Computational Approaches to Solving Some Partial Differential Equations and Neutrosophic Partial Differential with Variable Coefficients Using the Laplace Residual Power Series Method

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Abstract

We employ the Laplace Residual Power Series Method to approximate analytical solutions for differential equations and neutrosophic differential equations with associated parameters, including non-homogeneous equations and fractional formulas in partial differential equations (PDEs). This approach showcases the method's simplicity, effectiveness, and robustness in deriving analytical series solutions for PDEs that involve associated parameters, especially in the context of fractional differential equations. Several practical uses of LRPSM with an emphasis on non-homogeneous and partial differential equations and neutrosophic equations with fractions (PDEs). These applications are significant in a variety of scientific and engineering domains that simulate complicated dynamic system such as anomalous diffusion in physics, viscoelastic material modeling in engineering and signal processing.

Keywords: Effectiveness; Fractional formulas; Parameters; Analytical collection, Neutrosophic equation, Neutrosophic differential equation, Neutrosophic transformation

1. Introduction

Multiple physical events in the fields of science and technology may be explained using differential equations with partial derivatives. Take a look at various physical issues with non - homogeneous media, for instance [1–3]. Many false hypotheses on integral order differential equations have previously been employed to construct systems with storage and genetic features. By making such claims, some crucial data will be missed. Typically, without the need for further assumptions, numerical method is an effective tool for describing storage characteristics and hereditary traits of numerous procedures and materials. There is currently a lot of interest in the fractional differential equation in a number of disciplines, such biology, mathematics, technology, even financial and the sociology [4,5]. You can find many latest events in differential derivatives in [6–8]. In interdisciplinary field, the analytical serial solutions to differential equations play a crucial role. The Adomian decomposition approach [9], the partial real transform method [10], and the Laplace transform technique [11]are only a few of the numerical and analytical techniques that have been suggested. Even though many approaches have been proposed, researchers are continuously exploring for better solutions to particular issues, particularly for fractional formulas with changing parameters.

Numerous PDE have seen the extension of RPS, particularly fractional PDE. Examples include fractional of time-KdV-Burgers formulas [12], culturally homogenous time-fractional pulse equation [13], time-space fractional Boussinesq formulas [14] and time-fractional diffraction PDE [15,16].

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The technique of LRPS introduced also shown in [17], has been employed recently. The Laplace transform method and the RPS method are combined in the LRPS method that also offers precise and approximative solutions FPS by changing a core issue into space of Laplace and producing the answers to equations of new mathematical form. The difficulty of identifying can then be solved by reversing Laplace for the values obtained. In contrast to the FRPS approach, which relies on the fractional derivative and requires time-consuming computation of several fractional derivatives in stages to discover solutions, the unknown parameters in the proposed Laplace expansion may be described to use the idea of a limits.

In this paper to solve PDEs and neutrosophic PDEs (based on neutrosophic functions, see [20-22]) involving parameters associated, we establish solutions of approximative analytical series., such as heterogeneous equations, 2D partial formulas, and 3D partial formulas, using Laplace's residual power series approach.

2. Foundations of fractional calculus concepts

The fractional integrate with order is defined in a number of ways, and they are not all interchangeable. Riemann-definition Liouville's and Caputo's concept are the two used most types.

Definition 2.1: [18] The given definition of Mittag-Leffler formula:

$$E_{\beta}(z):=\sum_{\mathrm{j}=0}^{\infty}\frac{z^{j}}{\Gamma(\beta\mathrm{j}+1)},\beta>0 \quad (1)$$

Definition 2.2: [18] The integral of R-L time-fraction with $\alpha>0$ of the multidimensional function $\psi(x,t)$,$t>0$ is defined by:

$$J_{t}^{\alpha}\psi(x,t)=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{\alpha-1}\psi(x,\tau)d\tau, \quad (2)$$

Definition 2.3 [19]: The derivative of Caputo time-fraction with $m-1<\alpha<m$ of the multidimensional function $\psi(x,t)$ as mentioned:

$$D^{\alpha}\psi(x,t)=\begin{cases}J_{t}^{m-\alpha}\dfrac{\partial^{m}}{\partial t^{m}}\psi(x,t), & m-1<\alpha<m,\\ \dfrac{\partial^{m}}{\partial t^{m}}\psi(x,t), & \alpha=m.\end{cases} \quad (3)$$

Definition 2.4: (Hanna and Rowland, 1990) If an improper integral defined below:

$$\Psi(x,s)=\int_{0}^{\infty}e^{-st}\psi(x,t)dt, \quad (4)$$

exists for all s, then it is Laplace transform of $\psi(\mathrm{x},t)$, and indicated as $\mathcal{L}[\psi(x,t)](x,s)$.

