On single valued neutrosophic sets and neutrosophic $\aleph$-structures: Applications on algebraic structures (hyperstructures)

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Abstract

In this paper, we find a relationship between SVNS and neutrosophic $\aleph$-structures and study it. Moreover, we apply our results to algebraic structures (hyperstructures) and prove that the results on neutrosophic $\aleph$-substructure (subhyperstructure) of a given algebraic structure (hyperstructure) can be deduced from single valued neutrosophic algebraic structure (hyperstructure) and vice versa.
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1 Introduction

Neutrosophy,[19] a new branch of science that deals with indeterminacy, was launched by Smarandache in 1998. The theory of neutrosophy considers every notion or idea $< A >$ together with its opposite or negation $< antiA >$ and with their spectrum of neutralities $< neutA >$ in between them (i.e. notions or ideas supporting neither $< A >$ nor $< antiA >$). The $< neutA >$ and $< antiA >$ ideas together are referred to as $< nonA >$. Smarandach[20] defined neutrosophic sets as a generalization of the fuzzy sets introduced by Zadeh[22] in 1965 and as a generalization of intuitionistic fuzzy sets introduced by Atanassov[8] in 1986. Fuzzy sets allow gradual membership of an element in a set by assigning each element a degree of membership between 0 and 1 that are both inclusive. Whereas intutionistic fuzzy sets allow gradual membership as well as gradual non-membership of an element in a set by assigning each element a degree of membership and a degree of non-membership in a way that their sum is a real number in the unit interval $[0, 1]$. Single valued neutrosophic sets (SVNS)[24] generalize these two concepts so that each element has a truth value, indeterminacy value, and a falsity value and each of these values is a number in the unit interval $[0, 1]$. Sometimes we have negative information. As an example, "The rate increase in a certain bank depends on employees' performance. It increases by 3% annually if the employee's performance is outstanding (convincing many business women/men to invest their money in the bank), by 2% annually if the employee's performance is very good, by 1% annually if the employee's performance is good, and no increase if the employee's performance is average. Let's say that Sam convinces annually around twenty business women/men to invest their money in the bank, so he got the 3% annual increase as a result of his excellent job. And there is an employee "Bella" who comes always late to her work, leaves early, complains about the bank in public and as a result, she leads to the loss of some possible investors in the bank. So, in this case Bella is making the bank loses and as a result she does not deserve an annual increase but instead she should be given a decrease in her salary." In order to deal with such negative information, we need negative functions. So, by means of negative functions, neutrosophic $\aleph$-structures were introduced.[14,15] They are similar to SVNS where each element has a truth value, indeterminacy value, and a falsity value but each of these values is a number in the interval $[-1, 0]$, i.e., the truth, indeterminacy, and the falsity functions are negative-valued functions. Neutrosophy has many applications in different fields of Science. Many researchers[3],[5],[7],[14],[17],[21] worked on the connection between neutrosophy and algebraic structures (hyperstructures). More precisely, the connection between SVNS and algebraic structures (hypertsructures) and the connection between neutrosophic $\aleph$-structures and algebraic structures (hypertsructures) grabbed the attention of algebraist researchers. For example, Al-Tahan[5] studied single valued neutrosophic polygroups, Khan et al.[15] discussed neutrosophic $\aleph$-subsemigroups, Park studied

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neutrosophic ideals of subtraction algebras, and Al-Tahan and Davvaz[7] studied neutrosophic $\aleph$-ideals of subtraction algebras.

A question arises here:

"Is there a certain relationship between SVNS and neutrosophic $\aleph$-structures?"

Another question arises now:

"What would be the effect of such a relationship between SVNS and neutrosophic $\aleph$-structures on the study of both: single valued neutrosophic algebraic structures (hypertsructures) and neutrosophic $\aleph$-substructures (subhypertsructures)?"

This article answers the above two questions and it is constructed as follows: after an Introduction, in Section 2, we find a relationship between SVNS and neutrosophic $\aleph$-structures. In Section 3, we discuss the effect of such a relationship between SVNS and neutrosophic $\aleph$-structures on the study of both: single valued neutrosophic algebraic structures (hypertsructures) and neutrosophic $\aleph$-substructures (subhypertsructures) and we deal with some examples of algebraic structures (hypertsructures).

2 Relationship between SVNS and neutrosophic $\aleph$-structures

In this section, we find a relationship between SVNS and neutrosophic $\aleph$-structures and study it. Moreover, we illustrate our results by some examples.

Definition 2.1. [24] Let X be a space of points (objects), with a generic element in X denoted by x. A single valued neutrosophic set (SVNS) A on X is characterized by truth-membership T_A, indeterminacy-membership function I_A and falsity-membership function F_A. For each point $x \in X$, $T_A(x), I_A(x), F_A(x) \in [0,1]$.

