Proposing Triple fuzzy distribution based on the Quantile Function for monotonically decreasing failure data

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Abstract

In this paper, we propose a novel generalization for one parameter inverse Lindley distribution to fitting monotonically descending data named the T-ILD{Y} distribution class , T is one parameter inverse exponential distribution , R has an one parameter inverse Lindley distribution , and the variable Y is one parameter exponential distribution, the resulting distribution is inverse exponential- inverse Lindley- exponential (IEILDE). The theory of fuzzy sets are used by converting the distribution to fuzzy by using a fuzzy triangular distribution based on the quantile function (FIEILE), the maximum likelihood , and the maximum likelihood, and the maximum product spacing method were used estimate the parameters of the distribution. We conclude that at cutoff α=0.1, ML is better than the MPS, and at cutoff coefficients α=0.3, 0.5, 0.7, MPS was better than the ML, The higher the cutoff, the better the maximum likelihood method.

Keywords: fuzzy distribution; Quantile Function; monotonically; Lindley distribution Maximum Likelihood; Maximum Product spacing

1. Introduction

The study of life times of organisms, devices, structures, materials, etc. is of great importance in many applied sciences such as medicine, engineering, finance, biology, and engineering. Which are modelled in the form of a particular probability distribution? Each distribution has its own characteristics, specifically due to the shape of the failure rate function, which may be decreasing, increasing, or constant in its behaviour, it may be left skewed, right skewed, symmetric, asymmetric, heavy tailed, monotonically increasing, or monotonically decreasing, so that it often requires generalization of the basic distributions to make them more capable and flexible to deal with such data. Generalization of distributions is mainly based on adding more flexibility to known distributions which results from implanting a basic distribution in a more capable structure, and the literature of distribution theory is full of different techniques for generalizing distributions to enhance their capabilities in modelling real-world data. Many researchers have paid special attention to the Lindley distribution for its importance in modelling complex real-life data. Some researchers have gone the route of studying the Lindley distribution and its properties in more detail and have extended this distribution to fit various types of life-time data. Extended distributions, also known as generalized or flexible distributions, provide advantages and importance in statistical modelling and data analysis due to their ability to accommodate a wide range of shapes and properties. Extended distributions are designed to fit the diverse patterns observed in real-world data sets. They allow for a better fit to the observed data, leading to more accurate statistical models. We often encounter that some observations deviate from the rest of the sample observations, or that the observations of a single sample are divided into a set of parts (quantiles), each part representing a specific direction. The data may contain parts that are skewed to the right (Right Skewed) or skewed to the left (Left Skewed), or they may be symmetric (Symmetric) or heavy tailed (Heavy Tail), or they may be monotonically increasing (Monotonic Increasing) or monotonically decreasing (Monotonic Decreasing). As in oncology, tumors may decrease in size after the patient undergoes treatment, such as chemotherapy or radiation therapy. A data set that tracks tumor size over time may show a monotonous downward trend, or the tumor may increase in size even with treatment. Therefore, such a phenomenon must have a probability distribution that accurately represents it and accurately analyses and measures the probability of such a phenomenon.

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Researcher Lindley (1958) introduced a distribution named after him and studied its properties and its relationship to other life time distributions [1]. A new method for generating families of continuous distributions that uses the random variable X called the transformer to transform another random variable T called the transformer so that the resulting family is a family of T-X distributions. Alzaatreh & Lee [1] introduced a new family of distributions called the exponential T-X distribution. Alzaghal & Hamed [2] proposed a new family of distributions called T-Lomax{Y} using the transform-transformer methodology, known as the T-X formula. Hamed & Alzaghal [3] introduced a new generalized class of the (Lindley) distribution, which they called the T-Lindley{Y} class, based on the integration of quantile functions.

2. Quantile Function

The inverse cumulative distribution function (CDF) is a statistical concept used in probability and distribution theory that gives the value at which the cumulative distribution function (CDF) equals or exceeds a specified probability. The quantile function is particularly useful for understanding the distribution of data, estimating percentiles, and constructing confidence intervals. Many statistical distributions, such as the normal distribution, the exponential distribution, the Weibull distribution, etc., have specific quantile functions [4].

If X is a random variable with a probability distribution with a cumulative probability function (x)F, then the quantitative function is defined as follows [5]:

$$Q(p) = \inf\{x: F(x) \geq p\} \quad \forall\, 0 < p < 1 \qquad \dots(1)$$

3. Exponential distribution

It is a special case of the Weibull distribution when α = 1, then the probability density function of the exponential distribution is as follows [6]:

$$f(y, \lambda) = \lambda e^{-\lambda y} \quad ; \qquad y > 0 \qquad \dots(2)$$

$$F(y, \lambda) = P(Y \leq y) = \int_0^y f(u)du = 1 - e^{-\lambda y} \; ; \quad y \geq 0 \qquad \dots(3)$$

And $E(Y)= \frac{1}{\lambda}$, $E(\mathbf{Y^2}) = \frac{2}{\lambda^2}$, $\mathbf{Var}(\mathbf{Y}) = \frac{1}{\lambda^2}$

The inverse exponential distribution refers to the reciprocal of the exponential distribution. If Y has an exponential distribution, then T = 1/y for an inverse exponential distribution with the following probability density function:

$$f(t, \lambda) = \frac{1}{\lambda} e^{-\frac{t}{\lambda}} \quad ; \qquad x > 0 \qquad \dots(4)$$

If t~Iexp(1/λ), then its cumulative distribution function is:

$$F(t, \lambda) = P(T \leq t) = \int_0^t f(u)du = 1 - e^{-\frac{t}{\lambda}} \; ; \; t \geq 0 \qquad \dots(5)$$

and $E(T)= \lambda$, $E(T^2) = 2\lambda^2$, $Var(T) = \lambda^2$

4. Inverse Lindley Distribution

The Lindley distribution is only applicable to modelling data with monotonically increasing failure rate, but in the case of data that exhibit non-monotonic shapes such as bathtub and upside-down bathtub shapes (i.e. monotonically decreasing data), an extension of the Lindley distribution has been proposed by finding its inverse and is called the Inverse Lindley Distribution (IRD) which exhibits an inverted bathtub shape for its failure rate function.

If the random variable x has a Lindley distribution, then the random variable y = 1/x has an inverse Lindley distribution with the following probability density function:

$$f(x; \theta) = \frac{\theta^2}{(\theta + 1)} \left(\frac{1 + x}{x^3}\right) e^{-\frac{\theta}{x}} \quad ; \quad x > 0, \theta > 0 \qquad \dots(6)$$

Where the parameter θ represents the scale parameter.

