Fixed Point Theorems with its Applications in Fuzzy Complete Convex Fuzzy Metric Spaces

Jaafer Hmood Eidi[1,*], Ehsan M. Hameed[2], Jehad R. Kider[3]

[1]Department of Mathematics College of Education Mustansiriyah University, Iraq
[2]Department of Mathematics, College of Computer Science and Mathematics, University of Thi-Qar, Iraq
[3]Department of Applied Sciences, University of Technology, Iraq

Email: drjaffarmath@uomustansiriyah.edu.iq; ehsan.m@utq.edu.iq; jehad.r.kider@uotechnology.edu.iq

Abstract

In this paper, the basic properties of the convex fuzzy metric space will be presented. In particular, the proof of the fixed-point theorem for the fuzzy contraction single valued functions will be discussed. Furthermore, the solution system of linear equations, Volterra equations and Fredholm integral equations will be obtained as a direct application of the fixed-point theorem.

Keywords: Convex fuzzy metric space; Fuzzy contraction function; Fuzzy absolute value space; Fuzzy Lipschitz condition

1. Introduction and preliminaries

The common fixed-point theorems in fuzzy metric spaces are presented and proved in [1]. These theorems improve known results generalize and it was a generalization. In addition, some common fixed-point theorems for occasionally weakly compatible mappings in fuzzy metric spaces are stated and proved in [2]. The fixed-point theorem for complete fuzzy metric spaces is proved in the sense of George and Veeramani [3]. The notions of β-ψ-fuzzy contractive mapping and α-φ-fuzzy contractive mapping is introduced and proved two theorems, which ensure the existence and uniqueness of a fixed point for these two types of mappings [4]. These theorems extend, generalize and improve the corresponding known results. Further, the convex fuzzy distance from a point to a set as well as between two sets is introduced using the convex fuzzy metric. Moreover, some of basic theorems for convex fuzzy metric space are proved [5].

Here, we deal with new type of fuzzy metric space known as convex fuzzy metric space [6]. After that, we stated and proved fixed-point theorem for fuzzy contraction single valued function.

2. The convex fuzzy metric space

Definition 1 [6]:

If the fuzzy set $m: \mathcal{U} \times \mathcal{U} \to [0, 1]$ fulfills:

(i) $m(p, q) \in [0, 1]$;

(ii) $m(p, q) = 0 \Leftrightarrow p = q$;

(iii) $m(p, q) = m(q, p)$;

(iv) $\delta m(p, z) + \beta m(z, q) \geq m(p, q)$;

For all $\delta, \beta \in (0, 1)$ with $\delta + \beta = 1$ and $p, q, z \in \mathcal{U} \neq \emptyset$. Then $(\mathcal{U}, m)$ is **convex fuzzy metric space** (simply **c-FMS).**

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Remark 1 [6]:

If μ, $\sigma \in [0, 1]$ then $(\beta\mu + (1-\beta)\sigma) \in [0, 1]$ for any $\beta \in [0, 1]$, or $(\delta\mu + \beta\sigma) \in [0, 1]$ for all $\delta, \beta \in [0, 1]$ with $\delta + \beta = 1$.

In general if $\sigma_1, \sigma_2, \ldots, \sigma_k \in [0, 1]$ then $(\gamma_1\sigma_1 + \gamma_2\sigma_2 + \ldots + \gamma_k\sigma_k) \in [0, 1]$ for any $\gamma_1, \gamma_2,$

$\ldots, \gamma_k \in (0, 1)$ with $\gamma_1 + \gamma_2 + \ldots + \gamma_k = 1$.

Example 1 [6]:

Consider $m_d(y, w) = \frac{d(y,w)}{1+d(y,w)}$ for all $y, w \in \mathcal{U}$. If $(\mathcal{U}, d)$ is a MS then $(\mathcal{U}, m_d)$ is c-FMS.

Definition 2 [6]:

If $(\mathcal{U}, m)$ is c-FMS then

(1) $\mathfrak{B}(u, \gamma) = \{y \in \mathcal{U}: m(u, y) < \gamma\}$ is an **convex fuzzy open** (or simply **CFO**) ball;

(2) $\overline{\mathfrak{B}}(u, \gamma) = \{y \in \mathcal{U}: m(u, y) \leq \gamma\}$is **convex fuzzy closed** ball;

with radius $\gamma \in (0, 1)$ and center $u \in \mathcal{U}$.

Definition 3 [6]:

If $(\mathcal{U}, m)$ is a c-FMC then $\mathcal{W} \subseteq \mathcal{U}$ is convex fuzzy open (or simply CFO) set if $\mathfrak{B}(w, \gamma) \subseteq \mathcal{W}$, $\forall\, w \in \mathcal{W}$ and for some $\gamma \in (0, 1)$.

Definition 4 [6]:

If $(\mathcal{U}, m)$ is a c-FMC then

(1) If $\mathfrak{C} \subseteq \mathcal{U}$ and $\mathfrak{C}^C$ is CFO then $\mathfrak{C}$ is convex fuzzy closed;

(2) $\overline{\mathfrak{C}} = \cap\{\mathfrak{D}$ is convex fuzzy closed and $\mathfrak{C} \subseteq \mathfrak{D}\}$.

Definition 5 [6]:

The set $\mathcal{D}$ of $\mathcal{U}$ is convex fuzzy dense in $\mathcal{U}$ if $\overline{\mathcal{D}} = \mathcal{U}$ when $(\mathcal{U}, m)$ is a c-FMS.

Theorem 1 [6]:

In c-FMS $(\mathcal{U}, m)$ every CFO ball $\mathfrak{B}(w, \gamma)$ is a CFO set.

Definition 6 [6]:

In c-FMS $(\mathcal{U}, m)$, a sequence (u_k) is fuzzy converge to u in $\mathcal{U}$(or simply $u_k \to u$, or $\lim_{k\to\infty} u_k = u$) if for each $0 < \gamma < 1$, $\exists\ \mathcal{N}$ with $m(u_k, u) < \gamma$, $\forall$ k$\geq \mathcal{N}$.

