



On the Classification of 3-Cyclic Refined Real Vector Spaces and Related Inner Products

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Abstract

This paper is concerned with the study of the classification of 3-cyclic refined real vector spaces by semi-module isomorphisms, where we use this algebraic technique to find the algebraic relationship between real 3-cyclic refined vector spaces and classical vector spaces. Also, we define the inner products on these spaces and prove many related inequalities.

Keywords: 3-cyclic refined neutrosophic vector space; Semi-isomorphism, Orthogonality, Inner product

1. Introduction

The concept of n-cyclic refined neutrosophic algebraic structure was proposed in [12] as a novel generalization of classical structures, especially rings and modules. Many related substructures were studied by many different authors, where we can find in the literature many good results about Diophantine equations and the group of units in n-cyclic refined neutrosophic rings [1-4, 6-8]. On the other hand, n-cyclic refined neutrosophic sets were used in the study of groups and vector spaces [13, 18-19]. In [5], authors have proposed many open research questions that concerning the properties of n-cyclic refined neutrosophic vector spaces and matrices in general. Some of these problems were solved for 3-cyclic and 4-cyclic refined neutrosophic matrices with real entries [20].

In this work, we study of the classification of 3-cyclic refined real vector spaces by semi-module isomorphisms, where we use this algebraic technique to find the algebraic relationship between real 3-cyclic refined vector spaces and classical vector spaces. Also, we define the inner products on these spaces and prove many related inequalities.

2. Main Discussion

Definition [12, 13]

Let V be a vector space over the real field \mathbb{R} or a complex field C , the corresponding 3-cyclic refined vector space is defined as follows:

$$V_3(I) = \{V_0 + \sum_{i=1}^3 V_i I_i \ ; \ V_i \in V\}.$$

Theorem (1): [12, 13]

$(V_3(I), +, \cdot)$ is a module over $R_3(I)$ (not a space) or $C_3(I)$, with:

$$\begin{cases} (+): V_3(I) \times V_3(I) \rightarrow V_3(I) \\ (\cdot): R_3(I) \times V_3(I) \rightarrow V_3(I) \end{cases} \quad \text{such that:}$$

$$(V_0 + \sum_{i=1}^3 V_i I_i) + (V'_0 + \sum_{i=1}^3 V'_i I_i) = (V_0 + V'_0) + \sum_{i=1}^3 (V_i + V'_i) I_i,$$

$$(m_0 + \sum_{i=1}^3 m_i I_i) \cdot (V_0 + \sum_{i=1}^3 V_i I_i) = m_0 V_0 + (\sum_{i+j \equiv 1 \pmod{3}} m_i V_j) I_1 + (\sum_{i+j \equiv 2 \pmod{3}} m_i V_j) I_2 + (\sum_{i+j \equiv 3 \pmod{3}} m_i V_j) I_3 ;$$

$$V_i, V'_i \in V, m_i \in \mathbb{R}.$$

Remark:

If $V_3(I)$ is defined over $C_3(I)$, then $m_i \in \mathbb{C}$ and:

$$(\cdot): C_3(I) \times V_3(I) \rightarrow V_3(I).$$

Definition:

Let V be a vector space over \mathbb{R} , we denote by $V(\mathbb{C})$ to corresponding vector space defined over \mathbb{C} .

Example:

Consider $V = \mathbb{R}^2$ as a vector space over \mathbb{R} , then:

$$V(\mathbb{C}) = \{(x, y) ; x, y \in \mathbb{C}\} \text{ with: } (\cdot): \mathbb{C} \times V(\mathbb{C}) \rightarrow V(\mathbb{C}).$$

Theorem (2): [17]

The mapping $g: R_3(I) \rightarrow R \times R \times C$ such that:

$$g: (x_0 + \sum_{i=1}^3 x_i I_i) = (x_0, \sum_{i=0}^3 x_i, x_0 + x_3 + x_1 e^{\frac{2\pi}{3}i} + x_2 e^{\frac{4\pi}{3}i}) \text{ is a ring isomorphism.}$$

Theorem (3):

Let $V_3(I)$ be a 3-cyclic refined vector space over $R_3(I)$, then there exists a semi-module isomorphism:

$$h: V_3(I) \rightarrow V(\mathbb{R}) \times V(\mathbb{R}) \times V(\mathbb{C}).$$

Proof:

Define $h: V_3(I) \rightarrow V(\mathbb{R}) \times V(\mathbb{R}) \times V(\mathbb{C})$ with:

$$h(V_0 + \sum_{i=1}^3 V_i I_i) = (V_0, \sum_{i=0}^3 V_i, V_0 + V_3 + V_1 e^{\frac{2\pi}{3}i} + V_2 e^{\frac{4\pi}{3}i}).$$

First, we mention that $V(\mathbb{C}) = \{v + iw ; v, w \in V\}$.

