On The Numerical Solutions of the Neutrosophic One-Dimensional Sine-Gordon System

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Abstract

This paper uses finite difference methods to study the numerical solution for neutrosophic Sine-Gordon system in one dimension. We use the explicit method and Crank-Nicholson method. Also, an effective comparison between the results of the two methods has been made, where we obtain the result that Crank-Nicholson method is more accurate than the explicit method, but the explicit method is easier. We also study the stability analysis for each method by using Fourier (Von-Neumann) method and get that Crank-Nicholson method is unconditionally stable while the Explicit method is stable under the condition $r^2 \leq \frac{1}{c^2}$ and $r^2 \leq 1$.

Keywords: Neutrosophic Sine-Gordon system; Crank-Nicholson method; Fourier (Von-Neumann) method; Nonlinear differential equations

1. Introduction

Most physical phenomena, whether in the field of fluid flow, electricity, optics, heat transfer or wire vibration, can generally be described by partial differential equations, and the most realistic models in physics, biochemistry, biology and the like are of a non-linear nature, such as cracks in crystals, the movement of DNA stimuli and the transmission of nerve impulses through axons and how to calculate the partial weight of a macroscopic Silicon molecule. In general, many nonlinear problems have no known analytical solutions, so they must be solved by numerical methods [2].

Most of the wave equations and nonlinear evolution equations are special varieties of nonlinear differential equations that describe a physical quantity exhibiting different types of propagation or assembly properties. One of the important physical explanations is to solve the Partial Differential Equation by the method of solitary or traveling wave solutions. for the difficulty of these problems, there are few traveling wave solutions obtained by specific methods [10,11,12,13,14,15,16,17,18,19,20,21,22-27]. The Klein-Gordon equation was developed by the scientists Klein and Gordon in the twenties as a model of the nonlinear wave equation, and the scientist Kruskal was the first to name the Sine-Gordon to the Klein - Gordon Equation [1].

In (2003) S. Griffths, Grimshsw and Khusnutdinova used the system of binary Sine-Gordon equations for the study of certain solutions for the generalization of the continuity of periodic Energy Exchange in a dual system of pendulums. In (2004) S. Griffiths, Grimshaw and Khusnutdinova studied the instability of a pair of waves generated using the Sine-Gordon system and the energy exchange between the components of the system [6].

In this research, we will deal with a physical phenomenon, which is a generalization of sine-Gordon wave system (neutrosophic generalized version) based on neutrosophic numbers and functions [28-30], which is a system of neutrosophic nonlinear partial differential equations of the Hyperbola type. We will study the numerical solution of the Sine Gordon system in one dimension using two numerical methods, the explicit method and the Crank-Nicholson method, and compare them, then we will study the stability of the numerical solution of each of the two methods and compare them.

DOI: https://doi.org/10.54216/IJNS.250303
Received: February 7, 2024 Revised: May 5, 2024 Accepted: September 10, 2024

2. The Mathematical Model

The Sine-Gordon system has the following formula:

$$\left.\begin{aligned}\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=-\delta^2\sin(u-w)\\ \frac{\partial^2 w}{\partial t^2}-c^2\frac{\partial^2 w}{\partial x^2}=\sin(u-w)\end{aligned}\right\}. \qquad (1)$$

With elementary conditions. [9]:

$$u(x,0)=f(x)and\ \frac{\partial u(x,0)}{\partial t}=0,\ \ 0\le x\le 2\pi\ ,$$

$$w(x,0)=g(x)and\ \frac{\partial w(x,0)}{\partial t}=0,\ \ 0\le x\le 2\pi.$$

And boundary conditions:

$$u(0,t)=u(2\pi,t)=0\ ,$$

$$w(0,t)=w(2\pi,t)=0.$$

Whereas δ^2 represents the ratio of the density of the particles of the medium, and c^2 represents the ratio between the speed of wave u and wave w.

We define the neutrosophic version of it as:

$$\left.\begin{aligned}\frac{\partial^2 u+I}{\partial t^2}-\frac{\partial^2 u+I}{\partial x^2}=-\delta^2\sin(u-w)\\ \frac{\partial^2 w+I}{\partial t^2}-c^2\frac{\partial^2 w+I}{\partial x^2}=\sin(u-w)\end{aligned}\right\}. \qquad (1)$$

With elementary conditions:

$$(u+I)(x,0)=f(x)and\ \frac{\partial u(x,0)}{\partial t}=0,\ \ 0\le x\le 2\pi\ ,$$

$$(w+I)(x,0)=g(x)and\ \frac{\partial w(x,0)}{\partial t}=0,\ \ 0\le x\le 2\pi.$$

And boundary conditions:

$$(u+I)(0,t)=(u+I)(2\pi,t)=0\ ,$$

$$(w+I)(0,t)=(w+I)(2\pi,t)=0.$$

Whereas δ^2 represents the ratio of the density of the particles of the medium, and c^2 represents the ratio between the speed of wave u and wave w and *I* represents the indeterminacy element.

