Neutrosophic Lognormal Distribution with Applications in Complex Data Modeling

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Abstract

This study develops a new version of the lognormal distribution, the neutrosophic lognormal distribution (NLND), to address uncertainties commonly exist in reliability studies within the engineering field. The NLND is suitable for analyzing complex data with symmetrical or right-skewed patterns. The paper discusses the mathematical characteristics of the NLND, including concepts of reliability like mean time failure, hazard rate, cumulative failure rate, and reliability function. The model is based on real-life examples from life-test data and uses the maximum likelihood method to determine two key parameters. A simulation experiment was conducted to evaluate the accuracy of the estimated parameters, showing that maximum likelihood estimators can effectively estimate unknown parameters, especially with a large sample size. Finally, a real-world data is used to demonstrate the adequacy of the proposed model in a practical scenario.

Keywords: Neutrosophic probability; Neutrosophic distribution; Lognormal model; Estimation

1. Introduction

Reliability study of electronic components and devices necessitates the identification of failure distributions and the evaluation of failure mechanisms [1]. Diverse lifetime models, encompassing growing, decreasing, and continual failing patterns, are suggested to delineate product behavior [2]. The lognormal distribution is frequently employed in dependability fields because of its capacity to accommodate both symmetric and skewed data [3]. The lognormal model is employed to forecast the failure time of systems or components subjected to stress or fatigue, particularly in contexts involving chemical reactions or degradation, such as diffusion, corrosion, or migration [4]. The lognormal model has some improvements over previous software reliability models [5]. It is statistically classified within the Gaussian family and is well documented in reliability and statistical distribution literature for its accurate data assumptions and defining parameters [6]. There are occasions when the lifespan data derived from investigations does not conform to conventional lifetime models [7]. Consequently, it is essential to investigate different statistical distributions to precisely simulate the failure causes of diverse goods [8]. This work presents an innovative modification of the lognormal distribution to improve its use in empirical statistical research [9]. This expansion is inspired by Smarandache's investigation into neutrosophy. By integrating neutrosophic logic, scholars may now analyze propositions that exist between true and wrong, introducing a novel philosophy of complexity to their evaluations . In the realm of mathematics, indeterminacy exists across several disciplines, essentially functioning as a neutrosophic element [11]. Smarandache was among the pioneers in applying neutrosophic approaches to fields such as statistics, precalculus, and calculus to address data errors [12]. Neutrosophic statistics (NS) has been a prominent subject of interest among scholars due to its emphasis on analyzing and comprehending indeterminacy in statistical models [13-15]. Neutrosophic statistics demonstrates superior flexibility relative to classical statistics. When data and inference procedures are conclusive, neutrosophic statistics corresponds with classical statistics [16-18]. The notion of statistical modeling within a neutrosophic framework has just lately been examined in several works. Recent work also discusses methods for data description utilizing neutrosophic descriptive techniques and probability. Neutrosophy has demonstrated efficacy in quality control decision-making, as evidenced by several research [19-23]. Salama et al. were the pioneers in introducing

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neutrosophic algebraic structures into probability models [24]. Although the applications of neutrosophy have been the main emphasis, there is little examination of the algebraic structures of probability distributions in the subject [25]. This research aims to introduce a unique neutrosophic adaptation of the lognormal distribution.

This study elucidates the NLND idea by considering imprecise data on study variables. Incorporating unclear factors is essential for precise analysis in depicting a data-generating process. Prior research has not investigated the neutrosophic framework of the lognormal model, rendering this study distinctive in its methodology.

The study is organized as follows: In Section 2, the lognormal distribution is augmented to incorporate neutrosophic characteristics with some statistical properties. Section 3 delineates the process for ascertaining unknown NLND parameters, accompanied by a simulation study. Section 4 presents the derivation of the quantile function within a neutrosophic framework. Section 5 discusses an application of the proposed model on CO2 emissions data whereas Section 6 summarizes the study findings.