If $V_0 + \sum_{i=1}^3 V_i I_i = V'_0 + \sum_{i=1}^3 V'_i I_i$ then $V_i = V'_i$ for all $0 \leq i \leq 3$, and $h(V_0 + \sum_{i=1}^3 V_i I_i) = h(V'_0 + \sum_{i=1}^3 V'_i I_i)$.

For $X = (V_0 + \sum_{i=1}^3 V_i I_i), Y = (w_0 + \sum_{i=1}^3 w_i I_i) \in V_3(I)$, and $m = m_0 + \sum_{i=1}^3 m_i I_i$ from $R_3(I)$, we get:

$$h(X + Y) = h[(V_0 + w_0) + \sum_{i=1}^3 (V_i + w_i) I_i] = (V_0 + w_0, \sum_{i=0}^3 (V_i + w_i), V_0 + w_0 + V_3 + w_3 + (V_1 + w_1)e^{\frac{2\pi}{3}i} + (V_2 + w_2)e^{\frac{4\pi}{3}i}) = h(X) + h(Y).$$

$h(m \cdot X) = h[a + bI_1 + cI_2 + dI_3]$; where:

$$\begin{cases} a = m_0 x_0, b = m_0 V_1 + m_1 V_0 + m_1 V_3 + m_3 V_1 + m_2 V_2 \\ c = m_0 V_2 + m_2 V_0 + m_1 V_1 + m_2 V_3 + m_3 V_2 \\ d = m_0 V_3 + m_3 V_0 + m_3 V_3 + m_1 V_2 + m_2 V_1 \end{cases}$$

$$\text{Thus } h(mX) = (a, a + b + c + d, a + d + b e^{\frac{2\pi}{3}i} + c e^{\frac{4\pi}{3}i}),$$

We have:

$$a + b + c + d = (\sum_{i=0}^3 m_i)(\sum_{i=0}^3 V_i),$$

$$a + d + b e^{\frac{2\pi}{3}i} + c e^{\frac{4\pi}{3}i} = (m_0 V_0 + m_0 V_3 + m_3 V_0 + m_3 V_3 + m_1 V_2 + m_2 V_1) + e^{\frac{2\pi}{3}i}(m_0 V_1 + m_1 V_0 + m_1 V_3 + m_3 V_1 + m_2 V_2) + e^{\frac{4\pi}{3}i}(m_0 V_2 + m_2 V_0 + m_1 V_1 + m_2 V_3 + m_3 V_2),$$

$$\text{Also, } (m_0 + m_3 + m_1 e^{\frac{2\pi}{3}i} + m_2 e^{\frac{4\pi}{3}i})(V_0 + V_3 + V_1 e^{\frac{2\pi}{3}i} + V_2 e^{\frac{4\pi}{3}i}) = m_0 V_0 + m_0 V_3 + m_3 V_0 + m_3 V_3 + m_1 V_2 + m_2 V_1 + e^{\frac{2\pi}{3}i}(m_1 V_0 + m_1 V_3 + m_2 V_2 + m_0 V_1 + m_3 V_1) + e^{\frac{4\pi}{3}i}(m_0 V_2 + m_2 V_0 + m_1 V_1 + m_2 V_3 + m_3 V_2) = a + d + b e^{\frac{2\pi}{3}i} + c e^{\frac{4\pi}{3}i},$$

$$\text{So that: } h(mX) = (m_0, \sum_{i=0}^3 m_i, m_0 + m_3 + m_1 e^{\frac{2\pi}{3}i} + m_2 e^{\frac{4\pi}{3}i}) \cdot (V_0, \sum_{i=0}^3 V_i, V_0 + V_3 + V_1 e^{\frac{2\pi}{3}i} + V_2 e^{\frac{4\pi}{3}i}) = g(m) \cdot h(X).$$

Assume that: $h(X) = (0,0,0)$, then:

$$\begin{cases} V_0 = 0 \\ V_1 + V_2 + V_3 = 0 \\ V_3 + V_1 e^{\frac{2\pi}{3}i} + V_2 e^{\frac{4\pi}{3}i} = 0 \end{cases}$$

This means that:
$$\begin{cases} V_1 + V_2 + V_3 = 0 \\ V_3 + (\frac{-1}{2}V_1 - \frac{1}{2}V_2) = 0 \\ \frac{1}{2}V_1 - \frac{1}{2}V_2 = 0 \end{cases}$$

There for $V_1 = V_2 = V_3 = 0$, and $k_{er}(h) = \{0\}$.

