



## Characterization of various $(b, \ell)$ neutrosophic ideals of an ordered $\Gamma$ -semigroups

A. Rajalakshmi<sup>1</sup>, Nasreen Kausar<sup>2</sup>, Brikena Vrioni<sup>3,\*</sup>, K. Lenin Muthu Kumaran<sup>1</sup>, Nezir Aydin<sup>4,5</sup>,  
Murugan Palanikumar<sup>6</sup>

<sup>1</sup>Department of Mathematics, Shanmuga Industries Arts and Science College, Affiliated to Thiruvalluvar University, Tiruvannamalai, Tamil Nadu, 606603, India

<sup>2</sup> Department of Mathematics, Faculty of Arts and Science, Yildiz Technical University, Esenler, 34220, Istanbul, Turkey

<sup>3</sup> School of Arts and Sciences, American International University, Kuwait

<sup>4</sup>College of Science and Engineering Hamad Bin Khalifa University, 34110 Doha, Qatar

<sup>5</sup>Department of Industrial Engineering, Yildiz Technical University, Besiktas, 34349, Istanbul, Turkey

<sup>6</sup>Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai-602105, India

E-mails: pearlakshmi03@gmail.com; kausar.nasreen57@gmail.com; Brikena.vrioni@yahoo.com; leninmuthukumaran@gmail.com; naydin@hbku.edu.qa; palanimaths86@gmail.com

### Abstract

In this paper, we introduce the notion of  $b, \ell$ -neutrosophic subsemigroup (NSS), neutrosophic left ideal (NLI), neutrosophic right ideal (NRI), neutrosophic ideal (NI), neutrosophic bi-ideal (NBI),  $(\epsilon, \epsilon \vee q)$ -neutrosophic ideal, neutrosophic bi-ideal of an ordered  $\Gamma$ -semigroups and discuss some of their properties. The concept of  $b, \ell$ -neutrosophic ideal is a new extension of neutrosophic ideal over ordered  $\Gamma$ -semigroups  $\mathcal{Z}$ . A non-empty subset  $\xi_b$  is a  $(b, \ell)$ -NSS (NLI, NRI, NBI,  $(1,2)$ -ideal) of  $\mathcal{Z}$ . Then the lower level set  $\Delta_b$  is a subsemigroup  $[LI, RI, BI, (1,2) - ideal]$  of  $\mathcal{Z}$ , where  $\Delta_b = \{\varrho \in \mathcal{Z} | \Delta(\varrho) > b\}$ ,  $\Psi_b = \{\varrho \in \mathcal{Z} | \Delta(\varrho) > b\}$  and  $\mathcal{U}_b = \{\varrho \in \mathcal{Z} | \Delta(\varrho) < b\}$ . A subset  $\xi = [\Delta, \Psi, \mathcal{U}]$  is a  $(b, \ell) - NSS[NLI, NRI, NBI, (1,2) - ideal]$  of  $\mathcal{Z}$  if and only if each non-empty level subset  $\xi_t$  is a subsemigroup  $[LI, RI, BI, (1,2) - ideal]$  of  $\mathcal{Z}$  for all  $t \in (b, \ell]$ . Every  $(\epsilon, \epsilon \vee q)$ NBI of  $\mathcal{Z}$  is a  $(b, \ell)$ NBI of  $\mathcal{Z}$ , but converse need not be true and examples are provided to illustrate our results.

**Keywords:** Ordered  $\Gamma$ -semigroups; neutrosophic ideals; bi-ideals;  $(b, \ell)$  bi-ideals;  $(\epsilon, \epsilon \vee q)$  bi-ideals