3. Derivation of the formula of the Explicit Scheme of the system (Sine-Gordon)

In this method, we will calculate the value of $u_{n,m+1}$ and $w_{n,m+1}$ at time (t_{m+1}) based on the known values of $u_{n,m+1}$, $u_{n,m}$, $w_{n+1,m}$, $w_{n,m}$and $w_{n-1,m}$ at time t_m.

We divide the rectangle $R=\{(x,t): 0\le x\le 2\pi, 0\le t\le b\}$ into (*q-1*) and (*j-1*) rectangles whose side lengths $\Delta t=k$ and $\Delta x=h$ [8]:

The numerical approximation of the first and second partial derivatives of the function (u) relative to (*x*) and (*t*) obtained by us using Tyler's screwdriver is as follows [4]:

$$\left(\frac{\partial u}{\partial x}\right)_{n,m}=\frac{u_{n+1,m}-u_{n,m}}{h}\ , \qquad (2)$$

$$\left(\frac{\partial u}{\partial x}\right)_{n,m}=\frac{u_{n,m+1}-u_{n,m}}{k}\ , \qquad (3)$$

DOI: https://doi.org/10.54216/IJNS.250303
Received: February 7, 2024 Revised: May 5, 2024 Accepted: September 10, 2024

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{n,m} = \frac{u_{n+1,m} - 2u_{n,m} + u_{n-1,m}}{h^2} \quad , \qquad (4)$$

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{n,m} = \frac{u_{n,m+1} - 2u_{n,m} + u_{n,m-1}}{k^2} \quad . \qquad (5)$$

Using equations (4) and (5) for the variables u and w, System (1) becomes:

$$\frac{u_{n,m+1} - 2u_{n,m} + u_{n,m-1}}{k^2} - \frac{u_{n+1,m} - 2u_{n,m} + u_{n-1,m}}{h^2} = -\delta^2 \sin(u_{n,m} - w_{n,m}) \,, \qquad (6)$$

$$\frac{w_{n,m+1} - 2w_{n,m} + w_{n,m-1}}{k^2} - c^2 \left[\frac{w_{n+1,m} - 2w_{n,m} + w_{n-1,m}}{h^2}\right] = sin(u_{n,m} - w_{n,m}) \,. \qquad (7)$$

$$u_{n,m+1} - 2u_{n,m} + u_{n,m-1} = -k^2\delta^2 sin(u_{n,m} - w_{n,m}) + \frac{k^2}{h^2}(u_{n+1,m} - 2u_{n,m} + u_{n-1,m}),$$

$$w_{n,m+1} - 2w_{n,m} + w_{n,m-1} = k^2 sin(u_{n,m} - w_{n,m}) + c^2\frac{k^2}{h^2}(w_{n+1,m} - 2w_{n,m} + w_{n-1,m}).$$

Let be $r = {}^{k}/_{h}$

$$u_{n,m+1} = -k^2\delta^2 sin(u_{n,m} - w_{n,m}) - u_{n,m-1} + (2 - 2r^2)u_{n,m} + r^2(u_{n+1,m} + u_{n-1,m}) \,, \qquad (8)$$

$$w_{n,m+1} = k^2 sin(u_{n,m} - w_{n,m}) - w_{n,m-1} + (2 - 2c^2r^2)w_{n,m} + c^2r^2(w_{n+1,m} + w_{n-1,m}) \,. \qquad (9)$$

Equations (8) and (9) represent the approximation of the ending differences using the explicit method of the Gordon system, and equations (8) and (9) are used to calculate the first row *m+1* of the network using the known values of the rows *m,m-1*,as we note that this method explicitly calculates the unknown values $u_{n,m+1}, w_{n,m+1}$ in the indication of the known values $u_{n,m}$, $u_{n-1,m}$, $u_{n+1,m}$, $u_{n,m-1}$, $w_{n,m}$, $w_{n-1,m}$, ‘ $w_{n+1,m}$, $w_{n,m-1}$.

In order to find the calculations for these values, we need to find the values for the second line at $t = t_2$.