For each $(V_0, V_1, V_2 + V_3i) \in V(\mathbb{R}) \times V(\mathbb{R}) \times V(\mathbb{C})$, there exists:

$$X = V_0 + I_1 \left(\frac{1}{3} V_1 - \frac{1}{3} V_2 + \frac{1}{\sqrt{3}} V_3\right) + I_2 \left(\frac{1}{3} V_1 - \frac{1}{3} V_2 - \frac{1}{\sqrt{3}} V_3\right) + I_3 \left(\frac{1}{3} V_1 - V_0 + \frac{2}{3} V_2\right) \in V_3(I).$$

$h(X) = (V_0, V_1, V_2 + V_3i)$, hence (h) is a bijection and then it is a semi-module isomorphism.

Theorem:

Let V be a finite- dimensional vector space over \mathbb{R} , with $E = \{e_1, \dots, e_n\}$ as a basis , i.e. $\dim(V) = n$, then:

1] E is a basis of $V(\mathbb{C})$ over \mathbb{C} .

2] the basis of $V_3(I)$ over $R_3(I)$ is:

$$\{h^{-1}(e_k, e_s, e_l) ; k, s, l \in \{1, \dots, n\}\}$$

3] $\dim(V_3(I)) = n^3$.

Proof:

1] assume that $v + wi \in V(\mathbb{C}) ; v, w \in V$, then:

$$\begin{cases} v = \sum_{i=1}^n a_i e_i \\ w = \sum_{i=1}^n b_i e_i \end{cases} ; e_i \in E, a_i, b_i \in \mathbb{R}$$

So that: $V + wi = \sum_{i=1}^n a_i e_i + i \sum_{i=1}^n b_i e_i = (\sum_{i=1}^n (a_i + ib_i)) e_i$, hence E generates $V(\mathbb{C})$ over \mathbb{C} .

If $\sum_{i=1}^n (a_i + ib_i) e_i = 0$, then $a_i = b_i = 0$ and E is a basis of $V(\mathbb{C})$ over \mathbb{C} .

2] Let $X = (V_0 + \sum_{i=1}^3 V_i I_i) \in V_3(I)$, then:

$$h(X) = (V_0, \sum_{i=0}^3 V_i, s_1 + s_2i) ; s_1 + s_2i = V_0 + V_3 + V_1 e^{\frac{2\pi}{3}i} + V_2 e^{\frac{4\pi}{3}i},$$

Thus $h(X) = (\sum_{i=1}^n a_i e_i, \sum_{i=1}^n b_i e_i, \sum_{i=1}^n (c_i + id_i) e_i) ; a_i, b_i, c_i, d_i \in \mathbb{R}$.

Hence, $h(X) = \sum_{k,s,l,m=1}^n (a_k, b_s, c_l + d_m i) \cdot (e_k, e_s, e_l)$, then:

$X = \sum_{k,s,l,m=1}^n g^{-1}(a_k, b_s, c_l + d_m i) h^{-1}(e_k, e_s, e_l)$, thus $k = \{h^{-1}(e_k, e_s, e_l) ; k, s, l \in \{1, \dots, n\}\}$ generates $V_3(I)$.

Assume that $X = \sum_{i=1}^n T_i E_i \in V_3(I)$ such that:

$T_i \in R_3(I), E_i \in K$, then:

$h(X) = \sum_{i=1}^n g(T_i) h(E_i) = 0$, thus:

$$\left(\sum_{i=1}^n a_i e_i, \sum_{i=1}^n b_i e_i, \sum_{i=1}^n (c_i + id_i) e_i \right) = (0,0,0)$$

$$\text{And: } \begin{cases} \sum a_i e_i = 0 \\ \sum b_i e_i = 0 \\ \sum c_i e_i = 0 \\ \sum d_i e_i = 0 \end{cases} \quad \text{Thus } a_i = b_i = c_i = d_i = 0 \text{ and } X = 0.$$

Thus K is a basis of $V_3(I)$.

$$3] \dim(V_3(I)) = [\dim(V)]^3 = n^3.$$

Definition:

Let $X = x_0 + \sum_{i=1}^3 x_i I_i, Y = y_0 + \sum_{i=1}^3 y_i I_i \in V_3(I)$ with:

$$\begin{cases} h(X) = (x_0, \sum_{i=0}^3 x_i, x'_1 + x'_2 i) \\ h(Y) = (y_0, \sum_{i=0}^3 y_i, y'_1 + y'_2 i) \end{cases} \quad \text{where } \begin{cases} x'_1 + x'_2 i = x_0 + x_3 + x_1 e^{\frac{2\pi}{3}i} + x_2 e^{\frac{4\pi}{3}i} \\ y'_1 + y'_2 i = y_0 + y_3 + y_1 e^{\frac{2\pi}{3}i} + y_2 e^{\frac{4\pi}{3}i} \end{cases}$$

assume that $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{R}$ is an inner product on V , we define $\varphi: V_3(I) \times V_3(I) \rightarrow R_3(I)$ such that:

$$\varphi(X, Y) = g^{-1}[\langle x_0, y_0 \rangle, \langle \sum_{i=0}^3 x_i, \sum_{i=0}^3 y_i \rangle, \langle x'_1, y'_1 \rangle + \langle x'_2, y'_2 \rangle + i(\langle x'_2, y'_1 \rangle - \langle x'_1, y'_2 \rangle)], i \cdot e \cdot \varphi(X, Y) = g^{-1}(\langle h(X), h(Y) \rangle).$$

We say that (φ) is a 3-cyclic refined real inner product.