Definition 2.2. [14,15] Let X be a non-empty set. A neutrosophic $\aleph$-structure over X is defined as follows:

$$S_N = \{\frac{x}{(T_N(x), I_N(x), F_N(x))} : x \in X\}$$

where T_N, I_N, F_N are $\aleph$-functions on X (i.e. functions on X with codomain $[-1,0]$) which are called the negative truth membership function, the negative indeterminacy membership function and the negative falsity membership function, respectively, on X.

Definition 2.3. Let X be a non-empty set, $\alpha, \beta, \gamma \in [0,1]$, and A a SVNS over X. Then the (α, β, γ)-level set of A is defined as follows:

$$L_{(\alpha,\beta,\gamma)} = \{x \in X : T_A(x) \geq \alpha, I_A(x) \geq \beta, F_A(x) \leq \gamma\}.$$

Definition 2.4. Let X be a non-empty set, $\alpha, \beta, \gamma \in [-1,0]$, and S_N a neutrosophic $\aleph$-structure over X . Then the (α, β, γ)-level set of S_N is defined as follows:

$$\overline{L}_{(\alpha,\beta,\gamma)} = \{x \in X : T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma\}.$$

Definition 2.5. [24] Let X be a non-empty set and A, B be single valued neutrosophic sets over X defined as follows.

$$A = \{\frac{x}{(T_A(x), I_A(x), F_A(x))} : x \in X\}, B = \{\frac{x}{(T_B(x), I_B(x), F_B(x))} : x \in X\}$$

Then

- A is called a single valued neutrosophic subset of B and denoted as $A \subseteq B$ if $T_A(x) \leq T_B(x)$, $I_A(x) \leq I_B(x)$, and $F_A(x) \geq F_B(x)$ for all $x \in X$.
  If $A \subseteq B$ and $B \subseteq A$ then $A = B$.

- The union of A and B is defined to be the SVNS over X:

  $$A \cup B = \{\frac{x}{(T_{A\cup B}(x), I_{A\cup B}(x), F_{A\cup B}(x))} : x \in X\}.$$

  Where $T_{A\cup B}(x) = T_A(x) \vee T_B(x)$, $I_{A\cup B}(x) = I_A(x) \vee I_B(x)$, and $F_{A\cup B}(x) = F_A(x) \wedge F_B(x)$ for all $x \in X$.

- The intersection of A and B is defined to be the SVNS over X:

$$S_{A\cap B}=\{\frac{x}{(T_{A\cap B}(x),I_{A\cap B}(x),F_{A\cap B}(x))}:x\in X\}.$$

Where $T_{A\cap B}(x)=T_A(x)\wedge T_B(x)$, $I_{A\cap B}(x)=I_A(x)\wedge I_B(x)$, and $F_{A\cap B}(x)=F_A(x)\vee F_B(x)$ for all $x\in X$.

Definition 2.6. [15] Let X be a non-empty set and S_N, S_M be neutrosophic $\aleph$-structures over X defined as follows.

$$S_N=\{\frac{x}{(T_N(x),I_N(x),F_N(x))}:x\in X\}, S_M=\{\frac{x}{(T_M(x),I_M(x),F_M(x))}:x\in X\}$$

Then

- S_N is called a neutrosophic $\aleph$-substructure of S_M and denoted as $S_N\subseteq S_M$ if $T_N(x)\geq T_M(x)$, $I_N(x)\leq I_M(x)$, and $F_N(x)\geq F_M(x)$ for all $x\in X$.
If $S_N\subseteq S_M$ and $S_M\subseteq S_N$ then $S_N=S_M$.
- The union of S_N and S_M is defined to be the $\aleph$-structure over X:

$$S_{N\cup M}=\{\frac{x}{(T_{N\cup M}(x),I_{N\cup M}(x),F_{N\cup M}(x))}:x\in X\}.$$

Where $T_{N\cup M}(x)=T_N(x)\wedge T_M(x)$, $I_{N\cup M}(x)=I_N(x)\vee I_M(x)$, and $F_{N\cup M}(x)=F_N(x)\wedge F_M(x)$ for all $x\in X$.
- The intersection of S_N and S_M is defined to be the $\aleph$-structure over X:

$$S_{N\cap M}=\{\frac{x}{(T_{N\cap M}(x),I_{N\cap M}(x),F_{N\cap M}(x))}:x\in X\}.$$

Where $T_{N\cap M}(x)=T_N(x)\vee T_M(x)$, $I_{N\cap M}(x)=I_N(x)\wedge I_M(x)$, and $F_{N\cap M}(x)=F_N(x)\vee F_M(x)$ for all $x\in X$.

