On A Subclass of Analytic Functions Specified By Touchard Polynomials

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Abstract

In this investigation, we present a new collection of analytic functions that includes Touchard polynomials. We then aim to calculate the Maclaurin coefficients $|a_2|$ and $|a_3|$ and address the Fekete-Szegö functional problem within this specific subfamily. Additionally, we demonstrate several new outcomes by specifying the parameters used in our main findings.

Keywords: Analytic functions; Touchard Polynomials; Fekete-Szegö problem; Bi-univalent functions

1. preliminaries

Legendre made the initial discovery of orthogonal polynomials in 1784 [1]. Since then, these polynomials have been the subject of extensive research. They are commonly used in solving mathematical models to solve ordinary differential equations and fulfill model constraints. It is clear that orthogonal polynomials play a crucial role in modern mathematics and have numerous applications in physics and engineering. It is well known that these polynomials are crucial in matters pertaining to approximation theory. They are encountered in mathematical statistics and differential equation theory. Additionally, they are well-known are their applications to automated control, quantum physics, signal analysis, scattering theory, and axially symmetric potential theory [2, 3].

Touchard polynomials, named after French mathematician Jacques Touchard (see [4]), are a family of polynomials that have applications in combinatorics and the study of integer partitions. These polynomials are defined using exponential generating functions and are closely connected to Bell numbers (see [5], [6]), which are used to count the number of ways to partition a set. Touchard polynomials have applications in various areas, including probability theory, statistical mechanics, and the analysis of algorithms. They provide a useful tool for understanding the distribution of particles in certain physical systems and are employed in combinatorial problems involving set partitions. The study of Touchard polynomials contributes to a deeper understanding of the combinatorial structures and processes in mathematics. Also, comprise a polynomial sequence of binomial type that for X is a random variable via a Poisson distribution with an expected value δ, then its nth moment is $E(X\delta) = \mu(\delta, i)$, leading in the type:

DOI: https://doi.org/10.54216/IJNS.250312
Received: June 14, 2024 Revised: August 10, 2024 Accepted: September 24, 2024

$$\mu(\delta, i) = e^{\delta}\sum_{n=0}^{\infty}\frac{\delta^{n}(n^{i})}{n!}(k)^{n}$$

The coefficients of Touchard polynomials are used to present the outcome of the second force

$$\Lambda_{\delta}^{i}(k) = k + \sum_{n=0}^{\infty}\frac{\delta^{n-1}(n-1)^{i}}{e^{\delta}(n-1)!}k^{n}\,, (k \in \Delta) \quad (1.1)$$

where i ≥ 0; δ > 0 and we note that, by ratio test the radius of convergence of above series is infinity. Let Y be the class of functions analytic in the unit disk Δ= $\{k \in \mathbb{C}: |k| < 1\}$ of the form:

h(k)=k+$\sum_{n=2}^{\infty} d_n k^n, (k \in \Delta)$ (1.2)

We also let Ω consisting of functions which are normalized by $f(0) = f'(0) - 1 = 0$ *and* also univalent in Δ

Denote by Y the subclass of Ω consisting of functions of the form:

h(k)=k+$\sum_{n=2}^{\infty} d_n k^n, d_n \geq 0$ (1.3)

For functions h(k)=k+ $\sum_{n=2}^{\infty} d_n k^n$ and j(k)=k+$\sum_{n=2}^{\infty} c_n k^n$, we define the convolution of h and j by (h*j)(k)=k+ $\sum_{n=2}^{\infty} d_n c_n k^n$, (k∈ Δ)

Now, we define the linear operator $\omega(\iota, j, z)$: Y → Y by

$$\omega_h(\iota, j, \kappa) = \Lambda_{\delta}^{\iota}(\kappa) * h(\kappa) = \kappa + \sum_{n=2}^{\infty}\frac{\delta^{n-1}(n-1)^{\iota}}{e^{\delta}(n-1)!}d_n\kappa^{n}, (\kappa \in \Delta)$$