2. Proposed Model

A random variable Z said to follow NLND if the random variable $\tilde{T} = \ln \tilde{Z}$ follows a neutrosophic normal model. The probability density function of the NLND is provided by:

$$h_n(\tilde{t}) = \frac{1}{\sqrt{2\pi}\,\tilde{\sigma_n}\tilde{z}} exp\left(-\frac{(ln\tilde{z}- \theta_n)^2}{2\sigma^2{}_n}\right); \;\; \tilde{z}, \theta_n\,, \;\sigma_n > 0 \qquad (1)$$

The proposed model has two parameters with indeterminate components; $\theta_n = [\theta_l, \theta_u]$ is the neutrosophic location, $\sigma_n = [\sigma_l, \sigma_u]$. The parameters θ_n and σ_n known as the location and scale parameters of proposed model. For different values of the indeterminate parmeters the distribution of proposed NLND can be drawn. If we assume that $\theta_n \in [0, 0]$ and $\sigma_n \in [0.25, 1.3]$, shape of the Pdf can be depicted as seen in Figure 1.

The general statistical pattern of the Pdf is shown in Figure 1, where it is assumed that the l shape factors of the distribution are not known precisely. Due to errors in the values used to describe the distribution, the shaded area in Figure 1 shows the neutrosophic region. In Figure 1, the failure chance of a system that is thought to follow the NLND statistical pattern can be found between two points in the area under the neutrosophic curve.

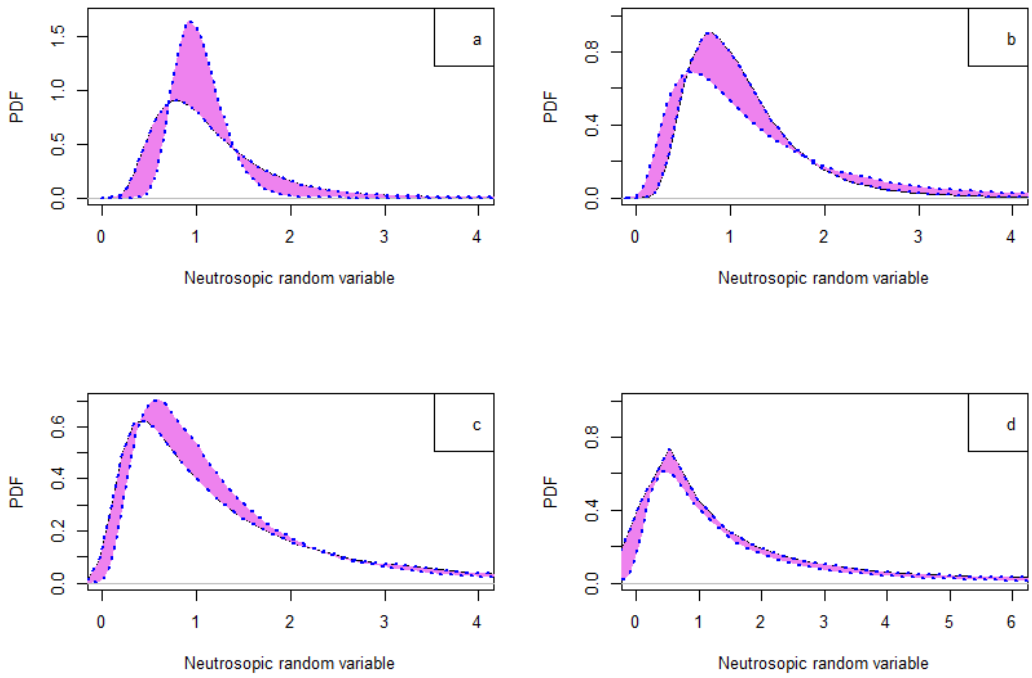

Figure 1. Density plot of the proposed NLND at (a) $\sigma_n = [0.25, 0.5]$, (b) $\sigma_n \in [0.5, 0.75]$, (c) $\sigma_n \in [0.75, 1]$, and (d) $\sigma_n \in [1, 1.3]$

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Furthermore, another significant role of the NLND is the neutrosophic cumulative function (CDF). The Cdf is a statistical measure used to calculate the likelihood that a certain number of operational objects will fail within the specified period. The critical value of the NLND is determined by the Cdf function given below:

$$\mathrm{H}(\tilde{z}) = \varphi\left(\frac{\ln(\tilde{z}) - \theta_n}{\sigma_n}\right) \quad (2)$$

Under zero indeterminacy, the cumulative function converts to classical structure of the CDF of the lognormal distribution.

