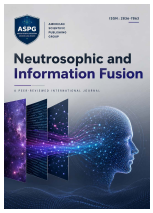
Neutrosophic and Information Fusion

♦ ISSN: 2836-7863 Vol: 04, No. 02, 2024 ♦ PP. 27–33

Article DOI: https://doi.org/10.54216/NIF.040205



FULL LENGTH ARTICLE

On the Problem of Inverting Discrete Self-Regression Models to Continuous Models

Taher Ahmed Jubbori[1,*]

[1] Computer Techniques Engineering Department, Al-Mustaqbal University, Babil, Iraq

Email: taherajubbori@mustaqbal-college.edu.iq

Received: January 17, 2024 Accepted: July 12, 2024 ⋆ Corresponding author

ABSTRACT

In this paper, we discuss the problem of converting auto-regression models at a discrete time into auto-regression models at continuous time, based on the idea of converting auto-regression models from first to second order. We study the general formula of AR(p) and its ability to convert from discrete to continuous time. Also, we use our model to study some real-life problems as a direct application of our approach.

Keywords: Discrete-time ▪ Continuous time ▪ Self-regression ▪ Mathematical model

1. INTRODUCTION

A time series is a set of observed values of a certain phenomenon at successive times, and time-series analysis is used to determine the reality of changes in the values of observations of the phenomenon and to predict future values for it. This is done by building a mathematical model of the series, either linear or non-linear, depending on the nature of the studied time series. Such models are usually expressed as differential equations with constant coefficients. When these models are represented in continuous time, they can be expressed as differential equations with constant coefficients.

Time series are among the important mathematical concepts adopted for scientific research and studies and are the main pillar of development plans, the development of planning methods, and a key input for resolutely addressing some of the existing problems and gaps in the medical, economic, military, and service aspects [3].

The importance of time series in scientific research and studies is highlighted by their dependence on the observed values of the phenomenon and the study of how these values change over equal time periods. This makes it possible to determine the reality of changes in the observed phenomenon, understand their causes and influencing factors, and build a suitable mathematical model to express its time series.

In this research, the process of reversing auto-regression models at discrete time into auto-regression models at continuous time was studied based on the basic idea of Priestley. In the applied aspect, we relied on a time-series study in the medical field. The research dealt with viral hepatitis Type-B, which is a health and social problem in many countries and threatens many people with death or disability [1, 5].

The available statistics on the number of infected people were obtained from the main blood bank in Nineveh Governorate from 1997 until 2006. The data were divided at the level of months into 120 observations representing the numbers of people with this disease. There are many studies in the field of embedding and converting, among which we mention the following.

Tong and Chan (1987) published their joint research on embedding auto-regression models at discrete time into auto-regression models at continuous time. They explained that it is not always possible to embed the real intermittent parameter values of the first-order auto-regression model AR(1) into the real continuous parameter values of a first-order auto-

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regression model [9].

He and Wang (1989) dealt with the immersion of intermittent ARMA(p,q) models into continuous ARMA(p',q') models. The researchers explained that the real intermittent parameter values for the causal autoregressive moving-average model ARMA(p,q), for every $q<p$, can be embedded in real continuous parameter values for an ARMA(p',q') model for every $q'<p'$ [7]. Brockwell and Brockwell (1999) showed that an autoregressive integrated moving-average model in which one root of the moving-average model lies on the unit circle cannot be embedded in any continuous-time autoregressive moving-average model [6].

In the field of converting discrete-time auto-regression models to continuous-time auto-regression models, Priestley (1981) demonstrated the mechanism of converting intermittent first-order AR(1) and second-order AR(2) auto-regression models to continuous first- and second-order auto-regression models, respectively [8].

2. THE THEORETICAL ASPECT

Regression models in intermittent time can be expressed as difference equations with constant coefficients, whereas continuous-time models are written as differential equations with constant coefficients [8]. Because most studies collect readings at intermittent times even though the phenomena occur continuously, we try in this paper to reverse intermittent-time auto-regression models into continuous-time auto-regression models.