Definition 2.5: Using the inverse Laplace transform described below, the original function $\psi(x,t)$ may be recovered from the Laplace transformation $\Psi(x,s)$:

$$\psi(\mathrm{x},t)=\int_{\zeta-i\infty}^{\zeta+i\infty}e^{st}\Psi(\mathrm{x},s)ds,\zeta=\mathrm{Re}(s), \quad (5)$$

Lemma 2.6: [19] Suppose that $\Psi(x,s)=\mathcal{L}[\psi(x,t)]$, and $\Omega(x,s)=\mathcal{L}[\phi(x,t)]$ and $\eta,\lambda,\beta,\sigma$ are elements. The following characteristics are thus met:

Linearity: The Laplace transform and its inverse are linear operators, that is, for η , σ and $\lambda\in\mathbb{R}$, then

$$\mathcal{L}[\eta\psi(x,t)+\lambda\phi(x,t)]=\eta\mathcal{L}[\psi(x,t)]+\lambda\mathcal{L}[\phi(x,t)]=\eta\Psi(x,s)+\lambda\Omega(x,s).$$

$$\mathcal{L}^{-1}[\eta\Psi(x,s)+\lambda\Omega(x,s)]=\eta\mathcal{L}^{-1}[\Psi(x,s)]+\lambda\mathcal{L}^{-1}[\Omega(x,s)]=\eta\psi(x,t)+\phi(x,t).$$

Shifting: $\mathcal{L}[e^{\sigma t}\phi(x,t)]=\Psi(s-\sigma)$.

$\mathcal{L}[\psi(\beta x,\beta t)]=\frac{1}{\beta}\Psi\left(\frac{x}{\beta},\frac{s}{\beta}\right),\beta>0.$

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$$\lim_{s\to\infty} s\Psi(x,s) = \psi(x,0).$$

$$\mathcal{L}[\partial_t^n \psi(x,t)] = s^n \Psi(x,s) - \sum_{k}^{n-1} s^{n-k-1}\, \partial_t^k \psi(x,0).$$

Definition 2.7: (El-Ajou et al,2015) A series expansion illustration of the structure:

$$\sum_{n=0}^{\infty} f_n(x)(t-t_0)^{n\alpha} = c_0 + c_1(t-t_0)^{\alpha} + c_2(t-t_0)^{2\alpha} + \cdots, \tag{6}$$

Where $t \geq t_0$, referred to as a series of fractional power (FPS, for short) regarding t_0, and $f_n(x)$ are functions of x.

Corollary 2.8: [19] Consider the $\Psi(x,s) = \mathcal{L}[\psi(\mathrm{x},\mathrm{t})]$ is as follows fractional Laurent expansion (FLE):

$$\Psi(\mathrm{x},s) = \sum_{n=0}^{\infty} \frac{f_n(x)}{s^{n\alpha+1}}, 0 < \alpha \leq 1, s > 0. \tag{7}$$

Then $\mathrm{f}_n(x) = (D_t^{n\alpha}\psi)(0)$.

