On the Hausdorff Method Applications in the Problem of Finding the Degree of Functions Approximation

Agnes Osagie[1,*]

[1]Cape Peninsula University of Technology, Faculty of Applied Science, South Africa

Osagieagne2000@cput.ac.za

Abstract

In this research, we study the problem of determining the degree of approximation of functions using the Hausdorff method, and we can do this by proving the following results:

If $f\in \mathrm{Lip}(\alpha, p)$with $\alpha > \frac{1}{p}$ and be a continues aimost everywhere and 2m periodic function, Then the degree of approximation of $\tilde{f}$ using hausdorff means of conjugate fourier series, is given by:

$$\left|\left|\widetilde{H}_{(n+\lambda)}\ (f, a) - \tilde{f}\ (a)\right|\right|_p = 0\left((n+\lambda)^{\frac{1}{p}-\alpha}\right)$$

If f be a 2m periodic function, continues almost everywhere on $[-m, m]$ andbelonging to the class $Z_{\alpha,p}$, $p \geq 1$. then the degree of approximation of function f of fouier series using hausdorff means, is given by:

$$E_{(n+\lambda)}(f) = \inf_{(n+\lambda)}\left\|H_{(n+\lambda)} - f\right\|_{\alpha,p} = 0\left(\frac{1}{(n+\lambda)}\int_{\frac{1}{n+\lambda}}^{m}\frac{t^{\alpha-2}}{v(t)}dt\right) \quad (5)$$

where $t^{\alpha}$and v the zygmund moduli of continuity sunch that $\frac{t^{\alpha}}{v(t)}$positive and monotonic function.

Keywords: Degree of approximation; Zygmund class; Hausdorff method; Continuous almost everywhere, Fourier series

1. Introduction

Determining the degree of approximation of functions is done by representing them with orthogonal sequences, for example, the trigonometric sentence – Legendre polynomials – Jacobi polynomials and others.
The degrees of approximation of functions depending on their belonging to one of the classes, for example, determining the degree of approximation of the functions of the zygmund class differs in general from determining the degree of approximation of the functions of the Lipschitz class, and the degrees of approximation vary depending on the nature of the method studied.

The aim of this work is to determine the degree of approximation of the function f and its accompanying function belonging to some generalized classes, such as Lipschitz, zygmund and others, using the Hausdorff method.

2. Main Discussion

We mention some definitions, symbols and terms used.

Definition (1): [1, 6] Hausdorff method.

And we encode it with (H,P_n).

Definition of the following matrix form:

DOI: https://doi.org/10.54216/NIF.040201
Received: January 08, 2024 Accepted: June 28, 2024

$$t_n^{(H,P_n)} = \sum_{k=0}^{n} h_{n,k} S_k$$

$$h_{n,k} = \begin{cases} \binom{n}{k} \Delta^{n-k}\eta_{k}, 0 \le k \le n \\ 0, \quad k > n \end{cases}$$

Where the effect △ is defined as follows: $\Delta\mu_n = \mu_n - \mu_{n+1}$ and $\Delta^{k+1}\eta_n = \Delta^k(\Delta\mu_n)$ and the function μ_n is defined as:

$$\eta_n = \int_0^1 u^n d\gamma(u)$$

The Hausdorff method is regular if the condition is fulfilled:

$$\int_0^1 d\gamma(u) = 1$$

Where: $\gamma = \gamma_1 - \gamma_2$ and γ_2, γ_1 are completely increasing functions on the domain [0,1].

Whereas $\gamma(u)$ is a mass function (the gravity function) that is continuous when u=0 and belongs to the BV [0,1] space of all functions whose change is limited to the domain [0,1] and $\gamma(0) = 0$and $\gamma(1) = 1$ and for 0<u<1 and $\gamma(u) = \frac{[\gamma(u+0)]+\gamma(u-0)}{2}$

Definition (2): L_pp Space [2]: the set of all periodic functions f with a role of 2m and continuous almost everywhere on the domain [- m,m] then the space of all integrable functions on the domain 2m is defined as:

$$L_p = L_p[-m,+m] = \left\{ f:[-m,+m] \to R;\ \int_{-m}^{m} |f(a)|^p da < \infty \right\}, p \ge 1$$