The shape of the CDF for some set of imprecise values of location and scale parameters of the proposed distribution is shown in Figure 2.

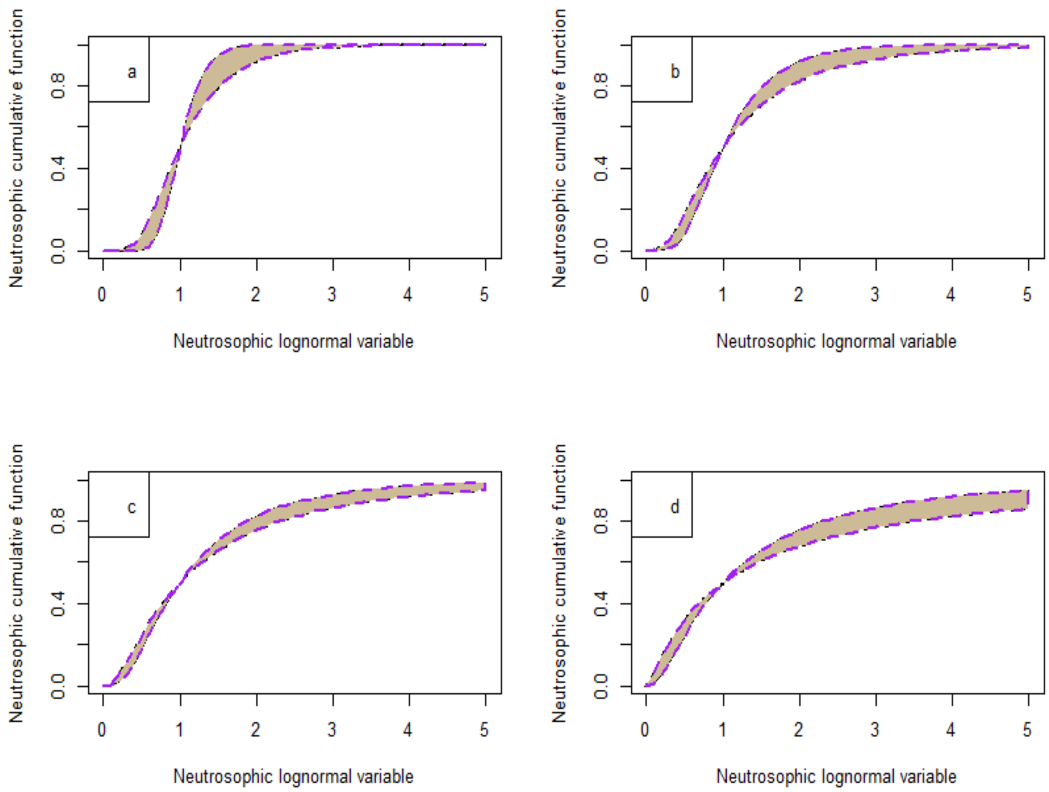

Figure 2. Cumulative function of the proposed distribution at a) $\sigma_n = [0.25, 0.5]$, (b) $\sigma_n \in [0.5, 0.75]$, (c) $\sigma_n \in [0.75, 1]$, and (d) $\sigma_n \in [1, 1.3]$

Figure 2 shows that curves are thick area that reflects the uncertainty, indeterminacy and potential contradiction. It provides a structure that can help in more complex situations modeling where traditional approach of CDF cannot capture the uncertain aspect of the situation.

Neutrosophic data is an extension of usual data that includes imprecise, ambiguous, or indeterminate values in some or all its observations. Broadly speaking, it may be expressed as:

$x = c + v,$

where c is constant part of the number and $v \in [x_L, x_U]$ is indeterminate part of the number for example, $10 + I$ where $I \in [4, 4.75]$. Apart from the distinct patterns of the components or system reliability that are most effectively NLD in the form PDF and CDF, a practitioner may also be interested in understanding other advantageous distributional characteristics of the NLD, which may be developed in theorems.