3. Constructing LRPS Solution of Period FPDEs

In this part, we provide new approach named LRPS for fractional partial differential equations FPDEs with given formula

$$D_t^{\alpha} u(x,t) + \delta u_{xx}(x,t) = \Omega(x) u(x,t). \tag{8}$$

$w$ith initial condition

$$u(x,0) = f(x). \tag{9}$$

First step for that is to applying Laplace transform to system (8) to convert it to

Laplace space as follows:

$$\mathcal{L}\{D_t^{\alpha} u(x,t)\} + \delta\mathcal{L}\{u_{xx}(x,t)\} - \Omega(x)\mathcal{L}\{u(x,t)\} = 0, \tag{10}$$

Considering that $U(x,s) = \mathcal{L}\{u(x,t)\}$, we obtain:

$$s^{\alpha} U(x,s) - s^{\alpha-1} f(x) + \delta U_{xx}(x,s) - \Omega(x) U(x,s) = 0. \tag{11}$$

Dividing Eq. (11) s^{α}gives a new form to it as follows:

$$U(x,s) - \frac{f(x)}{s} + \frac{\delta}{s^{\alpha}} U_{xx}(x,s) - \frac{\Omega(x)}{s^{\alpha}} U(x,s) = 0 \tag{12}$$

In the next step, we assume the solution of (12), $U(x,s)$ has expansions in fractional Lorient series form as follows:

$$U(x,s) = \sum_{i=0}^{\infty} \frac{f_i(x)}{s^{1+i\alpha}}, 0 < \alpha \leq 1, x \in I, s > 0, \tag{13}$$

The first coefficient of (13) following is:

$$U(x,s) = \frac{f(x)}{s} + \sum_{i=1}^{\infty} \frac{f_i(x)}{s^{1+i\alpha}}, 0 < \alpha \leq 1, x \in I, s > 0\,, \tag{14}$$

Now, we construction the LRFs of (12) as follows:

$$LRes_1(x,s) = U(x,s) - \frac{f(x)}{s} + \frac{\delta}{s^{\alpha}} U_{xx}(x,s) - \frac{\Omega(x)}{s^{\alpha}} U(x,s). \tag{15}$$

The kth-truncated series of (14) that provided via:

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$$U_k(x,s)=\frac{f(x)}{s}+\sum_{i=1}^{k}\frac{f_i(x)}{s^{1+i\alpha}}, s>0, \tag{16}$$

the kth-LRF may be defined:

$$LRes_k(x,s)=U_k(x,s)-\frac{f(x)}{s}+\frac{\delta}{s^\alpha}(U_k)_{xx}(x,s)-\frac{\Omega(x)}{s^\alpha}U_k(x,s) \tag{17}$$

It is clear that $\mathrm{Lim}_{k\to\infty}\mathrm{LRes}_k(s)=\mathrm{LRes}(s)$, $\mathrm{LRes}(s)=0$, and thus $s^k\mathrm{LRes}(s)=0$ for $s>0$ and $k=0,1,2,3,\ldots$. Thus, $\mathrm{Lim}_{s\to\infty}\left(s^k\mathrm{LRes}(s)\right)=0$ [19]:

$$\mathrm{Lim}_{s\to\infty}s^{k\alpha+1}\mathrm{LRes}_k(s)=0,\ \ \alpha>0,\ \ k=1,2,3,\ldots \tag{18}$$

The$1st$-LRF is:

$$LRes_1(x,s)=\frac{f(x)}{s}+\frac{f_1(x)}{s^{1+\alpha}}-\frac{f(x)}{s}+\frac{\delta}{s^\alpha}\left(\frac{f''(x)}{s}+\frac{f_1''(x)}{s^{1+\alpha}}\right)-\frac{\phi(x)}{s^\alpha}\left(\frac{f(x)}{s}+\frac{f_1(x)}{s^{1+\alpha}}\right) \tag{19}$$

Multiply both sides of Eq. (19) by $s^{\alpha+1}$to get:

$$s^{\alpha+1}LRes_1(x,s)=f_1(x)+\delta f''(x)+\delta\frac{f''(x)}{s^\alpha}-\phi(x)g(x)-\frac{\phi(x)g_1(x)}{s^\alpha}. \tag{20}$$