Proposition 1 [6]:

In c-FMS $(\mathcal{U}, m)$, $u_k \to u$ if and only if $m(u_k, u) \to 0$.

Definition 7 [6]:

In c-FMS $(\mathcal{U}, m)$, a sequence (u_k) is fuzzy Cauchy if $\forall\ 0 < \gamma < 1$, $\exists\ \mathcal{N}$ satisfying

$m(u_n, u_k) < \gamma$, $\forall$ k, n $\geq \mathcal{N}$.

Definition 8 [6]:

A c-FMS $(\mathcal{U}, m)$ is **fuzzy complete** if $\forall$ fuzzy Cauchy sequence (u_k) in $\mathcal{U}$ then $\exists\ u \in \mathcal{U}$ fulfills $u_k \to u$.

Theorem 2 [6]:

In c-FMS $(\mathcal{U}, m)$, if $u_k \to u \in \mathcal{U}$ then (u_k) is fuzzy Cauchy.

Definition 9 [6]:

In c-FMS $(\mathcal{U}, m)$

(1)The set $\mathcal{D} \neq \emptyset$ in $\mathcal{U}$ is **convex fuzzy bounded** (or simply **CFB**) if $\exists\ \gamma \in (0, 1)$ with $\mathcal{D} \subset \mathfrak{B}(u, \gamma)$ for some $u \in \mathcal{U}$.

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(2) The sequence (d_k) in $\mathcal{U}$ is **CFB** if $\exists\, \gamma \in(0, 1)$ satisfies $(d_k) \in \mathfrak{B}(u, \gamma)$, for some $u \in \mathcal{U}$.

Theorem 3 [6]:

In c-FMS $(\mathcal{U}, m)$

(1)if $p_k \to p \in \mathcal{U}$ with $(p_k) \in \mathcal{U}$ then (p_k) is CFB.

(2) if $p_k \to p \in \mathcal{U}$ and $p_k \to z \in \mathcal{U}$ with $(p_k) \in \mathcal{U}$ as k→ ∞ then p=z.

(3) if $(u_k) \in \mathcal{U}$ with $u_n \to u$ and $(d_k) \in \mathcal{U}$ with $m(u_k, d_k) \to 0$ as k→ ∞. Then $d_k \to u$.

Theorem 4 [6]:

In c-FMS $(\mathcal{U}, m)$ if $\mathfrak{D} \subset \mathcal{U}$ then

(1) $\exists\, (d_{\mathrm{k}}) \in \mathfrak{D}$ satisfying $d_{\mathrm{k}} \to d$;

(2) $d \in \overline{\mathfrak{D}}$;

Are equivalent.

Definition 10 [6]:

If $\forall\ 0 < \gamma < 1$, $\exists\ 0 < \sigma < 1$, satisfying $m_{\mathcal{V}}[\mathcal{T}(u), \mathcal{T}(b)] < \gamma$ as $b \in \mathcal{U}$ and when $m_{\mathcal{U}}(w, b) < \sigma$ then the function $\mathcal{T}$:$\mathcal{U}$→$\mathcal{V}$ is fuzzy continuous at $u \in \mathcal{U}$. Whenever $(\mathcal{U}, m_{\mathcal{U}})$ and $(\mathcal{V}, m_{\mathcal{V}})$ are two c-FMS.

Theorem 5 [6]:

If $(\mathcal{U}, m_{\mathcal{U}})$, $(\mathcal{V}, m_{\mathcal{V}})$ are two c-FMS and $\mathcal{T}$:$\mathcal{U}$→$\mathcal{V}$ is a function. Then

(1) $\mathcal{T}(u_{\mathrm{k}})$→ $\mathcal{T}(u)$ in $\mathcal{V}$ if $u_{\mathrm{k}} \to u$ in $\mathcal{U}$.
(2) $\mathcal{T}$:$\mathcal{U}$→$\mathcal{V}$ is fuzzy continuous at $u \in \mathcal{U}$

Are equivalent**.**

Theorem 6 [6]:

In c-FMS $(\mathcal{U}, m)$ if $(p_k) \in \mathcal{U}$ with $p_k \to p \in \mathcal{U}$ as k→ ∞ and $(y_k) \in \mathcal{U}$ with $y_k \to y \in \mathcal{U}$ as k→ ∞. Then $m(p_k, y_k) \to m(p, y)$ as k→ ∞.

Theorem 7 [6]:

If $(\mathcal{U}, m_{\mathcal{U}})$ and $(\mathcal{V}, m_{\mathcal{V}})$ are two c-FMS and $\mathcal{T}$:$\mathcal{U}$→$\mathcal{V}$ is a function. Then

(1)$\mathcal{T}^{-1}(\mathfrak{D})$ is CFO in $\mathcal{U}$, ∀ CFO subset $\mathfrak{D}$ of $\mathcal{V}$;

(2) $\mathcal{T}$:$\mathcal{U}$→$\mathcal{V}$ is fuzzy continuous on ;

(3) $\mathcal{T}^{-1}(\mathfrak{S})$ is convex fuzzy closed in $\mathcal{U}$ for all convex fuzzy closed subset $\mathfrak{S}$ of $\mathcal{V}$;

Are equivalent**.**

Theorem 8 [6]:

If $(\mathcal{U}, m_{\mathcal{U}})$, $(\mathcal{V}, m_{\mathcal{V}})$ and $(\mathcal{W}, m_{\mathcal{W}})$ are c-FMS and $\mathcal{T}$:$\mathcal{U}$→$\mathcal{V}$, $\mathcal{S}$:$\mathcal{V}$→$\mathcal{W}$ are fuzzy continuous functions. Then $\mathcal{S} \circ \mathcal{T}$: $\mathcal{U}$→$\mathcal{W}$ is a fuzzy continuous function.