The cumulative distribution function (CDF) of the inverse Lindley distribution is given by the following formula:

$$F(x; \theta) = 1 - \left[1 + \frac{\theta}{(\theta + 1)} \frac{1}{x}\right] e^{-\frac{\theta}{x}} \quad ; x > 0, \theta > 0 \qquad \dots(7)$$

The reliability function of the distribution is also known as:

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$$R(x) = \left[1 + \frac{\theta}{(\theta+1)}\frac{1}{x}\right] e^{-\frac{\theta}{x}} \qquad ; x > 0 , \theta > 0 \qquad \dots (8)$$

The moment of degree r about the origin is calculated using the following formula:

$$\mu'_r = \frac{\theta^r(\theta+1)}{r!\ (\theta+r+1)} \qquad \dots (9)$$

When r=1 we get the first moment about the origin:

$$\mu'_1 = \frac{\theta(\theta+1)}{\theta+2} \qquad \dots (10)$$

5. T-R(x) Class of Distribution

It was introduced by Alzaatreh et al., [1] and it is a structure for transforming a variable called the Transformer by means of a variable called the Transformer to generalize the Lindley distribution with a single parameter (One Parameter Lindley Distribution) which is called the T-Lindley{Y} distribution class [3,7] so that the distributions are generalized using the T-R{Y} class adding more parameters to the generalized distribution. Hence, there is greater flexibility in modeling life time data.

If we have three random variables T, R, Y, each of which has a distribution function $F_T(x) = P(T \le x)$, $F_R(x) = P(R \le x)$, $F_Y(x) = P(Y \le x)$. The probability density function for each variable is $f_Y(x)$ و $f_R(x)$, $f_T(x)$ and the quantitative function of the variable Y is $Q_Y(p) = \inf\{y: F_Y(x) = P(Y \ge x)\}$, The probability density function of the T-Lindley distribution class {Y} is [3]:

$$f_X(x) = f_R(x).\frac{f_T\left(Q_Y(F_R(x))\right)}{f_Y\left(Q_Y(F_R(x))\right)} \qquad \dots (11)$$

The distribution function of the T-Lindley distribution class {Y} is:

$$F_X(x) = \int_a^{Q_Y(F_R(x))} f_T(t)dt = F_T\left(Q_Y(F_R(x))\right) \qquad \dots (12)$$

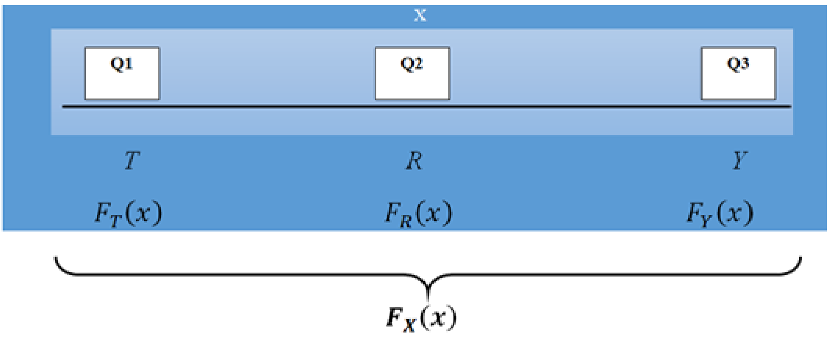

Figure 1. T-R(X) distribution class diagram

We notice from Figure (4-2) that the range of the random variable X (the new distribution variable) is divided into three parts, each part is represented by a quartile, which are Q1, Q2, and Q3, respectively, and each quartile is represented by a random variable with a cumulative probability density function, which are $F_T(x), F_R(x), F_Y(x)$, respectively, which are the components of the cumulative probability density function for the variable x, which is $\mathbf{F_X(x)}$. X (the new distribution variable) includes three variables (T, R, Y).

6. Proposed T-ILindley(Y) distribution class

The proposed T-ILindley{Y} distribution class is the inverse of the T-Lindley{Y} distribution class proposed by [3] for the purpose of modeling monotonically increasing life time data as follows:

If we have three random variables, which are:

The first random variable T: can be any continuous probability distribution

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The second random variable R is the inverse of Lindley with a single parameter, which has a probability density function defined by Equation (6) and a clustering function defined by Equation (7).

The third random variable Y: can be any continuous probability distribution

Therefore, the probability density function of the T-ILindley distribution class {Y} is defined by applying Equation (11) as follows:

$$f_X(x) = \frac{\theta^2}{(\theta+1)} \left(\frac{1+x}{x^3}\right) e^{-\frac{\theta}{x}} . \frac{f_T\left(Q_Y\left(1-\left[1+\frac{\theta}{(\theta+1)}\frac{1}{x}\right]e^{-\frac{\theta}{x}}\right)\right)}{f_Y\left(Q_Y\left(1-\left[1+\frac{\theta}{(\theta+1)}\frac{1}{x}\right]e^{-\frac{\theta}{x}}\right)\right)} \quad ... (13)$$

The cumulative probability density function of the T-ILindley distribution class {Y} is defined by applying Equation (12) as follows:

$$F_X(x) = \int_a^{Q_Y(F_R(x))} f_T(t)dt = F_T\left(Q_Y\left(1-\left[1+\frac{\theta}{(\theta+1)}\frac{1}{x}\right]e^{-\frac{\theta}{x}}\right)\right) \quad ... (13)$$

7. Fuzzy set

Let X be a universal set, then the fuzzy subset $\widetilde{A}$ of X that has a belonging function $\mu_{\widetilde{A}}(x)$ that produces values between [0,1] for all values of x in the fuzzy sample space, then the fuzzy set can be expressed as follows [8]:

$$\widetilde{A} = \{(x_i, \mu_{\widetilde{A}}(x_i)), x \in \Omega, i = 1,2,3, \dots \dots n, 0 \le \mu_{\widetilde{A}}(x) \le 1\} \quad \dots (14)$$

And the function that plots the degree of importance of the element (degree of belonging) in the comprehensive set to the fuzzy set, and it is a function with a positive value [Yadav & Yadav, 2019,120], [9]. One of the membership function is triangular membership function: It is one of the linear membership functions and is defined by three parameters: a lower limit (a), an upper limit (b), and a central value (m). Its formula is as follows:

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \le a \\ \frac{x-a}{m-a} & \text{if } a < x \le m \\ \frac{b-x}{b-m} & \text{if } m < x < b \\ 1 & \text{if } x \ge b \end{cases} \quad \dots (15)$$

Since a<m<b, and they are symmetric if the marginal values (Margin) b-m equal m-a, and Figure (1) (a) shows the symmetric trigonometric affiliation function (b) shows the general trigonometric affiliation function [10].

The cut-off of the fuzzy set $\widetilde{A}$ is the conventional set whose elements belongs to the fuzzy set $\widetilde{A}$ and have a degree of belonging greater than or equal to the coefficient α. It can be expressed mathematically as [11]:

$$A^{\alpha} = \{\tilde{x} \in X;\ \mu_{\widetilde{A}}(x) \ge \alpha\} \quad ... (16)$$

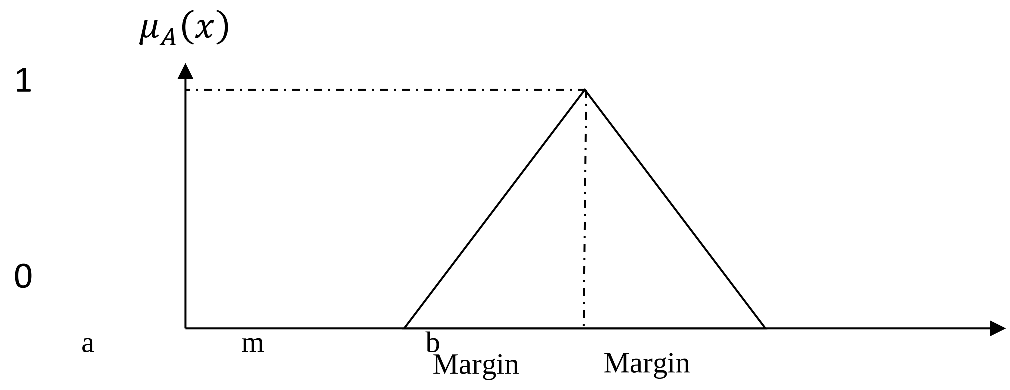

Figure 2. Trigonometric affiliation function

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8. Fuzzy probability distribution

