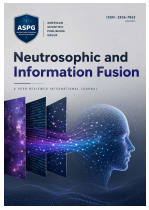
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FULL LENGTH ARTICLE

On the Numerical Solutions Based on the Exponential Finite Difference Method for the Kuramoto–Sivashinsky Equation and Numerical Stability Analysis

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ABSTRACT

In this paper, we solve the Kuramoto–Sivashinsky equation numerically by finite-difference methods using two different schemes: the fully implicit scheme and the exponential finite-difference scheme. Because of the fourth derivative in the equation, we suggest a treatment for the numerical solution of the two schemes by partitioning the mesh grid into five regions. The first region represents the first boundary condition, the second is at grid point x_1, the third represents grid points $x_2, x_3, \ldots, x_{n-2}$, the fourth represents grid point x_{n-1}, and the fifth is the second boundary condition. We also study numerical stability by the Fourier (Von Neumann) method for both schemes at all mesh points to ensure the stability of the points treated by the suggested approach. Two intervals with two initial conditions are used, and the numerical results obtained by these schemes are compared with the exact solution of the equation. Excellent agreement is found between the exact solution and the numerical solutions.

Keywords: Kuramoto–Sivashinsky equation ▪ Numerical stability ▪ Interval ▪ Exact solution ▪ Numerical solutions

1. INTRODUCTION

Partial differential equations have become a useful tool for describing natural phenomena in science and engineering models. It has therefore become necessary to solve these equations using numerical and sophisticated methods [15].

Partial differential equations appear in several branches of applied mathematics. Fluid dynamics, quantum mechanics, and electromagnetic theory are examples of areas described by partial differential equations. Analytical treatment of these equations is a very complex process that requires advanced mathematical methods; on the other hand, simple and generally efficient numerical methods make it easier to find sufficiently accurate approximate solutions [12].

Many researchers have studied the Kuramoto–Sivashinsky (K.S.) equation because of its complexity and its ability to describe a large variety of natural phenomena, such as the stability of sloping membrane fronts. Consequently, the equation has attracted considerable attention in the scientific community [3].

Kudryashov [7] used the simplest-equation method to search for exact solutions of nonlinear differential equations. Two basic ideas are employed: using general solutions of the simplest nonlinear differential equations and taking into account the possible singularities of the equation under study. The method was adjusted for the K.S. equation, which describes nonlinear waves in a heat-conducting fluid.

MacKenzie and Roberts [11] analyzed the nonlinear K.S. equation using centre-manifold theory. After dividing the physical domain into small elements and introducing internal boundary conditions that are later eliminated, they showed that holistic finite differences accurately model the dynamics.

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The K.S. equation was used to demonstrate that the method can be applied to nonlinear fourth-order equations and that the centre-manifold construction treats the evolution equation as a whole rather than as a sum of separate terms.

Al-Rawi and Al-Baker [1] solved the Korteweg–de Vries–Burgers equation using the fully implicit and exponential finite-difference methods. They studied numerical stability by the Fourier (Von Neumann) method at all mesh points, compared the numerical results with the exact solution, and found close agreement between the analytical and numerical solutions.

2. MATHEMATICAL MODEL

The Kuramoto–Sivashinsky equation is written as [10]

$$u_t = -uu_x - u_{xx} - u_{xxxx}, \qquad x \in [0, 32\pi]. \tag{1}$$

The boundary conditions are

$$\left.\begin{aligned} u(0,t) = u(32\pi,t) = 1, &\qquad t \geq 0,\\ u_{xx}(0,t) = u_{xx}(32\pi,t) = -\frac{1}{162}, &\qquad t \geq 0. \end{aligned}\right\} \tag{2}$$

For Example 1, the interval is $[0, 32\pi]$ and the initial condition is [10]

$$u(x,0) = \cos\left(\frac{x}{16}\right)\left[1 + \sin\left(\frac{x}{16}\right)\right]. \tag{3}$$

For Example 2, the interval is $[-30, 30]$ and the initial condition is [5]

$$u(x,0) = c + \frac{15}{19}\sqrt{\frac{11}{19}}\left[-9\tanh\big(k(x-x_0)\big) + 11\tanh^3\big(k(x-x_0)\big)\right], \tag{4}$$

where $x_0 = -12$, $c = 1.2$, and $k = \frac{1}{2}\sqrt{11/19}$.

3. NUMERICAL SOLUTION OF THE KURAMOTO–SIVASHINSKY EQUATION

The finite-difference method is one of the classical methods in the numerical analysis of differential equations and one of its most important applications. Finite differences also form the basis of other numerical techniques, including numerical differentiation and integration [13, 14].