Fore more details about operations on SVNS and operations on neutrosophic $\aleph$-structures, we refer to the papers.[14, 15, 24]

Proposition 2.7. *Let X be a non-empty set, A, S_N be defined as follows:*

$$A=\{\frac{x}{(T_A(x),I_A(x),F_A(x))}:x\in X\}, S_N=\{\frac{x}{(-T_A(x),I_A(x)-1,F_A(x)-1)}:x\in X\}.$$

Then A is a SVNS over X if and only if S_N is a neutrosophic $\aleph$-structure of X.

Proof. Let A be a SVNS of X. Then for every $x\in X$, $0\leq T_A(x), I_A(x), F_A(x)\leq 1$. The latter implies that $-1\leq -T_A(x), I_A(x)-1, F_A(x)-1\leq 0$. Thus, S_N is a neutrosophic $\aleph$-structure of X. Similarly, if S_N is a neutrosophic $\aleph$-structure of X then A is a SVNS of X. □

Example 2.8. Let $X=\{0,1,2\}$ and $A=\{\frac{0}{(0.1,0.9,0.3)},\frac{1}{(0.7,0.3,0.5)},\frac{2}{(0.8,0.5,0.3)}\}$ be a SVNS over X. Then $S_N=\{\frac{0}{(-0.1,-0.1,-0.7)},\frac{1}{(-0.7,-0.7,-0.5)},\frac{2}{(-0.8,-0.5,-0.7)}\}$ is a neutrosophic $\aleph$-structure of X.

Theorem 2.9. *Let A be a SVNS of X and $0\leq\alpha,\beta,\gamma\leq 1$. Then $L_{\alpha,\beta,\gamma}=\overline{L}_{-\alpha,\beta-1,\gamma-1}$.*

Proof. We have $L_{\alpha,\beta,\gamma}=\{x\in X: T_A(x)\geq\alpha, I_A(x)\geq\beta, F_A(x)\leq\gamma\}$ and $\overline{L}_{-\alpha,\beta-1,\gamma-1}=\{x\in X: T_N(x)\leq-\alpha, I_N(x)\geq\beta-1, F_N(x)\leq\gamma-1\}$. Having $T_A(x)\geq\alpha$, $I_A(x)\geq\beta$, and $F_A(x)\leq\gamma$ equivalent to $T_N(x)=-T_A(x)\leq-\alpha$, $I_N(x)=I_A(x)-1\geq\beta-1$, and $F_N(x)=F_A(x)-1\leq\gamma-1$ respectively implies that $L_{\alpha,\beta,\gamma}=\overline{L}_{-\alpha,\beta-1,\gamma-1}$. □

Proposition 2.10. *Let X be a non-empty set, S_N, A be defined as follows:*

$$S_N=\{\frac{x}{(T_N(x),I_N(x),F_N(x))}:x\in X\}, A=\{\frac{x}{(-T_N(x),I_N(x)+1,F_N(x)+1)}:x\in X\}.$$

Then A is a SVNS of X if and only if S_N is a neutrosophic $\aleph$-structure of X.

Proof. Let A be a SVNS of X. Then for every $x \in X$, $0 \leq -T_N(x), I_N(x)+1, F_N(x)+1 \leq 1$. The latter implies that $-1 \leq T_N(x), I_N(x), F_N(x) \leq 0$. Thus, S_N is a neutrosophic $\aleph$-structure of X. Similarly, if S_N is a neutrosophic $\aleph$-structure of X then A is a SVNS of X. $\square$

Example 2.11. Let $X = \{0,1,2\}$ and $S_N = \{\frac{0}{(-0.1,-0.9,-0.3)}, \frac{1}{(-0.7,-0.3,-0.5)}, \frac{2}{(0,-1,-0.3)}\}$ be a neutrosophic $\aleph$-structure over X. Then $A = \{\frac{0}{(0.1,0.1,0.7)}, \frac{1}{(0.7,0.7,0.5)}, \frac{2}{(0,0,0.7)}\}$ a SVNS over X.

Theorem 2.12. *Let A be a SVNS of X and $-1 \leq \alpha, \beta, \gamma \leq 0$. Then $L_{-\alpha,1+\beta,1+\gamma} = \overline{L}_{\alpha,\beta,\gamma}$*

Proof. The proof is similar to that of Theorem 2.9. $\square$

3 Applications to algebraic structures (hyperstructures)

In this section, we apply the relationship we found in Section 2 between SVNS and neutrosophic $\aleph$-structures on some algebraic structure (hypertsructures) and we present our main theorems in Subsection 3.4.

3.1 Applications to semigroups

In,[15] Khan et al. discussed neutrosophic $\aleph$-structures and applied it to semigroups. In this subsection, we deduce some of their results by applying the relationship that we found in Section 2 between SVNS and neutrosophic $\aleph$-structures.

A semigroup is a groupoid that satisfies the associative axiom. For example, the set of positive integers under standard addition, the set of negative integers under standard addition, the set of integers modulo a positive integer n under standard multiplication modulo n are semigroups.