Putting *m = 1* in equations (8) and (9) we get:

$$u_{n,2} = -k^2\delta^2 sin(u_{n,1} - w_{n,1}) - u_{n,0} + (2 - 2r^2)u_{n,1} + r^2(u_{n+1,1} + u_{n-1,1}) \qquad (10)$$

$$w_{n,2} = k^2 sin(u_{n,1} - w_{n,1}) - w_{n,0} + (2 - 2c^2r^2)w_{n,1} + c^2r^2(w_{n+1,1} + w_{n-1,1}) \,. \qquad (11)$$

Taking the central differentials of the initial condition as

$$\frac{\partial u}{\partial t} = 0 \,, \frac{\partial w}{\partial t} = 0,$$

we get:

$$\frac{u_{n,2} - u_{n,0}}{2k} = 0 \Rightarrow u_{n,2} = u_{n,0} \,,$$

$$\frac{w_{n,2} - w_{n,0}}{2k} = 0 \Rightarrow w_{n,2} = w_{n,0} \,.$$

Substituting equations (10) and (11), we get:

$$u_{n,2} = \frac{1}{2}\left[-k^2\delta^2 sin(u_{n,1} - w_{n,1}) + (2 - 2r^2)u_{n,1} + r^2(u_{n+1,1}, u_{n-1,1})\right], \qquad (12)$$

$$w_{n,2} = \frac{1}{2}\left[k^2 sin(u_{n,1} - w_{n,1}) + (2 - 2c^2r^2)w_{n,1} + c^2r^2(w_{n+1,1}, w_{n-1,1})\right] \,. \qquad (13)$$

DOI: https://doi.org/10.54216/IJNS.250303
Received: February 7, 2024 Revised: May 5, 2024 Accepted: September 10, 2024

From equations (12) and (13) we calculate the second row after we calculated the first row of the elementary condition, then from equations (8) and (9) we calculate the third row, the fourth, and so on.

By substituting every u and w by $u+I$ and $w+I$, we get the corresponding neutrosophic formulas.

4. Derivation of the formula of the Crank-Nicholson method for the Sine-Gordon system

In this method, the second partial derivative u_{xx} is replaced by the cloud rate of approximations of the central differences at time m+1 and m-1 ([8], [4]):

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{n,m} = \frac{u_{n,m+1} - 2u_{n,m} + u_{n,m-1}}{k^2}. \tag{15}$$

Substituting equations (14) and (15) for each of the variables u, w in System (1), we obtain:

$$\frac{u_{n,m+1} - 2u_{n,m} + u_{n,m-1}}{k^2} = -\delta^2 sin(u_{n,m} - w_{n,m}) + \frac{1}{2}\left[\frac{u_{n+1,m+1} - 2u_{n,m+1} + u_{n-1,m+1}}{h^2} + \frac{u_{n+1,m-1} - 2u_{n,m-1} + u_{n-1,m-1}}{h^2}\right]$$

$$\frac{w_{n,m+1} - 2w_{n,m} + w_{n,m-1}}{k^2} = sin(u_{n,m} - w_{n,m}) + \frac{c^2}{2}\left[\frac{w_{n+1,m+1} - 2w_{n,m+1} + w_{n-1,m+1}}{h^2} + \frac{w_{n+1,m-1} - 2w_{n,m-1} + w_{n-1,m-1}}{h^2}\right]$$

$$u_{n,m+1} - 2u_{n,m} + u_{n,m-1} = -k^2\delta^2 sin(u_{n,m} - w_{n,m}) + \frac{k^2}{2h^2}\left[u_{n+1,m+1} - 2u_{n,m+1} + u_{n-1,m+1} + u_{n+1,m-1} - 2u_{n,m-1} + u_{n-1,m-1}\right]$$

$$w_{n,m+1} - 2w_{n,m} + w_{n,m-1} = k^2 sin(u_{n,m} - w_{n,m}) + \frac{c^2k^2}{2h^2}\left[w_{n+1,m+1} - 2w_{n,m+1} + w_{n-1,m+1} + w_{n+1,m-1} - 2w_{n,m-1} + w_{n-1,m-1}\right]$$

$$(2-2r^2)u_{n,m+1} - r^2(u_{n+1,m+1} + u_{n-1,m+1}) = -2k^2\delta^2 sin(u_{n,m} - w_{n,m}) - (2-2r^2)u_{n,m-1} + 4u_{n,m} + r^2(u_{n+1,m-1} + u_{n-1,m-1}), \tag{16}$$

$$(2-2c^2r^2)w_{n,m+1} - c^2r^2(w_{n+1,m+1} + w_{n-1,m+1}) = 2k^2 sin(u_{n,m} - w_{n,m}) - (2-2c^2r^2)w_{n,m-1} + 4w_{n,m} + c^2r^2(w_{n+1,m-1} + w_{n-1,m-1}). \tag{17}$$

Equations (16) and (17) are approximations of the differences ending with the Crank-Nicholson method for the Sine-Gordon system, and in this method, we need to calculate the three unknown values on the three unknown side on the left side of equations (16) and (17) $u_{n-1,m+1}$ and $u_{n,m+1}$ and $u_{n+1,m+1}$ for equation (16) $w_{n-1,m+1}$ and $w_{n,m+1}$ and $w_{n+1,m+1}$ for equation (17) since all the terms on the right side of equations (16) and (17) are known, and lead to the formation of a linear algebraic system of three diagonals.