Miller and Mocanu [7] introduced the first differential subordination problem, for additionally details, see [8] and [9]

Every mathematical function h ∈ Ω has an inverse h^{-1} which is denoted by

$$h^{-1}(h(k)) = k \quad (k \in \Delta)$$

and

$$\mathcal{W} = h(h^{-1}(\mathcal{W})) \qquad \left(|\mathcal{W}| < r_0(h);\ r_0(h) \geq \frac{1}{4}\right)$$

where

$$q(\mathcal{W}) = h^{-1}(\mathcal{W}) = \mathcal{W} - d_2\mathcal{W}^2 + (-d_3 + 2d_2^2)\mathcal{W}^3 - (d_4 + 5d_2^3 - 5d_3d_2)\mathcal{W}^4 + \cdots \quad (1.4)$$

If both $h(k)$and $h^{-1}(k)$are univalent in Δ, a function is said to be bi-univalent in Δ.

Let Γ denote the class of bi-univalent functions in Δ given by (2). Example in the class Γ is $h(k) = \frac{k}{1-k}$ but$h(k) = \frac{k}{1-k^2}$ not members of Γ. For intriguing function classes in class Γ , see([10]*and* [11])

In recent years, various investigations have explored fundamental facets of geometric function theory, with a focus on estimating coefficients. Several subclasses of the class Γwere introduced and non-sharp estimates on the coefficients $|a2|$and $|a3|$ in the

Maclaurin series expansion (1.2) were obtented in ([12]-[42]).

We define a novel subclass of Γ involving the Touchard polynomials and derive bounds for the $|a2|$ and $|a3|$ Maclaurin coefficients and Fekete–Szegö functional problems [19]. Furthermore, number of new results are demonstrated to follow.

2. Bounds of the class $\otimes^{n}(\gamma,\beta,\delta,i,\mathcal{F})$

A definition of the new subclass $\otimes^{n}(\gamma,\beta,\delta,i,\mathcal{F})$ connected to Touchard polynomials is given at the start of this section

Definition 1. If the following subordinations are satisfied, a function $h \in \Omega$ as expressed in (1.2) is considered to be a member of the class $\otimes^{n}(\gamma,\beta,\delta,i,\mathcal{F})$ if

$$1+\frac{1}{\gamma}\left[(\omega_h(i,\delta,z))' + \beta z\left(\omega_h(i,\delta,z)\right)'' - 1\right] \prec \mathrm{F}(z) \quad (2.1)$$

and

$$1+\frac{1}{\gamma}\left[(\omega_q(i,\delta,w))' + \beta w(\omega_q(i,\delta,w))'' - 1\right] \prec F(w) \qquad (2.2)$$

where $\gamma \in \mathbb{C}\setminus\{0\};\ 0 \le \beta \le 1;\ \delta > 0;\ i \ge 0;\ z, w \in \mathbb{C},$ *and the function* $q = h^{-1}$ *is given by* (1.4)

Many subclasses can be found by taking special values for the parameters γ and β, for example but not limited we can state the following examples

Example 1. For $\gamma = 1$, we have $\otimes^{n}(1,\beta,\delta,i,F)$, *in which* $\otimes^{n}(1,\beta,\delta,i,F)$ *indicates by the functions* $h \in \Omega$ *and satisfying the crterion below*

$$\left(\omega_h(i,\delta,z)\right)' + \beta z\left(\omega_h(i,\delta,z)\right)'' \prec F(z)$$

and

$$\left(\omega_q(i,\delta,w)\right)' + \beta w\left(\omega_q(i,\delta,w)\right)'' \prec F(w)$$

where $0 \le \beta \le 1;\ \delta > 0;\ i \ge 0;\ z, w \in \mathbb{C}$ *and the function* $q = h^{-1}$ *is given by*(1.4)