### 1 Introduction

Theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, combinatorics, functional analysis, graph theory, Euclidean geometry, probability theory, commutative and non-commutative ring theory, optimization theory, discrete event dynamical systems, automata theory, mathematical modeling of quantum physics, parallel computation systems, and other fields all heavily rely on algebraic structures. Zadeh<sup>1</sup> created fuzzy set (FS) theory, which works best at handling ambiguity and uncertainty. An element in an FS is considered a member if it contains a single value inside the interval. However, since resistance can exist in real-world situations, the degree of non-membership might not necessarily equal one minus the degree of membership. As FS theory advances quickly, more and more hybrid fuzzy models

are being developed. Numerous uncertain theories, including FS,<sup>1</sup> intuitionistic FS (IFS),<sup>2</sup> Pythagorean FS (PFS),<sup>3</sup> and spherical FS (SFS),<sup>7</sup> have been developed as a result of the uncertainties. Sets with grades ranging from 0 to 1, referred to as MG, comprise an FS. IFS is classified as MG, despite Atanassov<sup>2</sup> stating that non-membership grades (NMG) can only have a value of 1. There is a chance that, in a decision-making process, the sum of MGs and NMGs will sometimes exceed 1. The generalized MG and NMG logic, which has a value not exceeding 1 and is determined by the square of the MGs and NMGs, was constructed by Yager<sup>3</sup> using PFS logic. These ideas are unable to explain the neutral state, which is neither positive nor negative. Recently Palanikumar et al. discussed many new research.<sup>4-6</sup>

He studied their properties on the parallel lines to set theory. In 1971, Rosenfeld<sup>8</sup> defined fuzzy subgroups and gave some of its properties. Kuroki<sup>9</sup> introduced fuzzy semigroups as a generalized of classical semigroups. Mordeson<sup>10</sup> obtained some characterization of fuzzy semigroups. Sen et al.<sup>11,12</sup> have introduced  $\Gamma$ -semigroups and their properties. Ordered  $\Gamma$ -semigroup was studied by Kehayopula.<sup>13</sup> Somsak Lekkoksung discussed  $Q$ -fuzzy ideals in ordered semigroups.<sup>14</sup> Kehayopula et al.<sup>15</sup> initiated the study of fuzzy ordered semigroups. Faiz Muhammad Khan et al.<sup>16</sup> introduced the concepts of  $(\lambda_1, \lambda_2)$ -fuzzy bi-ideal and  $(\lambda_1, \lambda_2)$ -fuzzy subsemigroup. Jun et al.<sup>17</sup> provided some results on ordered semigroups characterized by their  $(\epsilon, \epsilon \vee q)$ -fuzzy bi-ideals. Bhakat et al.<sup>18</sup> introduced the concepts of  $(\epsilon, \epsilon \vee q)$ -fuzzy subgroups using the notion "belongingness ( $\in$ )". Kazanci et al.<sup>19</sup> introduced the concept of a generalized fuzzy bi-ideal in semigroup and established some properties of fuzzy bi-ideals in terms of  $(\epsilon, \epsilon \vee q)$ -fuzzy bi-ideals. Smarandache<sup>20</sup> created the neutrosophic set (NS) to handle contradictory and unclear data. Recently, Palanikumar et al.<sup>21</sup> introduced an intuitionistic fuzzy normal subbisemiring of bisemiring. By using bipolar-valued neutrosophic normal sets, Palanikumar et al.<sup>22</sup> established the concept of bisemiring. We discuss that  $(b, \ell)$ -neutrosophic ideals and  $(\epsilon, \epsilon \vee q)$  neutrosophic ideals of ordered  $\Gamma$ -semigroup and discuss some properties with examples.

## 2 Preliminaries

**Definition 2.1.** An ordered  $\Gamma$ -semigroup  $\mathcal{Z}$  together with an order relation  $\leq$  such that  $x \leq y$  implies  $x\gamma z \leq y\gamma z$  and  $z\gamma x \leq z\gamma y$  for all  $x, y, z \in \mathcal{Z}$  and  $\gamma \in \Gamma$ .

**Definition 2.2.** Let  $M$  and  $N$  be two non empty subsets of  $\mathcal{Z}$ . We denote

1.  $[M] = \{t \in \mathcal{Z} \mid t \leq h \text{ for some } h \in M\}$ ,
2.  $M\Gamma N = \{a\gamma b : a \in M, b \in N \text{ and } \gamma \in \Gamma\}$ ,
3.  $M_x = \{(y, z) \in \mathcal{Z} \times S \mid x \leq y\gamma z\}$ .