Definition 11 [6]:

If $\forall\ 0 < \gamma < 1$, $\exists\ 0 < \sigma < 1$, satisfies $m_{\mathcal{V}}[\mathcal{T}(u), \mathcal{T}(b)] < \gamma$, when $m_{\mathcal{U}}(w, b) < \sigma$, $\forall\, u, b \in \mathcal{U}$ then the function $\mathcal{T}$:$\mathcal{U}$→$\mathcal{V}$ is **uniformly fuzzy continuous** on $\mathcal{U}$. Whenever $(\mathcal{U}, m_{\mathcal{U}})$ and $(\mathcal{V}, m_{\mathcal{V}})$ are two c-FMS.

Theorem 9 [6]:

Let $(\mathcal{U}, m_{\mathcal{U}})$, $(\mathcal{V}, m_{\mathcal{V}})$ be two c-FMS and $\mathcal{T}$:$\mathcal{U}$→$\mathcal{V}$ is uniformly fuzzy continuous on $\mathcal{U}$. If (u_{k}) is a fuzzy Cauchy sequence in $\mathcal{U}$ then so is $(\mathcal{T}(u_{\mathrm{k}}))$ in $\mathcal{V}$.

Definition 12 [6]:

If $A_{\mathbb{R}}$:$\mathbb{R}$ →[0, 1] is a fuzzy set fulfills

(i) $A_{\mathbb{R}}(\delta) \in [0, 1]$;

(ii) $A_{\mathbb{R}}(\gamma)$=0 ⟺ γ=0;

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(iii) $A_{\mathbb{R}}(\gamma) \,.\, A_{\mathbb{R}}(\delta) \geq A_{\mathbb{R}}(\gamma\delta)$;

(iv) $\sigma A_{\mathbb{R}}(\gamma) + \mu A_{\mathbb{R}}(\delta) \geq A_{\mathbb{R}}(\gamma+\delta)$;

For all $0 < \sigma, \mu < 1$ with $\sigma + \mu = 1$ and $\forall \gamma, \delta \in \mathbb{R}$. Then $(\mathbb{R}, A_{\mathbb{R}})$ is **convex fuzzy absolute value space (simply c-FAVS).**

Example 2 [6]:

Let $\mathcal{U} = C[a,b]$ where a, b ∈ ℝ, define, $m(\hbar, \ell) = max_{\alpha\in[a,b]} A_{\mathbb{R}}[\hbar(\alpha) - \ell(\alpha)]$ for all $\hbar, \ell \in \mathcal{U}$. Then $(\mathcal{U}, m)$ is c-FMS.

3. Results and Discussion

Definition 13:

Let $(\mathcal{U}, m)$ be c-FMS and $\mathcal{T}:\mathcal{U}\to\mathcal{U}$ be a function. A point $u \in \mathcal{U}$ is called a **fixed point** (FP) for $\mathcal{T}$ if $\mathcal{T}(u) = u$.

Definition 14:

Let $(\mathcal{U}, m)$ be c-FMS then the function $\mathcal{T}:\mathcal{U}\to\mathcal{U}$ is a **fuzzy contraction** (FC) on $\mathcal{U}$ if there exist $\alpha \in (0, 1)$ satisfying $m[\mathcal{T}(p), \mathcal{T}(y)] \leq \alpha\, m(p, y)$, for all $p, y \in \mathcal{U}$.

Theorem 10:

Let $(\mathcal{U}, m)$ be c-FMS then $\mathcal{T}:\mathcal{U}\to\mathcal{U}$ is fuzzy continuous if $\mathcal{T}$ is a FC on $\mathcal{U}$

Proof:

If $\mathcal{T}$ is fuzzy contraction function then $\mathcal{T}$ is uniform fuzzy continuous function this is clear. As well as if $\mathcal{T}$ is uniform fuzzy continuous function then $\mathcal{T}$ is fuzzy continuous function this follows directly. Hence fuzzy contraction of $\mathcal{T}$ implies fuzzy continuous of $\mathcal{T}$. █

Theorem 11:

The function $\mathcal{T}: \mathcal{U} \to \mathcal{U}$ has exactly one fixed point if $\mathcal{T}$ is a fuzzy contraction where $(\mathcal{U}, m)$ is fuzzy complete c-FMS as well as $\mathcal{U} \neq \emptyset$.

Proof:

Step 1: Existences

Choose $u_0 \in \mathcal{U}$ and define

$$\mathcal{T}(u_0) = u_1, \mathcal{T}(u_1) = \mathcal{T}^2(u_0) = u_2, \ldots, \mathcal{T}^k(u_0) = u_k. \quad (1)$$

Notice that

$m(u_{k+1}, u_k) = m[\mathcal{T}(u_k), \mathcal{T}(u_{k-1})] \leq \alpha\, m(u_k, u_{k-1}) = \alpha\, m[\mathcal{T}(u_{k-1}), \mathcal{T}(u_{k-2})]$

$\leq \alpha^2\, m(u_{k-1}, u_{k-2}) = \alpha^2\, m[\mathcal{T}(u_{k-2}), \mathcal{T}(u_{k-3})]$.

We have

$$m(u_{k+1}, u_k) \leq \alpha^k\, m(u_1, u_0). \quad (2)$$

Hence when $k > j \geq N$ with N ∈ ℕ, we get

$m(u_j, u_k) \leq \sigma_j m(u_j, u_{j+1}) + \sigma_{j+1} m(u_{j+1}, u_{j+2}) + \ldots + \sigma_{k-j}\, m(u_{k-1}, u_k)$,

Where

$\sigma_j + \sigma_{j+1} + \ldots + \sigma_{k-j} = 1$, with $\sigma_j, \sigma_{j+1}, \ldots \sigma_{k-j}\ \alpha \in (0, 1)$.