Example 2. *For* $\gamma = \beta = 1,$ *we have,* $\otimes^{n}(1,1,\delta,i,F)$ *in which* $\otimes^{n}(1,1,\delta,i,F)$ *indicates the functions* $h \in \Omega$ *given by* (2)*and satisfying the criterion below*

$$\left(\omega_h(i,\delta,z)\right)' + \beta z\left(\omega_h(i,\delta,z)\right)'' \prec F(z)$$

and

$$\left(\omega_q(i,\delta,w)\right)' + w\left(\omega_q(i,\delta,w)\right)'' \prec F(w)$$

where $\delta > 0;\ i \ge 0;\ z, w \in \mathbb{C},$ *and the function* $q = h^{-1}$ *is given by* (1.4).

First, we give the coefficient estimates for the class $\otimes^{n}(\gamma,\beta,\delta,i,F)$ given in Definition 1.

Theorem 1. *Let* $f \in \Omega$ *given by* (1.2) *belongs to the calss* $\otimes^{n}(\gamma,\beta,\delta,i,F)$ *.Then*

$$|a2| \le \frac{|\gamma|\sqrt{E_1^3}}{\sqrt{\delta^2 e^{-\delta}\left|\left[3.\,2^{i-1}\gamma(2\beta+1)E_1^2 + 4e^{-\delta}(\beta+1)^2(E_1-E_2)\right]\right|}},$$

and

$$|a3| \le \frac{|\gamma| e^{\delta} E_1}{4}\left(\frac{1}{3.\,2^{i-3}\delta^2(2\beta+1)} + \frac{|\gamma| e^{\delta} E_1}{(\beta+1)^2}\right)$$

where $\gamma \in \mathbb{C}\setminus\{0\};\ 0 \le \beta \le 1;\ \delta > 0;\ i \ge 0.$

Proof. Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \otimes^{n}(\gamma,\beta,\delta,i,F)$ *we cansider two functions* $r,$

DOI: https://doi.org/10.54216/IJNS.250312
Received: June 14, 2024 Revised: August 10, 2024 Accepted: September 24, 2024

$u:\Delta \to \Delta, whith\ r(0) = u(0) = 0\ and\ |r(z)| < 1, |u(w)| < 1\ for\ all, z, w \in \Delta. so\ from$ Definition 2.1, we can write

$$1+\frac{1}{\gamma}\left[\left(\omega_f(i,\delta,z)\right)' + \beta z\left(\omega_f(i,\delta,z)\right)'' - 1\right] = F\left(r(z)\right) \quad (2.3)$$

and

$$1+\frac{1}{\gamma}\left[\left(\omega_g(i,\delta,z)\right)' + \beta w\left(\omega_g(i,\delta,w)\right)'' - 1\right] = F\left(u(w)\right) \quad (2.4)$$

where $g = f^{-1}$

Definition the function s and d as following:

$$s(z) = \frac{1+r(z)}{1-r(z)} = 1 + t_1 z + t_2 z^2 + t_3 z^3 + \cdots, \left|t_j\right| \leq 2\ for\ all\ j \in \mathbb{N}\ (2.5)$$

and

$$d(w) = \frac{1+u(w)}{1-u(w)} = 1 + v_1 w + v_2 w^2 + v_3 w^3 + \cdots, \left|v_j\right| \leq 2\ for\ all\ j \in \mathbb{N}\ (2.6)$$

or equivalently,

$$r(z) = \frac{s(z)-1}{s(z)+1} = \frac{t_1}{2}z + \frac{1}{2}\left(t_2 - \frac{t_1^2}{2}\right)z^2 + \frac{1}{2}\left(t_3 + \frac{t_1}{2}\left(\frac{t_1^2}{2} - t_2\right) - \frac{t_1 t_2}{2}\right)z^3 + \cdots$$

and

$$u(w) = \frac{d(w)-1}{d(w)+1} = \frac{v_1}{2}w + \frac{1}{2}\left(v_2 - \frac{v_1^2}{2}\right)w^2 + \frac{1}{2}\left(v_3 + \frac{v_1}{2}\left(\frac{v_1^2}{2} - v_2\right) - \frac{v_1 v_2}{2}\right)w^3 + \cdots$$

