A Note on Some Generalized M-Flat Modules

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Abstract

Let I be a right (left) ideal of a ring R, then R/I is a right (left) generalized m – flat module (GmF – module) if and only if for each $a \in I$, there exists $b \in I$ and a fixed positive integer m such that $a^m = ba^m(a^m = a^m b)$. In this paper, we study the characterization and properties this class of flat modules, and we give the relation between this class and generalized m- flat modules and m– regular rings, reduced rings, reversible rings and uniform rings.

Keywords: Algebraic module; M-module; Flat module; Algebraic ring

1. Introduction

In this research, R is a neutral aggregation ring and the sizes are monotonic. For each $a \in R$ ، r(a) and $l(a)$ denote the right Corruptor and the left Corruptor of a respectively, and J(R) and N(R) denote the Jacobson Root and the set of all zero-power elements respectively. A ring R is said to be a right-handed uniform ring if every left-handed non-zero ideal in R is fundamental [6]. A ring R is said to be reduced if R contains no nonzero elements with zero powers, or in other words for every element $a \in R$ if $a^2 = 0$then a=0. A ring R is said to be reversible if ab=0 leads to ba=0 for each $a,b \in R$ [2]. A ring R is said to be a doubly right-dual ring (Weakly-duo), if for every $a \in R$ there exists a positive integer n such that $a^n R$ is ideally right-handed and left-handed in R [1]. A ring R is said to be a right (left) Quasi-duo ring, if a greater right (left) ideal is an ideal [7]. Let I be a right-handed (left-handed) ideal in the ring R, then R/I is a right-handed (left-handed) flat scalar if and only if for every $a \in I$, there exists a $b \in I$ such that a=ba (a=ab) [5]. A ring R is said to be a von Neumann regular ring if for every $a \in R$ there exists a $b \in R$ such that a=aba. In the source [4] McCoy gave a definition of the regular ring of the m- pattern and the regular of the π-pattern. A ring R is said to be a regular ring of the m-pattern (m-regular) if for every $a \in R$ there exists a constant positive integer m such that a^m is regular $(a^m = a^m b\, a^m)$for some $b \in R$. And if m is not constant then R is called a regular ring of the π-pattern.

2. Generalized flat modules of the m-pattern

In this article, a kind of generalized flat scalar of m-pattern is studied when m is a constant number with some of its basic properties.

Definition 2.1:

Let I be a right (left) ideal in ring R. R/I is a generalized right-handed (left-handed) planar dimension of the m-pattern (a dimension of the GmF-pattern) if and only if for every $a \in I$, there is a $b \in I$ constant positive integer m such that $(a^m = a^m b\,)a^m = b\, a^m$.

Example:

In the ring of integers norm 12, I={0,4,8}. The Z_{12}/I is a flat size, so it is a size of the G2F-type.

Case 2.2: [3]

Let R be an inverse ring, if for every $0 \neq a \in R$ and a positive integer n, then:

$$r(a^n) = l(a^n)$$

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The following case gives the relationship between the right and left sizes of the GmF-pattern.

Case 2.3:

Let R be an inverse ring, and Let I be an ideal in R. R/I is a left-hand dimension of the GmF-pattern if and only if R/I is a right-hand dimension of the GmF-pattern.

Proof: clear.

Case 2.4:

Let R be a ring and M be a right-greatest ideal in R. Let R / M be the right size of the GmF-pattern. Every element that is not a left zero divisor has a right inverse.

Proof:

We assume that $a \neq 0$is an element that is not a left zero divisor, and we assume that $aR \neq R$, so there is a right-greatest ideal M in R that contains aR, and since R/M is a right-hand measure of the GmF-pattern, there is b∈M and a constant positive integer m, so that $a^m = ba^m$ which leads to $(1-b)a^m = 0$, since a is an element that is not an Left zero divisor, then (1-b)=0, so we get b=1∈M, and this contradicts the assumption, if aR=R, then a has a right inverse.