Taking the limit $s\to\infty$ of both sides of (20) and using the fact in (18) After completing the resultant algebraic system, we are left with:

$$f_1(x)=\phi(x)f(x)-\delta f''(x). \tag{21}$$

The value of next unidentified parameter can be determined in a similar manner $U_2(x,s)$ substitute kth-truncated series of (16), $U_2(x,s)=\frac{f(x)}{s}+\frac{f_1(x)}{s^{1+\alpha}}+\frac{f_2(x)}{s^{1+2\alpha}}$ into the kth-LRF, $LRes_k(\mathrm{x},\mathrm{s})$ of (17), to get the following:

$$LRes_2(LRes_2(x,s)=\frac{f_1(x)}{s^{1+\alpha}}+\frac{f_2(x)}{s^{1+2\alpha}}+\frac{\delta}{s^\alpha}\left(\frac{f''(x)}{s}+\frac{f_1''(x)}{s^{1+\alpha}}+\frac{f_2''(x)}{s^{1+2\alpha}}\right)-\frac{\phi(x)}{s^\alpha}\left(\frac{f(x)}{s}+\frac{f_1(x)}{s^{1+\alpha}}+\frac{f_2(x)}{s^{1+2\alpha}}\right) \tag{22}$$

Again, multiply both sides of (22) by $s^{2\alpha+1}$to get:

$$s^{2\alpha+1}LRes_2(x,s)=f_2(x)+\delta f_1''(x)+\delta\frac{f_2''(x)}{s^\alpha}+\phi(x)f_1(x)+\frac{\phi(x)f_2(x)}{s^\alpha}. \tag{23}$$

Taking the limit $s\to\infty$ of both sides of (23) and using the fact in (18) and solving the resulting algebraic system, we can get the following simplified form:

$$f_2(x)=\phi(x)f_1(x)-\delta f_1''(x). \tag{24}$$

Furthermore, to determine the third unidentified parameters' value $U_3(x,s)$ substitute kth-truncated series of (16), $U_3(x,s)=\frac{f(x)}{s}+\frac{f_1(x)}{s^{1+\alpha}}+\frac{f_2(x)}{s^{1+2\alpha}}+\frac{f_3(x)}{s^{1+3\alpha}}$ into the kth-LRF, $LRes_k(\mathrm{x},\mathrm{s})$ of (18), to get the following:

$$LRes_3(x,s)=\frac{f_1(x)}{s^{1+\alpha}}+\frac{f_2(x)}{s^{1+2\alpha}}+\frac{f_3(x)}{s^{1+3\alpha}}+\frac{\delta}{s^\alpha}\left(\frac{f''(x)}{s}+\frac{f_1''(x)}{s^{1+\alpha}}+\frac{f_2''(x)}{s^{1+2\alpha}}+\frac{f_3''(x)}{s^{1+3\alpha}}\right)+\frac{\phi(x)}{s^\alpha}\left(\frac{f(x)}{s}+\frac{f_1(x)}{s^{1+\alpha}}+\frac{f_2(x)}{s^{1+2\alpha}}+\frac{f_3(x)}{s^{1+3\alpha}}\right). \tag{25}$$

We can get easily:

$$f_3(x)=\phi(x)f_2(x)-\delta f_2''(x). \tag{26}$$

Therefore, Eq. (16) has the following numerical solutions:

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$$U(x,s) = \frac{f(x)}{s} + \frac{\phi(x)f(x) - \delta f''(x)}{s^{1+\alpha}} + \frac{\phi(x)f_1(x) - \delta f_1''(x)}{s^{1+2\alpha}} + \frac{\phi(x)f_2(x) - \delta f_2''(x)}{s^{1+3\alpha}} + \cdots. \quad (27)$$

Using the reverse Laplace transformation of Eq. (27) yields the 4th-approximate LRPS result for Eqs. (8) and (9) in the next series structure:

$$u(x,t) = f(x) + \frac{\phi(x)f(x) - \delta f''(x)}{\Gamma(1+\alpha)} t^{\alpha} + \frac{\phi(x)f_1(x) - \delta f_1''(x)}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{\phi(x)f_2(x) - \delta f_2''(x)}{\Gamma(1+3\alpha)} t^{3\alpha} + \cdots. \quad (28)$$

4. Application

Example 4.1: Take a look at the single-dimensional linear physical equation below:

$$D_t^{\alpha} u(x,t) + 2u_{xx}(x,t) = 0, x \in \mathbb{R}, t \geq 0, 0 < \alpha \leq 1. \quad (29)$$