In (2022), Ali and Neamah proposed a method to transform any traditional probability distribution into a fuzzy probability distribution as follows [12]:

Assuming that the failure times $t \in T$ are imprecise and uncertain and are expressed as fuzzy numbers $\tilde{t} \in \tilde{T}$, such that $\tilde{t} = \{[0,\infty), \mu_{\tilde{t}}(t)\}$, Then the traditional sample observation vector that we can get from the fuzzy set, which represents all the elements that have a degree of belonging greater than or equal to the alpha cut (α-cut), which represents the degree of belonging of the elements that we are interested in, and those elements are expressed by the set $A^{(\alpha)}$such that:

$$A^{(\alpha)} = \{\tilde{t} = [0,\infty) \in \tilde{T}, \mu_{\tilde{t}}(t) = \alpha\,;\ \mu_{\tilde{t}}(t) \geq \alpha\} \qquad \ldots (17)$$

Where α is the cutting factor, 0<α<1. $\mu_{\tilde{t}}(t)$ the belonging function by which the belonging score is generated for each failure time in the sample space and can take any form of belonging functions.

The fuzzy cumulative distribution function ($\widetilde{CDF}$) at any value of the fuzzy sample space $A^{(\alpha)}$ and for any failure distribution can be obtained as follows:

$$\tilde{F}(\tilde{t}_{A^{(\alpha)}}) = \int_0^{\tilde{t}_{A^{(\alpha)}}} f(u)du \qquad \ldots (18)$$

By deriving formula (18) for $\tilde{t}_{A^{(\alpha)}}$we get the fuzzy probability distribution as follows:

$$\tilde{f}(\tilde{t}) = \frac{\partial \tilde{F}(\tilde{t}_{A^{(\alpha)}})}{\partial \tilde{t}_{A^{(\alpha)}}} = \frac{\partial}{\partial \tilde{t}_{A^{(\alpha)}}}\left[\int_0^{\tilde{t}_{A^{(\alpha)}}} f(u)du\right] \ ;\ 0 < \tilde{t}_{A^{(\alpha)}} < \infty \qquad \ldots (19)$$

Fuzzy triple distribution based on the inverse exponential–inverse Lindley–exponential (IEILE) Quantile function

The probability density function of the IEILE distribution is extracted by applying Equation (13) as follows:

$$f_X(x) = \frac{\lambda\beta\theta^2}{(\theta+1)}\left(\frac{1+x}{x^3}\right)\frac{e^{-\frac{\beta}{\lambda}\left[1+\frac{\theta}{(\theta+1)}\frac{1}{x}\right]e^{-\frac{\theta}{x}}}}{\left[1+\frac{\theta}{(\theta+1)}\frac{1}{x}\right]} \qquad \ldots (20)$$

$\lambda, \beta, \theta > 0$ Scale Parameters, we note When: β=1, λ=1 we get an inverse Lindley distribution with one parameter, When λ=1 we get an exponential-Lindley distribution with two parameters λ, θ. When λ=1 we get an inverse exponential-Lindley distribution with two parameters β, θ. This proves that the proposed distribution is a probability distribution because it is a combination of probability distributions.

Depending on the equation (19) to convert the proposed distribution into a fuzzy one, the resulting distribution is as follows:

$$\tilde{f}_{\tilde{X}}(\tilde{x}_{A^{(\alpha)}}) = \frac{\lambda\beta\theta^2}{(\theta+1)}\left(\frac{1+\tilde{x}_{A^{(\alpha)}}}{\tilde{x}_{A^{(\alpha)}}{}^3}\right)\frac{e^{-\frac{\beta}{\lambda}\left[1+\frac{\theta}{(\theta+1)}\frac{1}{\tilde{x}_{A^{(\alpha)}}}\right]e^{-\frac{\theta}{\tilde{x}_{A^{(\alpha)}}}}}}{\left[1+\frac{\theta}{(\theta+1)}\frac{1}{\tilde{x}_{A^{(\alpha)}}}\right]}\ ;0 < \tilde{x}_{A^{(\alpha)}} < \infty \quad \ldots (21)$$

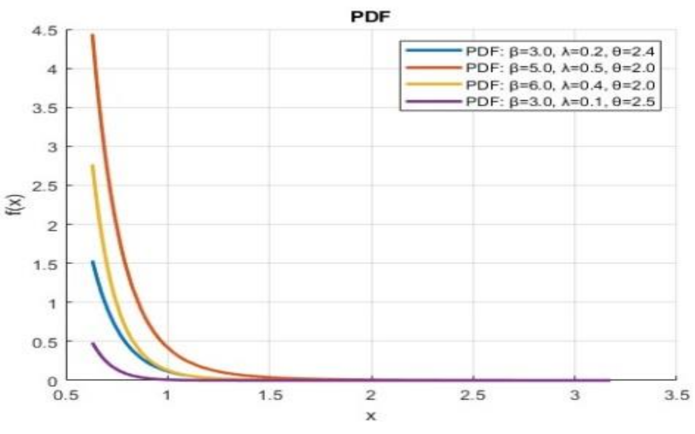

Figure 3. The probability density function curve of the new FIEILE distribution at different values of its parameters.

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The distribution function of the new distribution (FIEILE) can be extracted by applying Equation (20) as follows:

$$\tilde{F}_{\tilde{X}}\left(\tilde{x}_{A^{(\alpha)}}\right) = 1 - e^{-\left(\frac{\beta}{\lambda}\left[1+\frac{\theta}{(\theta+1)}\frac{1}{\tilde{x}_{A^{(\alpha)}}}\right]e^{-\frac{\theta}{\tilde{x}_{A^{(\alpha)}}}}\right)} \; ; 0 < \tilde{x}_{A^{(\alpha)}} < \infty \quad \dots (22)$$

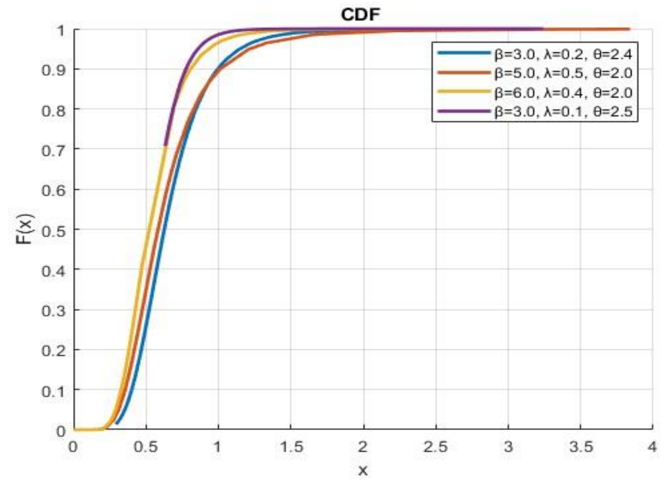

Figure 4. The curve of the cumulative probability density function of the FIEILI distribution at different values of its parameters.