Let us have the following form:

$$\sum_{i=0}^{n} a_i X_{t-i} = Y_t.$$

It can be written in the following form:

$$X_t = a_1 X_{t-1} - \cdots - a_n X_{t-n} + Y_t.$$

It is an auto-regression model of rank n, where $\{Y_t\}$ is white noise and t represents time for each $t = 0, \pm1, \pm2, \ldots$. The constants a_i represent the parameters of the model, where $i = 1,2,\ldots,n$ and $a_0 = 1$.

Priestley reverses the first- and second-order auto-regression models in discontinuous time to first- and second-order auto-regression models in continuous time. In this research, we take advantage of the same idea to reverse auto-regression models of rank p, where $p = 1,2,\ldots,k$.

2.1 Inversion of the Third-Order Auto-Regression Model: AR(3)

Let the third-order self-regression model be

$$X_n + a_1 X_{n-1} + a_2 X_{n-2} + a_3 X_{n-3} = Y_n,$$

where a_i are constants for $i = 1,2,3$, $\{Y_n\}$ is white noise, and $n = 0, \pm1, \pm2, \ldots$. Based on a formula similar to that used by Priestley for the inverse of AR(p), where $p = 1,2$, we obtain

$$\begin{aligned} -a_3\Delta^3 X_n + (a_2+3a_3)\Delta^2 X_n - (a_1+2a_2+3a_3)\Delta X_n \\ + (1+a_1+a_2+a_3)X_n = Y_n. \end{aligned} \tag{1}$$

Here, Δ denotes the difference operator and includes the backward-shift factor B:

$$\Delta = (1-B).$$

Let $t = n\Delta t$. Dividing both sides of Equation (1) by $-a_3(\Delta t)^3$ gives

$$\begin{aligned} \frac{\Delta^3 X_n}{(\Delta t)^3} - \frac{(a_2+3a_3)\Delta^2 X_n}{a_3(\Delta t)^3} + \frac{(a_1+2a_2+3a_3)\Delta X_n}{a_3(\Delta t)^3} \\ - \frac{1+a_1+a_2+a_3}{a_3(\Delta t)^3} X_n = -\frac{Y_n}{a_3(\Delta t)^3}, \end{aligned}$$

or, equivalently,

$$\begin{aligned} \frac{\Delta^3 X_n}{(\Delta t)^3} - \frac{a_2+3a_3}{a_3\Delta t}\frac{\Delta^2 X_n}{(\Delta t)^2} + \frac{a_1+2a_2+3a_3}{a_3(\Delta t)^2}\frac{\Delta X_n}{\Delta t} \\ - \frac{1+a_1+a_2+a_3}{a_3(\Delta t)^3} X_n = -\frac{Y_n}{a_3(\Delta t)^3}. \end{aligned}$$

We assume that

$$\begin{aligned} \varepsilon(t) &= -\frac{Y_n}{a_3(\Delta t)^3}, \\ \alpha_1 &= -\frac{a_2+3a_3}{a_3\Delta t}, \\ \alpha_2 &= \frac{a_1+2a_2+3a_3}{a_3(\Delta t)^2}, \\ \alpha_3 &= -\frac{1+a_1+a_2+a_3}{a_3(\Delta t)^3}. \end{aligned}$$

Taking Δt toward zero, we obtain the formal third-order differential equation

$$\dddot{X}(t) + \alpha_1 \ddot{X}(t) + \alpha_2 \dot{X}(t) + \alpha_3 X(t) = \varepsilon(t),$$

which represents a continuous third-order auto-regression model.

2.2 Inversion of the Fourth-Order Auto-Regression Model: AR(4)

Let us have the following fourth-order auto-regression model:

$$X_n + a_1 X_{n-1} + \cdots + a_4 X_{n-4} = Y_n.$$

Here, a_i are constants for $i = 1,2,3,4$, $\{Y_n\}$ is white noise, and $n = 0, \pm1, \pm2, \ldots$. Depending on the relations obtained for the inverse of AR(p), where $p = 1,2,3$, we obtain

$$\begin{aligned} a_4\Delta^4 X_n - (a_3+4a_4)\Delta^3 X_n + (a_2+3a_3+6a_4)\Delta^2 X_n \\ - (a_1+2a_2+3a_3+4a_4)\Delta X_n \\ + (1+a_1+a_2+a_3+a_4)X_n = Y_n. \end{aligned} \tag{2}$$