Thus

$m(u_j, u_k) \leq [\sigma_j \alpha^j + \sigma_{j+1} \alpha^{j+1} + \ldots + \sigma_{k-j}\, \alpha^{k-j}]\, m(u_1, u_0)$,

$m(u_j, u_k) \leq [\sigma \alpha^j + \sigma\, \alpha^{j+1} + \ldots + \sigma\, \alpha^{k-j}]\, m(u_1, u_0)$, where $\sigma = \max\{\sigma_j, \sigma_{j+1}, \ldots, \sigma_{k-j}\}$

$$m(u_j, u_k) \leq [\sigma \alpha^j \frac{(1-\alpha^{k-j})}{(1-\alpha)}]\, m(u_1, u_0). \quad (3)$$

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Now choose $\gamma \in (0, 1)$, so that $[\sigma\alpha^{j}\frac{(1-\alpha^{k-j})}{(1-\alpha)}]m(u_1, u_0) < \gamma$.

Thus $m(u_j, u_k) < \gamma, \forall\, n > j \geq N$. Therefore, (u_n) is a fuzzy Cauchy . $\exists\, u \in \mathcal{U}$ satisfies $u_k \to u$.

Using

$m(u, \mathcal{T}(u)) \leq \delta m(u, u_k) + \beta m(u_k, \mathcal{T}(u))$ [where , $\beta \in [0, 1]$ with $\delta + \beta = 1$]

$\leq \delta m(u, u_k) + \beta m(u_k, \mathcal{T}(u))$

$\leq \delta m(u, u_k) + \alpha\, \beta m(u_{k-1}, u)$.

Thus

$$\lim_{k\to\infty} m(u, \mathcal{T}(u)) \leq \delta \lim_{k\to\infty} m(u, u_k) + \alpha \lim_{k\to\infty} m(u_k, \mathcal{T}(u))$$

$$\leq \delta(0) + \alpha(0) = 0.$$

Hence $m(u, \mathcal{T}(u)) = 0$, we get $\mathcal{T}(u) = u$.

Step 2: The uniqueness

If $\mathcal{T}(p) = p$ and $\mathcal{T}(z) = z$ then

$m(p, z) = m(\mathcal{T}(p), \mathcal{T}(z)) \leq \alpha\, m(p, z)$.

Hence $m(p, z) = 0 \Longrightarrow p = z$ since $\alpha \in (0, 1)$. ∎

Corollary 1:

In Theorem 11, if $\mathcal{T}(u) = u$, then $u_0 \to u$. Thus the error estimates are the prior

$$m(u_j, \mathrm{u}) \leq [\frac{\sigma\alpha^{j}}{1-\alpha}]\, m(u_0, u_1). \qquad (4)$$

And the posterior estimates

$$m(u_j, \mathrm{u}) \leq [\frac{\sigma\alpha}{1-\alpha}]\, m(u_{j-1}, u_j) \qquad (5)$$

Proof:

From Eq. (3),

$m(u_j, u_k) \leq [\sigma\alpha^{j}\frac{(1-\alpha^{k-j})}{(1-\alpha)}]m(u_1, u_0)$.

Using $\alpha \in (0, 1)$ implies $(1 - \alpha^{k-j}) \in (0, 1)$, thus using $k \to \infty$ getting Eq. (4)

$m(u_j, \mathrm{u}) \leq [\frac{\sigma\alpha^{j}}{(1-\alpha)}]\, m(u_0, u_1)$, where $\sigma = \max\{\sigma_j, \sigma_{j+1}, \ldots, \sigma_{k-j}\}$.

Thus Eq. (5) becomes with $j = 1$ and putting y_0 for u_0 and y_1 for u_1, then Eq. (4) implies,

$m(y_1, \mathrm{u}) \leq [\frac{\sigma\alpha}{(1-\alpha)}]\, m(y_0, y_1)$.

Taking $y_0 = u_{j-1}$ implies $y_1 = \mathcal{T}(y_0) = u_j$, and hence Eq. (5) is achieved. ∎

Theorem 12:

If $m(u_0, \mathcal{T}(u_0)) < (1-\alpha)\mu$ and for $\alpha \in (0, 1)$ with $m[\mathcal{T}(u), \mathcal{T}(w)] \leq \alpha\, m(u, w)$ for all $u, w \in \overline{\mathfrak{B}}(u_0, \mu)$ then $(u_k) \to u \in \overline{\mathfrak{B}}(u_0, \mu)$, where $\mathcal{T}(u) = u$ and u is the unique FP of $\mathcal{T}$ in $\overline{\mathfrak{B}}(u_0, \mu)$. When $\mathcal{T}: \mathcal{U} \to \mathcal{U}$ is a function as well as $(\mathcal{U}, m)$ is c-FMS.

Proof:

In Eq. (3) put $j = 0$ and use $m(u_0, \mathcal{T}(u_0)) < (1-\alpha)\mu$ we have

$$m(u_0, u_n) \leq [\sigma/((1-\alpha))]\, m(u_0, u_1) < [\sigma/((1-\alpha))]\,(1-\alpha)\mu < \sigma\mu < \mu.$$

Hence $(u_k) \in \overline{\mathfrak{B}}(u_0, \mu)$ also, $u \in \overline{\mathfrak{B}}(u_0, \mu)$, since $(u_k) \to u$ and $\overline{\mathfrak{B}}(u_0, \mu)$ is CFC. Thus Theorem 11 implies u is the only FP of $\mathcal{T}$ in $\overline{\mathfrak{B}}(u_0, \mu)$. ∎

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Corollary 2:

Let $(\mathcal{U}, m)$ be fuzzy complete c-FMS and $\mathcal{T}:\mathcal{U}\rightarrow\mathcal{U}$ be a function. If $m[\mathcal{T}^k(p), \mathcal{T}^k(y)] \leq \alpha\, m(p, y)$, for all $p, y \in \mathcal{U}$, and some k$\in \mathbb{N}$. Then $\mathcal{T}$ has only one FP..