AX=B, VX=C.

A and *V* are the Crank-Nicholson matrices of three diagonals, *X* is a vertical vector containing the values of the unknowns on the left side, and *B* and *C* are two vertical vectors containing the known values on the right side.

Boundary conditions are used only in the first and last equations.

$$u_{1,m-1} = u_{1,m+1} = 0, u_{n,m-1} = u_{n,m+1} = 0,$$

$$w_{1,m-1} = w_{1,m+1} = 0, \quad w_{n,m-1} = w_{n,m+1} = 0.$$

The algebraic system resulting from the application of the Crank-Nicholson method to the Sine-Gordon system for the variable U is preferably expressed in the following form:

DOI: https://doi.org/10.54216/IJNS.250303
Received: February 7, 2024 Revised: May 5, 2024 Accepted: September 10, 2024

$$\begin{bmatrix} 2+2r^2 & -r^2 & & & & & \\ -r^2 & 2+2r^2 & -r^2 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & -r^2 & 2+2r^2 & -r^2 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -r^2 & 2+2r^2 & -r^2 \\ & & & & & -r^2 & 2+2r^2 \end{bmatrix} \begin{bmatrix} u_{2,m+1} \\ u_{3,m+1} \\ \vdots \\ u_{p,m+1} \\ \vdots \\ u_{n-2,m+1} \\ u_{n-1,m+1} \end{bmatrix} =$$

$$\begin{bmatrix} -2k^2\delta^2 sin(u_{2,m}-w_{2,m})-(2+2r^2)u_{2,m-1}+4u_{2,m}+r^2u_{3,m-1} \\ r^2u_{2,m-1}-2k^2\delta^2 sin(u_{3,m}-w_{3,m})-(2+2r^2)u_{3,m-1}+4u_{3,m}+r^2u_{4,m-1} \\ \vdots \\ r^2u_{p-1,m-1}-2k^2\delta^2 sin(u_{p,m}-w_{p,m})-(2+2r^2)u_{p,m-1}+4u_{p,m}+r^2u_{p+1,m-1} \\ \vdots \\ r^2u_{n-3,m-1}-2k^2\delta^2 sin(u_{n-2,m}-w_{n-2,m})-(2+2r^2)u_{n-2,m-1}+4u_{n-2,m}+r^2u_{n-2,m-1} \\ r^2u_{n-2,m-1}-2k^2\delta^2 sin(u_{n-1,m}-w_{n-1,m})-(2+2r^2)u_{n-1,m-1}+4u_{n-1,m} \end{bmatrix}.$$

As for the variable w, The Matrix is in the following form:

$$\begin{bmatrix} 2+2r^2 & -r^2 & & & & & \\ -r^2 & 2+2r^2 & -r^2 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & -r^2 & 2+2r^2 & -r^2 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -r^2 & 2+2r^2 & -r^2 \\ & & & & & -r^2 & 2+2r^2 \end{bmatrix} \begin{bmatrix} w_{2,m+1} \\ w_{3,m+1} \\ \vdots \\ w_{p,m+1} \\ \vdots \\ w_{n-2,m+1} \\ w_{n-1,m+1} \end{bmatrix} =$$

$$\begin{bmatrix} 2k^2 sin(u_{2,m}-w_{2,m})-(2+2c^2r^2)w_{2,m-1}+4w_{2,m}+c^2r^2w_{3,m-1} \\ c^2r^2w_{2,m-1}+2k^2 sin(u_{3,m}-w_{3,m})-(2+2c^2r^2)w_{3,m-1}+4w_{3,m}+c^2r^2w_{4,m-1} \\ \vdots \\ c^2r^2w_{p-1,m-1}+2k^2 sin(u_{p,m}-w_{p,m})-(2+2c^2r^2)w_{p,m-1}+4w_{p,m}+c^2r^2w_{p+1,m-1} \\ \vdots \\ c^2r^2w_{n-3,m-1}+2k^2 sin(u_{n-2,m}-w_{n-2,m})-(2+2c^2r^2)w_{n-2,m-1}+4w_{n-2,m}+c^2r^2w_{n-2,m-1} \\ c^2r^2w_{n-2,m-1}+2k^2 sin(u_{n-1,m}-w_{n-1,m})-(2+2c^2r^2)w_{n-1,m-1}+4w_{n-1,m} \end{bmatrix}$$

The above linear system is solved by one of the direct methods or by iterative methods.