3.1 Fully implicit finite-difference scheme

The method is called implicit because advancing the solution to level $j+1$ requires the solution of a linear system. The points at level $j+1$ depend on values at level j [4]. To find u_1^{j+1} at x_1, forward-difference approximations are used:

$$\frac{u_i^{j+1} - u_i^j}{k} = -u_i^j\left[\frac{-3u_i^{j+1} + 4u_{i+1}^{j+1} - u_{i+2}^{j+1}}{2h}\right] - \left[\frac{2u_i^{j+1} - 5u_{i+1}^{j+1} + 4u_{i+2}^{j+1} - u_{i+3}^{j+1}}{h^2}\right] - \left[\frac{3u_i^{j+1} - 14u_{i+1}^{j+1} + 26u_{i+2}^{j+1} - 24u_{i+3}^{j+1} + 11u_{i+4}^{j+1} - 2u_{i+5}^{j+1}}{h^4}\right].$$

Multiplying by k, setting $r = k/h^2$, and collecting the terms gives

$$\begin{aligned} u_i^j = &\left(1 + \frac{3hr}{2}u_i^j + 2r + \frac{3r}{h^2}\right)u_i^{j+1} \\ &+ \left(-2rhu_i^j - 5r - \frac{14r}{h^2}\right)u_{i+1}^{j+1} \\ &+ \left(-\frac{rh}{2}u_i^j + 4r + \frac{26r}{h^2}\right)u_{i+2}^{j+1} \\ &+ \left(-r - \frac{24r}{h^2}\right)u_{i+3}^{j+1} + \frac{11r}{h^2}u_{i+4}^{j+1} - \frac{2r}{h^2}u_{i+5}^{j+1}. \end{aligned} \tag{5}$$

For $i = 2, \ldots, n-2$, central-difference approximations are used. After simplification,

$$\begin{aligned} u_i^j = &\left(1 - 2r + \frac{6r}{h^2}\right)u_i^{j+1} + \left(\frac{rh}{2}u_i^j + r - \frac{4r}{h^2}\right)u_{i+1}^{j+1} \\ &+ \left(-\frac{rh}{2}u_i^j + r - \frac{4r}{h^2}\right)u_{i-1}^{j+1} + \frac{r}{h^2}u_{i+2}^{j+1} + \frac{r}{h^2}u_{i-2}^{j+1}. \end{aligned} \tag{6}$$

For u_{n-1}^{j+1}, backward-difference approximations produce

$$\begin{aligned} u_i^j = &\left(1 + \frac{3hr}{2}u_i^j + 2r + \frac{3r}{h^2}\right)u_i^{j+1} \\ &+ \left(-2rhu_i^j - 5r - \frac{14r}{h^2}\right)u_{i-1}^{j+1} \\ &+ \left(\frac{rh}{2}u_i^j + 4r + \frac{26r}{h^2}\right)u_{i-2}^{j+1} \\ &+ \left(-r - \frac{24r}{h^2}\right)u_{i-3}^{j+1} + \frac{11r}{h^2}u_{i-4}^{j+1} - \frac{2r}{h^2}u_{i-5}^{j+1}. \end{aligned} \tag{7}$$

3.1.1 Boundary conditions

To approximate the boundary conditions at $x = a$ and $x = b$, the exterior points are defined by

$$x_{-2} = x_0 - 2h, \qquad x_{-1} = x_0 - h,$$
$$x_{n+1} = x_n + h, \qquad x_{n+2} = x_n + 2h.$$

These points are outside the mesh, as illustrated in Figure 1.

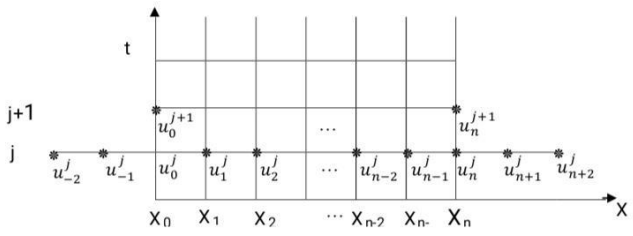


Figure 1. Points at which the mesh is located in plane j.

Using central differences for the second boundary condition gives

$$\frac{\partial^2 u(a,t)}{\partial x^2} = -\frac{1}{162}, \qquad \frac{\partial^2 u(b,t)}{\partial x^2} = -\frac{1}{162}.$$

At $i = 0$ and $i = n$, respectively, the second-order formula yields

$$u_{-1}^j = -u_1^j + 2u_0^j - \frac{h^2}{162}, \tag{8}$$

$$u_{n+1}^{j} = -u_{n-1}^{j} + 2u_{n}^{j} - \frac{h^2}{162}. \tag{9}$$

The fourth-order central formulas give

$$u_{-2}^{j} = -u_{2}^{j} + 16u_{1}^{j} - 30u_{0}^{j} + 16u_{-1}^{j} + \frac{12h^2}{162}, \tag{10}$$

and

$$u_{n+2}^{j} = -u_{n-2}^{j} + 16u_{n+1}^{j} - 30u_{n}^{j} + 16u_{n-1}^{j} + \frac{12h^2}{162}. \tag{11}$$