Proof:

Put $\mathcal{T}^k=\mathcal{L}$, we have $m[\mathcal{L}(p), \mathcal{L}(y)] \leq \alpha\, m(p, y)$, for all $p, y \in \mathcal{U}$, using Theorem 11, $\mathcal{L}(w)=w$. Therefore $\mathcal{L}^n(w)=w$.

Again using Th. 11, $\forall\, u \in \mathcal{U}$, $\mathcal{L}^n(u) \rightarrow w$ when n$\rightarrow \infty$.

Specially, $u=\mathcal{T}(w)$ because $\mathcal{L}^n=\mathcal{T}^{nk}$ implies

$$w=\lim_{n\rightarrow\infty} \mathcal{L}^n\mathcal{T}w = \lim_{n\rightarrow\infty} \mathcal{T}\mathcal{L}^n w = \lim_{n\rightarrow\infty} \mathcal{T}w=\mathcal{T}w.$$

Thus $\mathcal{T}(w)= w$.

But if $\mathcal{T}(w)= w$ implies $\mathcal{L}(w)= w$ hence w is the only fixed point of $\mathcal{T}$. ■

Theorem 13:

If m: $\mathbb{R}^2 \rightarrow$I defined by $m(s, t)= \mathrm{A}_{\mathbb{R}}(s-t)\ \forall\, s, t \in \mathbb{R}$, then $(\mathbb{R}, m)$ is c-FMS.

Proof:

(1) $m(s, t) \in$I, $\forall\, s, t \in \mathbb{R}$ since $\mathrm{A}_{\mathbb{R}}(s-t) \in$I.

(2) $m(s, t)=0 \Leftrightarrow \mathrm{A}_{\mathbb{R}}(s-t)=0 \Leftrightarrow s-t=0 \Leftrightarrow$ if $s=t$.

(3) $m(s, t) =\mathrm{A}_{\mathbb{R}}(s-t) =\mathrm{A}_{\mathbb{R}}(t-s) = m(t, s)$.

(4) $m(s, t)= \mathrm{A}_{\mathbb{R}}(s-t)= \mathrm{A}_{\mathbb{R}}(s-z+z-t) \leq \beta \mathrm{A}_{\mathbb{R}}(s-z) + \gamma \mathrm{A}_{\mathbb{R}}(z-t)$

$\leq \beta m(s, z) +\gamma\, m(z, t)$,

where $\beta, \gamma \in [0, 1]$ with $\beta+ \gamma=1$.

Hence $(\mathbb{R}, m)$ is c-FMS. ■

Theorem 14:

If $m(s, t)= \mathrm{A}_{\mathbb{R}}(s-t)$, $\forall\, s, t \in \mathbb{R}$ then c-FMS, $(\mathbb{R}, m)$ is a fuzzy complete.

Proof:

If (α_{k}) is a fuzzy Cauchy sequence in $\mathbb{R}$, then $A_{\mathbb{R}}\,(\alpha_{\mathrm{m}} - \alpha_{\mathrm{k}}) \leq \mu$, $\forall$ m, k $\geq$N and some $\mu \in(0, 1)$ as well as (α_{k}) has a monotonic subsequence $(\alpha_{k_{\mathrm{j}}})$ but (α_{k}) is a CFB. Hence $(\alpha_{k_{\mathrm{j}}})$ is a CFB and thus $\alpha_{k_{\mathrm{j}}} \rightarrow \alpha \in \mathbb{R}$ that is there is N $\in \mathbb{N}$ with $A_{\mathbb{R}}\,(\alpha_{k_{\mathrm{j}}} - \alpha) \leq \mu$.

Thus for $n \geq N$, $A_{\mathbb{R}}\,(\alpha_{k_{\mathrm{n}}} - \alpha_{\mathrm{n}}) \leq \mu$.

Hence $\forall\, n \geq N$

$$A_{\mathbb{R}}\,(\alpha_n - \alpha) \leq A_{\mathbb{R}}\,(\alpha_n - \alpha_{k_{\mathrm{n}}} + \alpha_{k_{\mathrm{n}}} - \alpha) \leq \beta\, A_{\mathbb{R}}\left(\alpha_n - \alpha_{k_{\mathrm{n}}}\right) +\gamma\, A_{\mathbb{R}}\,(\alpha_{k_{\mathrm{n}}} - \alpha)$$

$$\leq \beta\mu +\gamma\, \mu=(\beta+\gamma)\mu = \mu.$$

Therefore, $\alpha_n \rightarrow \alpha \in \mathbb{R}$. Thus $(\mathbb{R}, m)$ is a fuzzy complete. ■

Theorem 15:

If m: $\mathbb{R}^n \rightarrow$I is defined by $m(p, q)= max_k\, \mathrm{A}_{\mathbb{R}}(\sigma_k - \rho_k)$ for all $p =(\sigma_1, \sigma_2, \ldots, \sigma_n)$,

$q =(\rho_1, \rho_2, \ldots, \rho_n) \in \mathbb{R}^n$. Then $(\mathbb{R}^n, m)$ is c-FMS.

Proof:

(1) $m(p, q) \in$I for all $p, q \in \mathbb{R}^n$.

(2) $m(p, q)=0 \Leftrightarrow max_k\, \mathrm{A}_{\mathbb{R}}(\sigma_k - \rho_k)=0 \Leftrightarrow \mathrm{A}_{\mathbb{R}}(\sigma_k - \rho_k)=0$,

$\forall$ k=1, 2, …,n $\Leftrightarrow \sigma_k - \rho_k=0$, $\forall$ k=1, 2, …,n $\Leftrightarrow \sigma_k = \rho_k$, $\forall$ k=1, 2, …,n $\Leftrightarrow p = q$.

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(3) $m(p, q)$ $=max_k$ $A_{\mathbb{R}}(\sigma_k - \rho_k)$ $=max_k$ $A_{\mathbb{R}}(\rho_k - \sigma_k) = m(q, p)$.