Help case 2.5: [7]

If R is a right quasidiregular ring, then N(R)⊆J (R)

The following case shows the relationship between Jacobson J(R) and the set of all elements with zero powers N (R) in the right dimensions of the GmF-pattern.

Case 2.6:

Let R be a right quasidirectional ring, let a∈R, and since Ra/R is a right dimension of the GmF-pattern, then J(R) = N(R)

Proof:

We assume that $0 \neq a \in R$ since Ra/R is a right-hand measure of the GmF-pattern, there is b∈aR and a constant positive integer m, so that $a^m = ba^m$, and this leads $a^m = ara^m$to some r∈R, so $(1-ar)a^m = 0$, so (1-ar) has an inverse, therefore there is an element u∈R such that u(1-ar)=1 and multiplying it by $a^m$from the right we get $u(a^m - ara^m)a^m$that is, $a^m = 0$, if a∈N(R), So $J(R) \subseteq N(R)$‘, and since $N(R) \subseteq J(R)$ (by auxiliary case 2.6), we get $J(R)$=$N(R)$.

Theorem 2.7:

Let R be a weakly dual ring, then R/a^mR is a right-hand measure of the GmF-pattern, for some a∈R and for a constant positive integer m if and only if a^mR is a neutral ideal.

Proof:

We assume that I is ideal in the ring R, so that $I = a^mR$, where a∈R and a constant positive integer m, it is obvious that $I^2 \subseteq I$. On the other hand, since R/I is a right-hand dimension of the GmF-pattern, there exists a $b \in a^mR$ such that $a^m = ba^m = a^mra^m$, for some r∈R. Now, $a^m \in I$ and $a^m = a^m ra^m \in I^2$, so $I \subseteq I^2$. Hence $I^2 \subseteq I$.

To prove the converse, we assume that $I^2 = I$ and let be $a^m \in a^mR = (a^mR)^2$. Now, $(a^mR)^2 = a^mRa^mR$ which leads to $a^m = a^mca^m$, for some $c \in R$. Therefore, R/a^mR is a right-hand measure of the GmF-pattern.

3. The relationship between the flat modules of the GmF-pattern and other loops.

In this section, the relationship between the sizes of the GmF-pattern and the regular rings of the m-pattern, the divided ring, the uniform ring and the reduced ring is given.

The following case gives the relation between the uniform ring and the divided ring of the GmF-pattern.

Help case 3.1: [3]

Let R be a reduced ring, then for every $a \in R$ and for any positive integer n, then:

1. $r(a^n) = l(a^n)$

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2. $r(a) = l(a^n)$

3. $a^n R \cap r(a^n) = 0$

Theorem 3.2:

Let R be an inverse right-uniform ring and M be a right-greatest ideal in R, Where R / M is a right-dimensional of the GmF-pattern, R is a divided ring.

Proof:

We assume that $0 \neq a \in R$ and $aR \neq R$, so there is a right-greatest ideal m containing aR. Since R / M is a right-hand measure of the GmF-pattern, there exists $b \in aR \subseteq M$ and a constant positive integer m, such that $a^m = ba^m$, and this leads $a^m = ara^m$، for some $r \in R$. Since R is a right uniform ring, every right ideal in it is prime. Let $r(ar) \cap a^m R \neq 0$, since there is $00 \neq x \in r(ar) \cap a^m R$, and from it we get $arx = 0$and $x = a^m z$ for some z ∈ R, if $ara^m z = 0$ this leads to $a^m z = 0 = x$, so $r(ar) \cap a^m R = 0$, since R is a uniform ring and $a^m R \neq 0$, then $r(ar) = 0$. Since R is an inverse ring, $l(ar) = 0$, and (according to case 2.4), ar is an element that has a right inverse, so there is $v \in R$ such that $arv = 1$, so $a(rv) = 1 \in M$ and this is a contradiction, so $aR = R$. Now, let be $ar = 1$, and multiplying it by a from the right we get ara=a, and this leads to $(1 - ra) \in r(a) = l(a) \subseteq l(ar) = r(ar) = 0$, (by auxiliary case (2.2), so $(1 - ra) = 0$, if $ra = 1$ and therefore a has a left inverse and therefore R is a divided ring.