Theorem 1. The median of the NLD is $[\, exp(\mu_l),\ exp(\mu_u)]$.

Proof The median $(\widetilde{\mathcal{M}})$ of NLD can be found as:

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$$\int_0^m \mathrm{H}(\tilde{z})d\tilde{t} = \frac{1}{2}$$

$$\left[\int_0^m \mathrm{H}_l(z)d\tilde{z},\ \int_0^m \mathrm{H}_u(\tilde{z})d\tilde{z}\right] = \frac{1}{2} \quad (3)$$

with $\mathrm{H}_L(\tilde{z}) = \varphi\left(\frac{\ln(\tilde{z}) - \theta_L}{\sigma_L}\right)$ and $\mathrm{H}_U(z) = \varphi\left(\frac{\ln(z) - \theta_U}{\sigma_U}\right)$

Simplification further to (4) yielded:

$$m = [\, exp(\theta_L),\ \ exp(\theta_U)]$$

Theorem 2. Derive the expressions for first and third quartiles of the NLND

Proof The following expressions can be used to derived the first and third quartiles of the NLND:

$$\int_0^{q_1} \mathrm{H}(\tilde{z})d\tilde{z} = \frac{1}{4} \quad (4)$$

$$\int_0^{q_3} \mathrm{H}(\tilde{z})d\tilde{z} = \frac{3}{4} \quad (5)$$

Further simplification using the definition of CDF yielded:

$$q_1 = \ 1 - \int_0^{1/4} H(\tilde{z})d\tilde{z} = \left[exp\left(\theta_L + \sigma_L\varphi^{-1}(1/4)\right), exp\left(\theta_U + \sigma_U\varphi^{-1}(1/4)\right)\right]$$

$$q_3 = \ 1 - \int_0^{3/4} H(\tilde{z})d\tilde{z} = \left[exp\left(\theta_L + \sigma_L\varphi^{-1}(3/4)\right), exp\left(\theta_U + \sigma_U\varphi^{-1}(3/4)\right)\right]$$

which are required quantiles of the proposed distribution.

Theorem 3. Derive the mean of the proposed NLND

Proof By definition neutrosophic mean of the continuous random variable is defined as:

$$\mu_N = \int_0^{\infty}\{\tilde{t}h_U(\tilde{z}), \tilde{t}h_L(\tilde{z})\}d\tilde{z} \quad (6)$$

$$= \left\{\int_0^{\infty} \tilde{z}\frac{1}{\sqrt{2\pi}\,\sigma_L\tilde{z}}e^{\left(-\frac{(ln\tilde{z}-\theta_L)^2}{2\sigma^2{}_L}\right)}d\tilde{z},\ \int_0^{\infty} \tilde{z}\frac{1}{\sqrt{2\pi}\,\sigma_u\tilde{z}}e^{\left(-\frac{(ln\tilde{z}-\theta_U)^2}{2\sigma^2{}_U}\right)}d\tilde{z}\right\} \quad (7)$$

Solving (7) provides:

$$= \left[e^{\left(\theta_L+\frac{\sigma_L{}^2}{2}\right)}, e^{\left(\theta_U+\frac{\sigma_U{}^2}{2}\right)}\right]$$

$$\mu_N = e^{\left(\theta_N+\frac{\sigma_N^2}{2}\right)} \quad (8)$$

Theorem 4. The variance of the NLND is $\left(e^{\sigma_n^2} - 1\right)e^{2\mu_n+\sigma_n^2}$