(4) $m(p, q) = max_k \; A_{\mathbb{R}}(\sigma_k - \rho_k) = max_k \; A_{\mathbb{R}}(\sigma_k - \mu_k + \mu_k - \rho_k)$

$\leq max_k \; \beta \; A_{\mathbb{R}}(\sigma_k - \mu_k) + max_k \; \gamma \; A_{\mathbb{R}}(\mu_k - \rho_k)$

$= \beta \; max_k \; A_{\mathbb{R}}(\sigma_k - \mu_k) + \gamma \; max_k \; A_{\mathbb{R}}(\mu_k - \rho_k)$

$\leq \beta \; m(p, z) + \gamma \; m(z, q)$,

where $z = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n$. Hence $(\mathbb{R}^n, m)$ is c-FMS. ■

Theorem 16:

If m: $\mathbb{R}^n \to$I, is defined by $m(p, q) = max_k \; A_{\mathbb{R}}(\sigma_k - \rho_k)$ for all $p = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, $q=$

$(\rho_1, \rho_2, \ldots, \rho_n) \in \mathbb{R}^n$ then the c-FMS $(\mathbb{R}^n, m)$ is fuzzy complete

Proof:

Let (u_k) be a fuzzy Cauchy sequence in $\mathbb{R}^n$ and let $u_k = (\mu_{1k}, \mu_{2k}, \ldots, \mu_{nk})$, for $k = 1,2,\cdots$. Using (u_k) is fuzzy Cauchy then for all $\sigma \in (0, 1)$, there exist N∈ ℕ with

$$m(u_k, u_j) = max_i \; A_{\mathbb{R}}(\mu_{ik} - \mu_{ij}) < \sigma \qquad (6)$$

For all $k, j \geq N$ and $i = 1, 2, \ldots, n$. It follows $A_{\mathbb{R}}(\mu_{ik} - \mu_{ij}) < \sigma$.

Thus for i=1,2,… the sequence $(\mu_{i1}, \mu_{i2}, \ldots)$ is a fuzzy Cauchy sequence in $(\mathbb{R}, m)$. Using $\mathbb{R}$ is fuzzy complete by Theorem 14, then $\mu_{ik} \to \mu_i \in \mathbb{R}$ as k→ ∞.

Now, put $u = (\mu_1, \mu_2, \ldots, \mu_n)$ this implies $u \in \mathbb{R}^n$. From Eq. (6) by letting $j \to \infty$, we have

$m(u_k, u) = m(u_k, \lim_{j\to\infty} u_j) = max_i \; A_{\mathbb{R}}(\mu_{ik} - \mu_i) < \sigma$

Thus $u_k \to$ u. Therefore $(\mathbb{R}^n, m)$ is fuzzy complete. ■

Theorem 17:

If $U = C[a, b]$, then (U, m) is fuzzy complete where $m(g, h) = max_{t\in[a,b]} \; A_{\mathbb{R}}[g(t) - h(t)]$

Proof:

Consider a fuzzy Cauchy sequence (ℓ_k) in $C[a, b]$, therefore ∀ 0< μ <1, there exist $N \in$ ℕ satisfying for $k, n \geq N$

$m(\ell_k, \ell_n) = max_{t\in[a,b]} \; A_{\mathbb{R}}[\ell_k(t) - \ell_n(t)] < \mu$. (7)

Thus for fixed $= t_0 \in [a, b]$, $A_{\mathbb{R}}[\ell_k(t_0) - \ell_n(t_0)] < \mu$. Hence $(\ell_1(t_0), \ell_2(t_0), \ldots)$ is fuzzy Cauchy sequence in $(\mathbb{R}, m)$. Using $(\mathbb{R}, m)$ is fuzzy complete by Theorem 14, then $\ell_k(t_0) \to \ell(t_0)$. Hence ∀ $t \in [a, b]$ ∃ a unique $\ell(t) \in \mathbb{R}$. Hence ℓ : [a, b] → ℝ and to prove $\ell \in C[a, b]$ and $\ell_k \to \ell$. From Eq. (7) with $n \to \infty$ we have $max_{t\in[a,b]}$ $A_{\mathbb{R}}[\ell_k(t) - \ell(t)] < \mu$, ∀ $k \geq N$. Thus ∀ $t \in [a, b]$, $A_{\mathbb{R}}[\ell_k(t) - \ell(t)] < \mu$, ∀ $k \geq N$.

Therefore ℓ_k(t)→ ℓ(t) uniformly on $[a, b]$. Because all $(\ell_k) \in C[a, b]$, hence $\ell \in C[a, b]$. As well as $\ell_k \to \ell$. Therefore the prove is complete. ■

4. Applications

Definition 15:

If $(\mathbb{R}, A_{\mathbb{R}})$ is c-FAVS and $A_{\mathbb{R}}[h(\tau) - h(\rho)] \leq k \; A_{\mathbb{R}}(\tau - \rho)$, ∀ $\tau, \rho \in \mathbb{R}$ then the function h: ℝ → ℝ is said to be satisfying a fuzzy Lipschitz condition.

Theorem 18:

Let h:D→ $\mathbb{R}^2$ be fuzzy continuous function where $D = \{(t, u): |t - t_0| \leq \alpha, |u - u_0| \leq \varepsilon\}$and $A_{\mathbb{R}}(h(t, u)) \leq \mu$ for all $(t, u) \in D$. If $A_{\mathbb{R}}[h(t, u) - h(t, v)] \leq k \; A_{\mathbb{R}}(u - v)$, ∀ $(t, u), (t, v) \in D$ then the initial value problem $\frac{du}{dt} = h(t, u)$, $u(t_0) = u_0$ has only one solution in $[t_0 - \delta, t_0 + \delta]$ where $\delta < \min\{\alpha, \frac{\varepsilon}{\mu}, \frac{1}{k}\}$.

