1 - \tfrac{2}{3} I_1 + \tfrac{1}{3} I_2 + \tfrac{1}{3} I_3.$$

$$-X_2{}^{(1)} = -1 - I_1 + I_3.$$

$$-X_3{}^{(1)} = -\,1 - \tfrac{2}{3} I_1 - \tfrac{2}{3} I_2 + \tfrac{4}{3} I_3.$$

$$-X_4{}^{(1)} = -1 - I_2 + I_3.$$

$$-X_5{}^{(1)} = -\,1 + \tfrac{1}{3} I_1 - \tfrac{2}{3} I_2 + \tfrac{1}{3} I_3.$$

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Case (3):

$X_0{}^{(3)} = 1 + \left(-\frac{1}{3} - \frac{1}{3}\right) I_1 + \left(-\frac{1}{3} - \frac{1}{3}\right) I_2 + \left(\frac{2}{3} - \frac{4}{3}\right) I_2 = 1 - \frac{2}{3} I_1 - \frac{2}{3} I_2 - \frac{2}{3} I_3.$

$X_1{}^{(3)} = 1 + \left[-\frac{1}{3}\left(\frac{1}{2}\right) - \frac{1}{3} + \frac{1}{\sqrt{3}}(\frac{\sqrt{3}}{2})\right] I_1 + \left[-\frac{1}{3}\left(\frac{1}{2}\right) - \frac{1}{3} - \frac{1}{\sqrt{3}}(\frac{\sqrt{3}}{2})\right] I_2 + \left[\frac{2}{3}\left(\frac{1}{2}\right) - \frac{4}{3}\right] I_3 = 1 - I_2 - I_3\ .$

$X_2{}^{(3)} = 1 + \frac{1}{3} I_1 - \frac{4}{3} I_2 - \frac{5}{3} I_3.$

$X_3{}^{(3)} = 1 - 2I_3.$

$X_4{}^{(3)} = 1 - \frac{2}{3} I_1 + \frac{1}{3} I_2 - \frac{5}{3} I_3.$

$X_5{}^{(3)} = 1 - I_1 - I_3.$

Case (4):

$-X_0{}^{(3)} = -1 + \frac{2}{3} I_1 + \frac{2}{3} I_2 + \frac{2}{3} I_3.$

$-X_1{}^{(3)} = -1 + I_2 + I_3\ .$

$-X_2{}^{(3)} = -1 - \frac{1}{3} I_1 + \frac{4}{3} I_2 + \frac{5}{3} I_3.$

$-X_3{}^{(3)} = -1 + 2I_3.$

$-X_4{}^{(3)} = -1 + \frac{2}{3} I_1 - \frac{1}{3} I_2 + \frac{5}{3} I_3.$

$-X_5{}^{(3)} = -1 + I_1 + I_3.$

The algebraic structure of the real 3-refined n-th roots of unity:

Theorem:

Let n be an odd positive integer and $R_n$be the set of the n-th 3-cyclic refined roots of unity, then:(R_n, x) is a group and $R_n \cong C_n$.

Proof:

Let $x, y \in R_n$, then: $X^n = Y^n = 1$.

$(XY)^n = X^n Y^n = 1$, hence $XY \in R_n$.

$(X^{-1})^n = (X^n)^{-1} = 1$, hence $X^{-1} \in R_n$, thus (R_n, x) is a finite abelian group with $|R_n| = n$.

Define: $f: R_n \to C_n; f(X_k) = \beta_k;\ \beta_k = e^{\frac{2\pi k}{n} i}; 0 \leq k \leq n-1$

If $X_k = X_k{}'$, then $\beta_k = \beta_k{}'$, hence $f(X_k) = f(X_k{}')$.

$f(X_k . X_k{}') = \beta_k . \beta_k{}' = f(X_k) . f(X_k{}')$, thus, f is a group homomorphism.

$k_{er}(f) = \{X_k \in R_n\ ;\ f(X_k) = \beta_k = 1\} = \{1\}$.

Also, $|R_n| = |\{\beta_k\ ; 0 \leq k \leq n-1\}| = n$, thus f is an isomorphism and $R_n \cong \mathbb{C}_n$.

Remark:

Since $C_n \cong Z_n$, we get that $R_n \cong Z_n$.

Theorem:

Let n be an even positive integer and R_n be the set of the n-th 3-cyclic refined roots of unity, then: (R_n, x) is an abelian group with $R_n \cong Z_2 \times Z_2 \times Z_n$.

Proof:

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We can prove that (R_n, x) is abelian with $|R_n| = 4n$ by a similar argument of the previous theorem.

All elements of R_n have orders n at most, thus:

$R_n \cong Z_2 \times Z_2 \times Z_n$

3. Conclusion

In this paper we found all formulas that describe the 3-cyclic refined neutrosophic real solutions of the equation $X^n = 1$ which are called 3-cyclic refined real roots of unity. Also, we classified the algebraic group represented by these solutions as a direct product of some familiar finite abelian groups.

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