**Definition 2.3.** Let  $\xi$  be a fuzzy subset of  $\mathcal{Z}$  and  $t \in [0, 1]$ . The set  $\xi_t = \{x \in \mathcal{Z} \mid \xi(x) \geq t\}$  is called the level subset of  $\xi$ . Clearly  $\xi_t \subseteq \xi_s$  whenever  $t \geq s$ .

**Definition 2.4.** Let  $\xi$  be a fuzzy subset of  $\mathcal{Z}$ . A mapping  $\xi : \mathcal{Z} \rightarrow [0, 1]$  is called a fuzzy subsemigroup of  $\mathcal{Z}$  if  $\xi(xy) \geq \min\{\xi(x), \xi(y)\}$  for all  $x, y \in \mathcal{Z}$ .

**Definition 2.5.** A fuzzy  $\xi$  subset of  $\mathcal{Z}$  is called a fuzzy bi-ideal of  $\mathcal{Z}$  if

1.  $a \leq b \implies \xi(a) \geq \xi(b)$  and
2.  $\xi(xyz) \geq \min\{\xi(x), \xi(z)\}$  for all  $x, y, z \in \mathcal{Z}$ .

**Definition 2.6.** A  $(b, \ell)$  fuzzy subsemigroup  $\xi$  of  $\mathcal{Z}$  is called  $(1, 2)$ -ideal of  $\mathcal{Z}$  if  $\max\{\xi(p\tau_1q(rs)), b\} \geq \min\{\xi(p), \xi(r), \xi(s), \ell\}$ , for all  $p, q, r, s \in \mathcal{Z}$ .

**Definition 2.7.** A fuzzy subset  $\mu$  of an ordered  $\Gamma$ -semigroup  $S$  is called a fuzzy right (resp. left) ideal of  $S$  if

1.  $x \leq y \implies \mu(x) \geq \mu(y)$  for all  $x, y \in S$ ,
2.  $\mu(x\tau y) \geq \mu(x)$  (resp.  $\mu(x\tau y) \geq \mu(y)$ ) for all  $x, y \in S$  and  $\tau \in \Gamma$ ,

3. A fuzzy subset  $\mu$  of an ordered  $\Gamma$ -semigroup  $S$  is called a fuzzy ideal of  $S$ , if it is both fuzzy left ideal and fuzzy right ideal.

**Definition 2.8.** Let  $A$  be a fuzzy set, if  $\chi_A$  is the characteristic function of  $A$ , then  $(\chi_A)_{\tau}^{\beta}$  is defined as

$$(\chi_A)_{\tau}^{\beta}(x) := \begin{cases} \beta & \text{if } x \in A, \\ \tau & \text{if } x \notin A. \end{cases}$$

**Corollary 2.9.** <sup>16</sup> If  $\mathcal{Z}$  is regular if and only if every RI  $M$  and every LI  $N$  of  $\mathcal{Z}$ , then  $(M \cap N] = (M \circ_{\Gamma} N]$ .

### 3 $(b, \ell)$ neutrosophic ideals

In this section  $\mathcal{Z}$  denotes the ordered  $\Gamma$ -semigroup. In what follows that  $(b, \ell) \in [0, 1]$  be such that  $0 \leq b < \ell \leq 1$  both  $(b, \ell)$  are arbitrary fixed.

**Definition 3.1.** A fuzzy subset (FS)  $\xi = [\Delta_N, \Psi_N, \mathcal{U}_N]$  of  $\mathcal{Z}$  is called a  $(b, \ell)$  neutrosophic subsemigroup (NSS) of  $\mathcal{Z}$  if it satisfies the following conditions:

1.  $\varrho \leq \sigma \Rightarrow \Delta(\varrho) \geq \Delta(\sigma)$ .
2.  $\max\{\Delta(\varrho\tau\sigma), b\} \geq \min\{\Delta(\varrho), \Delta(\sigma), \ell\}$
3.  $\max\{\Psi(\varrho\tau\sigma), b\} \geq \min\{\Psi(\varrho), \Psi(\sigma), \ell\}$
4.  $\min\{\mathcal{U}(\varrho\tau\sigma), b\} \leq \max\{\mathcal{U}(\varrho), \mathcal{U}(\sigma), \ell\}$  for all  $\varrho, \sigma \in \mathcal{Z}, \tau \in \Gamma$ .