DOI: https://doi.org/10.54216/IJNS.250303
Received: February 7, 2024 Revised: May 5, 2024 Accepted: September 10, 2024

5. Stability analysis of the explicit method using the Fourier (Von-Neumann) method

The general principle of this method is to replace the solution by the method of differences ending at Time (t) by the amount $\xi^m e^{i\beta n\Delta x}$ for $u_{n,m}$ and the amount $\gamma^m e^{i\beta n\Delta x}$ for $w_{n,m}$ where $i=\sqrt{-1}$ ،$\beta>0$ ،$\gamma>0$ ، $k=\Delta t$ ،$h=\Delta x$ [11].

To apply the Von-Neumann method to System (1), we need to neglect the nonlinear limit of it, and we get the following [7]:

$$\left.\begin{aligned}\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=0\\ \frac{\partial^2 w}{\partial t^2}-c^2\frac{\partial^2 w}{\partial x^2}=0\end{aligned}\right\}. \quad (18)$$

Using the explicit method, we obtain:

$$u_{n,m+1}=-u_{n,m-1}+(2-2r^2)u_{n,m}+r^2\left(u_{n+1,m}+u_{n-1,m}\right), \quad (19)$$

$$w_{n,m+1}=-w_{n,m-1}+(2-2c^2r^2)u_{n,m}+c^2r^2\left(w_{n+1,m}+w_{n-1,m}\right). \quad (20)$$

By substituting $u_{n,m}=\xi^m e^{i\beta n\Delta x}$,$w_{n,m}=\gamma^m e^{i\beta n\Delta x}$ in equations (19) and (20) in order, we obtain:

$$\xi^{m+1}e^{i\beta n\Delta x}=-\xi^{m-1}e^{i\beta n\Delta x}+(2-2r^2)\xi^m e^{i\beta n\Delta x}+r^2(\xi^m e^{i\beta(n+1)\Delta x}+\xi^m e^{i\beta(n-1)\Delta x}),$$

$$\gamma^{m+1}e^{i\beta n\Delta x}=-\gamma^{m-1}e^{i\beta n\Delta x}+(2-2c^2r^2)\gamma^m e^{i\beta n\Delta x}+c^2r^2\left(\gamma^m e^{i\beta(n+1)\Delta x}+\gamma^m e^{i\beta(n-1)\Delta x}\right),$$

By dividing the above two equations by $\xi^m e^{i\beta n\Delta x}$ and $\gamma^m e^{i\beta n\Delta x}$ in order, we get:

$$\xi=-\xi^{-1}+(2-2r^2)+r^2\left(e^{i\beta\Delta x}+e^{-i\beta\Delta x}\right),$$

$$\gamma=-\gamma^{-1}+(2-2c^2r^2)+c^2r^2\left(e^{i\beta\Delta x}+e^{-i\beta\Delta x}\right),$$

$$\Rightarrow \xi+\xi^{-1}=2-2r^2+r^2\left(2cos(\beta\Delta x)\right),$$

$$\Rightarrow \gamma+\gamma^{-1}=2-2c^2r^2+c^2r^2\left(2cos(\beta\Delta x)\right),$$

$$\Rightarrow \xi+\xi^{-1}=2-2r^2+2r^2\left[1-2sin^2\left(\frac{\beta\Delta x}{2}\right)\right],$$

$$\Rightarrow \gamma+\gamma^{-1}=2-2c^2r^2+2c^2r^2\left[1-2sin^2\left(\frac{\beta\Delta x}{2}\right)\right],$$

$$\Rightarrow \xi+\xi^{-1}=2-4r^2sin^2\left(\frac{\beta\Delta x}{2}\right),$$

$$\Rightarrow \gamma+\gamma^{-1}=2-4c^2r^22sin^2\left(\frac{\beta\Delta x}{2}\right),$$

$$\Rightarrow \frac{\xi^2+1}{\xi}=2\left[1-2r^2sin^2\left(\frac{\beta\Delta x}{2}\right)\right],$$

$$\Rightarrow \frac{\gamma^2+1}{\gamma}=2\left[1-2c^2r^2sin^2\left(\frac{\beta\Delta x}{2}\right)\right],$$

$$A=\left[1-2r^2sin^2\left(\frac{\beta\Delta x}{2}\right)\right],$$

$$B=\left[1-2c^2r^2sin^2\left(\frac{\beta\Delta x}{2}\right)\right],$$

$$\Rightarrow \xi^2+1=2\xi A\Rightarrow \xi^2-2A\xi+1=0\Rightarrow \gamma^2-2B\gamma+1=0,$$

$$\Rightarrow \xi=A\pm\sqrt{A^2-1}\Rightarrow \gamma=B\pm\sqrt{B^2-1}.$$

In order to be $|\xi|\le 1$ and $|\gamma|\le 1$, it is necessary to be $|A|\le 1$ and $|B|\le 1$and in order $|A|\le 1$

