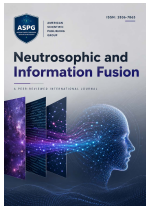
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FULL LENGTH ARTICLE

On the Hausdorff Method Applications in the Problem of Finding the Degree of Functions Approximation

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ABSTRACT

In this research, we study the problem of determining the degree of approximation of functions using the Hausdorff method, and we can do this by proving the following results. If $f \in \mathrm{Lip}(\alpha, p)$ with $\alpha > 1/p$ is a continuous almost everywhere and $2m$-periodic function, then the degree of approximation of f using Hausdorff means of the conjugate Fourier series is given by

$$\left\|\widetilde{H}_{n+\lambda}(f,a) - \widetilde{f}(a)\right\|_p = O\left((n+\lambda)^{\frac{1}{p}-\alpha}\right).$$

If f is a $2m$-periodic function, continuous almost everywhere on $[-m,m]$, and belongs to the class $Z_{\alpha,p}$, $p \geq 1$, then the degree of approximation of f by the Hausdorff means of its Fourier series is given by

$$E_{n+\lambda}(f) = \inf_{n+\lambda}\left\|H_{n+\lambda} - f\right\|_{\alpha,p} = O\left(\frac{1}{n+\lambda}\int_{\frac{1}{n+\lambda}}^{m}\frac{t^{\alpha-2}}{v(t)}\,dt\right), \tag{5}$$

where $t^{\alpha}/v(t)$ is a positive and monotonic function and v is the Zygmund modulus of continuity.

Keywords: Degree of approximation ▪ Zygmund class ▪ Hausdorff method ▪ Continuous almost everywhere ▪ Fourier series

1. INTRODUCTION

Determining the degree of approximation of functions is done by representing them with orthogonal sequences, for example, the trigonometric sequence, Legendre polynomials, Jacobi polynomials, and others.

The degrees of approximation of functions depend on their belonging to one of the classes. For example, determining the degree of approximation of functions of the Zygmund class differs in general from determining the degree of approximation of functions of the Lipschitz class, and the degrees of approximation vary depending on the nature of the method studied.

The aim of this work is to determine the degree of approximation of the function f and its conjugate function belonging to some generalized classes, such as Lipschitz, Zygmund, and others, using the Hausdorff method.

2. MAIN DISCUSSION

We mention some definitions, symbols, and terms used.

Definition 1 [1, 6] (Hausdorff method). We denote the method by (H,P_n). It has the following matrix form:

$$t_n^{(H,P_n)} = \sum_{k=0}^{n} h_{n,k}S_k,$$

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where

$$h_{n,k} = \begin{cases} \binom{n}{k}\Delta^{n-k}\eta_k, & 0 \le k \le n, \\ 0, & k > n. \end{cases}$$

The difference operator Δ is defined by $\Delta\mu_n = \mu_n - \mu_{n+1}$ and $\Delta^{k+1}\eta_n = \Delta^k(\Delta\mu_n)$, while

$$\eta_n = \int_0^1 u^n \, d\gamma(u).$$

The Hausdorff method is regular if

$$\int_0^1 d\gamma(u) = 1.$$

Here $\gamma = \gamma_1 - \gamma_2$, where γ_1 and γ_2 are completely increasing functions on $[0,1]$. Moreover, $\gamma(u)$ is a mass function that is continuous at $u = 0$, belongs to $BV[0,1]$, satisfies $\gamma(0) = 0$ and $\gamma(1) = 1$, and, for $0 < u < 1$,

$$\gamma(u) = \frac{\gamma(u+0) + \gamma(u-0)}{2}.$$

Definition 2 [2] (L_p space). Let f be a $2m$-periodic function that is continuous almost everywhere on $[-m,m]$. The space of all p-integrable functions is

$$L_p = L_p[-m,m] = \left\{ f : \begin{matrix} [-m,m] \to \mathbb{R}, \\ \int_{-m}^{m} |f(a)|^p \, da < \infty \end{matrix} \right\}, \qquad p \ge 1.$$