Definition 3.1. Let $(X, \circ)$ be a semigroup and A a SVNS over X. Then A is single valued neutrosophic semigroup over X if for all $x, y \in X$, the following conditions hold:

- $T_A(x \circ y) \geq T_A(x) \wedge T_A(y)$;
- $I_A(x \circ y) \geq I_A(x) \wedge I_A(y)$;
- $F_A(x \circ y) \leq F_A(x) \vee F_A(y)$.

Definition 3.2. [15] Let $(X, \circ)$ be a semigroup and S_N a neutrosophic $\aleph$-structure over X. Then S_N is neutrosophic $\aleph$-subsemigroup of X if for all $x, y \in X$, the following conditions hold:

- $T_N(x \circ y) \leq T_N(x) \vee T_N(y)$;
- $I_N(x \circ y) \geq I_N(x) \wedge I_N(y)$;
- $F_N(x \circ y) \leq F_N(x) \vee F_N(y)$.

Remark 3.3. Let a, b be any real numbers. Then

- $1 + (a \wedge b) = (1+a) \wedge (1+b)$;
- $1 + (a \vee b) = (1+a) \vee (1+b)$;
- if $c = a \wedge b$ then $-c = (-a) \vee (-b)$;
- if $d = a \vee b$ then $-d = (-a) \wedge (-b)$.

Theorem 3.4. *Let $(X, \circ)$ be a semigroup and S_N a neutrosophic $\aleph$-structure over X. Then S_N is neutrosophic $\aleph$-subsemigroup of X if and only if A is a single valued neutrosophic semigroup over X. Here,*

$$S_N = \{\frac{x}{(T_N(x), I_N(x), F_N(x))} : x \in X\}, A = \{\frac{x}{(-T_N(x), I_N(x)+1, F_N(x)+1)} : x \in X\}.$$

Proof. Let A be a single valued neutrosophic semigroup over X and $x, y \in X$. Then $-T_N(x \circ y) \geq (-T_N(x)) \wedge (-T_N(y))$, $1 + I_N(x \circ y) \geq (1 + I_N(x)) \wedge (1 + I_N(y))$, and $1 + F_N(x \circ y) \leq (1 + F_N(x)) \vee (1 + F_N(y))$. The latter implies that $T_N(x \circ y) \leq T_N(x) \vee T_N(y)$, $I_N(x \circ y) \geq I_N(x) \wedge I_N(y)$, and $F_N(x \circ y) \leq F_N(x) \vee F_N(y)$. Thus, S_N is neutrosophic $\aleph$-subsemigroup of X. Similarly, we can prove that if S_N is neutrosophic $\aleph$-subsemigroup of X then A is a single valued neutrosophic semigroup over X. $\square$

Theorem 3.5. *Let $(X, \circ)$ be a semigroup and A a SVNS over X. Then A is a single valued neutrosophic semigroup over X if and only if $L_{(\alpha,\beta,\gamma)}$ is either the empty set or a subsemigroup of X for all $0 \leq \alpha, \beta, \gamma \leq 1$.*

Proof. The proof is similar to that of Theorem 5.1.[5] □

Theorem 3.6. *Let $(X, \circ)$ be a semigroup and A a SVNS over X. Then A is single valued neutrosophic semigroup over X if and only if $\overline{L}_{(\alpha,\beta,\gamma)}$ is either the empty set or a subsemigroup of X for all $-1 \leq \alpha, \beta, \gamma \leq 0$.*

Proof. Let $-1 \leq \alpha, \beta, \gamma \leq 0$. Then there exist $0 \leq \alpha', \beta', \gamma' \leq 1$ such that $\alpha' = -\alpha$, $\beta' = \beta + 1$, and $\gamma' = \gamma + 1$. Theorem 3.5 asserts that $L_{(\alpha',\beta',\gamma')}$ is either the empty set or a subsemigroup of X. The latter and Theorem 2.12 imply that $\overline{L}_{(\alpha,\beta,\gamma)} = L_{(\alpha',\beta',\gamma')}$ is either the empty set or a subsemigroup of X.

Let $0 \leq \alpha', \beta', \gamma' \leq 1$. Then there exist $-1 \leq \alpha, \beta, \gamma \leq 0$ such that $\alpha' = -\alpha$, $\beta' = \beta+1$, and $\gamma' = \gamma+1$. But having $\overline{L}_{(\alpha,\beta,\gamma)}$ is either the empty set or a subsemigroup of X and that $L_{(\alpha',\beta',\gamma')} = \overline{L}_{(\alpha,\beta,\gamma)}$ imply that $L_{(\alpha',\beta',\gamma')}$ is either the empty set or a subsemigroup of X for all $0 \leq \alpha', \beta', \gamma' \leq 1$. Thus, A is single valued neutrosophic semigroup over X by Theorem 3.5. □

Theorem 3.7. *Let $(X, \circ)$ be a semigroup and S_N a neutrosophic $\aleph$-structure over X where,*

$$S_N = \{\frac{x}{(T_N(x), I_N(x), F_N(x))} : x \in X\}, A = \{\frac{x}{(-T_N(x), I_N(x)+1, F_N(x)+1)} : x \in X\}.$$

Then the following statements are equivalent.