Using last two equations in (2.3) and (2.4), we have

$$1+\frac{1}{\gamma}\left[\left(\omega_f(i,\delta,z)\right)' + \beta z\left(\omega_f(i,\delta,z)\right)'' - 1\right] = 1 + \frac{1}{2}E_1 t_1 z + \left(\frac{1}{2}E_1\left(t_2 - \frac{t_1^2}{2}\right) + \frac{1}{4}E_2 t_1^2\right)z^2 + \cdots \ (2.7)$$

and

$$1+\frac{1}{\gamma}\left[\left(\omega_g(i,\delta,w)\right)' + \beta w\left(\omega_g(i,\delta,w)\right)'' - 1\right] = 1 + \frac{1}{2}E_1 v_1 w + \left(\frac{1}{2}E_1\left(v_2 - \frac{v_2^1}{2}\right) + \frac{1}{4}E_2 V_1^2\right)w^2 + \cdots \ (2.8)$$

It follows by comparing the relevant coefficients in (2.7) and (2.8) that

$$\frac{2\delta e^{-1}}{\gamma}(\beta+1)a_2 = \frac{1}{2}E_1 t_1, \quad (2.9)$$

$$\frac{3.2^i\delta^2 e^{-\delta}}{2!\,\gamma}(2\beta+1)a_3 = \frac{1}{2}E_1\left(t_2 - \frac{t_1^2}{2}\right) + \frac{1}{4}E_2 t_1^2, \qquad (2.10)$$

$$-\frac{2\delta e^{-\delta}}{\gamma}(\beta+1)a_2 = \frac{1}{2}E_1 v_1, \quad (2.11)$$

and

$$\frac{3.2^i\delta^2 e^{-\delta}}{2!\,\gamma}(2\beta+1)(2a_2^2 - a_3) = \frac{1}{2}E_1\left(v_2 - \frac{v_1^2}{2}\right) + \frac{1}{4}E_2 v_1^2. \quad (2.12)$$

From (2.9) and (2.11), we get

$$t_1 = -v_1 \quad (2.13)$$

and

$$2^5\delta^2 e^{-2\delta}(\beta+1)^2 a_2^2 = \gamma^2 E_1^2(t_1^2 + v_1^2). \quad (2.14)$$

DOI: https://doi.org/10.54216/IJNS.250312
Received: June 14, 2024 Revised: August 10, 2024 Accepted: September 24, 2024

If we add (2.10) to (2.12), we get

$$\frac{3.2^{i}\delta^{2}e^{-\delta}}{\gamma}(2\beta+1)a_2^2=\frac{1}{2}E_1(t_2+v_2)+\frac{1}{4}(E_2-E_1)(t_1^2+v_1^2). \quad (2.15)$$

From both equations (2.14) and (2.15), we have

$$a_2^2=\frac{\gamma^2E_1^3(t_2+v_2)}{\delta^2e^{-\delta}[3.2^{i+1}\gamma E_1^2(2\beta+1)+2^4e^{-\delta}(\beta+1)^2(E_1-E_2)]} \quad (2.16)$$

Moreover, subtracting (2.12) from (2.10) and by view (2.13), we obtain

$$a_3=\frac{\gamma e^{\delta}E_1(t_2-v_2)}{3.2^{i+1}\delta^2(2\beta+1)}+\frac{\gamma^2e^{2\delta}E_1^2t_1^2}{2^4\delta^2(\beta+1)^2}. \quad (2.17)$$

Moreover computations using (2.16), (2.17) and by view (2.5) and (2.6), we find that

$$|a_2|\le\frac{|\gamma|\sqrt{E_1^3}}{\sqrt{\delta^2}e^{-\delta}|[3.2^{i-1}\gamma(2\beta+1)E_1^2+4e^{-\delta}(\beta+1)^2(E_1-E_2)]|}$$

and

$$|a_3|\le\frac{|\gamma|e^{\delta}E_1}{4}\left(\frac{1}{3.2^{i-3}\delta^2(2\beta+1)}+\frac{|\gamma|e^{\delta}E_1}{(\beta+1)^2}\right).$$

Which are asserted by the Theorem 2.2.