The norm on L_p is

$$\|f\|_p = \begin{cases} \left(\frac{1}{2m} \int_{-m}^{m} |f(a)|^p \, da \right)^{1/p}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{-m \le a \le m} |f(a)|, & p = \infty. \end{cases}$$

Definition 3 [3, 7] (Lipschitz class $\mathrm{Lip}(\alpha,p)$). Let f be a $2m$-periodic function. We say that f belongs to $\mathrm{Lip}(\alpha,p)$ if

$$\left(\int_{-m}^{m} |f(a+t) - f(a)|^p \, da \right)^{1/p} = O(t^\alpha),$$

where $0 < \alpha \le 1$ and $p \ge 1$.

Definition 4 [2, 8] (Zygmund class $Z_{\alpha,p}$). Let f be a $2m$-periodic function and let $1 \le p < \infty$. We say that f belongs to $Z_{\alpha,p}$ if

$$Z_{\alpha,p} = \left\{ f \in L_p : \frac{\sup\limits_{t \in [-m,m] \setminus \{0\}} \|f(a+t) + f(a-t) - 2f(a)\|_p}{t^\alpha} < \infty \right\}.$$

The corresponding norm is

$$\|f\|_{\alpha,p} = \|f\|_p + \sup_{t \in [-m,m] \setminus \{0\}} \frac{\|f(a+t) + f(a-t) - 2f(a)\|_p}{t^\alpha}, \qquad p \ge 1.$$

3. DISCUSSION AND RESULTS

Let f be a continuous almost everywhere function on $[-m,m]$ and periodic with period $2m$. The Fourier series of f is given by [4, 5, 12]

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{m} + b_n \sin\frac{n\pi x}{m} \right),$$

where

$$a_n = \frac{1}{m} \int_{-m}^{m} f(x) \cos\frac{n\pi x}{m} \, dx, \qquad n = 0,1,2,\ldots,$$
$$b_n = \frac{1}{m} \int_{-m}^{m} f(x) \sin\frac{n\pi x}{m} \, dx, \qquad n = 1,2,\ldots.$$

The conjugate Fourier series is [13, 14]

$$\sum_{n=1}^{\infty} \left(a_n \sin\frac{n\pi x}{m} - b_n \cos\frac{n\pi x}{m} \right).$$

The corresponding conjugate function is

$$\widetilde{f}(x) = -\frac{1}{2m} \lim_{\varepsilon \to 0} \int_{\varepsilon}^{m} \psi_x(t) \cot\left(\frac{t}{2}\right) dt,$$

where $\psi_x(t) = f(x+t) - f(x-t)$.

The sequence of partial sums of the Fourier series of order n is

$$s_n(f) = s_n(f;x) = \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k \cos\frac{k\pi x}{m} + b_k \sin\frac{k\pi x}{m} \right) = \sum_{k=0}^{n} u_k(f;x).$$

The partial sums of the conjugate Fourier series are

$$\widetilde{s}_n(f,x) = \sum_{k=1}^{n} \left(b_k \cos\frac{k\pi x}{m} - a_k \sin\frac{k\pi x}{m} \right) = -\int_0^m \psi_x(t) \frac{\cos(t/2) - \cos\left((n+\frac{1}{2})t\right)}{2\sin(t/2)} \, dt.$$

We denote the Hausdorff means of the Fourier series of f by

$$H_n(x) = H_n(f,x) = \sum_{k=0}^{n} h_{n,k} s_k(f,x), \qquad n = 0,1,2,\ldots,$$

and the Hausdorff means of the conjugate Fourier series by

$$\widetilde{H}_n(f,x) = \sum_{k=0}^{n} h_{n,k} \widetilde{s}_k(f,x), \qquad n = 0,1,2,\ldots.$$

Preliminary 1 [9]. Let

$$\widetilde{g}(u,t) = \mathrm{Re}\left[\sum_{k=0}^{n+\lambda} \binom{n+\lambda}{k} u^k (1-u)^{n+\lambda-k} e^{i(k+\frac{1}{2})t} \right].$$