With initial condition:

$$u(x,0) = \cos(5x). \quad (30)$$

By comparing (29) and (30) with (8) and (9), we find that, $\delta = 2$, $\phi(x) = 0$ and $f(x) = \cos 5x$. Thus, we can calculate the fourth-approximate LRPS solutions for (29) and (30). The first step for that is in order to use transform of Laplace to system (29) to transfer it to the Laplace space as follows:

$$\mathcal{L}\{D_t^{\alpha} u(x,t)\} + 2\mathcal{L}\{u_{xx}(x,t)\} = \mathcal{L}\{0\}. \quad (31)$$

considering that $U(x,s) = \mathcal{L}\{u(x,t)\}$ can be expressed as follows:

$$s^{\alpha} U(x,s) - s^{\alpha-1}\cos(5x) + 2U_{xx}(x,s) = 0. \quad (32)$$

Dividing Eq. (32) on s^{α} gives a new form to it as follows:

$$U(x,s) - \frac{\cos(5x)}{s} - \frac{2}{s^{\alpha}} U_{xx}(x,s) = 0. \quad (33)$$

We consider the answer to (33), $U(x,s)$ has expansions in fractional Lorient series form as following:

$$U(x,s) = \sum_{i=0}^{\infty} \frac{f_i(x)}{s^{1+i\alpha}}, 0 < \alpha \leq 1, s > 0, \quad (34)$$

The first coefficient of (34), can be rewritten as:

$$U(x,s) = \frac{\cos(5x)}{s} + \sum_{i=1}^{\infty} \frac{f_i(x)}{s^{1+i\alpha}}, 0 < \alpha \leq 1, s > 0. \quad (35)$$

Now, we construction LRPS solution is defining the LRF of the algebraic system (33) as follows:

$$LRes_1(x,s) = U(x,s) - \frac{\cos(5x)}{s} + \frac{1}{s^{\alpha}} U_{xx}(x,s), \quad (36)$$

If we considered the kth-truncated series of Eq (35) that is given by:

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$$U_k(x,s)=\frac{\cos(3x)}{s}+\sum_{i=1}^{k}\frac{f_i(x)}{s^{1+i\alpha}},0<\alpha\le 1,x\in I,s>0, \tag{37}$$

the kth-LRF may be defined:

$$LRes_k(x,s)=U_k(x,s)-\frac{\cos(5x)}{s}+\frac{1}{s^{\alpha}}(U_k)_{xx}(x,s), \tag{38}$$

Therefore, the solution of (38) in a series form is:

$$U_4(x,s)=\frac{\cos(5x)}{s}-50\frac{\cos(5x)}{s^{\alpha+1}}+12500\frac{\cos(5x)}{s^{2\alpha+1}}-1562500\frac{\cos(5x)}{s^{3\alpha+1}}+195312500\frac{\cos(5x)}{s^{4\alpha+1}} \tag{39}$$

The reverse Laplace transformation of Eq. (39) yields the 5th-approximate LRPS in the following series form:

$$u_4(x,t)=\cos(5x)-\frac{50t^{\alpha}\cos(5x)}{\Gamma(1+\alpha)}+\frac{12500t^{2\alpha}\cos(5x)}{\Gamma(1+2\alpha)}-\frac{1562500t^{3\alpha}\cos(5x)}{\Gamma(1+3\alpha)}+\frac{195312500t^{4\alpha}\mathrm{Cos}[5x]}{\Gamma(1+4\alpha)}. \tag{40}$$

The exact result is:

$$u(x,t)=\frac{t^{\alpha}\cos(5x)}{\Gamma[1+\alpha]}. \tag{41}$$

Examine a few numerical findings of the answer established in Eq. (40), then to assess the correctness of the approximation of the approach utilized, we will employ two types of mistakes, as follows:

$$Abs.Err.(x,t)=|\theta(x,t)-\theta_k(x,t)|, \tag{42}$$

and

$$Re.Err.(x,t)=\left|\frac{\theta(x,t)-\theta_k(x,t)}{\theta(x,t)}\right|, \tag{43}$$

where $\theta(x,t)$ is the precise answer, while $\theta_k(x,t)$ represents an approximation of the answer to type k.