DOI: https://doi.org/10.54216/IJNS.250303
Received: February 7, 2024 Revised: May 5, 2024 Accepted: September 10, 2024

$$\left|1 - 2r^2 sin^2\left(\frac{\beta\Delta x}{2}\right)\right| \le 1\,,$$

$$-1 \le 1 - 2r^2 sin^2\left(\frac{\beta\Delta x}{2}\right) \le 1\,.$$

Taking the right-hand side of the inequality:

$$\Rightarrow 1 - 2r^2 sin^2\left(\frac{\beta\Delta x}{2}\right) \le 1 \Rightarrow -2r^2 sin^2\left(\frac{\beta\Delta x}{2}\right) \le 0 \Rightarrow 2r^2 sin^2\left(\frac{\beta\Delta x}{2}\right) \ge 0\,.$$

For some values of β, let $sin^2\left(\frac{\beta\Delta x}{2}\right) = 1 \Rightarrow r^2 \ge 0$, which is always true.

Taking the left-hand side of the inequality:

$$-1 \le 1 - 2r^2 sin^2\left(\frac{\beta\Delta x}{2}\right),$$

$$-2 \le -2r^2 sin^2\left(\frac{\beta\Delta x}{2}\right),$$

$$1 \ge r^2 sin^2\left(\frac{\beta\Delta x}{2}\right),$$

$$r^2 \le \frac{1}{sin^2\left(\frac{\beta\Delta x}{2}\right)}.$$

Thus, the method is stable under the condition.

$$r^2 \le 1,$$

$$|B| \le 1\,,$$

$$\left|1 - 2c\, r^2 sin^2\left(\frac{\beta\Delta x}{2}\right)\right| \le 1\,,$$

$$-1 \le 1 - 2c^2 r^2 sin^2\left(\frac{\beta\Delta x}{2}\right) \le 1\,.$$

Taking the right-hand side of the inequality:

$$\Rightarrow 1 - 2c^2 r^2 sin^2\left(\frac{\beta\Delta x}{2}\right) \le 1\,,$$

$$\Rightarrow -2c^2 r^2 sin^2\left(\frac{\beta\Delta x}{2}\right) \le 0 \quad \Rightarrow 2c^2 r^2 sin^2\left(\frac{\beta\Delta x}{2}\right) \ge 0\,.$$

For some values of β, $sin^2\left(\frac{\beta\Delta x}{2}\right) = 1 \Rightarrow c^2 r^2 \ge 0$, which is always true.

And by taking the left end of the inequality:

$$-1 \le 1 - 2c^2 r^2 sin^2\left(\frac{\beta\Delta x}{2}\right),$$

$$-2 \le -2c^2 r^2 sin^2\left(\frac{\beta\Delta x}{2}\right),$$

$$1 \ge c^2 r^2 sin^2\left(\frac{\beta\Delta x}{2}\right),$$

$$r^2 \le \frac{1}{c^2 sin^2\left(\frac{\beta\Delta x}{2}\right)}.$$

Therefore, the method is stable under the condition:

$$r^2 \le \frac{1}{c^2}.$$

DOI: https://doi.org/10.54216/IJNS.250303
Received: February 7, 2024 Revised: May 5, 2024 Accepted: September 10, 2024

6. Stability analysis of the Crank-Nicholson method using the Fourier (Von-Neumann) method

Using the Crank-Nicholson method of System (1) after neglecting the nonlinear limit of it, we obtain:

$$(1+r^2)u_{n,m+1}-2u_{n,m} = -(1+r^2)u_{n,m-1} + \frac{r^2}{2}\left[u_{n+1,m+1}+u_{n-1,m+1}+u_{n+1,m-1}+u_{n-1,m-1}\right], \quad (21)$$

$$(1+c^2r^2)w_{n,m+1}-2w_{n,m} = -(1+c^2r^2)w_{n,m-1} + \frac{c^2r^2}{2}\left[w_{n+1,m+1}+w_{n-1,m+1}+w_{n+1,m-1}+w_{n-1,m-1}\right]. \quad (22)$$