For $0 < u < 1$, $0 \le t \le m$, and $\lambda \ge 1$,

$$\left| \int_0^1 \widetilde{g}(u,t) \, d\gamma(u) \right| = \begin{cases} O(1), & 0 < t \le \dfrac{m}{n+\lambda}, \\ O\left(\dfrac{1}{t(n+\lambda)} \right), & \dfrac{m}{n+\lambda} \le t \le m. \end{cases}$$

Preliminary 2 [10]. Let

$$g(u,t)=\operatorname{Im}\left[\sum_{k=0}^{n+\lambda}\binom{n+\lambda}{k}u^k(1-u)^{n+\lambda-k}e^{i(k+\frac{1}{2})t}\right].$$

For $0\le u\le 1$, $0\le t\le m$, and $\lambda\ge 1$,

$$\left|\int_0^1 g(u,t)\,d\gamma(u)\right|=\begin{cases}O\big((n+\lambda)t\big), & 0\le t\le \dfrac{1}{n+\lambda},\\ O\left(\dfrac{1}{t(n+\lambda)}\right), & \dfrac{1}{n+\lambda}\le t\le m.\end{cases}$$

Preliminary 3 [10]. Let

$$K_{n+\lambda}^{H}(t)=\frac{1}{2m\sin(t/2)}\int_0^1 g(u,t)\,d\gamma(u).$$

Then

$$|K_{n+\lambda}^{H}(t)|=\begin{cases}O(n+\lambda), & 0\le t\le \dfrac{1}{n+\lambda},\\ O\left(\dfrac{1}{t^2(n+\lambda)}\right), & \dfrac{1}{n+\lambda}\le t\le m.\end{cases}$$

Preliminary 4 [2]. Let $f\in Z_{\alpha,p}$. For $0<t\le m$, we have

$$\Delta_y^2\phi(a,t):=\phi(a+y,t)+\phi(a-y,t)-2\phi(a,t).$$

$$\|\phi(a,t)\|_p=O(t^\alpha), \tag{i}$$

$$\|\Delta_y^2\phi(a,t)\|_p=\begin{cases}O(t^\alpha),\\ O(y^\alpha),\end{cases} \tag{ii}$$

$$\|\Delta_y^2\phi(a,t)\|_p=O\left(v(y)\frac{t^\alpha}{v(t)}\right), \tag{iii}$$

where

$$\phi(a,t)=f(a+t)+f(a-t)-2f(a).$$

3.1 Proof of Theorem 1

We have

$$\begin{aligned}&\widetilde{H}_{n+\lambda}(f,a)-\widetilde{f}(a)\\&=\sum_{k=0}^{n+\lambda}h_{n+\lambda,k}\{\widetilde{s}_{n+k}(f,a)-\widetilde{f}(a)\}\\&=\frac{1}{2m}\int_0^m\frac{\psi_a(t)}{\sin(t/2)}\sum_{k=0}^{n+\lambda}h_{n+\lambda,k}\cos\big((k+\tfrac{1}{2})t\big)\,dt\\&=\frac{1}{2m}\int_0^m\frac{\psi_a(t)}{\sin(t/2)}\sum_{k=0}^{n+\lambda}\binom{n+\lambda}{k}\\&\qquad\times\Delta^{n+\lambda-k}\mu_k\cos\big((k+\tfrac{1}{2})t\big)\,dt.\end{aligned}$$

Using the moment representation, this becomes

$$\begin{aligned}&\widetilde{H}_{n+\lambda}(f,a)-\widetilde{f}(a)\\&=\frac{1}{2m}\int_0^m\frac{\psi_a(t)}{\sin(t/2)}\sum_{k=0}^{n+\lambda}\binom{n+\lambda}{k}\int_0^1u^k(1-u)^{n+\lambda-k}\,d\gamma(u)\\&\qquad\times\operatorname{Re}\left(e^{i(k+\frac{1}{2})t}\right)dt\\&=\frac{1}{2m}\int_0^m\frac{\psi_a(t)}{\sin(t/2)}\int_0^1\widetilde{g}(u,t)\,d\gamma(u)\,dt.\end{aligned}$$