Proof By definition, variance is

$$\sigma_N{}^2 = E(\tilde{z}^2) - \mu_N{}^2 \quad (9)$$

where $\sigma_N{}^2$ is statistical measure for finding variability of the neutrosophic data

$$E(\tilde{z}^2) = \int_0^{\infty} z^2h_n(\tilde{z})d\tilde{z} \quad (10)$$

Solving (10) provides:

$$E(\tilde{z}^2) = \left[\int_0^{\infty} \check{z}\frac{1}{\sqrt{2\pi}\,\sigma_L}e^{\left(-\frac{(ln\tilde{z}-\theta_L)^2}{2\sigma^2{}_L}\right)}d\tilde{z},\ \int_0^{\infty} \check{z}\frac{1}{\sqrt{2\pi}\,\sigma_U}e^{\left(-\frac{(ln\tilde{z}-\theta_U)^2}{2\sigma^2{}_U}\right)}d\tilde{z}\right] \quad (11)$$

Let $\tilde{Z} = \frac{ln(\tilde{Z})-\theta_N}{\sigma_n}$

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Solving (11) provides:

$$= \left[e^{\left(2\sigma_L{}^2+2\theta_L\right)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\ exp\left(\frac{-(\tilde{z}-2\sigma_L)^2}{2}\right) d\tilde{z}\ , e^{\left(2\sigma_u{}^2+2\theta_u\right)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\ exp\left(\frac{-(\tilde{z}-2\sigma_U)^2}{2}\right) d\tilde{z} \right]$$

$$= [exp\ (2\sigma_L{}^2+2\theta_L), exp\ (2\sigma_U{}^2+2\theta_U)]$$

$$= \ exp\ (2\sigma_N{}^2+2\theta_N)\ (12)$$

Using (12) in (10) yielded:

$$\sigma_N{}^2 = (\exp(\sigma_N^2)-1)\exp(2\theta_N+\sigma_N^2)\ (13)$$

which is required expression for variance. Likewise the others properties of the proposed model can be derived under the neutrosophic framework.

3. Estimation Framework

Numerous methods of estimation in classical statistics are used to find the unknown values of the underlying model. However, maximum likelihood (ML) approach is commonly recommended due to excellent theoretical properties of the estimated parameters. The Neutrosophic ML (NML) approach is an extension of the classical method, adapted to deal with uncertainty exist in analyzing data. This approach is adapted for scenarios where exact statistical figures are not available. Under this approach, the likelihood of observing data is maximized to find inexact values of unknown parameters. To estimate the distributional parameters of NLND, the NLM method has been devised in this section. Consider a subset of the NLND 's values. For a given sample, which neutrosophic parameter values are appropriate can be obtained by maximizing the following likelihood function:

$$\mathrm{L}_N\left(\theta_N, \sigma^2{}_N \middle| \widetilde{observed\ sample}\right) = \prod_{i=1}^{n} h_N(\tilde{z}_i) \qquad (13)$$

For lognormal realizations $\tilde{z}_i(i = 1, 2, , \ldots n)$, (13) further can be written as:

$$\mathrm{L}_N\left(\theta_N, \sigma^2{}_N \middle| \widetilde{observed\ sample}\right) = \frac{\sum_{i=1}^{n} ln(\tilde{z}_i)}{\sigma^2{}_n} - \frac{\sum_{i=1}^{n} ln(\tilde{z}_i{}^2)}{2\sigma^2{}_n} + \sum_{i=1}^{n} ln\ (\tilde{z}_i) - \frac{n\theta_n{}^2}{2\sigma^2{}_n} - \frac{nln(2\pi\sigma^2{}_n)}{2}\ (14)$$

The partial derivatives of (14) with respective to unknown parameters μ_N and $\sigma^2{}_N$ is given by:

$$\frac{\partial \mathrm{L}_N\left(\theta_N, \sigma^2{}_N \middle| \widetilde{observed\ sample}\right)}{\partial \mu_N} = -\frac{2n\theta_n}{2\sigma^2{}_n} + \frac{\sum_{i=1}^{n} ln(\tilde{z}_i)}{\sigma^2{}_n} \qquad (15)$$

$$\frac{\partial \mathrm{L}_N\left(\theta_N, \sigma^2{}_N \middle| \widetilde{observed\ sample}\right)}{\partial \sigma^2{}_N} = \frac{\sum_{i=1}^{n}(ln(\tilde{z}_i)-\theta_N)^2}{2(\sigma^2{}_N)^2} - \frac{n}{2\sigma^2{}_N} \qquad (16)$$