Example:

For $V = \mathbb{R}^2, \langle \cdot, \cdot \rangle$ be the ordinary Euclidean inner product defined on V . Let $V_3(I)$ be the corresponding 3-cyclic refined vector space over $R_3(I)$.

$$\text{Consider: } \begin{cases} X = (1,2) + (2,0)I_1 + (1,1)I_2 + (0,1)I_3 \\ Y = (-1,1) + (0,1)I_1 + (-1,0)I_2 + (1,1)I_3 \end{cases}$$

$$\text{Then: } \begin{cases} x_0 = (1,2), \sum_{i=0}^3 x_i = (4,4) \\ y_0 = (-1,1), \sum_{i=0}^3 y_i = (-1,3) \end{cases}$$

$$x_0 + x_3 + x_1 e^{\frac{2\pi}{3}i} + x_2 e^{\frac{4\pi}{3}i} = (1,3) + (2,0) \cdot \left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right) + (1,1) \left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right) = (1,3) + (-1,0) + (\sqrt{3}, 0)i + \left(\frac{-1}{2}, \frac{-1}{2}\right) + i\left(-\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right) = \left(\frac{-1}{2}, \frac{5}{2}\right) + i\left(\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right) = x'_1 + x'_2 i.$$

$$y_0 + y_3 + y_1 e^{\frac{2\pi}{3}i} + y_2 e^{\frac{4\pi}{3}i} = (0,2) + (0,1) \cdot \left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right) + (-1,0) \left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right) = (0,2) + \left(0, \frac{-1}{2}\right) + \left(0, \frac{\sqrt{3}}{2}\right)i + \left(\frac{1}{2}, 0\right) + \left(\frac{\sqrt{3}}{2}, 0\right)i = \left(\frac{1}{2}, \frac{3}{2}\right) + i\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right) = y'_1 + y'_2 i.$$

On the other hand, we have:

$$\langle x_0, y_0 \rangle = 1, \langle \sum_{i=0}^3 x_i, \sum_{i=0}^3 y_i \rangle = 8, \langle x'_1, y'_1 \rangle = \frac{14}{4} = \frac{7}{2}, \langle x'_2, y'_2 \rangle = 0, \langle x'_1, y'_2 \rangle = \sqrt{3}, \langle x'_2, y'_1 \rangle = -\frac{\sqrt{3}}{2}, \text{ so that:}$$

$$\varphi(X, Y) = g^{-1}\left(1, 8, \frac{7}{2}, \frac{-3\sqrt{3}}{2}i\right) = 1 + I_1 \left[\frac{1}{3}\left(8 - \frac{7}{2}\right) + \frac{1}{\sqrt{3}}\left(\frac{-3\sqrt{3}}{2}\right)\right] + I_2 \left[\frac{1}{3}\left(8 - \frac{7}{2}\right) - \frac{1}{\sqrt{3}}\left(\frac{-3\sqrt{3}}{2}\right)\right] + I_3 \left[\frac{8}{3} - 1 + \frac{2}{3}\left(\frac{7}{2}\right)\right] = 1 + I_1 \left(\frac{3}{2} - \frac{3}{2}\right) + I_2 \left(\frac{3}{2} + \frac{3}{2}\right) + I_3 \left(\frac{15}{3} - 1\right) = 1 + 3I_2 + 4I_3.$$

Definition:

Let $m = m_0 + \sum_{i=1}^3 m_i I_i \in R_3(I)$, we define:

$$\bar{m} = g^{-1}\left(m_0, \sum_{i=0}^3 m_i, m_0 + m_3 + m_1 e^{\frac{4\pi}{3}i} + m_2 e^{\frac{2\pi}{3}i}\right).$$

Example:

Take $m = 1 + I_1 + 2I_2 - 4I_3$, we have:

$$\begin{cases} m_0 = 1 \\ \sum_{i=0}^3 m_i = 0 \\ m_0 + m_3 + m_1 e^{\frac{2\pi}{3}i} + m_2 e^{\frac{4\pi}{3}i} = -\frac{9}{2} - \frac{\sqrt{3}}{2}i, \quad -\frac{9}{2} - \frac{\sqrt{3}}{2}i = -\frac{9}{2} + \frac{\sqrt{3}}{2}i \end{cases}$$

$$\bar{m} = g^{-1}\left(1, 0, -\frac{9}{2} + \frac{\sqrt{3}}{2}i\right) = 1 + I_1 \left[\frac{1}{3}\left(0 + \frac{9}{2}\right) + \frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}\right)\right] + I_2 \left[\frac{1}{3}\left(0 + \frac{9}{2}\right) - \frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}\right)\right] + I_3 \left[\frac{1}{3}(0) - 1 + \frac{2}{3}\left(\frac{-9}{2}\right)\right] = 1 + I_1 \left(\frac{3}{2} + \frac{1}{2}\right) + I_2 \left(\frac{3}{2} - \frac{1}{2}\right) + I_3(-1 - 3) = 1 + 2I_1 + I_2 - 4I_3.$$

Remark:

For $m = m_0 + \sum_{i=1}^3 m_i I_i$, then $\bar{m} = m_0 + m_2 I_1 + m_1 I_2 + m_3 I_3$.

Definition:

Let $m = m_0 + \sum_{i=1}^3 m_i I_i, n = n_0 + \sum_{i=1}^3 n_i I_i \in R_3(I)$, we define that:

$$m \geq n \text{ if and only if: } \begin{cases} g(m), g(n) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \\ m_0 \geq n_0, \quad \sum_{i=0}^3 m_i \geq \sum_{i=0}^3 n_i \\ m_0 + m_3 + m_1 e^{\frac{2\pi}{3}i} + m_2 e^{\frac{4\pi}{3}i} \geq n_0 + n_3 + n_1 e^{\frac{2\pi}{3}i} + n_2 e^{\frac{4\pi}{3}i} \end{cases}$$

Remark:

$m = m_0 + \sum_{i=1}^3 m_i I_i \geq 0$ if and only if:

$$\begin{cases} g(m) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \\ m_0 \geq 0, \quad \sum_{i=0}^3 m_i \geq 0 \\ m_0 + m_3 + m_1 e^{\frac{2\pi}{3}i} + m_2 e^{\frac{4\pi}{3}i} \geq 0 \end{cases}$$

Theorem:

Let $m = m_0 + \sum_{i=1}^3 m_i I_i \in R_3(I)$, then $g(m) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ if and only if $m_1 = m_2$.

Proof:

$g(m) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ if and only if: $m_0 + m_3 + m_1 e^{\frac{2\pi}{3}i} + m_2 e^{\frac{4\pi}{3}i} \in \mathbb{R}$, there for: $m_1 - m_2 = 0$, and $m_1 = m_2$.

On the other hand, we get $g(m) = (m_0, m_0 + m_3 + 2m_1, m_0 + m_3 - m_1)$.

Theorem:

(\leq) is partial order relation on $C(R_3(I))$, where $C(R_3(I))$ is the set of $m \in R_3(I)$ such that $m_1 = m_2$.

$$C(R_3(I)) = \{m_0 + m_1 I_1 + m_1 I_2 + m_3 I_3; m_0, m_1, m_3 \in \mathbb{R}\}.$$

Proof:

Let $m = m_0 + \sum_{i=1}^3 m_i I_i, n = n_0 + \sum_{i=1}^3 n_i I_i, t = t_0 + \sum_{i=1}^3 t_i I_i \in C(R_3(I))$, then we have:

$$m \leq m \text{ that is because: } \begin{cases} m_0 \leq m_0 \\ m_0 + 2m_1 + m_3 \leq m_0 + 2m_1 + m_3 \\ m_0 + m_3 - m_1 \leq m_0 + m_3 - m_1 \end{cases}$$

$$\text{If } (m \leq n) \text{ and } (n \leq m), \text{ then: } \begin{cases} m_0 = n_0 \\ m_0 + 2m_1 + m_3 = n_0 + 2n_1 + n_3 \\ m_0 + m_3 - m_1 = n_0 + n_3 - n_1 \end{cases}$$

$$\text{Thus } \begin{cases} m_1 = n_1 \\ m_3 = n_3 \end{cases} \Rightarrow m = n.$$

If $m \leq n$ and $n \leq t$, then:

$$\begin{cases} m_0 \leq n_0 \leq t_0 \\ m_0 + 2m_1 + m_3 \leq n_0 + 2n_1 + n_3 \leq t_0 + 2t_1 + t_3 \\ m_0 + m_3 - m_1 \leq n_0 + n_3 - n_1 \leq t_0 + t_3 - t_1 \end{cases}$$

Thus $m \leq t$.