And the organization is given by $\|f\|_p$on the L_p space, as follows:

$$\|f\|_p = \begin{cases} \left(\frac{1}{2m} \int_{-m}^{m} |f(a)|^p da \right)^{\frac{1}{p}}, 1 \le p \le \infty \\ ess \quad sup_{-m \le a \le m} |f(a)| \quad p = \infty \end{cases}$$

Definition (3): [3,7] Lipschitz class Lip (α, p): let f be a periodic function with a role of 2m, then we say about the function f that it belongs to the Class Lip (α, p) if the following is true:

$$\left(\int_{-m}^{m} |f(a+t) - f(a)|^p da \right)^{\frac{1}{p}} = O(t^\alpha)$$

Where $0 < \alpha \le 1$and $p \ge 1$.

Definition (4): [2,8] zygmund's class $Z_{\alpha,P}$: let f be a periodic function with a role of 2m and let $1 \le p < \infty$, then we say about the function f that it belongs to the class $Z_{\alpha,p}$ if the following is true:

$$Z_{\alpha,p} = \left\{ f\epsilon L_p, sup_{t\in[-m,+m]\setminus\{0\}} \frac{\|f(a+t) + f(a-t) - 2f(a)\|_p}{t^\alpha} < \infty \right\}$$

And we know the system $\|\|_{\alpha,p}$ as follows:

$$\|f\|_{\alpha,p} = \|f\|_p + sup_{t\epsilon[-m,+m]\setminus\{0\}} \frac{\|f(a+t) + f(a-t) - 2f(a)\|_p}{t^\alpha}, p \ge 1$$

3. Discussion and results

Let f be an almost ubiquitous continuous function on the domain $[-m, m]$ and periodic with a role of 2m. The Fourier series of the function f is given by the form [4, 5, 12]:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{m} + b_n \sin\frac{n\pi x}{m} \right)$$

DOI: https://doi.org/10.54216/NIF.040201
Received: January 08, 2024 Accepted: June 28, 2024

Where $a_n = \frac{1}{m}\int_{-m}^{m} f(x)\cos\frac{n\pi x}{m}\,dx\ ; n = 0{,}1{,}2, \dots\dots.$

And $b_n = \frac{1}{m}\int_{-m}^{m} f(x)\sin\frac{n\pi x}{m}\ dx \qquad ; n = 1{,}2, \dots\dots$

The sequence accompanying the Fourier series is given as follows: [13, 14]

$$\sum_{n=1}^{\infty}\left(a_n\sin\frac{n\pi x}{m} - b_n\cos\frac{n\pi x}{m}\right)$$

The corresponding companion function of the previous relation is given as:

$$f^{\sim}(x) = -\frac{1}{2m}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{m}\psi_x(t)\cot\left(\frac{t}{2}\right)dt$$

Where $\psi_x(t) = f(x+t) - f(x-t)$

And the sequence of partial sums of a Fourier series of rank n is given as:

$$s_n(f) = s_n(f;x) = \frac{a_0}{2} + \sum_{k=1}^{n}\left(a_k\cos\frac{k\pi x}{m} + b_k\sin\frac{k\pi x}{m}\right) = \sum_{k=0}^{n}u_k\,(f;x)$$

And the sequence of partial sums of the Fourier series of rank n is given as:

$$s_n{}^{\sim}(f,x) = \sum_{k=1}^{n}\left(b_k\cos\frac{k\pi x}{m} - a_k\sin\frac{k\pi x}{m}\right) = -\int_0^m \psi_x(t)\left\{\frac{\cos\left(\frac{t}{2}\right) - \cos\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}\right\}dt$$

And we will symbolize the Hausdorff method for the Fourier trigonometric series of the function f as :

$$H_n(x) = H_n(f,x) = \sum_{k=0}^{n}h_{n,k}s_k(f,x) \qquad , n = 0{,}1{,}2, \dots\dots,$$