**Example 3.2.** Let  $\mathcal{Z} = \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ , and  $\Gamma = \{\tau\}$  where  $\tau$  is defined on  $\mathcal{Z}$  with the following Cayley table:

$\tau$	$\epsilon_1$	$\epsilon_2$	$\epsilon_3$	$\epsilon_4$
$\epsilon_1$	$\epsilon_1$	$\epsilon_1$	$\epsilon_1$	$\epsilon_1$
$\epsilon_2$	$\epsilon_1$	$\epsilon_2$	$\epsilon_3$	$\epsilon_4$
$\epsilon_3$	$\epsilon_1$	$\epsilon_3$	$\epsilon_3$	$\epsilon_3$
$\epsilon_4$	$\epsilon_1$	$\epsilon_3$	$\epsilon_3$	$\epsilon_3$

$\leq := \{(\epsilon_1, \epsilon_1), (\epsilon_1, \epsilon_2), (\epsilon_1, \epsilon_3), (\epsilon_1, \epsilon_4), (\epsilon_2, \epsilon_2), (\epsilon_2, \epsilon_3), (\epsilon_2, \epsilon_4), (\epsilon_3, \epsilon_3), (\epsilon_4, \epsilon_3), (\epsilon_4, \epsilon_4)\}$ .  
 Define the mapping  $N = [\Delta_N, \Psi_N, \mathcal{U}_N] : \mathcal{Z} \times \Gamma \times \mathcal{Z} \rightarrow [0, 1]$ .

$$\Delta(\epsilon) = \begin{cases} 0.85 & \text{if } \epsilon = \epsilon_1 \\ 0.65 & \text{if } \epsilon = \epsilon_2 \\ 0.35 & \text{if } \epsilon = \epsilon_3 \\ 0.45 & \text{if } \epsilon = \epsilon_4 \end{cases} \quad \Psi(\epsilon) = \begin{cases} 0.60 & \text{if } \epsilon = \epsilon_1 \\ 0.40 & \text{if } \epsilon = \epsilon_2 \\ 0.15 & \text{if } \epsilon = \epsilon_3 \\ 0.20 & \text{if } \epsilon = \epsilon_4 \end{cases} \quad \mathcal{U}(\epsilon) = \begin{cases} 0.55 & \text{if } \epsilon = \epsilon_1 \\ 0.60 & \text{if } \epsilon = \epsilon_2 \\ 0.70 & \text{if } \epsilon = \epsilon_3 \\ 0.65 & \text{if } \epsilon = \epsilon_4 \end{cases}$$

Then  $N$  is a  $(0.75, 0.9)$  NSS of  $\mathcal{Z}$ .

**Definition 3.3.** A subset  $\xi$  of  $\mathcal{Z}$  is called a  $(b, \ell)$ -NBI of  $\mathcal{Z}$  if it satisfies the following conditions:

1. If  $\varrho \leq \sigma$ , then  $\Delta(\varrho) \geq \Delta(\sigma)$ ,  $\Psi(\varrho) \geq \Psi(\sigma)$  and  $\mathcal{U}(\varrho) \leq \mathcal{U}(\sigma)$ ,
2.  $\max\{\Delta(\varrho\tau_1\sigma), b\} \geq \min\{\Delta(\varrho), \Delta(\sigma), \ell\}$ ,  $\max\{\Psi(\varrho\tau_1\sigma), b\} \geq \min\{\Psi(\varrho), \Psi(\sigma), \ell\}$ ,  $\min\{\mathcal{U}(\varrho\tau_1\sigma), b\} \leq \max\{\mathcal{U}(\varrho), \mathcal{U}(\sigma), \ell\}$ ,
3.  $\max\{\Delta(\varrho\tau_1\sigma\tau_2\varpi), b\} \geq \min\{\Delta(\varrho), \Delta(\varpi), \ell\}$ ,  $\max\{\Psi(\varrho\tau_1\sigma\tau_2\varpi), b\} \geq \min\{\Psi(\varrho), \Psi(\varpi), \ell\}$ ,  $\min\{\mathcal{U}(\varrho\tau_1\sigma\tau_2\varpi), b\} \leq \max\{\mathcal{U}(\varrho), \mathcal{U}(\varpi), \ell\}$ , for  $\varrho, \sigma, \varpi \in \mathcal{Z}, \tau_1, \tau_2 \in \Gamma$  and .