Example:

For $m = 2 + 3I_1 + 3I_2 - I_3, n = 1 + I_1 + I_2 - 2I_3$, we have:

$$m \geq n, \text{ that is because: } \begin{cases} 2 \geq 1 \\ 7 \geq 1 \\ -2 \geq -2 \end{cases}$$

Theorem:

Let $(V_3(I), \varphi)$ be a 3-cyclic refined real inner product space, then:

- 1] $\varphi(X, X) \in C(R_3(I))$, and $\varphi(X, X) \geq 0$
- 2] If $\varphi(X, X) = 0$, then $X = 0$
- 3] $\varphi(X, Y) = \overline{\varphi(Y, X)}$
- 4] $\varphi(X + Y, Z) = \varphi(X, Z) + \varphi(Y, Z)$
- 5] $\varphi(m \cdot X, Y) = m \varphi(X, Y), \varphi(X, mY) = \overline{m} \varphi(X, Y)$
- 6] $\varphi(X, Y + Z) = \varphi(X, Y) + \varphi(X, Z); X, Y, Z \in V_3(I)$.

Proof:

Consider $X = x_0 + \sum_{i=1}^3 x_i I_i; x_i \in V$, then:

$$\langle h(X), h(Y) \rangle = (\|x_0\|^2, \|\sum_{i=0}^3 x_i\|^2, \|x'_1\|^2 + \|x'_2\|^2),$$

$\varphi(X, X) = g^{-1}(\langle h(X), h(X) \rangle) = \|x_0\|^2 + I_1 \left[\frac{1}{3} (\|\sum_{i=0}^3 x_i\|^2 - \|x'_1\|^2 - \|x'_2\|^2) \right] + I_2 \left[\frac{1}{3} (\|\sum_{i=0}^3 x_i\|^2 - \|x'_1\|^2 - \|x'_2\|^2) \right] + I_3 \left[\frac{1}{3} \|\sum_{i=0}^3 x_i\|^2 - \|x_0\|^2 + \frac{2}{3} (\|x'_1\|^2 + \|x'_2\|^2) \right] \in C(R_3(I))$, that is because the coefficients of I_1 and I_2 are equal.

$$\text{On the other hand: } \begin{cases} \|x_0\|^2 \geq 0 \\ m_0 + m_3 + 2m_1 = \|\sum_{i=0}^3 x_i\|^2 \geq 0 \\ m_0 + m_3 - m_1 = \|x'_1\|^2 + \|x'_2\|^2 \geq 0 \end{cases}$$

Thus $\varphi(X, X) \geq 0$.

$$2] \text{ for } \varphi(X, X) = 0, \text{ we get: } \begin{cases} \|x_0\| = 0 \\ \|\sum_{i=0}^3 x_i\| = 0 \\ \|x'_1\|^2 + \|x'_2\|^2 = 0 \end{cases} \text{ so that: } \begin{cases} x_0 = 0 \\ x'_1 = x'_2 = 0 \\ x_1 + x_2 + x_3 = 0 \end{cases} \text{ where } \begin{cases} x'_1 = x_0 + x_3 - \frac{1}{2}(x_1 + x_2) \\ x'_2 = \frac{\sqrt{3}}{2}(x_1 - x_2) \end{cases}$$

$$\text{Thus: } \begin{cases} x_0 = 0 \\ x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases} \text{ and } X = 0.$$

3] $\langle h(X), h(Y) \rangle = (a, b, c + di);$ where: $Y = y_0 + \sum_{i=1}^3 y_i I_i \in V_3(I)$

$$a = \langle x_0, y_0 \rangle, b = \langle \sum_{i=0}^3 x_i, \sum_{i=0}^3 y_i \rangle, c = \langle x'_1, y'_1 \rangle + \langle x'_2, y'_2 \rangle, d = \langle x'_2, y'_1 \rangle - \langle x'_1, y'_2 \rangle$$

Also, $\langle h(Y), h(X) \rangle = (a, b, c - di)$

So that $\varphi(X, Y) = g^{-1}(\langle h(X), h(Y) \rangle) = g^{-1}(\overline{\langle h(Y), h(X) \rangle}) = \overline{\varphi(Y, X)}$.

4] For $Z = z_0 + \sum_{i=1}^3 z_i I_i; z_i \in V$, we can write:

$$\varphi(X + Y, Z) = g^{-1}(\langle h(X + Y), h(Z) \rangle) = g^{-1}(\langle h(X) + h(Y), h(Z) \rangle) = g^{-1}(\langle h(X), h(Z) + h(Y), h(Z) \rangle) = g^{-1}(\langle h(X), h(Z) \rangle) + g^{-1}(\langle h(Y), h(Z) \rangle) = \varphi(X, Z) + \varphi(Y, Z).$$

$$5] \quad \varphi(m \cdot X, Y) = g^{-1}(\langle h(m \cdot X), h(Y) \rangle) = g^{-1}(\langle g(m) \cdot h(X), h(Y) \rangle) = g^{-1}(g(m) \cdot \langle h(X), h(Y) \rangle) = g^{-1}(g(m)) \cdot g^{-1}(\langle h(X), h(Y) \rangle) = m \varphi(X, Y).$$

$$\varphi(X, m \cdot Y) = g^{-1}(\langle h(X), h(m \cdot Y) \rangle) = g^{-1}(\langle h(X), g(m) \cdot h(Y) \rangle) = g^{-1}(g(\bar{m}) \langle h(X), h(Y) \rangle) = \bar{m} \varphi(X, Y).$$

6] It is similar to [4].