The reliability function of the proposed distribution (FIEILE) is as follows:

$$\tilde{R}_{\tilde{X}}\left(\tilde{x}_{A^{(\alpha)}}\right) = e^{-\left(\frac{\beta}{\lambda}\left[1+\frac{\theta}{(\theta+1)}\frac{1}{\tilde{x}_{A^{(\alpha)}}}\right]e^{-\frac{\theta}{\tilde{x}_{A^{(\alpha)}}}}\right)} \quad \dots (23)$$

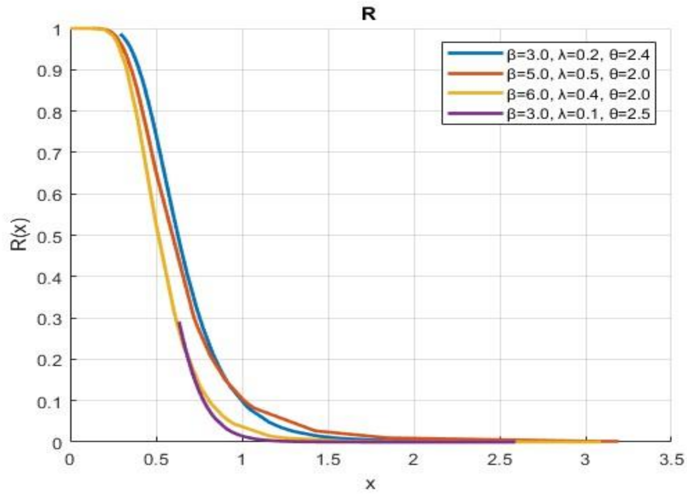

Figure 5. the reliability function curve of the proposed distribution FIEILI at different values of its parameters.

9. Maximum Likelihood Estimator for Fuzzy Triangular Distribution FIEILE

If we have a random sample $)A^{\alpha} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{\tilde{n}}$ from a FIEILI distribution with a probability density function as in Equation (49-2), then the probability function can be written as follows:

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$$L = \prod_{i=1}^{\tilde{n}} f(\tilde{x}_{iA^{(\alpha)}}, \lambda, \beta, \theta)$$

$$= \left(\frac{\lambda\beta\theta^2}{(\theta+1)}\right)^{\tilde{n}} \prod_{i=1}^{\tilde{n}} \left(\frac{1+\tilde{x}_{iA^{(\alpha)}}}{\tilde{x}_{iA^{(\alpha)}}{}^3}\right) \left(\frac{1}{\left[1+\frac{\theta}{(\theta+1)}\frac{1}{\tilde{x}_{iA^{(\alpha)}}}\right]}\right) e^{-\frac{\beta}{\lambda}\sum_{i=1}^{\tilde{n}}\left[1+\frac{\theta}{(\theta+1)}\frac{1}{\tilde{x}_{iA^{(\alpha)}}}\right] e^{-\theta\sum_{i=1}^{n}\frac{1}{\tilde{x}_{iA^{(\alpha)}}}}} \quad \dots (24)$$

$$\ln L = \tilde{n}\ln(\lambda) + \tilde{n}\ln(\beta) + 2\tilde{n}\ln(\theta) - \tilde{n}ln(\theta+1) + \sum_{i=1}^{\tilde{n}} ln\left(\left(\frac{1+\tilde{x}_{iA^{(\alpha)}}}{\tilde{x}_{iA^{(\alpha)}}{}^3}\right)\right) + \sum_{i=1}^{\tilde{n}} ln\left(\frac{1}{\left[1+\frac{\theta}{(\theta+1)}\frac{1}{\tilde{x}_{iA^{(\alpha)}}}\right]}\right) - \frac{\beta}{\lambda}\sum_{i=1}^{\tilde{n}}\left[1+\frac{\theta}{(\theta+1)}\frac{1}{\tilde{x}_{iA^{(\alpha)}}}\right] e^{-\theta\sum_{i=1}^{\tilde{n}}\frac{1}{\tilde{x}_{iA^{(\alpha)}}}} \quad \dots (25)$$

$$\frac{\partial \ln L}{\partial\theta} = \frac{2\tilde{n}}{\theta} - \frac{\tilde{n}}{\theta+1} + \sum_{i=1}^{\tilde{n}} -\frac{\frac{1}{(\theta+1)\tilde{x}_{iA^{(\alpha)}}}\frac{\theta}{(\theta+1)^2\tilde{x}_{iA^{(\alpha)}}}}{1+\frac{\theta}{(\theta+1)\tilde{x}_{iA^{(\alpha)}}}} - \frac{1}{\lambda}\left(\beta\left(\sum_{i=1}^{\tilde{n}} -\frac{\frac{1}{(\theta+1)\tilde{x}_{iA^{(\alpha)}}}\frac{\theta}{(\theta+1)^2\tilde{x}_{iA^{(\alpha)}}}}{\left(1+\frac{\theta}{(\theta+1)\tilde{x}_{iA^{(\alpha)}}}\right)^2}\right)\right) e^{-\theta\sum_{i=1}^{n}\frac{1}{\tilde{x}_{iA^{(\alpha)}}}} - \frac{1}{1+\frac{\theta}{(\theta+1)\tilde{x}_{iA^{(\alpha)}}}}\left(\left(\sum_{i=1}^{\tilde{n}}\frac{1}{\tilde{x}_{iA^{(\alpha)}}}\right) e^{-\theta\sum_{i=1}^{\tilde{n}}\frac{1}{\tilde{x}_{iA^{(\alpha)}}}}\right) \quad \dots (26)$$

$$\frac{\partial \ln L}{\partial\lambda} = \frac{\tilde{n}}{\lambda} + \frac{\beta\sum_{i=1}^{\tilde{n}}\frac{1}{1+\frac{\theta}{(\theta+1)\tilde{x}_{iA^{(\alpha)}}}} e^{-\theta\sum_{i=1}^{\tilde{n}}\frac{1}{\tilde{x}_{iA^{(\alpha)}}}}}{\lambda^2} \quad \dots (27)$$

$$\frac{\partial \ln L}{\partial\beta} = \frac{\tilde{n}}{\beta} - \frac{\beta\sum_{i=1}^{\tilde{n}}\frac{1}{1+\frac{\theta}{(\theta+1)\tilde{x}_{iA^{(\alpha)}}}} e^{-\theta\sum_{i=1}^{\tilde{n}}\frac{1}{\tilde{x}_{iA^{(\alpha)}}}}}{\lambda} \quad \dots (28)$$