By substituting $u_{n,m}=\xi^m e^{i\beta n\Delta x}$, $w_{n,m}=\gamma^m e^{i\beta n\Delta x}$ in equations (21) and (22) in order, we obtain:

$$(1+r^2)\xi^{m+1}e^{i\beta n\Delta x}-2\xi^m e^{i\beta n\Delta x} = -(1+r^2)\xi^{m-1}e^{i\beta n\Delta x} + \frac{r^2}{2}\left[\xi^{m+1}e^{i\beta(n+1)\Delta x}+\xi^{m+1}e^{i\beta(n-1)\Delta x}+\xi^{m-1}e^{i\beta(n+1)\Delta x}+\xi^{m-1}e^{i\beta(n-1)\Delta x}\right],$$

$$(1+c^2r^2)\gamma^{m+1}e^{i\beta n\Delta x}-2\gamma^m e^{i\beta n\Delta x} = -(1+c^2r^2)\gamma^{m-1}e^{i\beta n\Delta x} + \frac{c^2r^2}{2}\left[\gamma^{m+1}e^{i\beta(n+1)\Delta x}+\gamma^{m+1}e^{i\beta(n-1)\Delta x}+\gamma^{m-1}e^{i\beta(n+1)\Delta x}+\gamma^{m-1}e^{i\beta(n-1)\Delta x}\right].$$

By dividing the above two equations by $\xi^m e^{i\beta n\Delta x}$ and $\gamma^m e^{i\beta n\Delta x}$ in order, we get:

$$(1+r^2)(\xi+\xi^{-1})-2 = \frac{r^2}{2}\left[\xi(2cos(\beta\Delta x))+\xi^{-1}(2cos(\beta\Delta x))\right],$$

$$(1+c^2r^2)(\gamma+\gamma^{-1})-2 = \frac{c^2r^2}{2}\left[\gamma(2cos(\beta\Delta x))+\gamma^{-1}(2cos(\beta\Delta x))\right],$$

$$(1+r^2)(\xi+\xi^{-1})-2 = \frac{r^2}{2}\left[2cos(\beta\Delta x)(\xi+\xi^{-1})\right],$$

$$(1+c^2r^2)(\gamma+\gamma^{-1})-2 = \frac{c^2r^2}{2}\left[2cos(\beta\Delta x)(\gamma+\gamma^{-1})\right],$$

$$(1+r^2)(\xi+\xi^{-1})-2 = r^2\left(1-2sin^2\left(\frac{\beta\Delta x}{2}\right)\right)(\xi+\xi^{-1}),$$

$$(1+c^2r^2)(\gamma+\gamma^{-1})-2 = c^2r^2\left(1-2sin^2\left(\frac{\beta\Delta x}{2}\right)\right)(\gamma+\gamma^{-1}),$$

$$(1+r^2)-r^2+2r^2sin^2\left(\frac{\beta\Delta x}{2}\right) = \frac{2}{\xi+\xi^{-1}},$$

$$(1+c^2r^2)-c^2r^2+2c^2r^2sin^2\left(\frac{\beta\Delta x}{2}\right) = \frac{2}{\gamma+\gamma^{-1}},$$

$$1+2r^2sin^2\left(\frac{\beta\Delta x}{2}\right) = \frac{2}{\xi+\xi^{-1}},$$

$$1+2c^2r^2sin^2\left(\frac{\beta\Delta x}{2}\right) = \frac{2}{\gamma+\gamma^{-1}},$$

$$\xi+\xi^{-1} = \frac{2}{1+2r^2sin^2\left(\frac{\beta\Delta x}{2}\right)},$$

DOI: https://doi.org/10.54216/IJNS.250303
Received: February 7, 2024 Revised: May 5, 2024 Accepted: September 10, 2024

$$\gamma + \gamma^{-1} = \frac{2}{1 + 2c^2 r^2 sin^2\left(\frac{\beta\Delta x}{2}\right)},$$

$$\frac{\xi^2 + 1}{\xi} = \frac{2}{1 + 2r^2 sin^2\left(\frac{\beta\Delta x}{2}\right)},$$

$$\frac{\gamma^2 + 1}{\gamma} = \frac{2}{1 + 2c^2 r^2 sin^2\left(\frac{\beta\Delta x}{2}\right)}.$$

Then

$$D = \frac{1}{1 + 2r^2 sin^2\left(\frac{\beta\Delta x}{2}\right)},$$

$$E = \frac{1}{1 + 2c^2 r^2 sin^2\left(\frac{\beta\Delta x}{2}\right)},$$

$$\xi^2 - 2D\xi + 1 = 0\,,$$

$$\gamma^2 - 2E\gamma + 1 = 0\,.$$

In order for it to be $|\xi| \leq 1$ and $|\gamma| \leq 1$, it must be $|D| \leq 1$ and $|E| \leq 1$, respectively.

$$\left|\frac{1}{1 + 2r^2 sin^2\left(\frac{\beta\Delta x}{2}\right)}\right| \leq 1\,,$$

$$\left|\frac{1}{1 + 2c^2 r^2 sin^2\left(\frac{\beta\Delta x}{2}\right)}\right| \leq 1\,.$$

Then the Crank-Nicholson method will be unconditionally stable.