For a function $f\in\Omega$, Fekete and Szegö in 1933 [43] fined a sharp bound on the functional $|a_3-\epsilon a_2^2|$ Using the values of a_2^2 *and* a_3 we prove the functional $|a_3-\epsilon a_2^2|$ for class functions $\otimes^n$ $(\gamma,\beta,\delta,i,F)$

Theorem 2. Let $f\in\Omega$
given by (1.2) belongs to the class $\otimes^n$ $(\gamma,\beta,\delta,i,F)$. Then

$$|a_3-\epsilon a_2^2|\le\begin{cases}\dfrac{|\gamma|e^{\delta}E_1}{3.2^{I-1}\delta^2(2\beta+1)} & 0\le|\Theta(\epsilon)|\le\dfrac{e^{\delta}}{3.2^{i+1}\delta^2(2\beta+1)'}\\ 4|\gamma||\Theta(\epsilon)|E_1 & |\Theta(\epsilon)|\ge\dfrac{e^{\delta}}{3.2^{i+1}\delta^2(2\beta+1)}\end{cases}$$

where

$$\Theta(\epsilon)=\frac{\gamma e^{\delta}E_1^2(1-\epsilon)}{2^4\delta^2[3.2^{i-3}\gamma E_1^2(2\beta+1)+e^{-\delta}(\beta+1)^2(E_1-E_2)]}$$

where $\gamma\epsilon\mathbb{C}\setminus\{0\}; 0\le\beta\le1; \delta>0; i\ge0$

Proof . Subtracting (2.12) from (2.10) and by view (2.13), we have

$$a_3=\frac{\gamma E_1(t_2-v_2)}{3.2^{i+1}\delta^2e^{-\delta}(2\beta+1)}+a_2^2$$

which leads to writing

$$a_3-\epsilon a_2^2=\frac{\gamma E_1\,(t_2-v_2)}{3.2^{i+1}\delta^2e^{-\delta}(2\beta+1)}+(1-\epsilon)a_2^2$$

$$a_3-\epsilon a_2^2=\gamma E_1\left(\left[\Theta(\epsilon)+\frac{1}{3.2^{i+1}\delta^2e^{-\delta}(2\beta+1)}\right]t_2+\left[\Theta(\epsilon)-\frac{1}{3.2^{i+1}\delta^2e^{-\delta}(2\beta+1)}\right]v_2\right),$$

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Received: June 14, 2024 Revised: August 10, 2024 Accepted: September 24, 2024

where

$$\Theta(\epsilon)=\frac{\gamma e^{\delta}E_1^2\,(1-\epsilon)}{2^4\delta^2[3.2^{i-3}\gamma E_1^2(2\beta+1)+e^{-\delta}(\beta+1)^2(E_1-E_2)]'}$$

Then, in view (2.5) and (2.6), we conclude that

$$|a_3-\epsilon a_2^2|\le\begin{cases}\dfrac{|\gamma|e^{\delta}E_1}{3.2^{i-1}\delta^2(2\beta+1)} & 0\le|\Theta(\epsilon)|\le\dfrac{e^{\delta}}{3.2^{i+1}\delta^2(2\beta+1)'}\\ 4|\gamma||\Theta(\epsilon)|E_1 & |\Theta(\epsilon)|\ge\dfrac{e^{\delta}}{3.2^{i+1}\delta^2(2\beta+1)}\end{cases}$$

Which are asserted by the Theorem 2.3.