1. *S_N is a neutrosophic $\aleph$-subsemigroup of X;*
2. *A is a single valued neutrosophic semigroup over X;*
3. *$\overline{L}_{(\alpha,\beta,\gamma)}$ is either the empty set or a subsemigroup of X for all $-1 \leq \alpha, \beta, \gamma \leq 0$;*
4. *$L_{(\alpha,\beta,\gamma)}$ is either the empty set or a subsemigroup of X for all $0 \leq \alpha, \beta, \gamma \leq 1$.*

Proof. The proof follows from Theorem 3.4. Theorem 3.5, and Theorem 3.6. □

Example 3.8. Let $(\mathbb{Z}^+, +)$ be the semigroup of positive integers under standard addition. Let

$$(T_N(x), I_N(x), F_N(x)) = \begin{cases} (-0.6, -0.4, -0.7) & \text{if } x \text{ is a multiple of 2;} \\ (-0.5, -0.5, -0.6) & \text{otherwise.} \end{cases}$$

Then $S_N = \{\frac{x}{(T_N(x),I_N(x),F_N(x))} : x \in \mathbb{Z}^+\}$ is a neutrosophic $\aleph$-subsemigroup of $\mathbb{Z}^+$ as $A = \{\frac{x}{(T_A(x),I_A(x),F_A(x))} : x \in \mathbb{Z}^+\}$ is a single valued neutrosophic semigroup over $\mathbb{Z}^+$. Where

$$(T_A(x), I_A(x), F_A(x)) = \begin{cases} (0.6, 0.6, 0.3) & \text{if } x \text{ is a multiple of 2;} \\ (0.5, 0.5, 0.4) & \text{otherwise.} \end{cases}$$

3.2 Applications to polygroups

In,[5] Al-Tahan defined single valued neutrosophic polygroups and studied their properties. In this subsection, we use the result in[5] with the relationship we found in Section 2 between SVNS and neutrosophic $\aleph$-structures to prove some results on neutrosophic $\aleph$-subpolygroups.

Algebraic hyperstructures represent a natural generalization of classical algebraic structures and they were introduced by Marty[16] in 1934 at the eighth Congress of Scandinavian Mathematicians. Where he generalized the notion of a group to that of a hypergroup. He defined a hypergroup as a set equipped with associative and reproductive hyperoperation. In a group, the composition of two elements is an element whereas in a hypergroup, the composition of two elements is a set. Many researchers worked on hypertsructure theory and its applications. We refer to.[1,2,10,12] A certain subclasses of hypergroups were introduced such as polygroups. The latter were introduced by Comer,[9] where he emphasized their importance in connections to graphs, relations, Boolean and cylindric algebras. For more details about polygroups and their applications, we refer to.[4,6,11]

Definition 3.9. [11] Let P be a non-empty set. Then, a mapping $\circ : P \times P \to \mathcal{P}^*(P)$ is called a *binary hyperoperation* on P, where $\mathcal{P}^*(P)$ is the family of all non-empty subsets of P. The couple $(P, \circ)$ is called a *hypergroupoid.*

In the above definition, if A and B are two non-empty subsets of P and $x \in P$, then we define:

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b, \; x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

Definition 3.10. [9] A polygroup is a system $< P, \circ, e, ^{-1} >$, where $e \in P$, $^{-1} : P \to P$ is a unitary operation on P, "$\circ$" maps $P \times P$ into the non-empty subsets of P, and the following axioms hold for all $x, y, z \in P$:

1. $(x \circ y) \circ z = x \circ (y \circ z)$,
2. $e \circ x = x \circ e = \{x\}$,
3. $x \in y \circ z$ implies $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$.

Let $(P, \circ)$ be a polygroup and $K \subseteq P$. Then $(K, \circ)$ is a subpolygroup of $(P, \circ)$ if for all $a, b \in K$, we have that $a \circ b \subseteq K$ and $a^{-1} \in K$.

Example 3.11. Let $P = \{e, a, b\}$ and define the polygroup $(P_1, \circ_1)$ by Table 1.