10. Maximum Product of Spacing's Estimator

If we have a fuzzy random sample)$A^{\alpha} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{\tilde{n}})$ from the FIEILI distribution, let it be:

$$(\tilde{x}_{(1)}, \tilde{x}_{(2)}, \dots, \tilde{x}_{\tilde{n}})$$

The distribution sample is ordered skewed. Spaces indicate the differences between successive ordered statistics (ordered sample values) in the data set as follows:

$$D_i = \tilde{F}_{\tilde{X}}\left(\tilde{x}_{(i)A^{(\alpha)}}\right) - \tilde{F}_{\tilde{X}}\left(\tilde{x}_{(i-1)A^{(\alpha)}}\right) \quad ; i = 1, 2, \dots . \tilde{n}+1 \quad \dots (29)$$

Where $\tilde{F}_{\tilde{X}}\left(\tilde{x}_{(i)A^{(\alpha)}}\right)$ Fuzzy cumulative probability density function of the proposed distribution

The objective function for the MPS method is as follows:

$$MPS = \left(\prod_{i=1}^{\tilde{n}+1} D_i\right)^{\frac{1}{\tilde{n}+1}} \quad \dots (30)$$

Taking the natural logarithm of the equation (30), we get:

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$$\log(\text{MPS}) = \frac{1}{\tilde{n}+1}\sum_{i=1}^{\tilde{n}+1}\log(D_i)$$
$$= \frac{1}{\tilde{n}+1}\sum_{i=1}^{\tilde{n}+1}\log\left(\tilde{F}_{\tilde{X}}\left(\tilde{x}_{(i)A^{(\alpha)}}\right) - \tilde{F}_{\tilde{X}}\left(\tilde{x}_{(i-1)A^{(\alpha)}}\right)\right) \quad \dots (31)$$

The probability density function of the skewed distribution is as in Equation (19), and by substituting it in Equation (31), we get:

$$g(\text{MPS}) = \frac{1}{\tilde{n}+1}\sum_{i=1}^{\tilde{n}+1}\log\left(e^{-\left(\frac{\beta}{\lambda}\left[1+\frac{\theta}{(\theta+1)}\frac{1}{\tilde{x}_{(i-1)A^{(\alpha)}}}\right]e^{-\frac{\theta}{\tilde{x}_{(i-1)A^{(\alpha)}}}}\right)} - e^{-\left(\frac{\beta}{\lambda}\left[1+\frac{\theta}{(\theta+1)}\frac{1}{\tilde{x}_{(i)A^{(\alpha)}}}\right]e^{-\frac{\theta}{\tilde{x}_{(i)A^{(\alpha)}}}}\right)} \right) \quad \dots (32)$$

To find the estimators of the distribution parameters (λ, β, θ), the Nelder-Mead Simplex Method algorithm was used, which is a commonly used numerical technique for finding the minimum of an objective function in a multidimensional space. Unlike gradient-based optimization methods, it does not require the computation of derivatives, which makes it particularly useful for non-differentiable functions, as follows [13]:

11. Simulation study

Simulations experiments include choose default values for the proposed distribution parameters as the following table:

Table 1: Default values for IEILE distribution parameters

Parameter	1	2
λ	4	5
β	0.8	0.9
θ	4	3

Generating traditional data that follows the proposed distribution represented by the vector x according to generating a variable that follows a uniform distribution u~U(0,1), generating data that follows the IEILE distribution by applying the quantile function and using the Lambert function according to the following formula:

$$Q(x) = -\left[\frac{\left(1+\frac{1}{\theta}\right)}{\theta-1}W_{-1}\left(-u(1+\theta)\lambda\beta e^{-(1+\theta)}\right)\right]^{-1} \quad \dots (33)$$

where W_{-1} is a Lambert function, it is a mathematical function and is also known as the "logarithmic product function.".

The traditional sample vector $\underline{X} = (x_1, x_2, \dots, x_n)'$ of the proposed distribution is transformed into fuzziness by finding the degree of belonging corresponding to each observation of the traditional polluted sample vector using a trigonometric belonging function as follows:

$$\mu_A(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } x > b \end{cases} \quad \dots (34)$$

Where a represents the lowest value of the traditional sample observations and b represents the highest value of the traditional sample vector observations, which results in a fuzzy sample vector $\underline{\tilde{X}} = \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$ that includes each observation and its corresponding degree of affiliation, i.e. [12]:

$$\tilde{x}_i = \{(x_i, \mu_A(x_1)), (x_2, \mu_A(x_2)), \dots, (x_n, \mu_A(x_n))\} \quad \dots (35)$$

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Then the fuzzy set is obtained at the cutoff α , $\widetilde{A}_{\alpha}$ for distribution by choosing the elements in the fuzzy set that have a degree of belonging greater than or equal to the cutoff, α, i.e.:

$$\tilde{A}_{\alpha} = \{\tilde{x} \in T;\ \mu_{\tilde{A}}(x) \geq \alpha\} \quad \dots (36)$$

The cutting factor has been selected is α=0.1, 0.3, 0.5, 0,7 The estimation methods were compared using the mean square error (MSE) criterion and the average mean square error ((AMSE)) The simulation results were obtained using the (Matlab 2023) program, and all the results were displayed in special tables that we will show later.

First experiment: Theoretical parameters λ=4,β=0.8,θ=4

Table 2: Values of the actual probability density function and estimated by estimation methods and the mean square error for each method and the estimated values of the proposed distribution parameters for the first experiment

Alfa-cut	Real	MlE	MSE_MLE	MPS	MSE_MPS
0.1	0.96144	0.91331	0.00232	0.9104	0.00261
	0.93731	0.89773	0.00157	0.89147	0.00210
	0.90832	0.87859	0.00088	0.86852	0.00158
	0.89348	0.86863	0.00062	0.85669	0.00135
	0.88597	0.86354	0.00050	0.85068	0.00125
	0.87797	0.85808	0.00040	0.84427	0.00114
	0.83335	0.82699	0.00004	0.80815	0.00064
	0.82659	0.82218	0.00002	0.80262	0.00057
	0.80011	0.80309	0.00001	0.78088	0.00037
	0.79055	0.78787	0.00001	0.77444	0.00026
p.d.f. AMSE		0.00048		0.00496	
Estimates	$\hat{\lambda}$	4.13146	0.01728	3.55678	0.19644
	$\hat{\beta}$	0.92374	0.01531	0.85382	0.00290
	$\hat{\theta}$	3.87533	0.01554	3.61603	0.14743
Parameters AMSE		0.01605		0.11559	
Best		MLE			
Alfa-cut	Real	MlE	MSE_MLE	MPS	MSE_MPS
0.3	0.87326	0.93464	0.00377	0.87590	0.00001
	0.84635	0.90363	0.00328	0.84545	0.00000
	0.83331	0.88862	0.00306	0.83078	0.00001
	0.81110	0.86305	0.00270	0.8059	0.00003
	0.79388	0.84325	0.00244	0.78672	0.00005
	0.77272	0.81893	0.00214	0.76328	0.00009
	0.75771	0.80170	0.00194	0.74673	0.00012
	0.75346	0.79682	0.00188	0.74206	0.00013
	0.74315	0.78499	0.00175	0.73074	0.00015
	0.67899	0.77856	0.00991	0.69546	0.00027
p.d.f. AMSE		0.00329		0.00009	
AMSE					
Estimates	$\hat{\lambda}$	4.27049	0.07316	3.98933	0.00011
	$\hat{\beta}$	0.84521	0.00204	0.78041	0.00038
	$\hat{\theta}$	4.39225	0.15386	4.00787	0.00006
Parameters AMSE		0.07636		0.00019	
Best		MPS			
Alfa-cut	Real	MlE	MSE_MLE	MPS	MSE_MPS
0.5	0.98834	0.96590	0.00050	0.97059	0.00032
	0.92835	0.90545	0.00052	0.92911	0.00000
	0.92450	0.89978	0.00061	0.89507	0.00087
	0.88848	0.89591	0.00006	0.87842	0.00010
	0.86846	0.87672	0.00007	0.86714	0.00000
	0.84252	0.85328	0.00012	0.84198	0.00000
	0.83151	0.84673	0.00023	0.83091	0.00000