Substituting Equation (10) into Equation (6) at $i = 0$ gives

$$\begin{aligned} u_{0}^{j+1} = {} & \left(1 + 2r + \frac{24r}{h^2}\right) u_{0}^{j} + \left(-\frac{rh}{2}u_{0}^{j} - r - \frac{12r}{h^2}\right) u_{1}^{j} \\ & + \left(\frac{rh}{2}u_{0}^{j} - r - \frac{12r}{h^2}\right) u_{-1}^{j} - \frac{12r}{h^2}. \end{aligned} \tag{12}$$

Using Equation (8) in Equation (12) gives

$$u_{0}^{j+1} = \left(1 + rhu_{0}^{j} - \frac{rh^3}{2(162)}\right) u_{0}^{j} - \left(rhu_{0}^{j}\right) u_{1}^{j} + \frac{h^2}{162}. \tag{13}$$

Similarly, substituting Equation (11) at $i = n$ gives

$$\begin{aligned} u_{n}^{j+1} = {} & \left(1 + 2r + \frac{24r}{h^2}\right) u_{n}^{j} + \left(-\frac{rh}{2}u_{n}^{j} - r - \frac{12r}{h^2}\right) u_{n+1}^{j} \\ & + \left(\frac{rh}{2}u_{n}^{j} - r - \frac{12r}{h^2}\right) u_{n-1}^{j} - \frac{12r}{h^2}, \end{aligned} \tag{14}$$

and Equation (9) then gives

$$u_{n}^{j+1} = \left(1 - rhu_{n}^{j} + \frac{rh^3}{2(162)}\right) u_{n}^{j} + \left(rhu_{n}^{j}\right) u_{n-1}^{j} + \frac{rh^2}{162}. \tag{15}$$

The resulting banded linear system may be written compactly as

$$\begin{bmatrix} A_1 & B_1 & 0 & \cdots & 0 & & & \\ 0 & A_2 & B_2 & C_1 & D_1 & E_1 & F_1 & \cdots \\ T & H & A & B & C & 0 & \cdots & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ \cdots & F_2 & E_2 & D_2 & T_1 & H_1 & A_3 & 0 \\ 0 & \cdots & 0 & H_2 & A_4 & & & \end{bmatrix} \begin{bmatrix} u_0^{j+1} \\ u_1^{j+1} \\ u_2^{j+1} \\ \vdots \\ u_{n-2}^{j+1} \\ u_{n-1}^{j+1} \\ u_n^{j+1} \end{bmatrix} = \begin{bmatrix} \left(1 - \frac{rh^3}{2(162)}\right) u_0^j + \frac{rh^2}{162} \\ u_1^j \\ u_2^j \\ \vdots \\ u_{n-2}^j \\ u_{n-1}^j \\ \left(1 + \frac{rh^3}{2(162)}\right) u_n^j + \frac{rh^2}{162} \end{bmatrix}.$$

Here

$$A = 1 - 2r + \frac{6r}{h^2}, \qquad A_1 = 1 - rhu_{n}^{j},$$

$$A_4 = 1 + rhu_{n}^{j}, \qquad A_2 = 1 - \frac{3hr}{2}u_{i}^{j} + 2r + \frac{3r}{h^2},$$

$$A_3 = 1 + \frac{3hr}{2}u_{i}^{j} + 2r + \frac{3r}{h^2}.$$

$$B = \frac{hr}{2}u_{i}^{j} + r - \frac{4r}{h^2}, \qquad B_1 = rhu_{n}^{j},$$

$$B_2 = -2rhu_{i}^{j} - 5r - \frac{14r}{h^2}, \qquad C = \frac{r}{h^2},$$

$$C_1 = -\frac{rh}{2}u_{i}^{j} + 4r + \frac{26r}{h^2}, \qquad D_1 = -r - \frac{24r}{h^2}, \qquad D_2 = -r - \frac{26r}{h^2}.$$

$$E_1 = E_2 = \frac{11r}{h^2}, \qquad F_1 = F_2 = -\frac{2r}{h^2},$$

$$H = -\frac{rh}{2}u_{i}^{j} + r - \frac{4r}{h^2},$$

$$H_1 = -2rhu_{i}^{j} - 5r - \frac{14r}{h^2}, \qquad H_2 = -rhu_{n}^{j},$$

$$T = \frac{r}{h^2}, \qquad T_1 = \frac{rh}{2}u_{i}^{j} + 4r + \frac{26r}{h^2}.$$

The system can be solved by a direct or iterative method; the Gaussian elimination method is used here.

3.2 Exponential finite-difference scheme

This method was first introduced by Bhattacharya in 1985 for the unstable one-dimensional heat-conduction equation in Cartesian coordinates. The algorithm was later developed for the one-dimensional diffusion equation in cylindrical coordinates and applied to two- and three-dimensional problems. It is used to solve nonlinear partial differential equations in one and two Cartesian dimensions [6].