Let $t = n\Delta t$. Dividing Equation (2) by $a_4(\Delta t)^4$ and rearrang-

ing gives

$$\frac{\Delta^4 X_n}{(\Delta t)^4} - \frac{a_3+4a_4}{a_4\Delta t}\frac{\Delta^3 X_n}{(\Delta t)^3} + \frac{a_2+3a_3+6a_4}{a_4(\Delta t)^2}\frac{\Delta^2 X_n}{(\Delta t)^2} - \frac{a_1+2a_2+3a_3+4a_4}{a_4(\Delta t)^3}\frac{\Delta X_n}{\Delta t} + \frac{1+a_1+a_2+a_3+a_4}{a_4(\Delta t)^4}X_n = \frac{Y_n}{a_4(\Delta t)^4}.$$

Thus,

$$\begin{aligned}\varepsilon(t) &= \frac{Y_n}{a_4(\Delta t)^4},\\ \alpha_1 &= -\frac{a_3+4a_4}{a_4\Delta t},\\ \alpha_2 &= \frac{a_2+3a_3+6a_4}{a_4(\Delta t)^2},\\ \alpha_3 &= -\frac{a_1+2a_2+3a_3+4a_4}{a_4(\Delta t)^3},\\ \alpha_4 &= \frac{1+a_1+a_2+a_3+a_4}{a_4(\Delta t)^4}.\end{aligned}$$

Taking Δt toward zero gives the formal fourth-order differential equation

$$X^{(4)}(t)+\alpha_1 X^{(3)}(t)+\alpha_2 X^{(2)}(t)+\alpha_3 X^{(1)}(t)+\alpha_4 X(t)=\varepsilon(t),$$

which represents a continuous fourth-order auto-regression model.

2.3 Inversion of the Fifth-Order Auto-Regression Model: AR(5)

Let us have the fifth-order auto-regression model

$$X_n + a_1 X_{n-1} + a_2 X_{n-2} + \cdots + a_5 X_{n-5} = Y_n,$$

where a_i are constants for $i = 1,2,\ldots,5$, $\{Y_n\}$ is white noise, and $n = 0,\pm1,\pm2,\ldots$. Using the relations for the inverse of AR(p), where $p = 1,2,3,4$, we obtain

$$\begin{aligned}-a_5\Delta^5 X_n &+ (a_4+5a_5)\Delta^4 X_n\\ &- (a_3+4a_4+10a_5)\Delta^3 X_n\\ &+ (a_2+3a_3+6a_4+10a_5)\Delta^2 X_n\\ &- (a_1+2a_2+3a_3+4a_4+5a_5)\Delta X_n\\ &+ (1+a_1+a_2+a_3+a_4+a_5)X_n = Y_n.\end{aligned} \tag{3}$$

Let $t = n\Delta t$. Dividing Equation (3) by $-a_5(\Delta t)^5$ gives

$$\frac{\Delta^5 X_n}{(\Delta t)^5} - \frac{a_4+5a_5}{a_5\Delta t}\frac{\Delta^4 X_n}{(\Delta t)^4} + \frac{a_3+4a_4+10a_5}{a_5(\Delta t)^2}\frac{\Delta^3 X_n}{(\Delta t)^3} - \frac{a_2+3a_3+6a_4+10a_5}{a_5(\Delta t)^3}\frac{\Delta^2 X_n}{(\Delta t)^2} + \frac{a_1+2a_2+3a_3+4a_4+5a_5}{a_5(\Delta t)^4}\frac{\Delta X_n}{\Delta t} - \frac{1+a_1+a_2+a_3+a_4+a_5}{a_5(\Delta t)^5}X_n = -\frac{Y_n}{a_5(\Delta t)^5}.$$

Let

$$\begin{aligned}\varepsilon(t) &= -\frac{Y_n}{a_5(\Delta t)^5},\\ \alpha_1 &= -\frac{a_4+5a_5}{a_5\Delta t},\\ \alpha_2 &= \frac{3a_3+4a_4+10a_5}{a_5(\Delta t)^2},\\ \alpha_3 &= -\frac{a_2+3a_3+6a_4+10a_5}{a_5(\Delta t)^3},\\ \alpha_4 &= \frac{a_1+2a_2+3a_3+4a_4+5a_5}{a_5(\Delta t)^4},\\ \alpha_5 &= -\frac{1+a_1+a_2+a_3+a_4+a_5}{a_4(\Delta t)^5}.\end{aligned}$$