Proof:

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Consider the c-FMS $C(J)$ be the c-FMS when $J=[t_0-\delta, t_0+\delta]$ with $m(f, g)= max_{\tau\in J} A_{\mathbb{R}}[f(\tau)-g(\tau)]$. Then by Theorem 16 and Theorem 17, (C(J), m) is fuzzy complete c-FMS. Let $\acute{C}=\{\ \in C(J)$: $A_{\mathbb{R}}(u(t)-u_0)\le \mu\delta\ \}\subseteq C(J)$, as well as $\acute{C}$ is convex fuzzy closed in $C(J)$. $\acute{C}$ is fuzzy complete.

$\frac{du}{dt}= h(t,u)$ after integration is $u=\mathcal{T}(u)$ with $\mathcal{T}: \acute{C}\to\acute{C}$, $\mathcal{T}(u(t))=u_0+\int_{t_0}^{t} h(\tau,u(\tau))d\tau$. Hence $\mathcal{T}$ is defined $\forall\ u\in\acute{C}$ because $<\varepsilon$, therefore, if $u\ \in\acute{C}$ then $\rho\in J$ and $(\rho, u(\rho))\in\ D$.

Since $A_{\mathbb{R}}[\mathcal{T}(u(t))-u_0]\le\ A_{\mathbb{R}}[\int_{t_0}^{t} h(\tau,u(\rho)d\rho]\le\mu A_{\mathbb{R}}[t-t_0]\le\mu\delta$. Then, $\mathcal{T}: \acute{C}\to\acute{C}$.

And

$$A_{\mathbb{R}}[\mathcal{T}(u(t))-\mathcal{T}(v(t)] = A_{\mathbb{R}}[\int_{t_0}^{t}[h(\rho,u(\rho))-h(\rho,v(\rho)]d\rho$$

$$\le A_{\mathbb{R}}[t-t_0]\ max_{\rho\in J} k\ A_{\mathbb{R}}[u(\rho)-\ v(\rho)]$$

$$\le k\beta m(u,v).$$

Taking the maximum on the left, we get $m[\mathcal{T}(u),\mathcal{T}(v)]\le\lambda\ m(u,v)$ where $\lambda=k\beta$. By Theorem 11, $\mathcal{T}$ has one FP $\in\acute{C}$, $u=\ \mathcal{T}(u)$. Which means that

$$u(t) = u_0 + \int_{t_0}^{t} h(\tau,u(\tau)d\tau \qquad (8)$$

Because $(\rho,u(\rho))\in D$, then from Eq. (8) we have $\frac{du}{dt}= h$ (t, u), $u(t_0)= u_0$.

For the converse any solution of $\frac{du}{dt}= h(t,u)$, $u(t_0)= u_0$ must satisfy Eq. (8). ∎

Theorem 19:

If $u=Cu+d$, $(C=(c_{ik}))$, d is given) the system of n linear equations in

$\alpha_1, \alpha_2, \ldots, \alpha_n$, ($u=(\alpha_1, \alpha_2, \ldots, \alpha_n)$) full fits

$\sum_{k=1}^{n} A_{\mathbb{R}}\ (c_{ik}) <1\ (i=1,2,\ldots,n)$.

Then it has unique solution u when $u^{(i)}\to u$ and $u^{(i+1)}= Cu^{(i)}+ d$, ($i=0,1,\ldots$).

where

$m(u^{(i)},u)\le[(\frac{\sigma\alpha}{1-\alpha})]$ and $m(u^{(i-1)}, u^{(j)})\ \le[(\frac{\sigma\alpha^{i}}{1-\alpha})]m(u^{(0)}, u^{(1)})$

are error bounds.

Proof:

Let $p=(\mu_1, \mu_2, \ldots, \mu_n)$, $q=(\sigma_1, \sigma_2, \ldots, \sigma_n)\in\mathbb{R}^n$. From Theorem 15 and Theorem 16, we get $(\mathbb{R}^n, m)$ is fuzzy complete c-FMS. Let us defined $\mathcal{T}: \mathbb{R}^n\to\mathbb{R}^n$ by $p=\mathcal{T}(u)=Cu+b$ where $C=(c_{jk})\in M_n(\mathbb{R})$ and $d\in\mathbb{R}^n$.

To prove $\mathcal{T}$ is a FC, $\mu_i=\sum_{k=1}^{n} c_{ik}\ \alpha_i+\tau_i$ where $d=(\tau_1, \tau_2, \ldots, \tau_n)$. Take $S(w)=z$ with $z=(\delta_1, \delta_2, \ldots, \delta_n)$.

Now

$m(p, z)=\ m(\mathcal{T}(u),\mathcal{T}(w))=max_j A_{\mathbb{R}}(\mu_j-\delta_j) = max_j A_{\mathbb{R}}(\sum_{k=1}^{n} c_{jk}\ (\alpha_j-\delta_j)$

$\le max_j A_{\mathbb{R}}(\alpha_j-\delta_j)\ max_j(\sum_{k=1}^{n} A_{\mathbb{R}}(c_{jk}))$.

Thus $m(p, z)\le\alpha\ m(u, w)$ where $\alpha=max_j(\sum_{k=1}^{n} A_{\mathbb{R}}(c_{jk}))$.

Hence, the other steps holds by applying Theorem 11. ∎

Theorem 20:

If v in Volterra integral equation

$$v(t)\ = u(t) - \lambda\int_{a}^{t} k(t,\tau)u(\tau)d\tau \qquad (9)$$

$v: J\to\mathbb{R}$ where $J=[a,b]$ is fuzzy continuous and $k: R\to\mathbb{R}^2$ where $R=\{(t,\tau)$: $a\le\tau\le t, a\le t\le b\}$ is fuzzy continuous. Then Eq. (9) has one solution u on J for every λ.

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Proof:

At first, let us rewrite Eq. (9) by $u = \mathcal{T}(u)$ with $\mathcal{T}: C(J) \to C(J)$ defined by

$\mathcal{T}(u(t)) = v(t) + \lambda \int_a^t k(t,\tau)u(\tau)d\tau$.