**Example 3.4.** Let  $S = \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$  and  $\Gamma = \{\tau_1, \tau_2\}$  where  $\tau_1, \tau_2$  is defined on  $\mathcal{Z}$  with the following Cayley tables:

$\tau_1$	$\epsilon_1$	$\epsilon_2$	$\epsilon_3$	$\epsilon_4$	$\tau_2$	$\epsilon_1$	$\epsilon_2$	$\epsilon_3$	$\epsilon_4$
$\epsilon_1$	$\epsilon_1$	$\epsilon_1$	$\epsilon_1$	$\epsilon_1$	$\epsilon_1$	$\epsilon_1$	$\epsilon_1$	$\epsilon_1$	$\epsilon_1$
$\epsilon_2$	$\epsilon_1$	$\epsilon_2$	$\epsilon_3$	$\epsilon_4$	$\epsilon_2$	$\epsilon_1$	$\epsilon_2$	$\epsilon_3$	$\epsilon_4$
$\epsilon_3$	$\epsilon_1$	$\epsilon_3$	$\epsilon_3$	$\epsilon_3$	$\epsilon_3$	$\epsilon_1$	$\epsilon_3$	$\epsilon_3$	$\epsilon_3$
$\epsilon_4$	$\epsilon_1$	$\epsilon_3$	$\epsilon_3$	$\epsilon_3$	$\epsilon_4$	$\epsilon_1$	$\epsilon_2$	$\epsilon_3$	$\epsilon_4$

$\leq := \{(\epsilon_1, \epsilon_1), (\epsilon_1, \epsilon_2), (\epsilon_1, \epsilon_3), (\epsilon_1, \epsilon_4), (\epsilon_2, \epsilon_2), (\epsilon_2, \epsilon_3), (\epsilon_2, \epsilon_4), (\epsilon_3, \epsilon_3), (\epsilon_4, \epsilon_3), (\epsilon_4, \epsilon_4)\}$ . The mapping  $\xi = [\Delta, \Psi, \mathcal{U}] : \mathcal{Z} \times \Gamma \times \mathcal{Z} \rightarrow [0, 1]$

$$\Delta(\epsilon) = \begin{cases} 0.92 & \text{if } \epsilon = \epsilon_1 \\ 0.73 & \text{if } \epsilon = \epsilon_2 \\ 0.45 & \text{if } \epsilon = \epsilon_3 \\ 0.54 & \text{if } \epsilon = \epsilon_4 \end{cases} \quad \Psi(\epsilon) = \begin{cases} 0.90 & \text{if } \epsilon = \epsilon_1 \\ 0.71 & \text{if } \epsilon = \epsilon_2 \\ 0.43 & \text{if } \epsilon = \epsilon_3 \\ 0.52 & \text{if } \epsilon = \epsilon_4 \end{cases} \quad \mathcal{U}(\epsilon) = \begin{cases} 0.75 & \text{if } \epsilon = \epsilon_1 \\ 0.80 & \text{if } \epsilon = \epsilon_2 \\ 0.90 & \text{if } \epsilon = \epsilon_3 \\ 0.85 & \text{if } \epsilon = \epsilon_4 \end{cases}$$

Then  $\xi$  is a  $(0.78, 0.93)$  NBI of  $\mathcal{Z}$ .