Tables 4.1display Here are the numerical outcomes for Example 4.1's resolution for Eq. (40) as $\alpha=1$, together with the accompanying absolute as well as relative errors. The findings show the correctness of the tenth-approximate LRPS answers provided. It is clear from the results that the region of convergence varies with different α values so that the convergence region becomes smaller when the α value decreases. The table demonstrate that the convergence area is $R\times[0,0.4]$ for $\alpha=1$.

Figure 4.1 display the tenth approximation LRPS results graph; $u_{10}(x,t)$ and solution of exact which are provided in Eq. (39) furthermore, the precise answer when $\alpha=1$. This graph illustrates the actions for the remedies at $\alpha=1$, α and confirm the region of convergence shown in the tables. Figure 4.1 illustrate the agreement among the precise answer and tenth approximation at $\alpha=1$.

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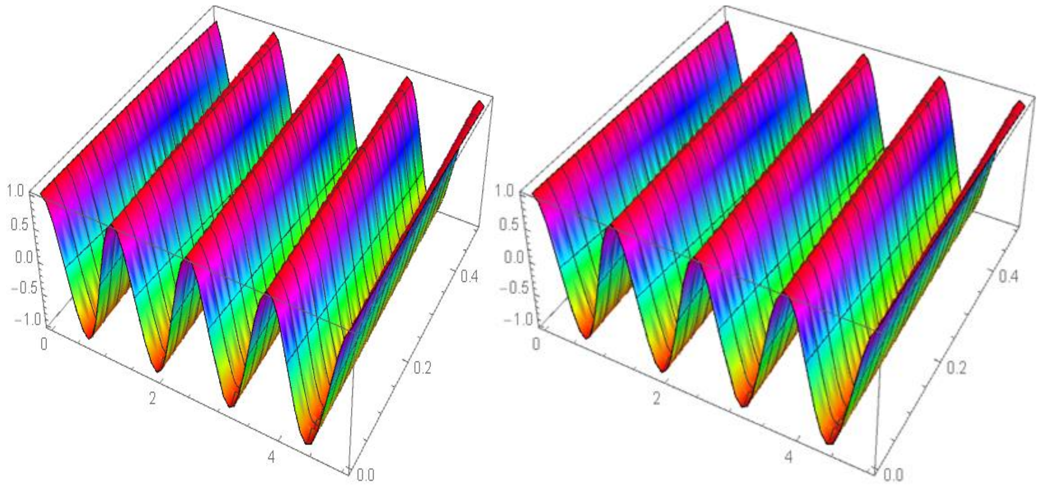
Table 1: Numerical comparisons between the exact value of $u(x,t)$ and the 10th-approximation of $u(x,t)$ at $\alpha = 1$

x	t	10th-approximation	$u(x,t)$	Re.Err.	Abs. Err.
0	0.1	0.995004	0.995004	7.4264×10^{-14}	5.5862×10^{-14}
2.5		0.999435	0.999435	4.3658×10^{-12}	3.5742×10^{-12}
5		0.999464	0.999464	7.6248×10^{-11}	8.67293×10^{-11}
0	0.2	0.980067	0.980067	9.36976×10^{-11}	7.7365×10^{-11}
2.5		0.991085	0.991085	5.23765×10^{-10}	4.9625×10^{-11}
5		0.997739	0.997739	2.7529×10^{-10}	1.76492×10^{-10}
0	0.3	0.955336	0.955336	6.83686×10^{-9}	6.27531×10^{-10}
2.5		0.972833	0.972833	9.29763×10^{-9}	5.29633×10^{-9}
5		0.986045	0.986045	2.86341×10^{-8}	5.28621×10^{-8}
0	0.4	0.921061	0.921061	5.2863×10^{-7}	7.27536×10^{-9}
2.5		0.94486	0.94486	4.6205×10^{-6}	7.28528×10^{-8}
5		0.964499	0.964498	7.28643×10^{-6}	8.29631×10^{-7}



(a)$u_{10}(x,t;\alpha = 1)$ (b)$u(x,t;\alpha = 1)$

Figure 1. The surface graphs of the 10th approximations of $u(x,t)$, and exact solution at $\alpha = 1$.