Definition:

Let $X, Y \in V_3(I)$, then we say:

1] $X \perp Y$ if and only if $\varphi(X, Y) = 0$.

2] $\|X\|^2 = \varphi(X, X)$.

Theorem:

Let $X = x_0 + \sum_{i=1}^3 x_i I_i, Y = y_0 + \sum_{i=1}^3 y_i I_i \in V_3(I)$, then:

$$1] X \perp Y \text{ if and only if: } \begin{cases} x_0 \perp y_0, \sum_{i=0}^3 x_i \perp \sum_{i=0}^3 y_i \\ \langle x'_1, y'_1 \rangle + \langle x'_2, y'_2 \rangle = 0 \\ \langle x'_1, y'_2 \rangle - \langle x'_2, y'_1 \rangle = 0 \end{cases}.$$

2] If $X \perp Y$, then $\|X + Y\|^2 = \|X\|^2 + \|Y\|^2$.

Proof:

1] $X \perp Y$ if and only if $\varphi(X, Y) = 0$, so that:

$$\langle h(X), h(Y) \rangle = (0,0,0), \text{ hence: } \begin{cases} \langle x_0, y_0 \rangle = 0 \Rightarrow x_0 \perp y_0 \\ \langle \sum_{i=0}^3 x_i, \sum_{i=0}^3 y_i \rangle = 0 \Rightarrow \sum_{i=0}^3 x_i \perp \sum_{i=0}^3 y_i \\ \langle x'_1, y'_1 \rangle + \langle x'_2, y'_2 \rangle = 0 \\ \langle x'_1, y'_2 \rangle - \langle x'_2, y'_1 \rangle = 0 \end{cases}$$

$$\text{Where } \begin{cases} x'_1 = x_0 + x_3 - \frac{1}{2}(x_1 + x_2) \\ x'_2 = \frac{\sqrt{3}}{2}(x_1 - x_2) \\ y'_1 = y_0 + y_3 - \frac{1}{2}(y_1 + y_2) \\ y'_2 = \frac{\sqrt{3}}{2}(y_1 - y_2) \end{cases}$$

2] $\|X + Y\|^2 = \varphi(X + Y, X + Y) = \varphi(X, X) + \varphi(X, Y) + \overline{\varphi(X, Y)} + \varphi(Y, Y) = \|X\|^2 + \|Y\|^2$.

Remark:

$$\|X\| = g^{-1}(\|x_0\|, \|\sum_{i=0}^3 x_i\|, \sqrt{\|x'_1\|^2 + \|x'_2\|^2}) = \|x_0\| + I_1 \left[\frac{1}{3} (\|\sum_{i=0}^3 x_i\| - \sqrt{\|x'_1\|^2 + \|x'_2\|^2}) \right] + I_2 \left[\frac{1}{3} (\|\sum_{i=0}^3 x_i\| - \sqrt{\|x'_1\|^2 + \|x'_2\|^2}) \right] + I_3 \left[\frac{1}{3} \|\sum_{i=0}^3 x_i\| - \|x_0\| + \frac{2}{3} \sqrt{\|x'_1\|^2 + \|x'_2\|^2} \right].$$

Example:

Consider $V = \mathbb{R}^2$, and $X = (1,0) + (0,1)I_1 + (0,0)I_2 + (0,2)I_3 \in V_3(I)$, we have:

$$\begin{cases} x_0 = (1,0), \|x_0\| = 1 \\ \sum_{i=0}^3 x_i = (1,3), \|\sum_{i=0}^3 x_i\| = \sqrt{10} \end{cases}, \begin{cases} x'_1 = \left(1, \frac{3}{2}\right), \|x'_1\|^2 = \frac{13}{4} \\ x'_2 = \left(0, \frac{\sqrt{3}}{2}\right), \|x'_2\|^2 = \frac{3}{4} \end{cases}$$

$$\|X\| = 1 + I_1 \left[\frac{1}{3} (\sqrt{10} - 2) \right] + I_2 \left[\frac{1}{3} (\sqrt{10} - 2) \right] + I_3 \left[\frac{1}{3} \sqrt{10} - 1 + \frac{2}{3} (2) \right] = 1 + \frac{\sqrt{10}-2}{3} I_1 + \frac{\sqrt{10}-2}{3} I_2 + \frac{\sqrt{10}+1}{3} I_3.$$