**Theorem 3.5.** A non-empty subset  $\xi_b$  is a  $\Delta_b$  is a  $(b, \ell)$ -NSS (NLI, NRI, NBI,  $(1, 2)$ -ideal) of  $\mathcal{Z}$ . Then the lower level set  $\Delta_b$  is an subsemigroup (LI, RI, BI,  $(1, 2)$  – ideal) of  $\mathcal{Z}$ , where  $\Delta_b = \{\varrho \in \mathcal{Z} | \Delta(\varrho) > b\}$ ,  $\Psi_b = \{\varrho \in \mathcal{Z} | \Psi(\varrho) > b\}$  and  $\mathcal{U}_b = \{\varrho \in \mathcal{Z} | \mathcal{U}(\varrho) < b\}$

**Proof.** Suppose that  $\xi_b$  is a  $(b, \ell)$ -NSS of  $\mathcal{Z}$ . Let  $\varrho, \sigma \in \mathcal{Z}$  and  $\tau \in \Gamma$  such that  $\varrho, \sigma \in \Delta_b$ . Then  $\Delta(\varrho) > b, \Delta(\sigma) > b$ . Therefore  $\max\{\Delta(\varrho\tau\sigma), b\} \geq \min\{\Delta(\varrho), \Delta(\sigma), \ell\} > \min\{b, b, \ell\} = b$ . Hence  $\Delta(\varrho\tau\sigma) > b$ . It shows that  $\varrho\tau\sigma \in \Delta_b$ . Therefore  $\Delta_b$  is a subsemigroup of  $\mathcal{Z}$ . Let  $\varrho, \sigma \in \mathcal{Z}$  and  $\tau \in \Gamma$  such that  $\varrho, \sigma \in \Psi_b$ . Then  $\Psi(\varrho) > b, \Psi(\sigma) > b$ . Therefore  $\max\{\Psi(\varrho\tau\sigma), b\} \geq \min\{\Psi(\varrho), \Psi(\sigma), \ell\} > \min\{b, b, \ell\} = b$ . Hence  $\Psi(\varrho\tau\sigma) > b$ . It shows that  $\varrho\tau\sigma \in \Psi_b$ . Therefore  $\Psi_b$  is a subsemigroup of  $\mathcal{Z}$ . Let  $\varrho, \sigma \in \mathcal{Z}$  and  $\tau \in \Gamma$  such that  $\varrho, \sigma \in \mathcal{U}_b$ . Then  $\mathcal{U}(\varrho) < b, \mathcal{U}(\sigma) < b$ . Therefore  $\min\{\mathcal{U}(\varrho\tau\sigma), b\} \leq \max\{\mathcal{U}(\varrho), \mathcal{U}(\sigma), \ell\} < \max\{b, b, \ell\} = \ell$ . Hence  $\mathcal{U}(\varrho\tau\sigma) < b$ . It shows that  $\varrho\tau\sigma \in \mathcal{U}_b$ . Therefore  $\mathcal{U}_b$  is a subsemigroup of  $\mathcal{Z}$ . Therefore  $\xi_b$  is a subsemigroup of  $\mathcal{Z}$ .

**Theorem 3.6.** A non-empty subset  $M$  of  $\mathcal{Z}$  is a SS [LI, RI, BI,  $(1, 2)$  – ideal] of  $\mathcal{Z}$  if and only if the neutrosophic subset  $\xi = [\Delta, \Psi, \mathcal{U}]$  of  $\mathcal{Z}$  is defined as

$$\Delta(\varrho) = \begin{cases} \geq \ell & \text{for all } \varrho \in (M) \\ b & \text{for all } \varrho \notin (M) \end{cases} \quad \Psi(\varrho) = \begin{cases} \geq \ell & \text{for all } \varrho \in (M) \\ b & \text{for all } \varrho \notin (M) \end{cases} \quad \mathcal{U}(\varrho) = \begin{cases} \leq \ell & \text{for all } \varrho \in (M) \\ b & \text{for all } \varrho \notin (M) \end{cases}$$