When $\Delta t \to 0$, the fifth-order differential equation becomes

$$X^{(5)}(t)+\alpha_1 X^{(4)}(t)+\alpha_2 X^{(3)}(t)+\alpha_3 X^{(2)}(t) + \alpha_4 X^{(1)}(t)+\alpha_5 X(t)=\varepsilon(t).$$

Here, $X^{(5)}(t)$ is the rank of the equation, which represents a continuous fifth-order auto-regression model.

2.4 Inversion of the Auto-Regression Model of Rank p: AR(p)

By continuing to find the limits of the displacement factor Δ^i, for each $i = 1,2,3,4,5$, it is possible to design a table for quickly calculating the expansion coefficients of Δ^i, for each $i = 1,2,\ldots,p$, and to obtain the numerical values $Z_{i,j}$ of the coefficients associated with Δ^i. Table 1 gives these values for every $i = 1,2,\ldots,p$ and $j = 1,2,\ldots,i$.

The columns represent the coefficients of $X_n, X_{n-1},\ldots$, and the rows represent the difference operators $\Delta^0,\Delta^1,\Delta^2,\ldots$. Here, $\Delta = (1-B)$ and B is the backward-shift operator, previously defined by $B^r X_n = X_{n-r}$. The quantity $Z_{i,j}$ is represented by

$$Z_{i,j} = \sum_{f=i-1}^{0} \left|Z_{f,j-1}\right|.$$

Continuing in a similar way gives AR(p). The inverse operation can be generalized to all p as shown in Table 2.

From Tables 1 and 2, the following relations are deduced:

$$\left.\begin{array}{ll} Z_{i,j} = -i & \text{for } j = 1,\\ Z_{i,j} = (-1)^i & \text{for } i = j,\\ Z_{i,j} = 1 & \text{for } j = 0,\\ Z_{i,j} = |Z_{i-1,j}| + |Z_{i-1,j-1}| & \text{for } i > 1,\ j \geq 2,\\ Z_{i,j} = 0 & \text{for } i < j,\ j \geq 1,\\ a_0 = 1. & \end{array}\right\} \tag{4}$$

Based on Table 1 and the conclusions in Equation (4), the general inversion formula from discontinuous models to continuous models can be obtained. The general formula for finding AR(p) in continuous time is

$$\text{AR}(p): \quad \sum_{i=0}^{p} S_i = Y_n, \tag{5}$$

Table 1. Calculation of the coefficients of the difference operator $\Delta(\Delta^0,\Delta^1,\Delta^2,\ldots)X_{n-j}$.

	X_n	X_{n-1}	X_{n-2}	X_{n-3}	X_{n-4}	X_{n-5}	$\cdots$	X_{n-p}
$a_0\Delta^0$	$Z_{0,0}=1$	0	0	0	0	0	$\cdots$	0
$a_1\Delta^1$	$Z_{1,0}=1$	-1	0	0	0	0	$\cdots$	0
$a_2\Delta^2$	$Z_{2,0}=1$	-2	1	0	0	0	$\cdots$	0
$a_3\Delta^3$	$Z_{3,0}=1$	-3	3	-1	0	0	$\cdots$	0
$a_4\Delta^4$	$Z_{4,0}=1$	-4	6	-4	1	0	$\cdots$	0
$a_5\Delta^5$	$Z_{5,0}=1$	-5	10	-10	5	-1	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$

Table 2. Representation and generalization of the inverse process X_{n-j}.