Also,

$$\widetilde{s}_{n+\lambda}(f,a)-\widetilde{f}(a)=\frac{1}{2m}\int_0^m\psi_a(t)\frac{\cos\big((n+\lambda+\frac{1}{2})t\big)}{\sin(t/2)}\,dt.$$

Therefore,

$$\begin{aligned}&\big\|\widetilde{H}_{n+\lambda}(f,a)-\widetilde{f}(a)\big\|\\&\le\int_0^m\frac{|\psi_a(t)|}{t}\left|\int_0^1\widetilde{g}(u,t)\,d\gamma(u)\right|dt\\&=\int_0^{\frac{m}{n+\lambda}}\frac{|\psi_a(t)|}{t}\left|\int_0^1\widetilde{g}(u,t)\,d\gamma(u)\right|dt\\&\quad+\int_{\frac{m}{n+\lambda}}^m\frac{|\psi_a(t)|}{t}\left|\int_0^1\widetilde{g}(u,t)\,d\gamma(u)\right|dt\\&=I_1+I_2.\end{aligned}\tag{10}$$

Here we used $1/\sin(t/2)\le m/t$ for $0<t\le m$.

By Preliminary 1 and Hölder's inequality,

$$\begin{aligned}I_1&=O(1)\int_0^{\frac{m}{n+\lambda}}\frac{\psi_a(t)}{t^\alpha}t^{\alpha-1}\,dt\\&=O(1)\left\{\int_0^{\frac{m}{n+\lambda}}\left(\frac{|\psi_a(t)|}{t^\alpha}\right)^p dt\right\}^{1/p}\\&\quad\times\left\{\lim_{\varepsilon\to0}\int_\varepsilon^{\frac{m}{n+\lambda}}(t^{\alpha-1})^q dt\right\}^{1/q}\\&=O\left(\left(\frac{m}{n+\lambda}\right)^\alpha\right).\end{aligned}\tag{11}$$

This uses the relations (2), (3), and $1/p+1/q=1$.

Similarly,

$$\begin{aligned}I_2&=O\left(\int_{\frac{m}{n+\lambda}}^m\frac{t^{-\delta-\alpha}|\psi_a(t)|}{n+\lambda}t^{\alpha+\delta-2}\,dt\right)\\&=O\left\{\frac{1}{n+\lambda}\int_{\frac{m}{n+\lambda}}^m\big(t^{-\delta-\alpha}|\psi_a(t)|\big)^p dt\right\}^{1/p}\\&\quad\times\left\{\int_{\frac{m}{n+\lambda}}^m\big(t^{\alpha+\delta-2}\big)^q dt\right\}^{1/q}\\&=O\left[(n+\lambda)^\delta\left(\frac{m}{n+\lambda}\right)^\alpha(n+\lambda)^{1-\delta-1/q}\right]\\&=O\Big((n+\lambda)^{1-\alpha-1/q}\Big)=O\Big((n+\lambda)^{\frac{1}{p}-\alpha}\Big).\end{aligned}\tag{12}$$

Here $0<\delta<1/p$. Using Preliminary 1, Hölder's inequality, the middle-value theorem for integration, and the relations (1) and (4), the sum of (10)–(12) gives

$$\big|\widetilde{H}_{n+\lambda}(f,a)-\widetilde{f}(a)\big|=O\Big((n+\lambda)^{\frac{1}{p}-\alpha}m^\alpha\Big).$$

Finally, since $(m/(n+\lambda))^{\alpha-1}\le(1/(n+\lambda))^{\alpha-1}$, we obtain

$$\big\|\widetilde{H}_{n+\lambda}(f,a)-\widetilde{f}(a)\big\|_p=O\Big((n+\lambda)^{\frac{1}{p}-\alpha}\Big).$$

3.2 Proof of Theorem 2

Let

$$s_k(f,a)-f(a)=\frac{1}{2m}\int_0^m \phi(a,t)\frac{\sin\left((k+\frac{1}{2})t\right)}{\sin(t/2)}\,dt.$$