7. Numerical results

For the purpose of numerical solution, we take the Sine-Gordon system in one dimension represented by the System (1):

$$\frac{\partial^2 u + I}{\partial t^2} - \frac{\partial^2 u + I}{\partial x^2} = -\delta^2 sin(u - w),$$

$$\frac{\partial^2 w + I}{\partial t^2} - c^2 \frac{\partial^2 w + I}{\partial x^2} = sin(u - w).$$

Initial and boundary conditions [9]:

$$(u + I)(x, 0) = f(x) \text{ and } \frac{\partial (u + I)(x, 0)}{\partial t} = 0,\ \ 0 \leq x \leq 2\pi\,,$$

$$(w + I)(x, 0) = g(x)\text{and } \frac{\partial (w + I)(x, 0)}{\partial t} = 0,\ \ 0 \leq x \leq 2\pi.$$

$$(u + I)(0, t) = u(2\pi, t) = 0\,,$$

$$(w + I)(0, t) = w(2\pi, t) = 0.$$

If we used two methods of finite difference methods in one dimension, the first method is the explicit Scheme and the second is the Crank-Nicholson method, and the two methods were compared when:

$$(u + I)(x, 0) = sin(x),$$

$$(w + I)(x, 0) = sin(x).$$

DOI: https://doi.org/10.54216/IJNS.250303
Received: February 7, 2024 Revised: May 5, 2024 Accepted: September 10, 2024

Table 1: Comparison of the Explicit and Crank-Nicholson methods

Explicit Method ***x=1.884955592 , h=0.314159265 k=0.314159265 , c=0.5 ,*** $\delta = 1$		**Crank-Nicholson Method** ***x=1.884955592 , h=0.314159265 k=0.314159265 , c=0.5 , d*** $\delta = 1$	
u(x, t)	*w(x, t)*	*u(x, t)*	*w(x, t)*
0	0	0	0
0.2912	0.2998	0.2915	0.3052
0.5540	0.5703	0.5544	0.5805
0.7625	0.7849	0.7631	0.7990
0.8964	0.9227	0.8971	0.9393
0.9425	0.9702	0.9432	0.9877
0.8964	0.9227	0.8971	0.9393
0.7625	0.7849	0.7631	0.7900
0.5540	0.5730	0.5544	0.5805
0.2912	0.2998	0.2915	0.3052
0	0	0	0
-0.2912	-0.2998	-0.2915	-0.3052
-0.5540	-0.5703	-0.5544	-0.5805
-0.7625	-0.7849	-0.7631	-0.7990
-0.8964	-0.9227	-0.8971	-0.9393
-0.9425	-0.9702	-0.9432	-0.9877
-0.8964	-0.9227	-0.8971	-0.9393
-0.7625	-0.7849	-0.7631	-0.7900
-0.5540	-0.5730	-0.5544	-0.5805
-0.2912	-0.2998	-0.2915	-0.3052
0	0	0	0

By observing Table (1), we conclude that the Crank-Nicholson method is better than the explicit method in one dimension, and the numerical solution is periodic and symmetrical, the solution is the same for each period, therefore, we need less calculations and therefore less time.

8. Conclusions

By comparing the solution of the neutrosophic Sine-Gordon system in one dimension of the explicit method and the implicit method (Grank-Nicholson), it turned out that the explicit method is faster and easier to use than the implicit method, while the implicit method is more accurate than the explicit method and that the solution in both methods is a corresponding solution, which leads to calculations and saves a lot of time and effort, and the consistency of each of the two methods used to solve the Sine-Gordon system in one dimension and using the Fourier (von-Neumann) shows that an explicit method is stable if $r^2 \leq 1$ so, $(\Delta t)^2 \leq (\Delta x)^2$. It is conditionally stable while the implicit method is stable For all values were r^2 so, It is unconditionally stable.

DOI: https://doi.org/10.54216/IJNS.250303
Received: February 7, 2024 Revised: May 5, 2024 Accepted: September 10, 2024

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DOI: https://doi.org/10.54216/IJNS.250303
Received: February 7, 2024 Revised: May 5, 2024 Accepted: September 10, 2024