AR(p)	Δ^0X_n	Δ^1X_n	Δ^2X_n	Δ^3X_n	Δ^4X_n	Δ^5X_n	$\cdots$
a_0	$Z_{0,0}$	0	0	0	0	0	$\cdots$
a_1	$Z_{1,0}$	$Z_{1,1}$	0	0	0	0	$\cdots$
a_2	$Z_{2,0}$	$Z_{2,1}$	$Z_{2,2}$	0	0	0	$\cdots$
a_3	$Z_{3,0}$	$Z_{3,1}$	$Z_{3,2}$	$Z_{3,3}$	0	0	$\cdots$
a_4	$Z_{4,0}$	$Z_{4,1}$	$Z_{4,2}$	$Z_{4,3}$	$Z_{4,4}$	0	$\cdots$
a_5	$Z_{5,0}$	$Z_{5,1}$	$Z_{5,2}$	$Z_{5,3}$	$Z_{5,4}$	$Z_{5,5}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

where

$$S_i = a_i \sum_{j=0}^{i} Z_{i,j}\Delta^j X_n, \tag{6}$$

and

$$Z_{i,j} = \sum_{f=i-1}^{0} \left|Z_{f,j-1}\right|. \tag{7}$$

By dividing Equation (5) by $(\Delta t)^p$ multiplied by the coefficient of $\Delta^p X_n$, we arrive at the general formula AR(p). To verify that the formula is correct, it is applied to first- and second-order discontinuous-time auto-regression models.

2.4.1 Continuous-Time Auto-Regression Model of the First Order: AR(1)

Let the discrete-time first-order auto-regression model be

$$X_n - a_1X_{n-1} = Y_n. \tag{8}$$

Applying Equation (5) for $p=1$ gives

$$\text{AR}(1): \quad \sum_{i=0}^{1} S_i = Y_n,$$

and hence

$$S_0 + S_1 = Y_n. \tag{9}$$

Applying Equations (4) and (6) to Equation (9) gives

$$S_0 = a_0 \sum_{j=0}^{0} Z_{0,j}\Delta^j X_n = a_0 Z_{0,0}\Delta^0 X_n, \qquad \therefore S_0 = X_n.$$

We find S_1 as follows:

$$S_1 = a_1\left(Z_{1,0}\Delta^0X_n + Z_{1,1}\Delta^1X_n\right) = a_1X_n - a_1\Delta X_n.$$

Substituting S_i, for $i=0,1$, into Equation (9) gives

$$\text{AR}(1): \quad X_n + a_1X_n - a_1\Delta X_n = Y_n,$$

or

$$\text{AR}(1): \quad (1+a_1)X_n - a_1\Delta X_n = Y_n. \tag{10}$$

Dividing Equation (10) by $-a_1\Delta t$ gives

$$\frac{\Delta X_n}{\Delta t} - \frac{1+a_1}{a_1\Delta t}X_n = -\frac{Y_n}{a_1\Delta t},$$

which represents the first-order auto-regression equation in continuous time.

2.4.2 Continuous-Time Auto-Regression Model of the Second Order: AR(2)

Let the second-order difference equation be

$$X_n + a_1X_{n-1} + a_2X_{n-2} = Y_n. \tag{11}$$

Substituting $p=2$ into Equation (5) gives

$$\text{AR}(2): \quad \sum_{i=0}^{2} S_i = Y_n,$$

and

$$S_0 + S_1 + S_2 = Y_n. \tag{12}$$

For AR(1), both S_0 and S_1 were found. Applying Equations (4), (6), and (7) to Equation (12) gives

$$\begin{aligned} S_2 &= a_2\big((-1)^0Z_{2,0}\Delta^0X_n + (-1)^1Z_{2,1}\Delta^1X_n \\ &\qquad + (-1)^2Z_{2,2}\Delta^2X_n\big) \\ &= a_2X_n - 2a_2\Delta X_n + a_2\Delta^2X_n. \end{aligned}$$