Since $\sum_{k=0}^{n+\lambda} h_{n+\lambda,k}=1$ for every $n+\lambda$ in any regular Hausdorff matrix H, we have

$$\begin{aligned}H_{n+\lambda}(a)-f(a)&=\frac{1}{2m}\int_0^m \frac{\phi(a,t)}{\sin(t/2)}\\&\quad\times\sum_{k=0}^{n+\lambda}h_{n+\lambda,k}\sin\left((k+\tfrac{1}{2})t\right)dt\\&=\frac{1}{2m}\int_0^m\frac{\phi(a,t)}{\sin(t/2)}\\&\quad\times\int_0^1\mathrm{Im}\Big[\\&\qquad\sum_{k=0}^{n+\lambda}\binom{n+\lambda}{k}u^k\\&\qquad\times(1-u)^{n+\lambda-k}\\&\qquad\times e^{i(k+\frac{1}{2})t}\Big]d\gamma(u)\,dt\\&=\frac{1}{2m}\int_0^m\frac{\phi(a,t)}{\sin(t/2)}\int_0^1 g(u,t)\,d\gamma(u)\,dt.\end{aligned}$$

Put

$$q_{n+\lambda}(a)=H_{n+\lambda}(a)-f(a)=\int_0^m\phi(a,t)K^H_{n+\lambda}(t)\,dt.$$

For compactness, write

$$\Delta_y^2 q_{n+\lambda}(a):=q_{n+\lambda}(a+y)+q_{n+\lambda}(a-y)-2q_{n+\lambda}(a).$$

Then

$$\Delta_y^2 q_{n+\lambda}(a)=\int_0^m\Delta_y^2\phi(a,t)K^H_{n+\lambda}(t)\,dt.$$

Using the generalized Minkowski inequality [11], we obtain

$$\begin{aligned}&\|\Delta_y^2 q_{n+\lambda}(a)\|_p\\&=\left\{\frac{1}{2m}\int_{-m}^m\left|\int_0^m\Delta_y^2\phi(a,t)K^H_{n+\lambda}(t)\,dt\right|^p da\right\}^{1/p}\\&\le\int_0^m\left\{\frac{1}{2m}\int_{-m}^m|\Delta_y^2\phi(a,t)K^H_{n+\lambda}(t)|^p da\right\}^{1/p}dt\\&=\int_0^m\|\Delta_y^2\phi(a,t)\|_p|K^H_{n+\lambda}(t)|\,dt.\end{aligned}$$

Splitting the last integral yields

$$\begin{aligned}&\|\Delta_y^2 q_{n+\lambda}(a)\|_p\\&=\int_0^{\frac{1}{n+\lambda}}\|\Delta_y^2\phi(a,t)\|_p|K^H_{n+\lambda}(t)|\,dt\\&\quad+\int_{\frac{1}{n+\lambda}}^m\|\Delta_y^2\phi(a,t)\|_p|K^H_{n+\lambda}(t)|\,dt\\&=I_1+I_2.\end{aligned}\tag{13}$$

By Preliminaries 3 and 4(iii), and the monotonicity of $t^\alpha/v(t)$,

$$\begin{aligned}I_1&=O\left(\int_0^{\frac{1}{n+\lambda}}v(y)\frac{t^\alpha}{v(t)}(n+\lambda)\,dt\right)\\&=O\left((n+\lambda)v(y)\frac{(1/(n+\lambda))^\alpha}{v(1/(n+\lambda))}\int_0^{\frac{1}{n+\lambda}}dt\right)\\&=O\left(v(y)\frac{(1/(n+\lambda))^\alpha}{v(1/(n+\lambda))}\right).\end{aligned}\tag{14}$$

Also,

$$\begin{aligned}I_2&=O\left(\int_{\frac{1}{n+\lambda}}^m v(y)\frac{t^\alpha}{v(t)}\frac{1}{t^2(n+\lambda)}\,dt\right)\\&=O\left(\frac{v(y)}{n+\lambda}\int_{\frac{1}{n+\lambda}}^m\frac{t^{\alpha-2}}{v(t)}\,dt\right).\end{aligned}\tag{15}$$