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	0.75426	0.76206	0.00006	0.74382	0.00011
	0.73577	0.75074	0.00022	0.72552	0.00011
	0.72868	0.74546	0.00028	0.71835	0.00011
p.d.f. AMSE		0.00385		0.00008	
Estimates	$\hat{\lambda}$	3.65063	0.12206	3.99037	0.00009
	$\hat{\beta}$	0.78866	0.00013	0.77954	0.00042
	$\hat{\theta}$	3.44735	0.30542	4.00746	0.00006
Parameters AMSE		0.14254		0.00019	
Best		MPS			
Alfa-cut	Real	MlE	MSE_MLE	MPS	MSE_MPS
0.7	0.97334	0.97447	0.00000	0.97372	0.00000
	0.96137	0.96414	0.00001	0.96219	0.00000
	0.90200	0.91889	0.00029	0.90585	0.00001
	0.89917	0.89864	0.00000	0.89615	0.00001
	0.89828	0.89952	0.00000	0.897231	0.00000
	0.84704	0.85374	0.00004	0.84618	0.00000
	0.82977	0.83617	0.00004	0.83851	0.00008
	0.82562	0.84193	0.00027	0.83651	0.00012
	0.7557	0.78952	0.00114	0.75452	0.00000
	0.74544	0.77239	0.00073	0.74577	0.00000
p.d.f. AMSE		0.00185		0.00023	
Estimates	$\hat{\lambda}$	4.20657	0.04267	4.01467	0.00022
	$\hat{\beta}$	0.83222	0.00104	0.81432	0.00021
	$\hat{\theta}$	4.11047	0.01220	4.00672	0.00005
Parameters AMSE		0.01864		0.00016	
Best		MPS			

We note from Table (3) and Figures (6), (7), (9) and (8) when the cutoff α=0.1, the method of the maximum likelihood (MLE) is better than the maximum spacing (MPS) because the probability density function estimated by this method achieved the lowest mean square error rate of (0.00048) compared to the mean square error rate of the (MPS) method, which amounted to (0.00496). Also, the mean square error rate for the parameters estimated by this method, which amounted to (0.01605), is less than the mean square error rate for the parameters estimated by the (MPS) method, which amounted to (0.11559). At cutoff coefficients α = 0.3, 0.5, 0.7, the maximum parsimony (MPS) method was better than the maximum likelihood (MLE) method, as the probability density function estimated by this method achieved the lowest mean square error rate of (0.00009, 0.00008, 0.00023) respectively, compared to the mean square error rate of the (MLE) method, which reached (0.09538, 0.18785, 0.00552) respectively. Also, the mean square error rate of the parameters estimated by this method, which reached (0.00019, 0.00019, 0.00016), was less than the mean square error rate of the parameters estimated by the (MPS) method, which reached (0.07636, 0.14254, 0.01864) respectively.

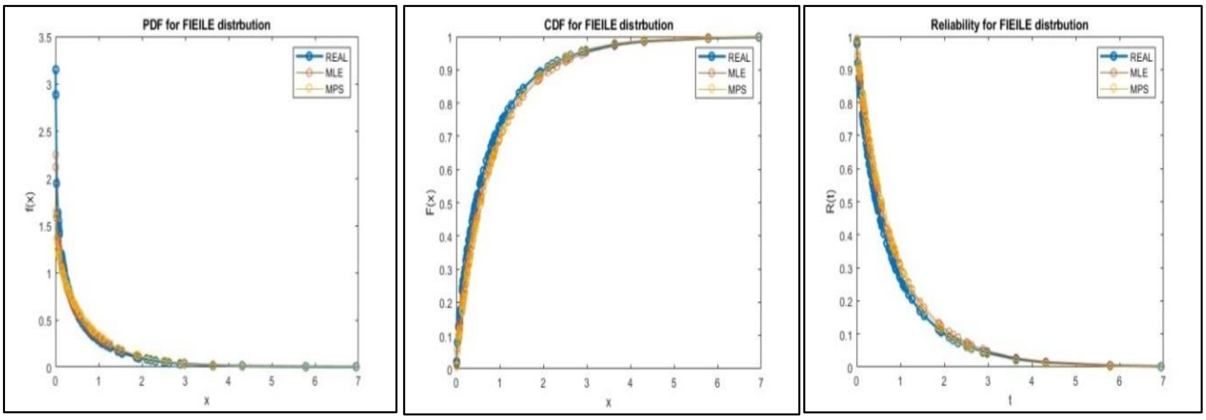

Figure 6. (a) Probability density function curve (b) Cumulative density function curve (c) Reliability function curve at cut-off 0.1

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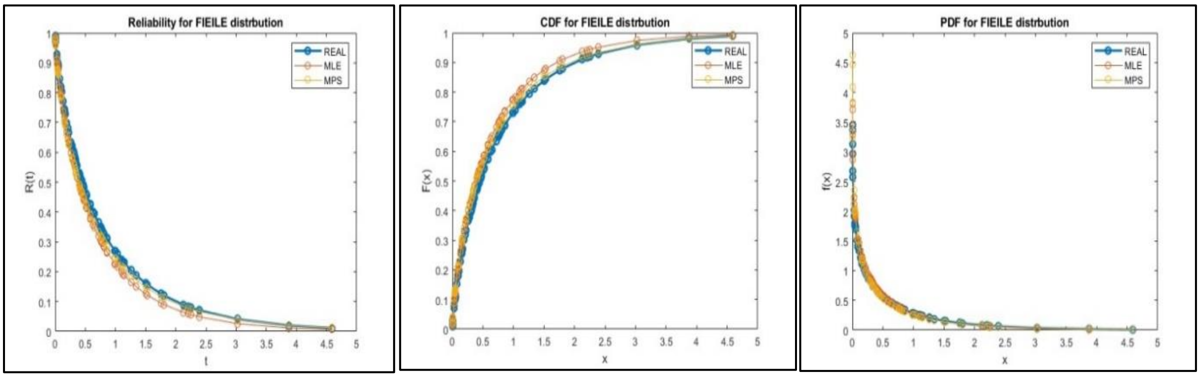

Figure 7. (a) Probability density function curve (b) Cumulative density function curve (c) Reliability function curve at cut-off 0.3