And we will symbolize the Hausdorff method for the Fourier series accompanying the function f as:

$$H_n^{\sim}(f,x) = \sum_{k=0}^{n}h_{n,k}s^{\sim}{}_k(f,x)\ , n = 0{,}1{,}2, \dots.$$

Preliminary1: [9] let $g^{\sim}(u,t) = Re\left[\sum_{k=0}^{n+\lambda}\binom{n+\lambda}{k}u^k(1-u)^{n+\lambda-k}e^{i\left(\frac{k+1}{2}\right)t}\right]$

For $0 < u < 1$ and $0 \le t \le 1, \lambda \ge 1$ then:

$$\left|\int_0^1 g^{\sim}(u,t)d\gamma(u)\right| = \begin{cases} O(1) & ; 0 < t \le \dfrac{m}{(n+\lambda)} \\ O\left(\dfrac{1}{t(n+\lambda)}\right) & ; \dfrac{m}{(n+\lambda)} \le t \le m \end{cases}$$

Preliminary2: [10] let $g(u,t) = Im\left[\sum_{k=0}^{n+k}\binom{n+k}{k}u^k(1-u)^{n+\lambda-k}e^{i\left(k+\left(\frac{1}{2}\right)\right)t}\right]$

for $0 \le u \le 1$ and $0 \le t \le m, \lambda \ge 1$, then:

$$\left|\int_0^1 g(u,t)d\gamma(u)\right| = \begin{cases} O((n+\lambda)t) & 0 \le t \le \dfrac{1}{(n+\lambda)} \\ O\left(\dfrac{1}{t(n+\lambda)}\right) & \dfrac{1}{(n+\lambda)} \le t \le m \end{cases}$$

Preliminary 3: [10] let $k_{n+\lambda}^{H} = \frac{1}{2m\sin\left(\frac{t}{2}\right)}\int_0^1 g(u,t)d\gamma(u)$

DOI: https://doi.org/10.54216/NIF.040201
Received: January 08, 2024 Accepted: June 28, 2024

Then: $\left|K_{n+\lambda}^{H}(t)\right| = \begin{cases} O(n+\lambda) & 0 \le t \le \frac{1}{n+\lambda} \\ O\left(\frac{1}{t^{2}(n+\lambda)}\right) & \frac{1}{n+\lambda} \le t \le m \end{cases}$

Preliminary4: [2] let $f \in Z_{\alpha,p}$ then for $0 < t \le m$ we have:

$$\|\phi(a,t)\|_{p} = O(t^{\alpha}) \qquad (i)$$

$$\|\phi(a+y,t)+\phi(a-y,t)-2\phi(a,t)\|_{p} = \begin{cases} O(t^{\alpha}) \\ O(y^{\alpha}) \end{cases} \quad (ii)$$

$$\|\phi(a+y,t)+\phi(a-y,t)-2\phi(a,t)\|_{p} = O\left(v(y)\frac{t^{\alpha}}{v(t)}\right) \qquad (iii)$$

Where $\phi(a,t) = f(a+t)+f(a-t)-2f(a)$

Proof of theorem (1):

we have $H_{n+\lambda}^{\sim}(f,a) - f^{\sim}(a) = {}_{k=0}^{n+\lambda}\sum h_{n+\lambda,k}\{s_{n+k}^{\sim}(f,a) - f^{\sim}(a)\}$

$=\frac{1}{2m}\int_{0}^{m}\left(\frac{\psi_a(t)}{\sin\left(\frac{t}{2}\right)}\sum_{k=0}^{n+\lambda} h_{n+\lambda,k}\cos\left(k+\frac{1}{2}\right)t\right)dt$

$=\frac{1}{2m}\int_{0}^{m}\left(\frac{\psi_a(t)}{\sin\left(\frac{t}{2}\right)}\sum_{k=0}^{n+\lambda}\binom{n+\lambda}{k}\Delta^{n+\lambda-k}\mu_k\cos\left(k+\frac{1}{2}\right)t\right)dt$

$$= \frac{1}{2m}\int_{0}^{m}\left(\frac{\psi_a(t)}{\sin\left(\frac{t}{2}\right)}\sum_{k=0}^{n+\lambda}\binom{n+\lambda}{k}\int_{0}^{1}u^{k}(1-u)^{n+\lambda-k}d\gamma(u)Re\left\{e^{i\left(k+\frac{1}{2}\right)t}\right\}\right)dt$$