It follows from (13)–(15) that

$$\begin{aligned}&\sup_{y\neq0}\frac{\|\Delta_y^2 q_{n+\lambda}(a)\|_p}{v(y)}\\&=O\left(\frac{(1/(n+\lambda))^\alpha}{v(1/(n+\lambda))}\right)+O\left(\frac{1}{n+\lambda}\int_{\frac{1}{n+\lambda}}^m\frac{t^{\alpha-2}}{v(t)}\,dt\right).\end{aligned}\tag{16}$$

By Preliminaries 3 and 4 again,

$$\begin{aligned}\|q_{n+\lambda}(a)\|_p&\le\left(\int_0^{\frac{1}{n+\lambda}}+\int_{\frac{1}{n+\lambda}}^m\right)\|\phi(a,t)\|_p|K^H_{n+\lambda}(t)|\,dt\\&=O\left((n+\lambda)\int_0^{\frac{1}{n+\lambda}}t^\alpha\,dt\right)\\&\quad+O\left(\frac{1}{n+\lambda}\int_{\frac{1}{n+\lambda}}^m t^{\alpha-2}\,dt\right)\\&=O\big((n+\lambda)^{-\alpha}\big)\\&\quad+O\left(\frac{1}{n+\lambda}\int_{\frac{1}{n+\lambda}}^m t^{\alpha-2}\,dt\right).\end{aligned}\tag{17}$$

Thus, from (16) and (17),

$$\begin{aligned}\|q_{n+\lambda}(a)\|_{\alpha,p}&=\|q_{n+\lambda}(a)\|_p+\sup_{y\neq0}\frac{\|\Delta_y^2 q_{n+\lambda}(a)\|_p}{v(y)}\\&=O\left(\frac{1}{(n+\lambda)^\alpha}\right)\\&\quad+O\left(\frac{1}{n+\lambda}\int_{\frac{1}{n+\lambda}}^m t^{\alpha-2}\,dt\right)\\&\quad+O\left(\frac{(1/(n+\lambda))^\alpha}{v(1/(n+\lambda))}\right)\\&\quad+O\left(\frac{1}{n+\lambda}\int_{\frac{1}{n+\lambda}}^m\frac{t^{\alpha-2}}{v(t)}\,dt\right)\\&=\sum_{i=1}^4 O(J_i).\end{aligned}\tag{18}$$

We now write J_1 in terms of J_3, and J_2 and J_3 in terms of J_4.

From the monotonicity of $v(t)$,

$$t^{\alpha} = \frac{t^{\alpha}}{v(t)} v(t) \leq v(m) \frac{t^{\alpha}}{v(t)} = O\left(\frac{t^{\alpha}}{v(t)}\right), \qquad 0 < t \leq m.$$

For $t = 1/(n+\lambda)$, this gives $J_1 = O(J_3)$. Moreover,

$$J_2 = \frac{1}{n+\lambda} \int_{\frac{1}{n+\lambda}}^{m} t^{\alpha-2}\, dt \leq \frac{v(m)}{n+\lambda} \int_{\frac{1}{n+\lambda}}^{m} \frac{t^{\alpha-2}}{v(t)}\, dt = O(J_4). \tag{20}$$

Since $t^{\alpha}/v(t)$ is positive and monotonic,

$$\begin{aligned} J_4 &= \frac{1}{n+\lambda} \int_{\frac{1}{n+\lambda}}^{m} \frac{t^{\alpha-2}}{v(t)}\, dt \\ &\geq \frac{(n+\lambda)^{-\alpha}}{v(1/(n+\lambda))} \frac{1}{n+\lambda} \int_{\frac{1}{n+\lambda}}^{m} t^{-2}\, dt \\ &= \frac{(n+\lambda)^{-\alpha}}{v(1/(n+\lambda))} \frac{1}{n+\lambda} \left(n+\lambda-\frac{1}{m}\right) \\ &\geq \frac{(n+\lambda)^{-\alpha}}{2v(1/(n+\lambda))}. \end{aligned} \tag{21}$$