Let $F(u)$ be a continuously differentiable function [2]. Multiplying Equation (1) by $F'(u)$ yields

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial t} = F'(u)\left(-u\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4}\right). \tag{16}$$

Thus,

$$\frac{\partial F}{\partial t} = F'(u)\left(-u\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4}\right). \tag{17}$$

Using a forward time difference gives

$$\begin{aligned} \frac{F(u_i^{j+1}) - F(u_i^j)}{k} = F'(u_i^j)\Bigg[& -u_i^j \left(\frac{\partial u}{\partial x}\right)_i^j - \left(\frac{\partial^2 u}{\partial x^2}\right)_i^j \\ & - \left(\frac{\partial^4 u}{\partial x^4}\right)_i^j \Bigg]. \end{aligned} \tag{18}$$

For $F(u) = \ln u$, the exponential-difference formula is

$$u_i^{j+1} = u_i^j \exp\left\{\frac{k}{u_i^j}\left[-u_i^j \left(\frac{\partial u}{\partial x}\right)_i^j - \left(\frac{\partial^2 u}{\partial x^2}\right)_i^j - \left(\frac{\partial^4 u}{\partial x^4}\right)_i^j\right]\right\}. \tag{19}$$

At the forward node, substitution of the forward-difference formulas into Equation (19) gives

$$\begin{aligned} u_i^{j+1} = u_i^j \exp\Bigg\{\frac{k}{u_i^j}\Bigg[& -u_i^j \frac{-3u_i^j + 4u_{i+1}^j - u_{i+2}^j}{2h} \\ & - \frac{2u_i^j - 5u_{i+1}^j + 4u_{i+2}^j - u_{i+3}^j}{h^2} \\ & - \frac{3u_i^j - 14u_{i+1}^j + 26u_{i+2}^j - 24u_{i+3}^j + 11u_{i+4}^j - 2u_{i+5}^j}{h^4}\Bigg]\Bigg\}. \end{aligned} \tag{20}$$

For $i = 2, 3, \ldots, n-2$, the central-difference formula is

$$\begin{aligned} u_i^{j+1} = u_i^j \exp\Bigg\{\frac{k}{u_i^j}\Bigg[& -u_i^j \frac{u_{i+1}^j - u_{i-1}^j}{2h} - \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} \\ & - \frac{u_{i+2}^j - 4u_{i+1}^j + 6u_i^j - 4u_{i-1}^j + u_{i-2}^j}{h^4}\Bigg]\Bigg\}. \end{aligned} \tag{21}$$

At $i = n-1$, the backward-difference formula becomes

$$u_i^{j+1} = u_i^j \exp\left\{\frac{k}{u_i^j}\left[-u_i^j\frac{3u_i^j - 4u_{i-1}^j + u_{i-2}^j}{2h} - \frac{2u_i^j - 5u_{i-1}^j + 4u_{i-2}^j - u_{i-3}^j}{h^2} - \frac{3u_i^j - 14u_{i-1}^j + 26u_{i-2}^j - 24u_{i-3}^j + 11u_{i-4}^j - 2u_{i-5}^j}{h^4}\right]\right\}. \tag{22}$$

3.2.1 Boundary conditions

Using the same boundary approximations as in the implicit method gives, at $i = 0$,

$$u_0^{j+1} = u_0^j \exp\left\{\frac{k}{u_0^j}\left[-u_0^j\frac{u_1^j - u_{-1}^j}{2h} - \frac{u_1^j - 2u_0^j + u_{-1}^j}{h^2} - \frac{12u_1^j - 24u_0^j + 12u_{-1}^j}{h^4} - \frac{12}{162h^2}\right]\right\}, \tag{23}$$

which, after using Equation (8), becomes

$$u_0^{j+1} = u_0^j \exp\left\{\frac{k}{u_0^j}\left[\frac{u_0^j}{2h}\left(2u_1^j + 2u_0^j - \frac{h^2}{162}\right) - \frac{1}{162}\right]\right\}. \tag{24}$$

At $i = n$,

$$u_n^{j+1} = u_n^j \exp\left\{\frac{k}{u_n^j}\left[-u_n^j\frac{u_{n+1}^j - u_{n-1}^j}{2h} - \frac{u_{n+1}^j - 2u_n^j + u_{n-1}^j}{h^2} - \frac{12u_{n+1}^j - 24u_n^j + 12u_{n-1}^j}{h^4} - \frac{12}{162h^2}\right]\right\}, \tag{25}$$

and Equation (9) produces

$$u_n^{j+1} = u_n^j \exp\left\{\frac{k}{u_n^j}\left[\frac{u_n^j}{2h}\left(2u_n^j - 2u_{n-1}^j - \frac{h^2}{162}\right) - \frac{1}{162}\right]\right\}. \tag{26}$$

4. NUMERICAL STABILITY OF THE KURAMOTO–SIVASHINSKY EQUATION

The numerical stability theory of partial differential equations with general boundary conditions is often difficult because of the interaction between the discretized boundary conditions and the discretized differential equation [9].