Equating the gradient to zero yield the following solution:

$$\widehat{\sigma}^2{}_n = \frac{\sum_{i=1}^{n}\left(ln(\tilde{z}_i) - \frac{\sum_{j=1}^{n} ln(\tilde{z}_i)}{n}\right)^2}{n} \qquad (17)$$

$$\hat{\theta}_N = \frac{\sum_{i=1}^{n} ln(\tilde{z}_i)}{n} \qquad (18)$$

where $\widehat{\sigma}^2{}_N = [\widehat{\sigma}^2{}_L, \widehat{\sigma}^2{}_U]$ and $\hat{\theta}_N = [\hat{\theta}_L, \hat{\theta}_U]$are two respective estimators of μ_n and $\sigma^2{}_N$.

The efficacy of these unknown parameters is assessed by the neutrosophic average biased and neutrosophic root mean square error as delineated below:

$$MSE = \sqrt{\frac{\sum_{i=1}^{N}\left(\hat{\vartheta}_i - \vartheta_N\right)^2}{N}} \qquad (19)$$

$$Average\ Bias = \frac{\sum_{i=1}^{N}(\hat{\vartheta}_j - \vartheta_N)}{N} \qquad (20)$$

where ϑ_N is the true value of estimator in repeated sampling and $\hat{\vartheta}_i$ is estimated value of neutrosophic parameter for ith run $\{i = 1,2, \ldots N\}$.

An example of a Monte Carlo simulation is carried out with varying sample sizes and constant values for the parameters. This simulation is carried out with the computer language R using $\theta_N = [0.25,0.75]$ and shape parameter $\sigma_N^2 = [0, 0]$. A total of 10000 simulation runs were performed. After that, the performance measures of the unknown parameters are computed and presented in Table 1.

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Table 1: Estimation results from a simulated data from the proposed model

Sample size	Location estimation		Scale estimation	
	Average Bias	MSE	Average Bias	MSE
15	[4.79,35.37]	[6.29,46.45]	[0.12, 0.20]	[0.13,0.53]
30	[4.81,35.51]	[5.62,41.53]	[0.12, 0.15]	[0.13,0.42]
50	[4.81,35.54]	[5.31 39.26]	[0.12, 0.13]	[0.12,0.36]
80	[4.80,35.46]	[5.12,37.85]	[0.11,0.12]	[0.12,0.30]
120	[4.78 35.34]	[5.00,36.93]	[0.10,0.12]	[0.12,0.26]
300	[4.79,35.36]	[4.87,36.01]	[0.9,0.12]	[0.12,0.18]

The results presented in Table 1 demonstrate that both average bias and MSE decrease as the sample size increased. Simulations are run both for location and scale parameters with values $\theta_N = [0\ ,0]$ and $\sigma_N^2 = [0.25, 0.75]$ and $\theta_N = [2\ ,4]$ and $\sigma_N^2 = [1, 1]$. This illustrates that neutrosophic estimators offer superior reliability and efficiency with larger sample sizes. This means that sample size selection plays a key role in the reliable estimation of required parameters.

4. Random Sample Generation

This is a process of producing numbers from the data generated model in a way that cannot reasonably generate through a specified model but comes through random chance. We do this random generation using simulation approach and sampling method. In statistical applications most random generations are carried out using algorithms. Such types of random generations are also known as pseudo random generation. Commonly approach of such pseudo random generations based on random sampling from the specified model and Monte Carlo simulation or bootstrapping. However, bootstrapping is commonly used in estimation setup. It is repeated sampling method from the same dataset to estimate the variability of estimators. There are many mathematical approaches for pseudo random generations, however inverse transform sampling method is commonly used to generate random samples.

The inverse CDF method in neutrosophic context is similar approach accounting for the presence of indeterminacy or incomplete information. It incorporates the neutrosophic aspects of truth, false and indeterminacy.