3. Corollaries

If we set $\gamma=e^{i\phi}cos\phi,\left(-\frac{\pi}{2}<\phi<\frac{\pi}{2}\right) and$

$$F(z)=\frac{1+(1-2\varepsilon)z}{1-z}=1+2(1-\varepsilon)z+2(1-\varepsilon)z^2+\cdots(0\le\varepsilon<1)$$

which gives $E_1=E_2=2(1-\varepsilon); in\ Theorem\ 2.2\ and\ Theorem\ 2.3, we\ get\ the\ following$ corollaries

Corollary 1 let$f\in\Omega$ given by $(1.2)\ belongs\ to\ the\ class\ \otimes\ ^n\left(e^{i\phi}cos\phi,\beta,\delta,i,\frac{1+(1-2\varepsilon)z}{1-z}\right).then$

$$|a_2|\le\sqrt{\frac{e^{\delta}(1-\varepsilon)cos\phi}{3.2^{i-2}\delta^2(2\beta+1)}},|a_3|\le e^{\delta}(1-\varepsilon)\left(\frac{1}{3.2^{i-2}\delta^2(2\beta+1)}+\frac{e^{\delta}(1-\varepsilon)cos\phi}{(\beta+1)^2}\right)cos\phi$$

and

$$|a_3-\epsilon a_2^2|\le\begin{cases}\dfrac{e^{\delta}(1-\varepsilon)cos\phi}{3.2^{i}\delta^2(2\beta+1)} & 0\le1-\epsilon\le1\\ \dfrac{e^{\delta}(1-\varepsilon)|1-\epsilon|cos\phi}{3.2^{i-2}\delta^2(2\beta+1)} & \epsilon\le0\end{cases}$$

For $\beta=0; Corollary\ 1\ simplifies\ to\ the\ following\ Corollary$

Corollary 2. $Let\ f\in\Omega\ given\ by\ (1.2) belongs\ to\ the\ class\ \ \otimes^n\left(e^{i\phi}cos\phi,0,\delta,i,\frac{1+(1-2\varepsilon)z}{1-z}\right).Then$

$$|a_2|\le\sqrt{\frac{e^{\delta}(1-\varepsilon)cos\phi}{3.2^{i-2}\delta^2}},|a_3|\le e^{\delta}(1-\varepsilon)\left(\frac{1}{3.2^{i-2}\delta^2}+e^{\delta}(1-\varepsilon)cos\phi\right)cos\phi$$

and

$$|a_3-\epsilon a_2^2|\le\begin{cases}\dfrac{e^{\delta}(1-\varepsilon)cos\phi}{3.2^{i}\delta^2} & 0\le1-\epsilon\le1\\ \dfrac{\delta^2(1-\varepsilon)|1-\epsilon|cos\phi}{3.2^{i-2}\delta^2} & \epsilon\le0\end{cases}$$

II) If we set $\gamma=1,\left(-\frac{\pi}{2}<\phi<\frac{\pi}{2}\right) and$

$$F(z)=\left(\frac{1+z}{1-z}\right)^{\varphi}=1+2\varphi z+2\varphi^2z^2+\cdots(0\le\varphi\le1)$$

which gives $E_1=2\varphi\ and\ E_2=2\varphi^2, in\ theorems\ 1\ and\ theorem\ 2, we\ get\ the\ following$ corollaries

Corollary 3. Let $f\in\Omega\ given\ by\ (1.2)\ belongs\ to\ the\ class\ \otimes^n\left(1,\beta,\delta,i,\left(\frac{1+z}{1-z}\right)^{\varphi}\right).then$