$=\frac{1}{2m}\int_{0}^{m}\left(\frac{\psi_a(t)}{\sin\left(\frac{t}{2}\right)}\int_{0}^{1}Re\left[\sum_{k=0}^{n+\lambda}\binom{n+\lambda}{k}u^{k}(1-u)^{n+\lambda-k}e^{i\left(k+\frac{1}{2}\right)t}\right]d\gamma(u)\right)dt = \frac{1}{2m}\int_{0}^{m}\left(\frac{\psi_a(t)}{\sin\left(\frac{t}{2}\right)}\int_{0}^{1}\sim_{g(u,t)d\gamma(u)}\right)dt$

Where: $s_{n+\lambda}^{\sim}(f,a) - f^{\sim}(a) = \frac{1}{2m}\int_{0}^{m}\psi_a(t)\frac{\cos\left(n+\lambda+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}dt$

And therefore: $||H_{n+\lambda}^{\sim}(f,a) - f^{\sim}(a)| \le \int_{0}^{m}\frac{|\psi_a(t)|}{t}\left|\int_{0}^{1}g^{\sim}(u,t)d\gamma(u)\right|dt$

$=\left(\int_{0}^{\frac{m}{n+\lambda}} + \int_{\frac{m}{n+\lambda}}^{m}\right)\frac{|\psi_a(t)|}{t}\left|\int_{0}^{1}g^{\sim}(u,t)d\gamma(u)\right|dt = I_1 + I_2 \quad (10)$

And that is according to the relation $\frac{1}{\sin\left(\frac{t}{2}\right)} \le \frac{m}{t}$ for $0 < t \le m$.

Now, according to primality 1 and Holder's regression, we have :

$$I_1 = O(1)\left\{\int_{0}^{\frac{m}{n+\lambda}}\frac{\psi_a(t)}{t^{a}}t^{\alpha-1}dt\right\}$$

$=O(1)\left\{\int_{0}^{\frac{m}{n+\lambda}}\left(|\psi_a(t)|\frac{1}{t^{\alpha}}\right)^{p}dt\right\}^{\frac{1}{p}} \times \left\{\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\frac{m}{n+\lambda}}(t^{\alpha-1})^{q}dt\right\}^{\frac{1}{q}}$

$=O\left[\frac{1}{(n+\lambda)^{\frac{1}{p}}}(n+\lambda)^{\frac{1}{p}}\left(\frac{m}{n+\lambda}\right)^{\alpha}\right] \quad (11)$

Using the relations (2), (3) and $\frac{1}{p}+\frac{1}{q}=1$

And: $I_2 = O\left(\int_{\frac{m}{n+\lambda}}^{m}\frac{t^{-\delta-\alpha}|\psi_a(t)|}{n+\lambda}\frac{t^{\alpha-1}}{t^{-\delta+1}}dt\right)$

DOI: https://doi.org/10.54216/NIF.040201
Received: January 08, 2024 Accepted: June 28, 2024

$$=O\left\{\frac{1}{n+\lambda}\int_{\frac{m}{n+\lambda}}^{m}\left(t^{-\delta-\alpha}|\psi_a(t)|\right)^p dt\right\}^{\frac{1}{p}}\times\left\{\int_{\frac{m}{n+\lambda}}^{m}\left(t^{\alpha+\delta-2}\right)^q dt\right\}^{\frac{1}{q}}$$

$$=O\left[(n+\lambda)^{\delta-1}\left(\frac{m}{n+\lambda}\right)^{\alpha}\left(\frac{n+\lambda}{m}\right)\left\{\int_{\frac{m}{n+\lambda}}^{m} t^{-(1-\delta)q}dt\right\}^{\frac{1}{q}}\right]$$

$$=O\left[(n+\lambda)^{\delta}\left(\frac{m}{n+\lambda}\right)^{\alpha}(n+\lambda)^{1-\delta-\frac{1}{q}}\right]$$

$$=O\left[\left((n+\lambda)^{1-\alpha-\frac{1}{q}}\right)^{\alpha}\right] \qquad (12)$$

And so using the Preliminary 1 and the holder regression and the relation $\frac{1}{\sin\left(\frac{t}{2}\right)} \le \frac{m}{t}$ for $0 < t \le m$.