4.1 Stability of the fully implicit method

The Fourier (Von Neumann) method is used to study the numerical stability of the fully implicit scheme. The nonlinear term is removed to obtain a linear equation. At the second node, $i = 1$, the equation is

$$\frac{u_i^{j+1} - u_i^j}{k} = -\frac{2u_i^{j+1} - 5u_{i+1}^{j+1} + 4u_{i+2}^{j+1} - u_{i+3}^{j+1}}{h^2} - \frac{3u_i^{j+1} - 14u_{i+1}^{j+1} + 26u_{i+2}^{j+1} - 24u_{i+3}^{j+1} + 11u_{i+4}^{j+1} - 2u_{i+5}^{j+1}}{h^4}. \tag{27}$$

Substitute $u_i^j = \varphi(t)e^{i\beta x}$, where $\varphi(t) \neq 0$, $\beta > 0$, and $i = \sqrt{-1}$. With $r = k/h^2$, the amplification ratio can be rearranged as

$$\frac{\varphi(t+k)}{\varphi(t)} = \frac{1}{a_2},$$
$$a_2 = 1 + r(1-e^{i\beta h})^3\left[1 + \frac{2}{h^2}(1-e^{i\beta h})^2\right] + r(1-e^{i\beta h})^2\left[1 + \frac{1}{h^2}(1-e^{i\beta h})^2\right]. \tag{28}$$

Writing the result in real and imaginary parts gives

$$\frac{\varphi(t+k)}{\varphi(t)} = \frac{1}{A+iB} = \psi. \tag{29}$$

Here

$$A = 1 + r\left[8\sin^6\frac{\beta h}{2} - 6\sin^2(\beta h)\sin^2\frac{\beta h}{2} + \frac{64}{h^2}\sin^{10}\frac{\beta h}{2} - \frac{160}{h^2}\sin^2(\beta h)\sin^6\frac{\beta h}{2} + \frac{20}{h^2}\sin^4(\beta h)\sin^2\frac{\beta h}{2} + 4\sin^4\frac{\beta h}{2} - \sin^2(\beta h) + \frac{16}{h^2}\sin^8\frac{\beta h}{2} - \frac{24}{h^2}\sin^2(\beta h)\sin^4\frac{\beta h}{2} + \frac{1}{h^2}\sin^4(\beta h)\right],$$

and

$$B = r\sin(\beta h)\left[-12\sin^4\frac{\beta h}{2} + \sin^2(\beta h) - \frac{160}{h^2}\sin^8\frac{\beta h}{2} + \frac{80}{h^2}\sin^2(\beta h)\sin^4\frac{\beta h}{2} - \frac{2}{h^2}\sin^4(\beta h) - 4\sin^2\frac{\beta h}{2} - \frac{32}{h^2}\sin^6\frac{\beta h}{2} + \frac{8}{h^2}\sin^2(\beta h)\sin^2\frac{\beta h}{2}\right].$$

The necessary and sufficient stability condition is

$$\left|\frac{\varphi(t+k)}{\varphi(t)}\right| = |\psi| \leq 1.$$

For values satisfying $\sin^2(\beta h/2) = 1$ [8],

$$A = 1 + r\left(5 - \frac{83}{h^2}\right), \qquad B = r\sin(\beta h)\left(-15 - \frac{106}{h^2}\right),$$