Table 1: The polygroup $(P_1, \circ_1)$

$\circ_1$	e	a	b
e	e	a	b
a	a	{e,b}	{a,b}
b	b	{a,b}	{e,a}

Definition 3.12. [5] Let $(P, \circ)$ be a polygroup and A a SVNS over X. Then A is single valued neutrosophic polygroup of P if for all $x, y \in P$, the following conditions hold:

- $T_A(x \circ y) \geq T_A(x) \wedge T_A(y)$;
- $I_A(x \circ y) \geq I_A(x) \wedge I_A(y)$;
- $F_A(x \circ y) \leq F_A(x) \vee F_A(y)$;
- $T_A(x^{-1}) \geq T_A(x), I_A(x^{-1}) \geq I_A(x), F_A(x^{-1}) \leq F_A(x)$.

Definition 3.13. Let $(P, \circ)$ be a polygroup and S_N a neutrosophic $\aleph$-structure over P. Then S_N is neutrosophic $\aleph$-subpolygroup of P if for all $x, y \in P$, the following conditions hold:

- $T_N(x \circ y) \leq T_N(x) \vee T_N(y)$;
- $I_N(x \circ y) \geq I_N(x) \wedge I_N(y)$;
- $F_N(x \circ y) \leq F_N(x) \vee F_N(y)$;
- $T_N(x^{-1}) \leq T_N(x), I_N(x^{-1}) \geq I_N(x), F_N(x^{-1}) \leq F_N(x)$.

Theorem 3.14. *Let $(P, \circ)$ be a polygroup and S_N a neutrosophic $\aleph$-structure over P. Then S_N is neutrosophic $\aleph$-subpolygroup of P if and only if A is a single valued neutrosophic polygroup over X. Here,*

$$S_N = \{\frac{x}{(T_N(x), I_N(x), F_N(x))} : x \in P\}, A = \{\frac{x}{(-T_N(x), I_N(x) + 1, F_N(x) + 1)} : x \in P\}.$$

Proof. Let A be a single valued neutrosophic polygroup over P and $x, y \in P$. Then $-T_N(x \circ y) \geq (-T_N(x)) \wedge (-T_N(y)), 1 + I_N(x \circ y) \geq (1 + I_N(x)) \wedge (1 + I_N(y))$, and $1 + F_N(x \circ y) \leq (1 + F_N(x)) \vee (1 + F_N(y))$. The latter implies that $T_N(x \circ y) \leq T_N(x) \vee T_N(y)$, $I_N(x \circ y) \geq I_N(x) \wedge I_N(y)$, and $F_N(x \circ y) \leq F_N(x) \vee F_N(y)$. Moreover, having $-T_N(x^{-1}) \geq -T_N(x)$, $I_N(x^{-1}) - 1 \geq I_N(x) - 1$, $F_N(x^{-1}) - 1 \leq F_N(x) - 1$ implies that $T_N(x^{-1}) \leq T_N(x)$, $I_N(x^{-1}) \geq I_N(x)$, $F_N(x^{-1}) \leq F_N(x)$.Thus, S_N is neutrosophic $\aleph$-subpolygroup of P. Similarly, we can prove that if S_N is neutrosophic $\aleph$-subpolygroup of P then A is a single valued neutrosophic polygroup over P. □

Theorem 3.15. [5] *Let $(P, \circ)$ be a polygroup and A a SVNS over P. Then A is single valued neutrosophic polygroup over X if and only if $L_{(\alpha,\beta,\gamma)}$ is either the empty set or a subpolygroup of P for all $0 \leq \alpha, \beta, \gamma \leq 1$.*

Theorem 3.16. *Let $(P, \circ)$ be a polygroup and A a SVNS over X. Then A is single valued neutrosophic polygroup over X if and only if $\overline{L}_{(\alpha,\beta,\gamma)}$ is either the empty set or a subpolygroup of P for all $-1 \leq \alpha, \beta, \gamma \leq 0$.*

Proof. The proof is similar to the proof of Theorem 3.6. □

Theorem 3.17. *Let $(P, \circ)$ be a polygroup and S_N a neutrosophic $\aleph$-structure over P where,*

$$S_N = \{\frac{x}{(T_N(x), I_N(x), F_N(x))} : x \in P\}, A = \{\frac{x}{(-T_N(x), I_N(x)+1, F_N(x)+1)} : x \in P\}.$$

Then the following statements are equivalent.

1. *S_N is a neutrosophic $\aleph$-subpolygroup of P;*
2. *A is a single valued neutrosophic polygroup over P;*
3. *$\overline{L}_{(\alpha,\beta,\gamma)}$ is either the empty set or a subpolygroup of X for all $-1 \leq \alpha, \beta, \gamma \leq 0$;*
4. *$L_{(\alpha,\beta,\gamma)}$ is either the empty set or a subpolygroup of X for all $0 \leq \alpha, \beta, \gamma \leq 1$.*

Proof. The proof follows from Theorem 3.4, Theorem 3.14, and Theorem 3.15. □

Example 3.18. Let $(P_1, \circ_1)$ be the polygroup defined in Example 3.11. Then

$$S_N = \{\frac{e}{(-0.7,-0.4,-0.9)}, \frac{a}{(-0.6,-0.6,-0.8)}, \frac{b}{(-0.6,-0.6,-0.8)}\}$$

is a neutrosophic $\aleph$-subpolygroup of P_1 as $A = \{\frac{e}{(0.7,0.6,0.1)}, \frac{a}{(0.6,0.4,0.2)}, \frac{b}{(0.6,0.4,0.2)}\}$ is a single valued neutrosophic polygroup over P_1.