In neutrosophic framework the inverse function of the proposed model is given by:

$$X_i = \varphi^{-1}(p_i) \quad (21)$$

where p_i follows the universe distribution, $p_i \sim U[0,1]$.

Using the proposed model we have:

$$X_i = \frac{\ln(\tilde{Z}_i) - \theta_N}{\sigma_N},\ i = 1,2, \dots \quad (22)$$

Statistically, the (21) is utilized to create and enhance quantile analogs of traditional moment-based descriptive measures. The quantile function can be utilized to provide random data that aligns with the density outlined in (1). The NLND can be readily simulated in R program to simulate the random samples for specified parameters. These random samples can be used to assess the validity of theory-derived outcomes. If we assume that $\theta_N = [0,0]$ and $\sigma_N^2 = [0.25, 0.75]$. These random samples are shown in Table 2

Table 2: Random samples from the proposed model with specified parameters

Random Samples				
[0.65,0.86]	[1.22,1.82]	[0.84,0.9]	[1.34,2.44]	[1.47,3.21]
[0.28,0.65]	[1.01,1.05]	[1.36,2.5]	[1.03,1.10]	[0.92,0.97]
[1.53,3.62]	[0.92,0.97]	[1.12,1.4]	[1.05,1.15]	[0.38,0.72]

Results in Table 2 show that random samples values are interval forms showing indeterminacies in values due to inexact values of the parameters. Note that these random samples are generated for a specific set of assumed parameters values. Different seed settings in algorithms allow the users to generate different samples. For the same random generation, we must provide the fixed seed number. These random samples at large simulation runs can be utilized to prove the theoretical results of the proposed model.

5. Real Data Application

In this section, a real data application is provided to illustrate the practical usefulness of the proposed model. The proposed NLND model in this study implemented on CO2 emissions data calculated from Austria for the period of 2001 to 2020. The data is taken from publicly available source [26]. The calculation of CO2 emissions is usually

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based on burning fossil fuels for each country and is typically carried out by various research organizations within the country. The amount of CO2 emissions is based on fuel such as coal, gas, oil etc. Each fuel type has a different emissions factor. These are not only the source of CO2 emissions but other sources such as factories and transport are also playing major role in emission values. According to different energy consumption patterns and energy production, every county has different CO2 emissions. Some international organizations such as International Energy Agency (IEA), United Nations Framework Convention on Climate Change (UNFCCC) and World Resources Institute (WRI) report the CO2 emissions for every country because these emissions are significantly affected the world globally. The CO2 emissions from original source is crisp values. To check if the lognormal distribution is adequately fitted CO2 emission data or not, basic probability plots are depicted in the Figure 3.