and hence, with

$$D = 1 + 2r(5 - 83/h^2) + r^2(5 - 83/h^2)^2 + r^2(-15 - 106/h^2)^2,$$

we have

$$|\psi| = \frac{1}{\sqrt{A^2+B^2}} = \frac{1}{\sqrt{D}} \leq 1.$$

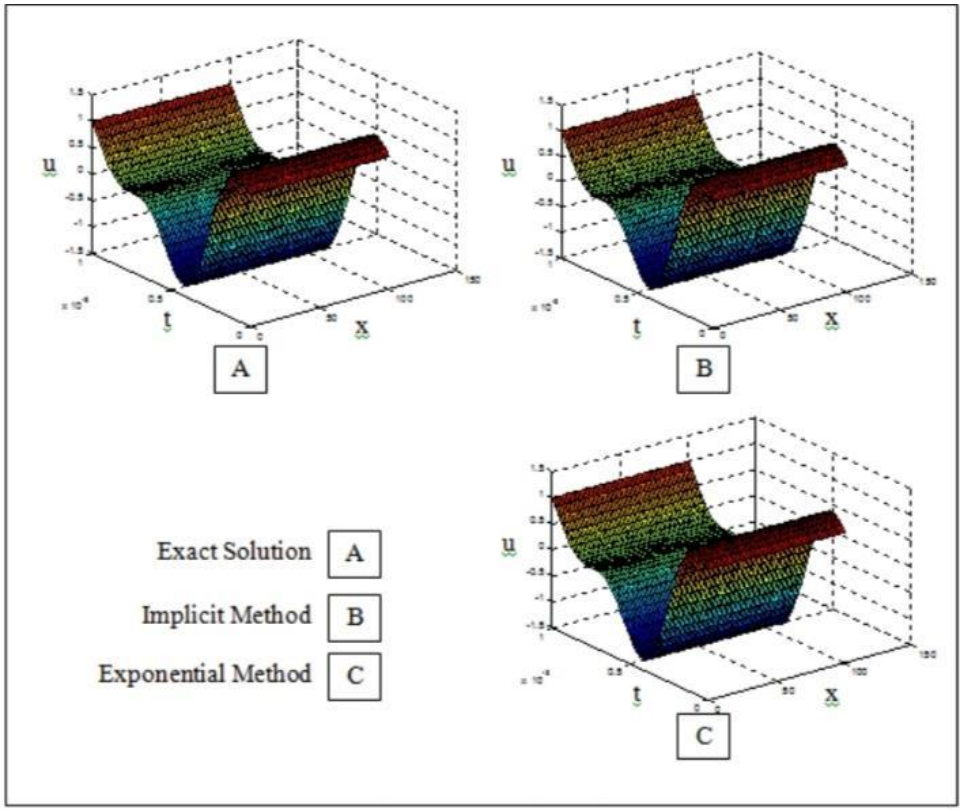


Figure 2. Numerical solutions of the two methods and the exact solution on $[0, 32\pi]$ with the initial condition in Equation (3).

For the interior nodes $i = 2, 3, \ldots, n-2$, the same substitution in the central-difference equation gives

$$\begin{aligned} \frac{\varphi(t+k)}{\varphi(t)} &= \frac{1}{a_1} = \frac{1}{A} = \psi, \\ a_1 &= 1 + 4r\sin^2\frac{\beta h}{2} - \frac{8r}{h^2}\sin^4\frac{\beta h}{2} \\ &\quad + \frac{2r}{h^2}\sin^2(\beta h) - \frac{8r}{h^2}\sin^2\frac{\beta h}{2}. \end{aligned} \tag{30}$$

When $\sin^2(\beta h/2) = 1$,

$$|\psi| = \frac{1}{\sqrt{1 + r(-8 + 28/h^2) + r^2(4 - 14/h^2)^2}} \leq 1.$$

For the node $i = n-1$, the backward-difference equation is

$$\begin{aligned} \frac{u_i^{j+1} - u_i^j}{k} &= -\frac{2u_i^{j+1} - 5u_{i-1}^{j+1} + 4u_{i-2}^{j+1} - u_{i-3}^{j+1}}{h^2} \\ &\quad - \frac{3u_i^{j+1} - 14u_{i-1}^{j+1} + 26u_{i-2}^{j+1} - 24u_{i-3}^{j+1} + 11u_{i-4}^{j+1} - 2u_{i-5}^{j+1}}{h^4}. \end{aligned} \tag{31}$$

After substituting the Fourier mode and simplifying,

$$\begin{aligned} \frac{\varphi(t+k)}{\varphi(t)} &= \frac{1}{a_2}, \\ a_2 &= 1 + r(1 - e^{-i\beta h})^3\left[1 + \frac{2}{h^2}(1 - e^{-i\beta h})^2\right] \\ &\quad + r(1 - e^{-i\beta h})^2\left[1 + \frac{1}{h^2}(1 - e^{-i\beta h})^2\right]. \end{aligned} \tag{32}$$

Thus,

$$\frac{\varphi(t+k)}{\varphi(t)} = \frac{1}{A + iB} = \psi, \tag{33}$$

where the real part A is the same as for the forward node and the sign of B is reversed. For $\sin^2(\beta h/2) = 1$,

$$A = 1 + r\left(5 - \frac{83}{h^2}\right), \qquad B = r\sin(\beta h)\left(15 + \frac{106}{h^2}\right).$$

Therefore, defining

$$D = 1 + 2r(5 - 83/h^2) + r^2(5 - 83/h^2)^2 + r^2(15 + 106/h^2)^2,$$

gives $|\psi| = 1/\sqrt{D} \leq 1$. The fully implicit method is therefore unconditionally stable at all mesh points.

4.2 Stability of the exponential finite-difference method

To study the stability of the exponential finite-difference method, first write the scheme including the nonlinear term as

$$u_i^{j+1} = u_i^j \exp\left\{\frac{k}{u_i^j}\left[-u_i^j\left(\frac{\partial u}{\partial x}\right)_i^j - \left(\frac{\partial^2 u}{\partial x^2}\right)_i^j - \left(\frac{\partial^4 u}{\partial x^4}\right)_i^j\right]\right\}. \tag{34}$$

Removing the nonlinear term gives

$$u_i^{j+1} = u_i^j \exp\left\{\frac{k}{u_i^j}\left[-\left(\frac{\partial^2 u}{\partial x^2}\right)_i^j - \left(\frac{\partial^4 u}{\partial x^4}\right)_i^j\right]\right\}. \tag{35}$$

At the second node, $i = 1$, the forward formula is

$$\begin{aligned} u_i^{j+1} = u_i^j \exp\Bigg\{\frac{k}{u_i^j}\Bigg[&-\frac{2u_i^j - 5u_{i+1}^j + 4u_{i+2}^j - u_{i+3}^j}{h^2} \\ &- \frac{3u_i^j - 14u_{i+1}^j + 26u_{i+2}^j - 24u_{i+3}^j + 11u_{i+4}^j - 2u_{i+5}^j}{h^4}\Bigg]\Bigg\}. \end{aligned} \tag{36}$$