Remark 3.19. Theorem 3.17 implies that the results known for single valued neutrosophic polygroups in[5] hold also for neutrosophic $\aleph$-subpolygroups.

3.3 Applications to subtraction algebras

Park in,[17] Al-Tahan and Davvaz in[7] defined neutrosophic ideals and $\aleph$-ideals of subtraction algebras respectively and studied their properties. In this subsection, we use the results in[17] with the relationship we found in Section 2 between SVNS and neutrosophic $\aleph$-structures to some results on neutrosophic $\aleph$-ideals of subtraction algebras that were proved in.[7]

Subtraction algebra was introduced by Shein in 1992[18] and some results about it can be found in.[13,23]

Definition 3.20. [23] An algebra $(X, -)$ is called a subtraction algebra if the single binary operation "-" satisfies the following identities: for any $x, y, z \in X$,

1. $x - (y - x) = x$;
2. $x - (x - y) = y - (y - x)$;
3. $(x - y) - z = (x - z) - y$.

Definition 3.21. [13] A non-empty subset I of a subtraction algebra X is called an ideal of X if it satisfies the following conditions.

1. $a - x \in I$for all $a \in I$ and $x \in X$;
2. for all $a, b \in I$, whenever $a \vee b$ exists in X then $a \vee b \in I$.

Example 3.22. Let $X_1 = \{0, 1, 2\}$ and define the subtraction algebra $(X_1, -_1)$ by Table 2.

Definition 3.23. [17] Let $(X, -)$ be a subtraction algebra and A a SVNS over X. Then A is single valued neutrosophic ideal of X if for all $x, y \in X$, the following conditions hold:

- $T_A(x - y) \geq T_A(x)$;
- $I_A(x - y) \geq I_A(x)$;

Table 2: The subtraction algebra $(X_1, -_1)$

$-_1$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

- $F_A(x-y) \leq F_A(x)$;
- if $x \vee y$ exists in X then $T_A(x \vee y) \geq T_A(x) \wedge T_A(y)$, $I_A(x \vee y) \geq I_A(x) \wedge I_A(y)$, and $F_A(x \vee y) \leq F_A(x) \vee F_A(y)$.

Definition 3.24. [7] Let $(X, \circ)$ be a subtraction algebra and S_N a neutrosophic $\aleph$-structure over X. Then S_N is neutrosophic $\aleph$-ideal of X if for all $x, y \in X$, the following conditions hold:

- $T_N(x-y) \leq T_N(x)$;
- $I_N(x-y) \geq I_N(x)$;
- $F_N(x-y) \leq F_N(x)$;
- if $x \vee y$ exists in X then $T_N(x \vee y) \leq T_N(x) \vee T_N(y)$, $I_N(x \vee y) \geq I_N(x) \wedge I_N(y)$, and $F_N(x \vee y) \leq F_N(x) \vee F_N(y)$.

Theorem 3.25. *Let $(X, -)$ be a subtraction algebra and S_N a neutrosophic $\aleph$-structure over X. Then S_N is neutrosophic $\aleph$-ideal of X if and only if A is a neutrosophic ideal of X. Here,*

$$S_N = \{\frac{x}{(T_N(x), I_N(x), F_N(x))} : x \in X\}, A = \{\frac{x}{(-T_N(x), I_N(x)+1, F_N(x)+1)} : x \in X\}.$$

Proof. The proof is similar to the proof of Theorem 3.14. □

Theorem 3.26. [17] *Let $(X, \circ)$ be a subtraction algebra and A a SVNS over X. Then S_N is neutrosophic ideal of X if $L_{(\alpha,\beta,\gamma)}$ is either the empty set of ideal of X for all $0 \leq \alpha, \beta, \gamma \leq 1$.*

Theorem 3.27. *Let $(X, -)$ be a subtraction algebra and A a SVNS over X. Then A is neutrosophic ideal of X if and only if $\overline{L}_{(\alpha,\beta,\gamma)}$ is either the empty set or an ideal of X for all $-1 \leq \alpha, \beta, \gamma \leq 0$.*

Proof. The proof is similar to the proof of Theorem 3.6. □

Theorem 3.28. *Let $(X, -)$ be a subtraction algebra and S_N a neutrosophic $\aleph$-structure over X where,*

$$S_N = \{\frac{x}{(T_N(x), I_N(x), F_N(x))} : x \in X\}, A = \{\frac{x}{(-T_N(x), I_N(x)+1, F_N(x)+1)} : x \in X\}.$$

. Then the following statements are equivalent.