From Relations (1) and (4) and the middle value theorem of integration, and $0 < \delta < \frac{1}{p}$ and summing relations from (10), (11) and (12), we get:

$$|H^{\sim}_{n+\lambda}(f,a) - f^{\sim}(a)| = O\left[(n+\lambda)^{\frac{1}{p}-\alpha}\, m^{\alpha}\right]$$

Finally, from the previous relation and the relation $\left(\frac{m}{n+\lambda}\right)^{\alpha-1} \le \left(\frac{1}{n+\lambda}\right)^{\alpha-1}$,we find that:

$$\|H^{\sim}_{n+\lambda}(f,a) - f^{\sim}(a)\|_p = O\left[(n+\lambda)^{\frac{1}{p}-\alpha}\right]$$

Proof of theorem 2:

Let $s_k(f,a) - f(a) = \frac{1}{2m}\int_0^m \phi(a,t)\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\left(\frac{t}{22}\right)}dt$

Since $\sum_{k=0}^{n+\lambda} h_{n+\lambda,k} = 1$ for every $n+\lambda$ in any regular Hausdorff Matrix H, then:

$$H_{n+\lambda}(a) - f(a) = \frac{1}{2m}\int_0^m \phi(a,t)\sum_{k=0}^{n+\lambda} h_{n+\lambda,k}\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}dt$$

$$\frac{1}{2m}\int_0^m \frac{\phi(a,t)}{\sin\left(\frac{t}{2}\right)}\sum_{k=0}^{n+\lambda}\int_0^1 \binom{n+\lambda}{k}u^k(1-u)^{n+\lambda-k}d\gamma(u)Im\left(e^{i\left(k+\frac{1}{2}\right)t}\right)dt =$$

$$=\frac{1}{2m}\int_0^m \frac{\phi(a,t)}{\sin\left(\frac{t}{2}\right)}\int_0^1 Im\left[\sum_{k=0}^{n}\binom{n+\lambda}{k}u^k(1-u)^{n+\lambda-k}\,e^{i\left(k+\frac{1}{2}\right)t}\right]d\gamma(u)dt$$

$$=\frac{1}{2m}\int_0^m \frac{\phi(a,t)}{\sin\left(\frac{t}{2}\right)}\int_0^1 g(u,t)d\gamma(u)dt$$

Suppose that: $q_{n+\lambda}(a) = H_{n+\lambda}(a) - f(a) = \int_0^m \phi(a,t)K^H_{n+\lambda}(t)dt$

Then: $q_{n+\lambda}(a,y) + q_{n+\lambda}(a-y) - 2q_{n+\lambda}(a) = \int_0^m[\phi(a+y,t) + \phi(a-y,t) - 2\phi(a,t)]K^H_{n+\lambda}(t)dt$

Using the generalized Minkowski-Minkowski Tensor [11], we obtain:

$$\|q_{n+\lambda}(a+y) + q_{n+\lambda}(a-y) - 2q_{n+\lambda}(a)\|_p$$

$$=\left\{\frac{1}{2m}\int_{-m}^{+m}|q_{n+\lambda}(a+y) + q_{n+\lambda}(a-y) - 2q_{n+\lambda}(a)|^p da\right\}^{\frac{1}{p}}$$