Consider $Y = (0,1) + \left(-1, \frac{1}{3}\right) I_1 + \left(-1, \frac{1}{3}\right) I_2 + \left(-1, \frac{-2}{3}\right) I_3$,

$$\text{We have: } \begin{cases} y_0 = (0,1), \langle x_0, y_0 \rangle = 0 \\ \sum_{i=0}^3 y_i = (-3,1), \langle \sum_{i=0}^3 x_i, \sum_{i=0}^3 y_i \rangle = 0 \end{cases}$$

$$\text{And: } \begin{cases} y'_1 = \left(-1, \frac{1}{3}\right) - \frac{1}{2} \left(-2, \frac{2}{3}\right) = \left(-1, \frac{1}{3}\right) - \left(-1, \frac{1}{3}\right) = (0,0) \\ y'_2 = \frac{\sqrt{3}}{2} (0,0) = (0,0) \end{cases}$$

$\langle x'_1, y'_1 \rangle + \langle x'_2, y'_2 \rangle = 0$, $\langle x'_1, y'_2 \rangle - \langle x'_2, y'_1 \rangle = 0$, hence $X \perp Y$.

Definition:

Let $m = m_0 + \sum_{i=1}^3 m_i I_i \in R_3(I)$, we define:

$$|m| = g^{-1}(|m_0|, |\sum_{i=0}^3 m_i|, |m_0 + m_3 + m_1 e^{\frac{2\pi}{3}i} + m_2 e^{\frac{4\pi}{3}i}|).$$

Remark:

Since $m_i \in \mathbb{R}$, then $|m_0| = \sqrt{m_0^2}$, $|\sum_{i=0}^3 m_i| = \sqrt{(\sum_{i=0}^3 m_i)^2}$, $|m_0 + m_3 + m_1 e^{\frac{2\pi}{3}i} + m_2 e^{\frac{4\pi}{3}i}| = |m'_1 + im'_2| = \sqrt{(m'_1 + im'_2) \cdot (m'_1 - im'_2)}$.

So that: $|m| = \sqrt{g^{-1}[(m_0^2, (\sum_{i=0}^3 m_i)^2, (m'_1 + im'_2) \cdot (m'_1 - im'_2))]} = \sqrt{m \times \bar{m}}$.

From this point of view, we can get the following results:

- 1) $|m| = 0 \Leftrightarrow m = 0$
- 2) $|m|^2 = m \times \bar{m}$
- 3) $|m|^2 = 1 \Rightarrow m = \frac{1}{\bar{m}}$
- 4) $|m + n| \leq |m| + |n|$, $|m \cdot n| = |m| \cdot |n|$; $n \in R_3(I)$.

Theorem:

Let $X = x_0 + \sum_{i=1}^3 x_i I_i$, $Y = y_0 + \sum_{i=1}^3 y_i I_i \in V_3(I)$, $m = m_0 + \sum_{i=1}^3 m_i I_i \in R_3(I)$, then:

- 1) $\|m \cdot X\| = |m| \cdot \|X\|$
- 2) $\|X + Y\| \leq \|X\| + \|Y\|$
- 3) $|\varphi(X, Y)| \leq \|X\| \cdot \|Y\|$

Proof:

- 1) $\|m \cdot X\|^2 = \varphi(m \cdot X, m \cdot X) = m \cdot \bar{m} \varphi(X, X) = |m|^2 \cdot \|X\|^2 \Rightarrow \|m \cdot X\| = |m| \cdot \|X\|$.
- 2) $\|X + Y\|^2 = \|X\|^2 + \|Y\|^2 + \varphi(X, Y) + \overline{\varphi(X, Y)} \leq \|X\|^2 + \|Y\|^2 + 2\|X\| \cdot \|Y\|$

Thus $\|X + Y\|^2 \leq (\|X\| + \|Y\|)^2$, and $\|X + Y\| \leq \|X\| + \|Y\|$.

3) According to classical Cauchy-Schwartz inequality, we have:

$$\begin{cases} |\langle x_0, y_0 \rangle| \leq \|x_0\| \cdot \|y_0\| \\ \left| \left\langle \sum_{i=0}^3 x_i, \sum_{i=0}^3 y_i \right\rangle \right| \leq \left\| \sum_{i=0}^3 x_i \right\| \cdot \left\| \sum_{i=0}^3 y_i \right\| \\ |\langle x'_1 + x'_2 i, y'_1 + y'_2 i \rangle| \leq \|x'_1 + x'_2 i\| \cdot \|y'_1 + y'_2 i\| \end{cases}$$

There for, $|\varphi(X, Y)| \leq \|X\| \cdot \|Y\|$.

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