Case 3.3:

Let I be an ideal in the ring R, if R is a regular ring of m-pattern, then R / I is a right (left) scalar of GmF-pattern.

Proof: clear

Definition 3.4:

An ideal I in the ring R is said to be semi-Prime complete (Completely semi-prime) If $a \in R$ and a positive integer n such that $a^n \in I$, leads to $a \in I$.

Theorem 3.5:

Let R be a ring and I a perfect semi-Prime complement in R. R/I is left-dimensional of the GmF-pattern and for every $x \in I$, if and only if $I + r(x^m) = R$ for a constant positive integer m.

Proof:

Suppose that R / I is a left dimension of the GmF-pattern, and also suppose $x \in I$, there exists $b \in I$ and a constant positive integer m, so that $x^m = x^m b$, then $x^m(1 - b) = 0$, and then $(1 - b) \in r(x^m)$, being $1 = b + (1 - b)$, therefore $R = I + r(x^m)$.

To prove the converse, we assume that $R = I + r(x^m)$, then $1 = b + c$, where $b \in I$and $c \in r(x^m)$, multiplying by $x^m$from the left we get $x^m = x^m b + x^m c$, this leads to $x^m = x^m b$, that is, $x^m \in I$. Since I is an integral quasi-Prime Ideal, $x \in I$ and $x^m = x^m b$ and this leads to the fact that R / I is a left dimension of the GmF-pattern.

From case 3.3 and theorem 3.5 we get the following result:

Result 3.6:

Let R be a regular ring of the m-pattern and I is a perfect semi-Prime complement in R, then $R = I + r(a^m)$ for every $a \in I$ and for a constant positive integer m.

Case 3.7:

Let R be a ring and M be a greater right ideal in R, Where R / M is a right dimension of the GmF-pattern. R is a reduced ring if $l(a^m) \subseteq r(a)$ and for a constant positive integer m.

Proof:

We assume that a is a nonzero element of R, so that $a^2 = 0$. $a \in r(a)$, now, let M be a right-greatest ideal containing $r(a)$. Since $a \in r(a) \subset M$, then $a \in M$ and since R/M is a right-hand measure of the GmF-pattern, then there is $b \in M$ and a constant positive integer m, so that $a^m = ba^m$and therefore $(1 - b)a^m = 0$, this leads to $(1 - b) \in l(a^m) \subseteq r(a)$, if $(1 - b) \in r(a) \subset M$, from which $1 \in M$ results, and this is a contradiction of the fact $M \neq R$, so a=0, that R is a reduced loop.

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Case 3.8:

Let R be a ring and M be a greater ideal where R / M is a right - hand measure of the GmF-pattern. The $r(a^m)$compound is a direct summation in R. If $l(a^m) \subseteq r(a)$for each $a \in R$ and m is a constant positive integer.

Proof:

We must prove $r(a^m)$ direct summation complex. First, we claim that $a^m R + r(a^m) = R$. If this claim is not true there exists a right-greatest ideal m containing $a^m R + r(a^m)$, Now, R/M is a right-hand measure of the GmF-pattern, then $(a^m)^n = b(a^m)^n$, for some $b \in M$ and for a constant positive integer n this results in $(1 - b) \in l(a^{mn}) = r(a^m) \subseteq M$, (by the case of R 3.6 reduced ring), that is, 1∈M, and this is a contradiction. So $a^m R + r(a^m) = R$. Now, we observe $a^m R \cap r(a^m) = 0$, (by auxiliary case 3.1). Therefore $r(a^m)$is a direct summation compound in R.

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