Here

$$\frac{1}{n+\lambda}\left(n+\lambda-\frac{1}{m}\right) > \frac{1}{n+\lambda}(n+\lambda) \geq \frac{1}{2}.$$

Therefore $J_3 = O(J_4)$. Adding (18) and (21), we obtain

$$\|q_{n+\lambda}(a)\|_{\alpha,p} = O(J_4) = O\left(\frac{1}{n+\lambda} \int_{\frac{1}{n+\lambda}}^{m} \frac{t^{\alpha-2}}{v(t)}\, dt\right).$$

Consequently,

$$E_{n+\lambda}(f) = \inf_{n+\lambda} \|q_{n+\lambda}(a)\|_{\alpha,p} = O\left(\frac{1}{n+\lambda} \int_{\frac{1}{n+\lambda}}^{m} \frac{t^{\alpha-2}}{v(t)}\, dt\right).$$

REFERENCES

[1] B. E. Rhoades, "The degree of approximation of functions, and their conjugates, belonging to several general lipschitz classes by hausdorff matrix means of the fourier series and conjugate series of a fourier series," *Tamkang Journal of Mathematics*, vol. 45, no. 4, pp. 389–395, 2014.

[2] S. Lal and Shireen, "Best approximation of functions of generalized zygmund class by matrix–euler summability mean of fourier series," *Bulletin of Mathematical Analysis and Applications*, vol. 5, no. 4, pp. 1–13, 2013.

[3] U. Singh and S. Sonkar, "Trigonometric approximation of signals function belonging to weighted lip(t,p) class by hausdorff means," *Journal of Applied Functional Analysis*, vol. 8, pp. 37–44, 2013.

[4] H. K. Nigam and A. Sharma, "On approximation of conjugate of functions belonging to different classes by product means," *International Journal of Pure and Applied Mathematics*, vol. 76, no. 2, pp. 303–316, 2012.

[5] V. N. Mishra, K. Khatri, and L. N. Mishra, "Product summability transform of conjugate series of fourier series," *International Journal of Mathematics and Mathematical Sciences*, vol. 2012, pp. 1–13, 2012.

[6] B. E. Rhoades, K. Ozkoklu, and I. Albayrak, "On the degree of approximation of functions belonging to a lipschitz class by hausdorff means of its fourier series," *Applied Mathematics and Computation*, vol. 217, no. 16, pp. 6868–6871, 2011.

[7] H. K. Nigam and A. Sharma, "On approximation of conjugate of a function belonging to Lip(t,r) class by product summability means of conjugate series of fourier series," *International Journal of Contemporary Mathematical Sciences*, vol. 5, pp. 2673–2683, 2010.

[8] F. Móricz, "Enlarged lipschitz and zygmund classes of functions and fourier transforms," *East Journal on Approximations*, vol. 16, no. 3, pp. 259–271, 2010.

[9] B. E. Rhoades, "On the degree of approximation of functions belonging to a lipschitz class by hausdorff means of its fourier series," *Tamkang Journal of Mathematics*, vol. 34, no. 3, pp. 245–247, 2003.

[10] G. Das, A. Nath, and B. K. Ray, "An estimate of the rate of convergence of fourier series in the generalized hölder metric," *Analysis and Applications*, pp. 43–60, 2002.

[11] A. Zygmund, *Trigonometric Series*, 3rd ed. London: Cambridge University Press, 2002.

[12] G. Bachman, L. Narici, and E. Beckenstein, *Fourier and Wavelet Analysis*. New York: Springer-Verlag, 2000.

[13] K. Qureshi, "On the degree of approximation of functions belonging to the Lip(α,p) by means of a conjugate series," *Indian Journal of Pure and Applied Mathematics*, vol. 13, no. 5, pp. 560–563, 1982.

[14] K. Qureshi, "On the degree of approximation of functions belonging to the class of Lip α," *Indian Journal of Pure and Applied Mathematics*, vol. 13, no. 8, pp. 898–903, 1982.