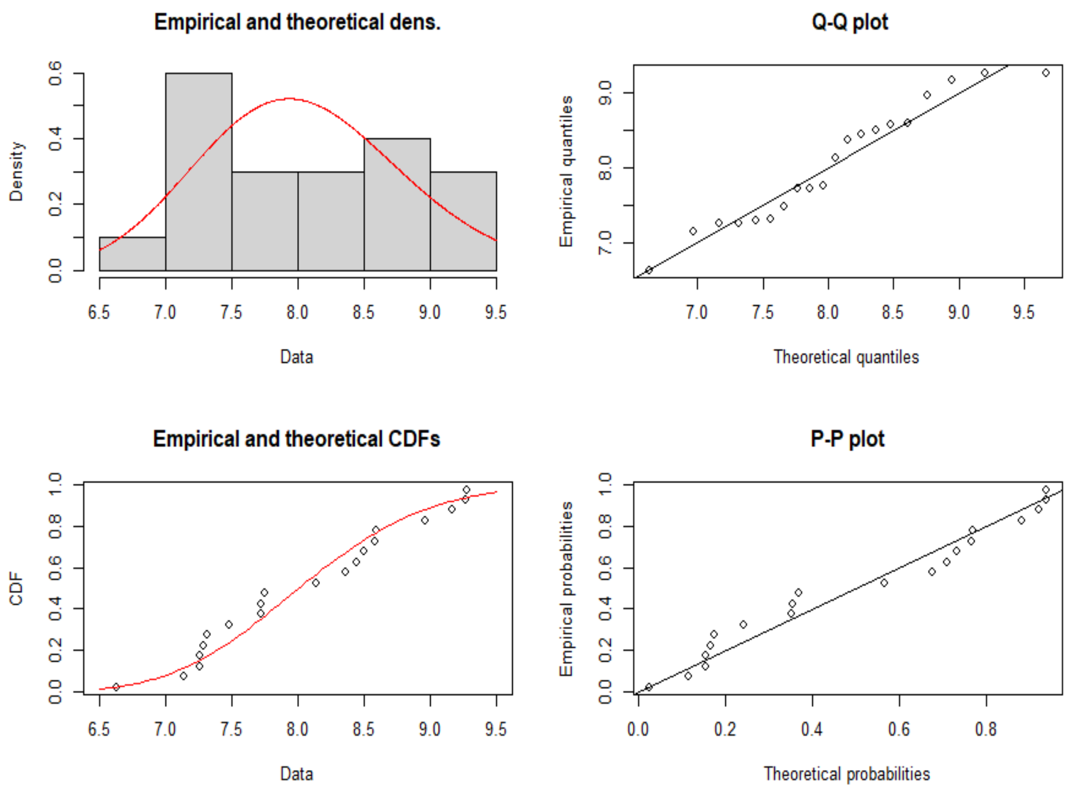

Figure 3. Fitted CO2 emission data using proposed model

Figure 3 shows that lognormal distribution perfectly fits the emission values as most of data values are closely related to theoretical lines. This shows that CO2 emissions data with indeterminacies can be utilized for further analysis using the proposed model. The CO2 emissions by considering some indeterminacies are shown in Table 3. These indeterminacies are created considering twenty random values from the uniform distribution with a specific seed. The randomly generated values are then subtracted and added to the original values to create data in interval forms.

Table 3: CO2 emissions data for Austria for the period 2001-2020

[8.13, 8.75]	[8.47, 9.445]	[7.51, 8.76]	[6.62, 7.95]
[8.32, 8.84]	[7.76, 9.40]	[6.84, 8.60]	[7.28, 7.69]
[8.61, 9.72]	[8.13, 8.87]	[7.47, 8.03]	[6.78, 7.49]
[9.21, 9.33]	[7.17, 8.26]	[6.86, 7.65]	[6.90, 7.62]
[8.79, 9.73]	[8.19, 8.53]	[6.55, 8.08]	[5.94, 7.32]

Results in Table 3 clearly indicate that existing model of lognormal distribution cannot be employed for data involving indeterminacies. However, proposed model can easily be fitted to uncertain situations with more flexible properties. Results of proposed model for CO2 emissions data is presented in Table 4

Table 4: Summary statistics of CO2 emissions data using proposed model

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Descriptive Measures	
θ_N	[2.019, 2.13]
σ_N	[0.087, 0.11]
Mean	[7.58, 8.45]
Standard deviation	[0.74, 0.84]

Results in Table 4 show that descriptive measures of the NLND are interval forms due to assuming uncertainty in the processing data. Note that if we consider the high vagueness in the observed data, uncertainties will be high in the descriptive measures of the proposed model.

6. Conclusions

In this study a novel extension of the lognormal distribution, known as a NLND has been presented. The proposed model can easily address uncertainties in statistical data across various fields. The suggested distribution is suitable to analyze data with symmetrical or right-skewed patterns. Mathematical characteristics of the NLND such as mean, variance, cumulative function, median and quartile of the NLND are derived and discussed from an application viewpoint. The estimation method of inverse cumulative function under uncertain environment function has been developed. Assessment metrics for estimators have been established. Random data generation procedure from the proposed model are explain via simulation analysis. Simulation results indicated that more reliable results are predicted from the proposed model with larger sample sizes. The CO2 emissions data has been used to demonstrate the real application of the proposed in practical scenarios. Results from real analysis also show that the NLND outperforms in real world problems as compared to existing model.

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Conflicts of Interest: The authors declare no conflict of interest.

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