Due to k is fuzzy continuous on $\mathbb{R}$ and $\mathbb{R}$ is convex fuzzy closed also, $A_{\mathbb{R}}(k(t,\tau)) \leq \varepsilon$ for all $(t,\tau) \in \mathbb{R}$. Thus $\forall\, u, v \in C(J)$

$$A_{\mathbb{R}}[\mathcal{T}u(t) - \mathcal{T}v(t)] = A_{\mathbb{R}}(\lambda)\, A_{\mathbb{R}}[\int_a^t k(t,\tau)(u(\tau) - v(\tau))d\tau]$$

$$\leq A_{\mathbb{R}}(\lambda)\, \varepsilon\, m(u,v) \int_a^t d\tau = A_{\mathbb{R}}(\lambda)\, \varepsilon\, m(u,v)\,(t-a) \qquad (10)$$

By induction, we can prove that

$$A_{\mathbb{R}}\,[\mathcal{T}^k u(t), \mathcal{T}^k v(t)] \leq A_{\mathbb{R}}(\lambda)^{\,k} \varepsilon^k \frac{(t-a)^k}{k!} m(u,v) \qquad (11)$$

When putting $k = 1$ in the equation above, we can get Eq. (10).

Assume that the equation (11) holds for any k we obtain

$$A_{\mathbb{R}}\,[\mathcal{T}^{k+1}u(t), \mathcal{T}^{k+1}v(t)] = A_{\mathbb{R}}(\lambda)\, A_{\mathbb{R}}[\int_a^t k(t,\tau)(\mathcal{T}^k u(\tau) - \mathcal{T}^k v(\tau))d\tau]$$

$$\leq A_{\mathbb{R}}(\lambda)\, \varepsilon \int_a^t A_{\mathbb{R}}(\lambda)^{\,k} \varepsilon^k \frac{(t-a)^k}{k!} d\tau\; m(u,v)$$

$$\leq A_{\mathbb{R}}(\lambda)^{\,k+1} \varepsilon^{k+1} \frac{(t-a)^{k+1}}{(k+1)!} m(u,v).$$

Then

$m(\mathcal{T}^k u, \mathcal{T}^k v) \leq r_k m(u,v)$ where $r_k = A_{\mathbb{R}}(\lambda)^{\,k} c^k \frac{(t-a)^k}{k!}$.

Follows by taking the maximum over $t \in J$ on the left we obtain from Eq. (11)

For λ fixed and k large implies $r_k < 1$. Thus $\mathcal{T}^k$ is a FC on $C(J)$. The other steps of the proof holds by Theorem 11.

∎

Theorem 21:

Let $k: J \times J \to \mathbb{R}$ and $v : J \times J \to \mathbb{R}$ in the Fredholm integral equation

$$v(t) = u(t) - \mu \int_d^b k(t,\tau)u(\tau)d\tau \qquad (12)$$

be fuzzy continuous where $J = [d,b]$ and let μ satisfies

$$A_{\mathbb{R}}(\mu) < \frac{1}{\varepsilon(b-d)}, \qquad (13)$$

Where

$$A_{\mathbb{R}}(k(t,\tau)) \leq \varepsilon, \forall\, (t,\tau) \in G. \qquad (14)$$

This implies that Eq. (12) has one solution u on J, where $(u_0, u_1, \ldots) \to u$, when $u_0 : J \to \mathbb{R}$ is any fuzzy continuous for $n = 0, 1, \ldots$.

$$u_{n+1}(t) = v(t) + \mu \int_d^b k(t,\tau)u_n(\tau)d\tau \qquad (15)$$

Proof:

Let $v \in C[d,b]$ rewriting Eq. (12) by $\mathcal{T}(u) = u$ where

$$\mathcal{T}(u) = v(t) + \mu \int_d^b k(t,\tau)u(\tau)d\tau \qquad (16)$$

Hence

$$m(\mathcal{T}(u), \mathcal{T}(y)) = max_{t\in J}\, A_{\mathbb{R}}[\mathcal{T}u(t) - \mathcal{T}y(t)]$$

$$= A_{\mathbb{R}}(\mu)\, max_{t\in J} A_{\mathbb{R}}[\int_d^b k(t,\rho)[u(\rho) - y(\rho)]d\rho$$

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$$\leq A_{\mathbb{R}}(\mu)\ max_{t\in J} \int_d^b A_{\mathbb{R}}\{k(t,\rho)[u(\rho)-y(\rho)]\}d\rho$$

$$\leq A_{\mathbb{R}}(\mu)\ max_{t\in J} \int_d^b A_{\mathbb{R}}\{k(t,\rho)\}A_{\mathbb{R}}\{[u(\rho)-y(\rho)]\}d\tau$$

$$\leq A_{\mathbb{R}}(\mu)\ \varepsilon\ max_{\theta\in J}\ A_{\mathbb{R}}[u(\theta) - y(\theta)] \int_d^b d\rho$$

$$\leq A_{\mathbb{R}}(\mu)\ \varepsilon\ m(u,y)(b-d).$$

Rewrite this by $m(\mathcal{T}(u), \mathcal{T}(y)) \leq r\ m(u,y)$ where $r = A_{\mathbb{R}}(\mu)\ \varepsilon(b-d)$. Thus $\mathcal{T}$ becomes a fuzzy contraction from Eq. (13). Using Theorem 11, $\mathcal{T}$ has a one FP u on J. Where the iterative sequence $(u_0, u_1, \ldots)$ converge to u where u_0 is any initial solution on J for $n = 0, 1, \ldots$

$u_{n+1}(t) = v(t) + \mu \int_d^b k(t,\tau)u_n(\tau)d\tau.$

5. Conclusion

Here we have proved that the fixed point theorem for a single valued function in the convex fuzzy metric space. We also applied this theorem to find the solution for Fredholm integral equation and Volterra integral equation. For future work the authors can be prove the fixed theorem for multi-variable function in this space.

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