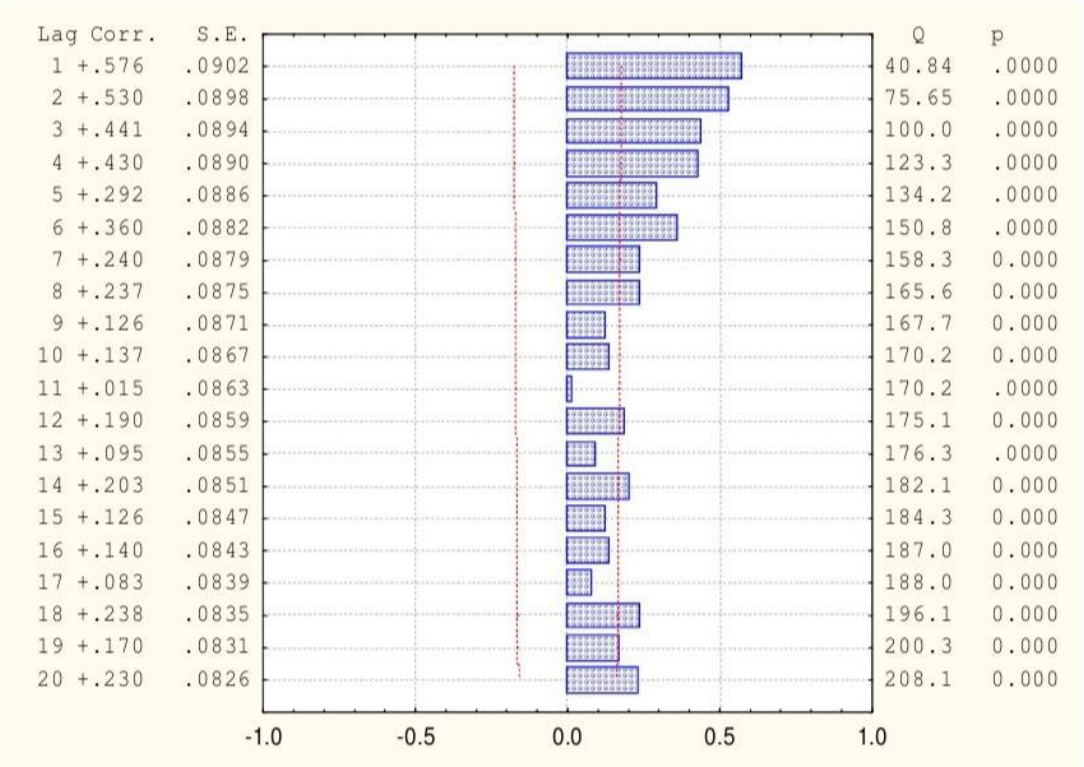


Figure 1. The autocorrelation function of the time series of the number of cases of viral hepatitis disease pattern-B for the period 1997–2006.

Therefore,

$$\sum_{i=0}^{2} S_i = X_n + a_1 X_n - a_1 \Delta X_n + a_2 X_n - 2a_2 \Delta X_n + a_2 \Delta^2 X_n,$$

which yields

$$\text{AR}(2): \quad a_2 \Delta^2 X_n - (a_1 + 2a_2)\Delta X_n + (1 + a_1 + a_2) X_n = Y_n. \tag{13}$$

Dividing Equation (13) by $a_2(\Delta t)^2$ gives

$$\frac{\Delta^2 X_n}{(\Delta t)^2} - \frac{a_1 + 2a_2}{a_2 \Delta t}\frac{\Delta X_n}{\Delta t} + \frac{1 + a_1 + a_2}{a_2 (\Delta t)^2} X_n = \frac{Y_n}{a_2 (\Delta t)^2}.$$

This represents the second-order auto-regression equation in continuous time and is identical to Priestley's inverse equation.

3. THE PRACTICAL PART

In this research, the time series of viral hepatitis Type-B was studied. It represents the number of people infected with this disease. The data were obtained from the virus division of the main blood bank for the years 1997–2006 at the monthly level. Figure 1 represents the autocorrelation function of the studied series.

From the autocorrelation function in Figure 1, we find that most of the autocorrelation coefficients fall outside the bounds $\mp 1.96/\sqrt{n}$, where n represents the number of observations. This indicates that the series is interconnected but close to the normal distribution, as shown in Figure 2.

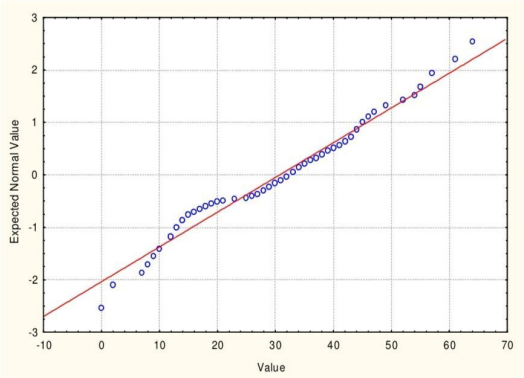


Figure 2. Natural time-series graph of the number of viral hepatitis C infections pattern-B for the period 1997–2006.