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Received: June 14, 2024 Revised: August 10, 2024 Accepted: September 24, 2024

$$|a_2| \leq \frac{2\varphi}{\sqrt{\delta^2 e^{-\delta}(3.2^{i-2}(2\beta+1)\varphi + e^{-\delta}(\beta+1)^2(1-\varphi)}}$$

$$|a_3| \leq e^{\delta}\varphi\left(\frac{1}{3.2^{i-2}\delta^2(2\beta+1)} + \frac{e^{\delta}\varphi}{(\beta+1)^2}\right)$$

and

$$|a_3 - \epsilon a_2^2| \leq \begin{cases} \dfrac{e^{\delta}\varphi}{3.2^{i-2}\delta^2(2\beta+1)} & 0 \leq \Theta(\epsilon) \leq \dfrac{e^{\delta}}{3.2^{i+1}\delta^2(2\beta+1)} \\ \dfrac{2e^{\delta}\varphi^2|1-\epsilon|}{\delta^2(3.2^{i-2}(2\beta+1)\varphi + e^{-\delta}(\beta+1)^2(1-\varphi))} & |\Theta(\epsilon)| \geq \dfrac{e^{\delta}}{3.2^{i+1}\delta^2(2\beta+1)} \end{cases}$$

where

$$\Theta(\epsilon) = \frac{e^{\delta}(1-\epsilon)\varphi}{2^3\delta^2(3.2^{i-2}(2\beta+1)\varphi + e^{-\delta}(\beta+1)^2(1-\varphi))}$$

For $\beta = 0; Corollary\ 1\ simplifies\ to\ the\ following\ corollary$

Corollary 4. *Let* $f \in \Omega$ *given by* (2)*belongs to the class* $\otimes^n \left(1,0,\delta,i,\left(\frac{1+z}{1-z}\right)^{\varphi}\right)$*.then*

$$|a_2| \leq \frac{2\varphi}{\sqrt{\delta^2 e^{-\delta}\left(3.2^{i-2}\varphi + e^{-\delta}(1-\varphi)\right)}}, |a_3| \leq e^{\delta}\varphi\left(\frac{1}{3.2^{i-2}\delta^2} + e^{\delta}\varphi\right)$$

and

$$|a_3 - \in a_2^2| \leq \begin{cases} \dfrac{e^{\delta}\varphi}{3.2^{i-2}\delta^2} & 0 \leq |\Theta(\epsilon)| \leq \dfrac{1}{3.2^{i+1}\delta^2 e^{-\delta}}, \\ \dfrac{2e^{\delta}\varphi^2|1-\epsilon|}{\delta^2\left(3.2^{i-1}\varphi + e^{-\delta}(1-\varphi)\right)} & \left|\Theta\left(\epsilon \geq \dfrac{1}{3.2^{i+1}\delta^2 e^{-\delta}}\right)\right| \end{cases}$$

where

$$\Theta(\epsilon) = \frac{e^{\delta}\varphi(1-\epsilon)}{2^3\delta^2(3.2^{i-2}\varphi + e^{-\delta}(1-\varphi))}$$

4. Concluding Remark

In this present investigation, we have presented and investigated the coeficient-related issues

Associated with each of the three newly introduced subclasses $\otimes^n\ (\gamma,\beta,\delta,i,\mathcal{F})$, $\otimes^n\ (1,\beta,\delta,i,\mathcal{F})$,

and $\otimes^n\ (1,1,\delta,i,\mathcal{F})$ within the class of bi-univalent functions defined in the open unit disk. The corresponding definitions for these bi-univalent function classes are provided in Definition 2.1. Our analysis includes the computation of estimates for the Fekete-Szeg"o functional problems and the Maclaurin coefficients ja2j and ja3j for functions belonging to each of these three bi-univalent function classes. Numerous additional novel findings emerge when we specialize the parameters involved in our primary results.

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DOI: https://doi.org/10.54216/IJNS.250312
Received: June 14, 2024 Revised: August 10, 2024 Accepted: September 24, 2024

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DOI: https://doi.org/10.54216/IJNS.250312
Received: June 14, 2024 Revised: August 10, 2024 Accepted: September 24, 2024

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DOI: https://doi.org/10.54216/IJNS.250312
Received: June 14, 2024 Revised: August 10, 2024 Accepted: September 24, 2024