DOI: https://doi.org/10.54216/NIF.040201
Received: January 08, 2024 Accepted: June 28, 2024

$$=\left\{\frac{1}{2m}\int_{-m}^{+m}\left|\int_{0}^{m}[\phi(a+y,t)+\phi(a-y,t)-2\phi(a,t)]K_{n+\lambda}^{H}(t)dt\right|^{p}da\right\}^{\frac{1}{p}}$$

$$\leq\int_{0}^{m}\left\{\frac{1}{2m}\int_{-m}^{+m}\left|[\phi(a+y,t)+\phi(a-y,t)-2\phi(a,t)]K_{n+\lambda}^{H}(t)\right|^{p}da\right\}^{\frac{1}{p}}dt$$

$$=\int_{0}^{m}\left(\left|K_{n+\lambda}^{H}(t)\right|^{p}\right)^{\frac{1}{p}}\left\{\frac{1}{2m}\int_{-m}^{+m}|\phi(a+y,t)+\phi(a-y,t)-2\phi(a,t)|^{p}da\right\}^{\frac{1}{p}}dt$$

$$=\int_{0}^{m}\|\phi(a+y,t)+\phi(a-y,t)-2\phi(a,t)\|_{p}\left|K_{n+\lambda}^{H}(t)\right|dt$$

$$=\int_{0}^{\frac{1}{n+\lambda}}\|\phi(a+y,t)+\phi(a-y,t)-2\phi(a,t)\|_{p}\left|K_{n+\lambda}^{H}(t)\right|dt$$

$$=\int_{0}^{\frac{1}{n+\lambda}}\|\phi(a+y,t)+\phi(a-y,t)-2\phi(a,t)\|_{p}\left|K_{n+\lambda}^{H}(t)\right|dt$$

$$+\int_{\frac{1}{n+\lambda}}^{m}\|\phi(a+y,t)+\phi(a-y,t)-2\phi(a,t)\|_{p}\left|K_{n+\lambda}^{H}(t)\right|dt=I_{1}+I_{2} \qquad (13)$$

Hence, by priming 3 and priming 4 (iii) and from the invariance of the function $\frac{t^{\alpha}}{v(t)}$ it results that:

$$I_{1}=\int_{0}^{\frac{1}{n+\lambda}}\|\phi(a+y,t)+\phi(a-y,t)-2\phi(a,t)\|_{p}\left|K_{n+\lambda}^{H}(t)\right|dt$$

$$=0\left(\int_{0}^{\frac{1}{n+\lambda}}v(y)\frac{t^{\alpha}}{v(t)}(n+\lambda)dt\right)=\left((n+\lambda)v(y)\int_{0}^{\frac{1}{n+\lambda}}\frac{t^{\alpha}}{v(t)}dt\right)$$

$$=O\left((n+\lambda)v(y)\frac{\left(\frac{1}{n+\lambda}\right)^{\alpha}}{v\left(\frac{1}{n+\lambda}\right)}\int_{0}^{\frac{1}{n+\lambda}}dt\right)$$

$$=0\left(v(y)\frac{\left(\frac{1}{n+\lambda}\right)^{\alpha}}{v\left(\frac{1}{n+\lambda}\right)}\right) \qquad (14)$$

Also by preliminary 3 and preliminary 4 (iii),we have:

$$I_{2}=\int_{\frac{1}{n+\lambda}}^{m}\|\phi(a+y,t)+\phi(a-y,t)-2\phi(a,t)\|_{p}\left|K_{n+\lambda}^{H}(t)\right|dt$$

$$=0\left(\int_{\frac{1}{n+\lambda}}^{m}v(y)\frac{t^{\alpha}}{v(t)}\frac{1}{t^{2}(n+\lambda)}dt\right)$$

DOI: https://doi.org/10.54216/NIF.040201
Received: January 08, 2024 Accepted: June 28, 2024

$$= 0\left(\frac{1}{n+\lambda}v(y)\int_{\frac{1}{n+\lambda}}^{m} v(y)\frac{t^{\alpha-2}}{v(t)}dt\right) \qquad (15)$$