3.1 Modeling

Because the series is unstable, the first difference of the original series was taken, as shown in Figure 3, to convert it into a stable series; further details are given in [2].

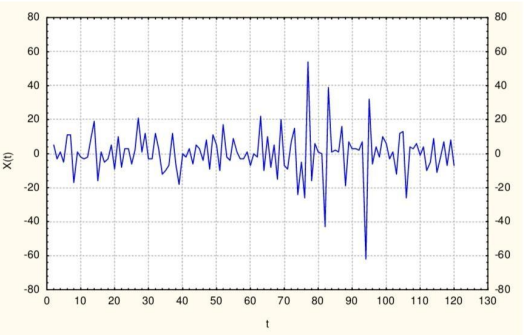


Figure 3. Altered time-series graph of the number of viral hepatitis disease infections pattern-B for the period 1997–2006.

The Box–Jenkins method was used to analyze and model the

time series of the number of cases of the disease. The method consists of three basic stages: identification, estimation, and diagnostic checking [4].

Using the Statistica-99 program to estimate the parameters of the diagnosed model by the approximate maximum-likelihood method, an auto-regression model of the first rank, AR(1), was obtained. The AIC and BIC (Bayesian Information Criterion) were used as follows:

$$\mathrm{AIC}(m) = n\ln(\sigma_z^2) + 2m,$$

where m is the number of parameters in the model, n is the number of observations, and σ_z^2 is the residual variance. The best rank is selected by the lowest value of AIC(m).

The BIC criterion is a development of the AIC criterion and is defined as follows [10]:

$$\mathrm{BIC}(m) = n\ln(\sigma_z^2) + m\ln(n).$$

The AR(1) model has the lowest value for the two criteria, so it was selected to represent the series. Table 3 shows the models fitted to the series according to the AIC, BIC, and MSE criteria.

Table 3. Convenient mathematical models of the series ARIMA(p,d,q).

p	d	MSE	AIC	BIC
1	1	144	550	556
2	1	140	552	560
3	1	136	554	565
4	1	137	556	570
5	1	131	558	574
6	1	132	560	579

The AR(1) form is expressed by

$$X_t - 0.4498X_{t-1} = Z_t. \tag{14}$$

The standard error is 0.08269, and BIC = 556.

3.2 Inverting the Discrete-Time Model to a Continuous-Time Model

Using Tables 1 and 2 and Equations (4)–(7), and because the obtained model is an auto-regression model in discontinuous time, we obtain the following continuous-time auto-regression model:

$$X^{(1)}(t) + \alpha_1 X(t) = \varepsilon(t).$$

Assuming that $\Delta t = 0.1$, Equation (14) gives

$$a_0 = 1, \qquad a_1 = -0.4498.$$

Therefore,

$$\alpha_1 = \frac{1+a_1}{a_1(\Delta t)} = -12.2321, \qquad \varepsilon(t) = \frac{Y_n}{a_1(\Delta t)}.$$

The stationarity condition of the proposed AR(1) model in discontinuous time is $|\mu| < 1$. The characteristic equation of the discrete-time AR(1) model is

$$\lambda - 0.4498 = 0,$$

where the root $\lambda_1 = \mu_1$. Thus,

$$\lambda_1 = 0.4498.$$

We note that $|\mu_1| < 1$, which indicates that the series represented by the AR(1) model is stable.

To test the stationarity of the continuous-time auto-regression model, we find its characteristic equation:

$$\lambda - 12.2321 = 0.$$

Here, $\lambda_1 = \mu_1$ represents the root, giving

$$\lambda_1 = 12.2321.$$

The stationarity condition for the inverse AR(1) model in continuous time is that the real part of the root be negative, that is, $\mathrm{Re}(\mu_1) < 0$. The real part of the obtained root is not negative, which indicates that the inverse model is unstable.

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