5. Discussion and results

To delve more deeply into the efficacy of the Laplace Residual Power Series Method (LRPSM) in solving Equation (40), particularly under varying conditions of the fractional order parameter α, it is insightful to expand upon several critical areas of analysis:

In-Depth Analysis:

Convergence Analysis

- Influence of α on Convergence: Explore in detail how changes in α affect the region of convergence. This analysis should include theoretical considerations of the method's stability and how fractional order variations impact convergence rates and error magnitudes.

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- Convergence Rate Variations: Examine how the rate at which the series solution converges to the exact solution varies with different $\alpha\alpha$ values. Highlight the dynamics for $\alpha = 1$ and compare these with other α values to understand rate differentials.
- Error Analysis
- Detailed Error Insights: Delve into the absolute and relative errors presented in Table 4.1, analyzing why error magnitudes may fluctuate with changes in α. This discussion should cover potential practical implications of these errors when utilizing LRPSM.
- Enhanced Error Metrics: Propose integrating additional error metrics such as mean squared error (MSE) or normalized root mean square error (NRMSE) to enrich the evaluation of approximation accuracy.

Graphical Interpretations

- Analysis of Figure 4.1: Enhance interpretation by discussing the behavior of the series solutions versus the exact solutions graphically depicted. Emphasize notable trends or points that particularly showcase the effectiveness of the LRPS method for $\alpha = 1$.
- Visual Comparison Enhancements: Recommend the inclusion of more graphical comparisons at varying α levels for a more intuitive grasp of α's impact on solution behavior.
- Theoretical and Practical Implications
- Theoretical Foundations: Elaborate on the mathematical rationale behind observed phenomena, especially the dependency of convergence regions on α. Explore the characteristics of fractional differential operators and their influence on solution stability.
- Real-World Applications: Discuss how these findings might be applied in fields such as materials science, control theory, and signal processing, where fractional differential equations are prevalent.

Neutrosophic Application

6. Application

Example 6.1: Take a look at the single-dimensional linear physical equation below:

$$D_t^{\alpha} u(x+\mathrm{yI},t) + 2u_{xx}(x+\mathrm{yI},t) = 0, x \in \mathbb{R}, t \geq 0, 0 < \alpha \leq 1. \quad (29j)$$

With initial condition:

$$u(x+\mathrm{yI},0) = \cos(5x+5\mathrm{yI})\,. \quad (30j)$$

By comparing (29j) and (30j) with (8) and (9), we find that,$\delta = 2$, $\phi(x+\mathrm{yI}) = 0$ and $f(x+\mathrm{yI}) = \cos 5(x+\mathrm{yI})$.Thus, we can calculate the fourth-approximate LRPS solutions for (29j) and (30j). The first step for that is in order to use transform of Laplace to system (29) to transfer it to the Laplace space as follows:

$$\mathcal{L}\{D_t^{\alpha} u(x+\mathrm{yI},t)\} + 2\mathcal{L}\{u_{x+\mathrm{yI}x+\mathrm{yI}}(x+\mathrm{yI},t)\} = \mathcal{L}\{0\}. \quad (31j)$$

considering that $U(x+\mathrm{yI},s) = \mathcal{L}\{u(x+\mathrm{yI},t)\}$ can be expressed as follows:

$$s^{\alpha} U(x+\mathrm{yI},s) - s^{\alpha-1}\cos(5(x+\mathrm{yI})) + 2U_{x+\mathrm{yI}x+\mathrm{yI}}(x+\mathrm{yI},s) = 0. \quad (32j)$$

Dividing Eq. (32) on s^{α}gives a new form to it as follows:

$$U(x+\mathrm{yI},s) - \frac{\cos(5(x+\mathrm{yI}))}{s} - \frac{2}{s^{\alpha}} U_{x+\mathrm{yI}x+\mathrm{yI}}(x+\mathrm{yI},s) = 0\,. \quad (33j)$$

We consider the answer to (33), $U(x+\mathrm{yI},s)$ has expansions in fractional Lorient series form as following:

$$U(x+\mathrm{yI},s) = \sum_{i=0}^{\infty} \frac{f_i(x+\mathrm{yI})}{s^{1+i\alpha}}, 0 < \alpha \leq 1, s > 0, \quad (34j)$$

The first coefficient of (34), can be rewritten as:

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$$U(x+\mathrm{yI},s)=\frac{\cos(5(x+\mathrm{yI}))}{s}+\sum_{i=1}^{\infty}\frac{f_i(x+\mathrm{yI})}{s^{1+i\alpha}},0<\alpha\leq 1,s>0. \quad (35j)$$

Now, we construction LRPS solution is defining the LRF of the algebraic system (33) as follows:

$$LRes_1(x+\mathrm{yI},s)=U(x+\mathrm{yI},s)-\frac{\cos(5(x+\mathrm{yI}))}{s}+\frac{1}{s^\alpha}U_{x+\mathrm{yI}x+\mathrm{yI}}(x+\mathrm{yI},s), \quad (36j)$$

If we considered the kth-truncated series of Eq (35) that is given by:

$$U_k(x+\mathrm{yI},s)=\frac{\cos(3(x+\mathrm{yI}))}{s}+\sum_{i=1}^{k}\frac{f_i(x+\mathrm{yI})}{s^{1+i\alpha}},0<\alpha\leq 1,x\in I,s>0, \quad (37j)$$

the kth-LRF may be defined:

$$LRes_k(x+\mathrm{yI},s)=U_k(x+\mathrm{yI},s)-\frac{\cos(5(x+\mathrm{yI}))}{s}+\frac{1}{s^\alpha}(U_k)_{x+\mathrm{yI}x+\mathrm{yI}}(x+\mathrm{yI},s), \quad (38j)$$

Therefore, the solution of (38j) in a series form is:

$$U_4(x,s)=\frac{\cos(5x)}{s}-50\frac{\cos(5x)}{s^{\alpha+1}}+12500\frac{\cos(5x)}{s^{2\alpha+1}}-1562500\frac{\cos(5x)}{s^{3\alpha+1}}+195312500\frac{\cos(5x)}{s^{4\alpha+1}} \quad (39j)$$

The reverse Laplace transformation of Eq. (39j) yields the 5th-approximate LRPS in the following series form:

$$u_4(x,t)=\cos(5x)-\frac{50t^\alpha\cos(5x)}{\Gamma(1+\alpha)}+\frac{12500t^{2\alpha}\cos(5x)}{\Gamma(1+2\alpha)}-\frac{1562500t^{3\alpha}\cos(5x)}{\Gamma(1+3\alpha)}+\frac{195312500t^{4\alpha}\mathrm{Cos}[5x]}{\Gamma(1+4\alpha)}. \quad (40j)$$

The exact result is:

$$u(x+\mathrm{yI},t)=\frac{t^\alpha\cos(5(x+\mathrm{yI}))}{\Gamma[1+\alpha]}. \quad (41j)$$

Examine a few numerical findings of the answer established in Eq. (40j), then to assess the correctness of the approximation of the approach utilized, we will employ two types of mistakes, as follows:

$$Abs.Err.(x+\mathrm{yI},t)=|\theta(x+\mathrm{yI},t)-\theta_k(x+\mathrm{yI},t)|, \quad (42j)$$

and

$$Re.Err.(x+\mathrm{yI},t)=\left|\frac{\theta(x+\mathrm{yI},t)-\theta_k(x+\mathrm{yI},t)}{\theta(x+\mathrm{yI},t)}\right|, \quad (43j)$$

where $\theta(x+\mathrm{yI},t)$ is the precise answer, while $\theta_k(x+\mathrm{yI},t)$ represents an approximation of the answer to type k.

7. Conclusions

We build analytical precise and approximate solutions to differential equations with unknown parameters, such as non - homogenous equations, fractional equations, and LRPS technique to fractional equations. It illustrates that LRPS is a straightforward, effective approach that may be broadly used to solve many different PDEs with complex coefficients. The Laplace Residual Power Series (LRPS) method presents a powerful approach for addressing partial differential equations (PDEs) featuring unknown parameters, non-homogeneous elements, and

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fractional dimensions. Its direct and efficient implementation across a range of complex scenarios underscores its effectiveness. To further explore the capabilities of LRPS, it is insightful to compare this technique with other traditional methods used in solving differential equations, including the Finite Element Method (FEM), Finite Difference Method (FDM), and the Adomian Decomposition Method (ADM).

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