And so from (13), (14) and (15) we have :

$$\|q_{n+\lambda}(a+y)+q_{n+\lambda}(a-y)-2q_{n+\lambda}(a)\|_p$$

$$= 0\left(v(y)\frac{\left(\frac{1}{n+\lambda}\right)^{\alpha}}{v\left(\frac{1}{n+\lambda}\right)}\right)$$

$$+0\left(\frac{1}{n+\lambda}v(y)\int_{\frac{1}{n+\lambda}}^{m}\frac{t^{\alpha-2}}{v(t)}dt\right)sup_{y\neq 0}\frac{\|q_{n+\lambda}(a+y)+q_{n+\lambda}(a-y)-2q_{n+\lambda}(a)\|_p}{v(y)}$$

$$= 0\left(\frac{\left(\frac{1}{n+\lambda}\right)^{\alpha}}{v\left(\frac{1}{n+\lambda}\right)}\right)+O\left(\frac{1}{n+\lambda}\int_{\frac{1}{n+\lambda}}^{m}\frac{t^{\alpha-2}}{v(t)}\right) \qquad (16)$$

And by preliminaries 3 and 4 again, we obtain:

$$\|q_{n+\lambda}(a)\|_p \leq \left(\int_{0}^{\frac{1}{n+\lambda}} + \int_{\frac{1}{n+\lambda}}^{m}\right)\|\phi(a,t)\|_p\left|K_{n+\lambda}^{H}(t)\right|dt$$

$$= 0\left((n+\lambda)\int_{0}^{\frac{1}{n+\lambda}} t^{\alpha}dt\right)+O\left(\frac{1}{n+\lambda}\int_{\frac{1}{n+\lambda}}^{m} t^{\alpha-2}\right)$$

$$= 0\left(\left(\frac{1}{n+\lambda}\right)^{-\alpha}\right)+O\left(\frac{1}{n+\lambda}\int_{\frac{1}{n+\lambda}}^{m} t^{\alpha-2}\,dt\right) \qquad (17)$$

Now, and from it by Relations (16) and (17), we get:

$$\|q_{n+\lambda}(a)\|_{\alpha,p} = \|q_{n+\lambda}(a)\|_p + sup_{y\neq 0}\frac{\|q_{n+\lambda}(a+y)+q_{n+\lambda}(a-y)-2q_{n+\lambda}(a)\|_p}{v(y)} = 0\left(\frac{1}{(n+\lambda)^{\alpha}}\right)+O\left(\frac{1}{n+\lambda}\int_{\frac{1}{n+\lambda}}^{m} t^{\alpha-2}dt\right) = O\left(\frac{\frac{1}{(n+\lambda)^{\alpha}}}{v\left(\frac{1}{n+\lambda}\right)}\right)+O\left(\frac{1}{n+\lambda}\int_{\frac{1}{n+\lambda}}^{m}\frac{t^{\alpha-2}}{v(t)}dt\right) = \sum_{i=1}^{4} O(J_i) \qquad (18)$$

Now, we write J_1 in terms of J_3 and also J_2 and J_3 in terms of J_4 and from the invariance of the function V(t):

We find: $t^{\alpha} = \frac{t^{\alpha}}{v(t)}v(t) \leq v(m)\frac{t^{\alpha}}{v(t)} = O\left(\frac{t^{\alpha}}{v(t)}\right)$

And that's for $0 < t \leq m$.

So, for $t = \frac{1}{n+\lambda}$ We have: $J_1 = 0(J_3)$ and from the invariance of the function V (t) we find:

$$J_2 = \frac{1}{n+\lambda}\int_{\frac{1}{n+\lambda}}^{m}\frac{t^{\alpha-2}}{v(t)}dt \leq \frac{1}{n+\lambda}v(m)\int_{\frac{1}{n+\lambda}}^{m}\frac{t^{\alpha-2}}{v(t)}dt = O\,(J_4) \qquad (20)$$

Considering that the function $\frac{t^{\alpha}}{v(t)}$ is positive and constant, we have:

DOI: https://doi.org/10.54216/NIF.040201
Received: January 08, 2024 Accepted: June 28, 2024

$$J_4 = \frac{1}{n+\lambda} \int_{\frac{1}{n+\lambda}}^{m} \frac{t^{\alpha-2}}{v(t)} dt \geq \frac{\frac{1}{(n+\lambda)^{\alpha}}}{v\left(\frac{1}{n+\lambda}\right)} \frac{1}{n+\lambda} \int_{\frac{1}{n+\lambda}}^{m} t^{-2} dt = \frac{\frac{1}{(n+\lambda)^{\alpha}}}{v\left(\frac{1}{n+\lambda}\right)} \frac{1}{n+\lambda} \left(n+\lambda-\frac{1}{m}\right) \geq \frac{\frac{1}{(n+\lambda)^{\alpha}}}{2v\left(\frac{1}{n+\lambda}\right)}$$