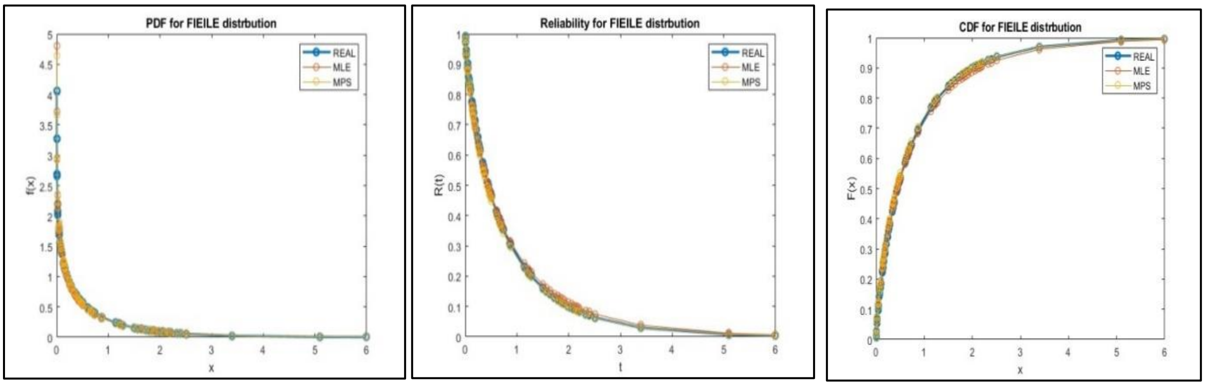

Figure 8. (a) Probability density function curve (b) Cumulative density function curve (c) Reliability function curve at cut-off 0.5

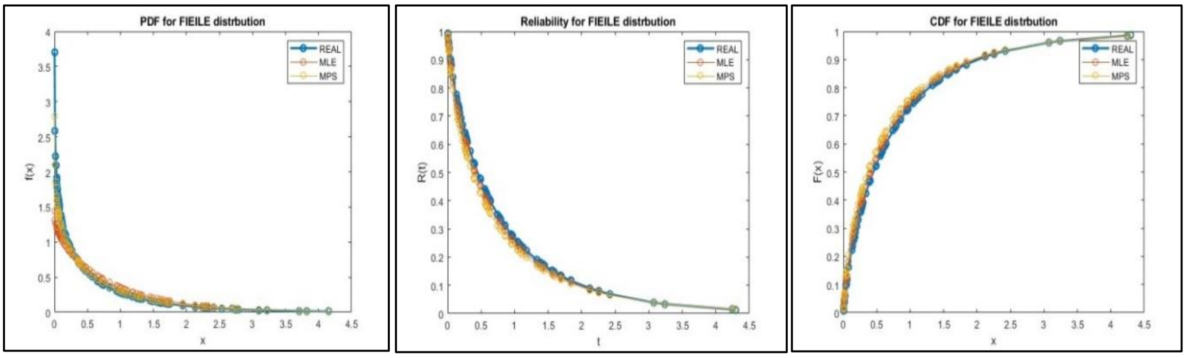

Figure 9. (a) Probability density function curve (b) Cumulative density function curve (c) Reliability function curve at cutoff 0.7

Second experiment: Theoretical parameters $\lambda=5,\beta=0.9,\theta=3$

Table 3: Values of the actual probability density function and the estimated value using estimation methods, the mean square error for each method, and the estimated values of the proposed distribution parameters for the second experiment.

Alfa-cut	Real	MlE	MSE_MLE	MPS	MSE_MPS
0.1	0.97640	0.96974	0.00004	0.92825	0.00232
	0.91623	0.90917	0.00005	0.96606	0.00248
	0.89570	0.88854	0.00005	0.94470	0.00240
	0.88959	0.8824	0.00005	0.93833	0.00238
	0.88463	0.87742	0.00005	0.93315	0.00235
	0.87888	0.87165	0.00005	0.92714	0.00233

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	0.87692	0.86968	0.00005	0.92510	0.00232
	0.86155	0.85425	0.00005	0.90901	0.00225
	0.84231	0.83497	0.00005	0.88882	0.00216
	0.75442	0.79587	0.00172	0.74702	0.00005
p.d.f. AMSE		0.00005		0.00031	
Estimates	$\hat{\lambda}$	5.19269	0.84438	4.98403	0.00026
	$\hat{\beta}$	0.94776	0.00228	0.89029	0.00009
	$\hat{\theta}$	3.23693	1.90771	3.02361	0.00056
Parameters AMSE		0.00030		0.03185	
Best		MLE			
Alfa-cut	Real	MlE	MSE_MLE	MPS	MSE_MPS
0.3	0.98091	0.99044	0.00009	0.95234	0.00082
	0.97128	0.98275	0.00013	0.94346	0.00077
	0.90664	0.93014	0.00055	0.88384	0.00052
	0.90036	0.92493	0.00060	0.87804	0.00050
	0.89869	0.92355	0.00062	0.8765	0.00049
	0.85906	0.89035	0.00098	0.83987	0.00037
	0.84703	0.88014	0.00110	0.82874	0.00033
	0.81585	0.85343	0.00141	0.79987	0.00026
	0.8127	0.85071	0.00144	0.79696	0.00025
	0.80709	0.84584	0.00150	0.79175	0.00024
p.d.f. AMSE		0.09538		0.04168	
Estimates	$\hat{\lambda}$	5.39313	0.15455	4.97751	0.00051
	$\hat{\beta}$	0.98496	0.00722	0.89505	0.00002
	$\hat{\theta}$	3.33848	0.11457	3.17759	0.03154
Parameters AMSE		0.09211		0.01069	
Best		MPS			
Alfa-cut	Real	MlE	MSE_MLE	MPS	MSE_MPS
0.5	0.87779	0.93146	0.00288	0.89271	0.00022
	0.87257	0.92597	0.00285	0.88757	0.00023
	0.80715	0.85671	0.00246	0.82289	0.00025
	0.66293	0.70166	0.00150	0.67891	0.00026
	0.6497	0.68729	0.00141	0.66561	0.00025
	0.64061	0.67741	0.00135	0.65646	0.00025
	0.60282	0.63623	0.00112	0.61835	0.00024
	0.60057	0.63377	0.00110	0.61608	0.00024
	0.57919	0.6104	0.00097	0.59446	0.00023
	0.57814	0.60925	0.00097	0.59339	0.00023
p.d.f. AMSE		0.18785		0.06843	
Estimates	$\hat{\lambda}$	5.36604	0.13398	5.22777	0.05188
	$\hat{\beta}$	0.95280	0.00279	0.94196	0.00176
	$\hat{\theta}$	3.63860	0.40782	3.35703	0.12747
Parameters AMSE		0.08312		0.02130	
Best		MPS			
Alfa-cut	Real	MlE	MSE_MLE	MPS	MSE_MPS
0.7	0.96836	0.9304	0.00144	0.96772	0.00000
	0.96123	0.92329	0.00144	0.96015	0.00000
	0.95629	0.91839	0.00144	0.95492	0.00000
	0.83143	0.79544	0.00130	0.83459	0.00001
	0.8211	0.78535	0.00128	0.82396	0.00001
	0.81263	0.77711	0.00126	0.80528	0.00005
	0.78285	0.74818	0.00120	0.78485	0.00000
	0.76555	0.73142	0.00116	0.76727	0.00000
	0.73893	0.70571	0.00110	0.73034	0.00007
	0.71262	0.68038	0.00104	0.71387	0.00000
p.d.f. AMSE		0.00552		0.00008	