1. *S_N is a neutrosophic $\aleph$-subpolygroup of P;*
2. *A is a single valued neutrosophic polygroup over P;*
3. *$\overline{L}_{(\alpha,\beta,\gamma)}$ is either the empty set or a subpolygroup of X for all $-1 \leq \alpha, \beta, \gamma \leq 0$;*
4. *$L_{(\alpha,\beta,\gamma)}$ is either the empty set or a subpolygroup of X for all $0 \leq \alpha, \beta, \gamma \leq 1$.*

Proof. The proof follows from Theorem 3.25, Theorem 3.26, and Theorem 3.27. □

The authors proved in[7] the following theorem which can be deduced from Theorem 3.28.

Theorem 3.29. [7] *Let $(X, -)$ be a subtraction algebra and S_N a neutrosophic $\aleph$-structure over X. Then S_N is neutrosophic $\aleph$-ideal of X if and only if $\overline{L}_{(\alpha,\beta,\gamma)}$ is either the empty set of ideal of X for all $-1 \leq \alpha, \beta, \gamma \leq 0$.*

Example 3.30. Let $(X_1, -_1)$ be the subtraction algebra defined in Example 3.22. Then

$$S_N = \{\frac{0}{(-0.7, -0.4, -0.9)}, \frac{1}{(-0.7, -0.4, -0.9)}, \frac{2}{(-0.6, -0.6, -0.8)}\}$$

is a neutrosophic $\aleph$-ideal of X_1 as $A = \{\frac{0}{(0.7,0.6,0.1)}, \frac{1}{(0.7,0.6,0.1)}, \frac{2}{(0.6,0.4,0.2)}\}$ is a neutrosophic ideal of X_1.

3.4 Generalization to any algebraic structure (hyperstructure)

We can deduce from the work presented in the previous subsections that neutrosophic substructures (subhyperstructures) and neutrosophic ℵ-substructures (subhyperstructures) are connected. The following two theorems generalize our work.

Theorem 3.31. *Let X be any algebraic structure (hyperstructure) and S_N a neutrosophic ℵ-structure over X. Then S_N is neutrosophic ℵ-substructure (subhyperstructure) of X if and only if A is a single valued neutrosophic algebraic structure (hyperstructure) over X. Here,*

$$S_N = \{\frac{x}{(T_N(x), I_N(x), F_N(x))} : x \in X\}, A = \{\frac{x}{(-T_N(x), I_N(x)+1, F_N(x)+1)} : x \in X\}.$$

Theorem 3.32. *Let X be any algebraic structure (hyperstructure) and S_N a neutrosophic ℵ-structure over X where,*

$$S_N = \{\frac{x}{(T_N(x), I_N(x), F_N(x))} : x \in X\}, A = \{\frac{x}{(-T_N(x), I_N(x)+1, F_N(x)+1)} : x \in X\}.$$

Then the following statements are equivalent.

1. *S_N is a neutrosophic ℵ-substructure (subhyperstructure) of X;*
2. *A is a single valued neutrosophic algebraic structure (hyperstructure) over P;*
3. *$\overline{L}_{(\alpha,\beta,\gamma)}$ is either the empty set or a substructure (subhyperstructure) of X for all $-1 \leq \alpha, \beta, \gamma \leq 0$;*
4. *$L_{(\alpha,\beta,\gamma)}$ is either the empty set or a substructure (subhyperstructure) of X for all $0 \leq \alpha, \beta, \gamma \leq 1$.*

Remark 3.33. Theorem 3.32 implies that if some results are known for single valued algebraic structures (hyperstructures) such as single valued neutrosophic groups, rings, hypergroups, hyperrings, etc., then these results hold also for neutrosophic ℵ-substructures (subhyperstructures) of these algebraic structures (hyperstructures).

4 Conclusion and discussion

SVNS and neutrosophic ℵ-structures grabbed the attention of neutrosophic researchers. In this paper, we found a relationship between the two concepts. And we used this relation to prove that there is a connection between neutrosophic substructures (subhyperstructures) and neutrosophic ℵ-substructures (subhyperstructures). Moreover, we presented examples on this connection by dealing with specific algebraic substructures (subhyperstructures) such as semigroups, polygroups, and subtraction algebras. As a result, we were able to deduce that by defining a new single valued neutrosophic structures (hyperstructures) over a given algebraic structure (hyperstructure) and working on it, we can immediately define neutrosophic ℵ-substructures (subhyperstructures) of the same algebraic structure (hyperstructure) and the results that we get for SVNS will be applicable for neutrosophic ℵ-structures.
For future work, it will be interesting to find more applications on SVNS and to project the relationship between SVNS and neutrosophic ℵ-structures we found in this paper on the new applications.

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