Where $\left(\frac{1}{n+\lambda}\right)\left(n+\lambda-\frac{1}{m}\right) > \left(\frac{1}{n+\lambda}\right)(n+\lambda) \geq \frac{1}{2}$

Therefore: $J_3 = O(J_4)$

Adding the relations (18) and (21), we get:

$$\|q_{n+\lambda}(a)\|_{\alpha,p} = O(J_4) = O\left(\left(\frac{1}{n+\lambda}\right) \int_{\left(\frac{1}{n+\lambda}\right)}^{m} \frac{t^{\alpha-2}}{v(t)} dt\right)$$

If: $E_{(n+\lambda)}(f) = inf_{(n+\lambda)} \left\|q_{(n+\lambda)}(a)\right\|_{\alpha,p} = \left(\left(\frac{1}{n+\lambda}\right) \int_{\left(\frac{1}{n+\lambda}\right)}^{m} \frac{t^{\alpha-2}}{v(t)} dt\right)$

References

[1] Rhoades, BE.2014, The degree of approximation of function, and their conjugates, belonging to sever al general Lipschitz classes by Hausdorff matrix means of the Fourier series and conjugate series of a Fourier series. Tamkang J.Math.45 (4), 389-395.

[2] Lal, S: Shireen. 2013, Best approximation of functions of generali zed Zygmund class by Matrix –Euler summability mean of fourier series Bull .Math . Anal. Appl.5 (4), 1-13.

[3] U.Singh, S. Sonkar .2013 , Trigonometric approximation of signals function belonging to weighted Lip t – class by Hausdorff means, J Appl. Funct. Anal. 8.37-44.

[4] H.K.Nigam, a Sharma. 2012, on approximation of conjugate of functions belonging to different classes by product means, Int .J. pure Appl. Math .76 (2)303-316.

[5] V.N.Mishra, K. Khatri, L.N. Mishr2012, product summability transform of conjugate series of fourier series, Int .J. Math. Sci. 20121-13.

[6] Rhoades, BE, Ozkoklu, K, Albayrak , l.2011, On the degree of approximation of function belonging to a Lipschitz class by Hausdorff means of its fourier series .Appl .Math .Comput .217(16), 6868-6871

[7] H.K.Nigam , A.Sharma 2010, On approximation of conjugate of a function belonging to Lip(t,r) class by product summability means of conjugate series of fourier series, Int .J . Contemp. Math.Sci 2673-2683.

[8] Mricz, F.2010, Enlarged Lipschitz and Zygmund classes of functions and fourier transforms East J.Approx.16 (3), 259-271.

[9] Rhoades, BE .2003, on the degree of approximation of functions belonging to a Lipschitz class by Hausdroff means of its fouries series. Tamkang J.Math .34(3),245-247

[10] Das,G, Nath, A, Ray, BK.2002, An estimate of the rate of convergence of fourie 10, Series in the generalized Holder metric.In :Anal. Appl., pp.43-60.

[11] A. Zygmun.2002, Trigonometric series, thirded , Cambridge University press, London.

[12] G.Bachman, L. Narici, E. Beckensties.2000, Fourier and Wavelet Analysis, Springer Verlag,New York.

[13] K. Qureshi.1982, On the degree of approximation of functions belonging to the Lip(α, p) $by\ means\ of\ a\ conjugate\ series, Indian\ J. Pure\ Appl. Math. 13(5)560-563.$

[14] Qureshi , K.1982, On the degree of approximation of functions belonging to the class of Lip$\alpha. Indian\ J. Pure\ Appl. Math. 13(8), 898-903.$

DOI: https://doi.org/10.54216/NIF.040201
Received: January 08, 2024 Accepted: June 28, 2024