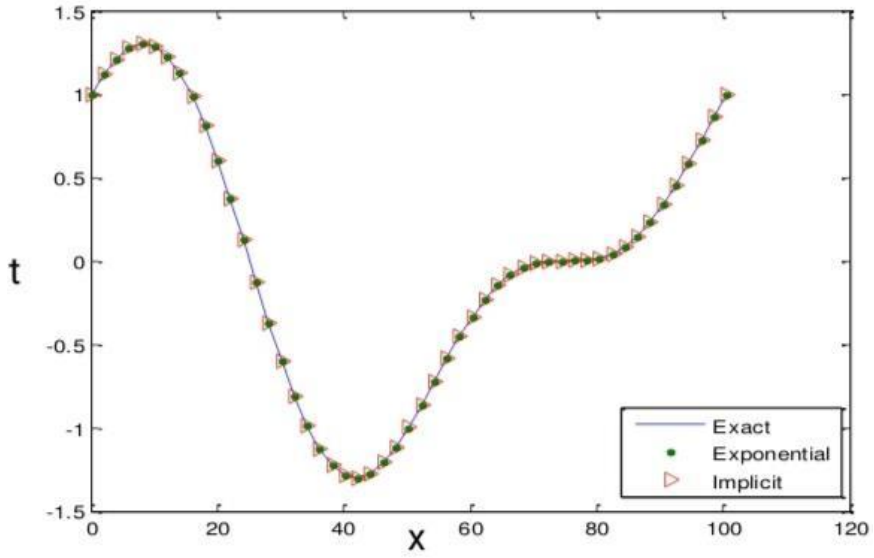


Figure 3. Comparison of the two numerical solutions with the exact solution on $[0, 32\pi]$ with the initial condition in Equation (3).

Substitution of $u_i^j = \varphi(t)e^{i\beta x}$ and simplification gives

$$\frac{\varphi(t+k)}{\varphi(t)} = \exp(A+iB) = \psi, \tag{37}$$

where

$$A = r\left(-5+\frac{83}{h^2}\right), \qquad B = r\sin(\beta h)\left(15+\frac{106}{h^2}\right).$$

The stability condition is

$$|\psi| = |\exp(A+iB)| = e^A \le 1,$$

which requires $A \le 0$, or

$$-5+\frac{83}{h^2} \le 0 \quad \Longrightarrow \quad h \ge \sqrt{\frac{83}{5}}.$$

For $i = 2, \ldots, n-2$, the central formula is

$$u_i^{j+1} = u_i^j \exp\left\{\frac{k}{u_i^j}\left[-\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} - \frac{u_{i+2}^j - 4u_{i+1}^j + 6u_i^j - 4u_{i-1}^j + u_{i-2}^j}{h^4}\right]\right\}. \tag{38}$$

The Fourier substitution gives

$$\frac{\varphi(t+k)}{\varphi(t)} = \exp\left[4r\sin^2\frac{\beta h}{2} - \frac{8r}{h^2}\sin^4\frac{\beta h}{2} + \frac{2r}{h^2}\sin^2(\beta h) - \frac{8r}{h^2}\sin^2\frac{\beta h}{2}\right] = e^A = \psi. \tag{39}$$

For $\sin^2(\beta h/2) = 1$, the condition $A \le 0$ becomes

$$r\left(4-\frac{14}{h^2}\right) \le 0 \quad \Longrightarrow \quad h \ge \sqrt{\frac{14}{4}}.$$

At $i = n-1$, the backward formula is

$$u_i^{j+1} = u_i^j \exp\left\{\frac{k}{u_i^j}\left[-\frac{2u_i^j - 5u_{i-1}^j + 4u_{i-2}^j - u_{i-3}^j}{h^2} - \frac{3u_i^j - 14u_{i-1}^j + 26u_{i-2}^j - 24u_{i-3}^j + 11u_{i-4}^j - 2u_{i-5}^j}{h^4}\right]\right\}. \tag{40}$$

After the Fourier substitution,

$$\frac{\varphi(t+k)}{\varphi(t)} = \exp(A+iB) = \psi, \tag{41}$$

where

$$A = r\left(-5+\frac{83}{h^2}\right), \qquad B = r\sin(\beta h)\left(-15-\frac{106}{h^2}\right).$$

Thus $|\psi| = e^A \le 1$ when

$$-5+\frac{83}{h^2} \le 0 \quad \Longrightarrow \quad h \ge \sqrt{\frac{83}{5}}.$$

The exponential finite-difference method for the K.S. equation is therefore stable under the stated mesh restrictions for all points.

5. NUMERICAL RESULTS

This section presents the numerical results of the fully implicit and exponential finite-difference schemes. Equation (1), subject to the boundary conditions in Equation (2), is solved on $[0, 32\pi]$ with the initial condition in Equation (3) and on $[-30, 30]$ with the initial condition in Equation (4).