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Estimates	$\hat{\lambda}$	5.24224	0.05868	5.12777	0.01633
	$\hat{\beta}$	0.94355	0.00190	0.92196	0.00048
	$\hat{\theta}$	3.4345	0.18879	3.21703	0.04710
Parameters AMSE		0.05473		0.00685	
Best		MPS			

we note from Table (3) and Figures (10), (11) , (12), and (13) ,when the cutoff α=0.1, the method of the maximum likelihood (MLE) method is better than the maximum spacing (MPS) method, since the probability density function estimated by this method achieved the lowest mean square error rate of (0.00005) compared to the mean square error rate of the (MPS) method, which amounted to (0.00031). Also, the mean square error rate for the parameters estimated by this method, which amounted to (0.00030), is less than the mean square error rate for the parameters estimated by the (MPS) method, which amounted to (0.03185). At cutoff coefficients α = 0.3, 0.5, 0.7, the maximum parsimony (MPS) method was better than the maximum likelihood (MLE) method, as the probability density function estimated by this method achieved the lowest mean square error rate of (0.04168, 0.02130, 0.00685) respectively, compared to the mean square error rate of the (MLE) method, which reached (0.09538, 0.18785, 0.00552) respectively. Also, the mean square error rate of the parameters estimated by this method, which reached (0.01069, 0.02130, 0.00685), was less than the mean square error rate of the parameters estimated by the (MPS) method, which reached (0.09538, 0.08312, 0.05473) respectively.

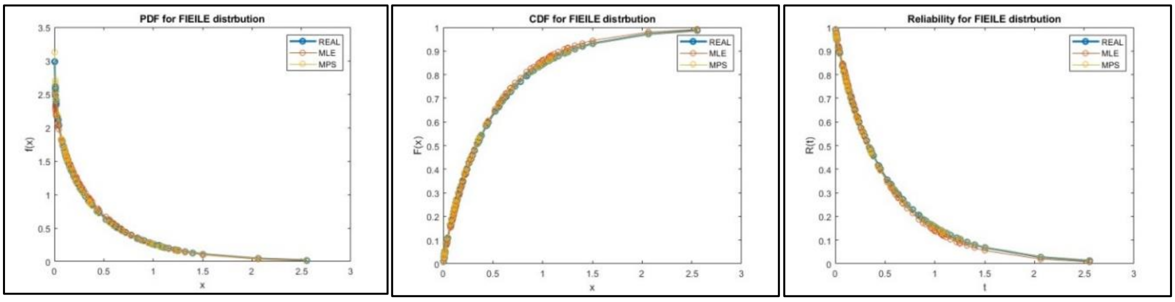

Figure 10. (a) Probability density function curve (b) Cumulative density function curve (c) Reliability function curve at cut-off 0.1

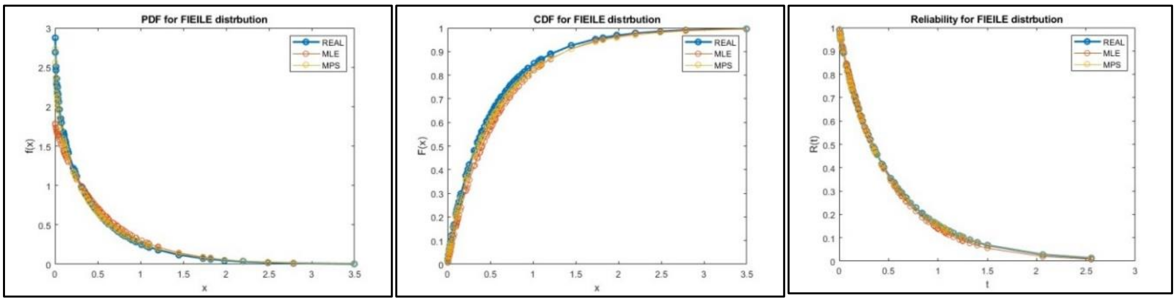

Figure 11. (a) Probability density function curve (b) Cumulative density function curve (c) Reliability function curve at cut-off 0.3

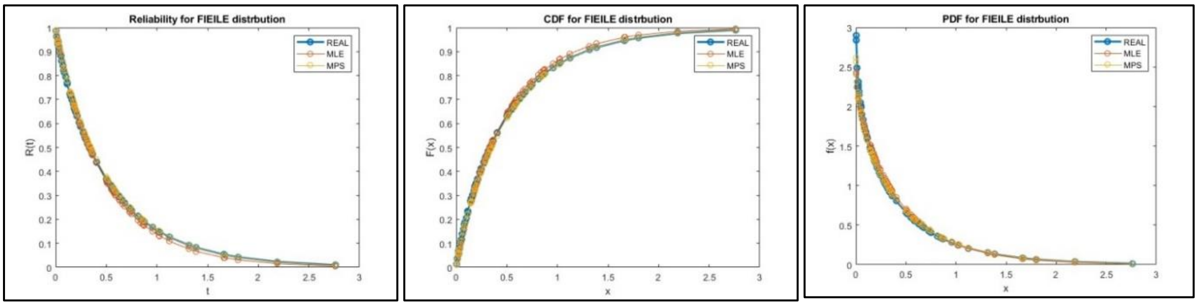

Figure 12. (a) Probability density function curve (b) Cumulative density function curve (c) Reliability function curve at cut-off 0.5

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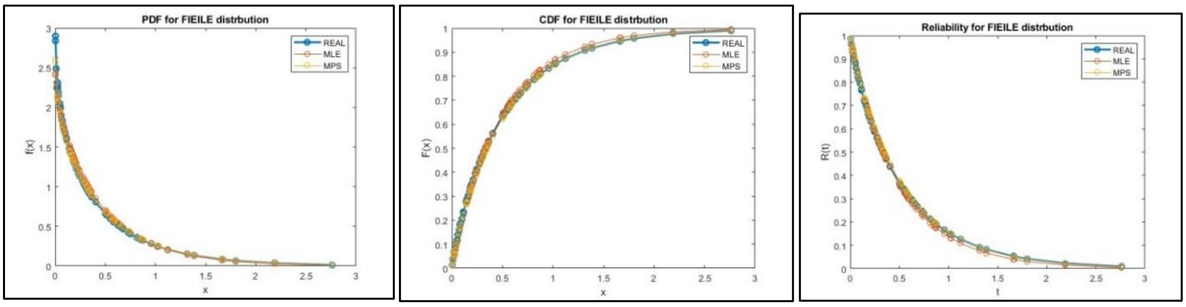

Figure 13. (a) Probability density function curve (b) Cumulative density function curve (c) Reliability function curve at cut-off 0.7

12. Conclusion

at cut-off α=0.1, the maximum likelihood method (MLE) is better than the maximum likelihood method (MPS), and at cut-off coefficients α=0.3, 0.5, 0.7, the maximum likelihood method (MPS) was better than the maximum likelihood method (MLE) because the probability density function estimated by this method achieved the lowest mean square error rate. The higher the cut-off, the better the maximum likelihood method.

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