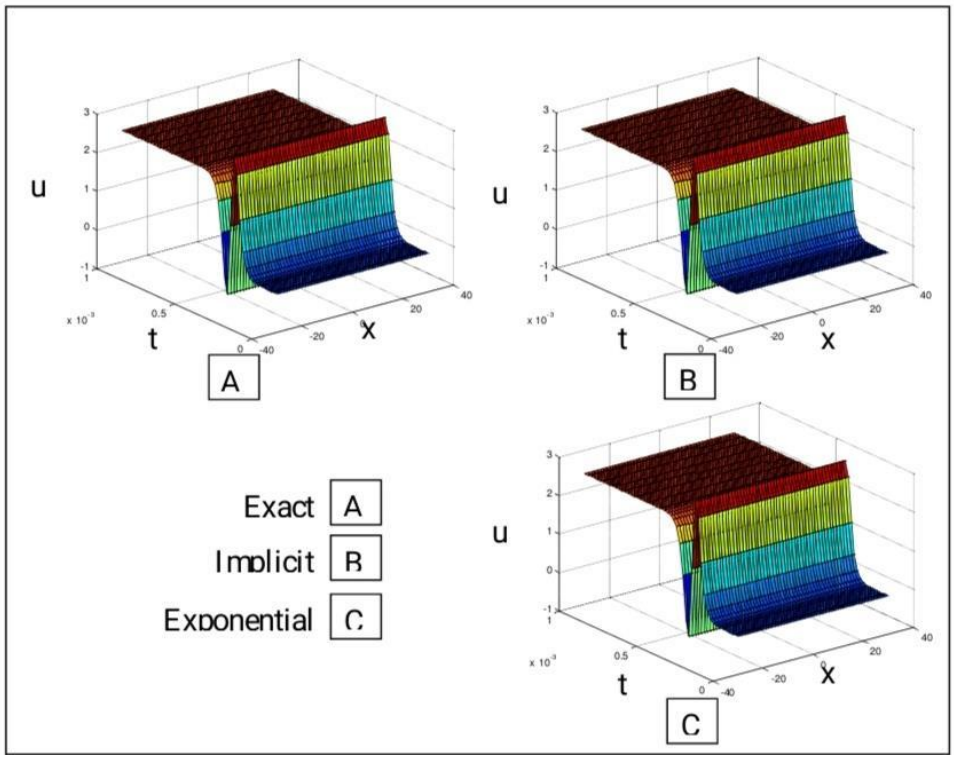


Figure 4. Numerical solutions of the two methods and the exact solution on $[-30, 30]$ with the initial condition in Equation (4).

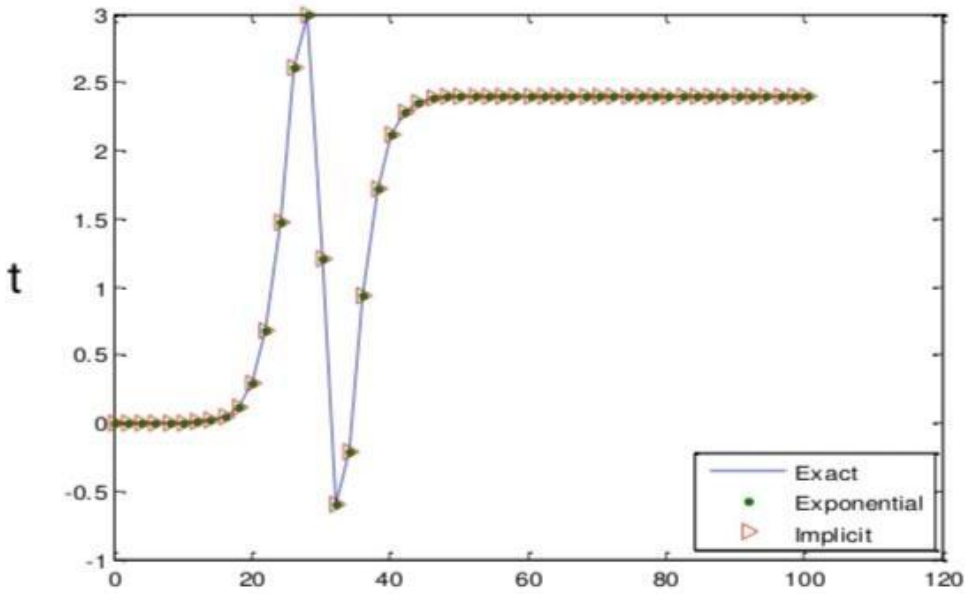


Figure 5. Comparison of the two numerical solutions with the exact solution on $[-30, 30]$ with the initial condition in Equation (4).

6. CONCLUSION

The K.S. equation is an important thermal equation in engineering and physical applications. It is nonlinear, requires considerable effort to solve numerically, and demands particular care in the treatment of the fourth derivative. The study shows that the fully implicit method is unconditionally stable at all mesh points. The exponential finite-difference method also provides stable numerical solutions under the derived mesh conditions. The numerical results show close agreement between both methods and the exact solutions for the two test problems.

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