is a  $(b, \ell)$ NSS[NLI, NRI, NBI,  $N(1, 2)$  – ideal] of  $\mathcal{Z}$

**Proof.** Suppose that  $M$  is an subsemigroup of  $\mathcal{Z}$ . Let  $\varrho, \sigma \in \mathcal{Z}$  be such that  $\varrho, \sigma \in (M)$  then  $\varrho\tau\sigma \in (M)$  and  $\tau \in \Gamma$ . Hence  $\Delta(\varrho\tau\sigma) \geq \ell, \Psi(\varrho\tau\sigma) \geq \ell$  and  $\mathcal{U}(\varrho\tau\sigma) \leq \ell$ . Thus  $\max\{\Delta(\varrho\tau\sigma), b\} \geq \ell = \min\{\Delta(\varrho), \Delta(\sigma), \ell\}, \max\{\Psi(\varrho\tau\sigma), b\} \geq \ell = \min\{\Psi(\varrho), \Psi(\sigma), \ell\}$  and  $\min\{\mathcal{U}(\varrho\tau\sigma), b\} \leq \ell = \max\{\mathcal{U}(\varrho), \mathcal{U}(\sigma), \ell\}$ . If  $\varrho \notin (M)$  or  $\sigma \notin (M)$ , then  $\min\{\Delta(\varrho), \Delta(\sigma), \ell\} = b, \min\{\Psi(\varrho), \Psi(\sigma), \ell\} = b$  and  $\max\{\mathcal{U}(\varrho), \mathcal{U}(\sigma), \ell\} = \ell$ . That is  $\max\{\Delta(\varrho\tau\sigma), b\} \geq \min\{\Delta(\varrho), \Delta(\sigma), \ell\}, \max\{\Psi(\varrho\tau\sigma), b\} \geq \min\{\Psi(\varrho), \Psi(\sigma), \ell\}$  and  $\min\{\mathcal{U}(\varrho\tau\sigma), b\} \leq \max\{\mathcal{U}(\varrho), \mathcal{U}(\sigma), \ell\}$ . Therefore  $\xi$  is a  $(b, \ell)$  NSS of  $\mathcal{Z}$ .

Conversely assume that  $\xi = [\Delta, \Psi, \mathcal{U}]$  is a  $(b, \ell)$ -NSS of  $\mathcal{Z}$ . Let  $\varrho, \sigma \in (M)$ . Then  $\Delta(\varrho) \geq \ell, \Delta(\sigma) \geq \ell, \Psi(\varrho) \geq \ell, \Psi(\sigma) \geq \ell$  and  $\mathcal{U}(\varrho) \leq \ell, \mathcal{U}(\sigma) \leq \ell$ . Now  $\xi = [\Delta, \Psi, \mathcal{U}]$  is a  $(b, \ell)$ -NSS of  $\mathcal{Z}$ . Therefore  $\max\{\Delta(\varrho\tau\sigma), b\} \geq \min\{\Delta(\varrho), \Delta(\sigma), \ell\} \geq \min\{\ell, \ell, \ell\} = \ell, \max\{\Psi(\varrho\tau\sigma), b\} \geq \min\{\Psi(\varrho), \Psi(\sigma), \ell\} \geq \min\{\ell, \ell, \ell\} = \ell$  and  $\min\{\mathcal{U}(\varrho\tau\sigma), b\} \leq \max\{\mathcal{U}(\varrho), \mathcal{U}(\sigma), \ell\} \leq \max\{\ell, \ell, \ell\} = \ell$ . It follows that  $\varrho\tau\sigma \in (M)$ . Therefore  $M$  is a subsemigroup of  $\mathcal{Z}$ .

**Theorem 3.7.** A subset  $\xi = [\Delta, \Psi, \mathcal{U}]$  is a  $(b, \ell)$  – NSS[NLI, NRI, NBI,  $(1, 2)$  – ideal] of  $\mathcal{Z}$  if and only if each non-empty level subset  $\xi_t$  is a subsemigroup [LI, RI, BI,  $(1, 2)$  – ideal] of  $\mathcal{Z}$  for all  $t \in (b, \ell]$ .

**Proof.** Assume that  $\xi_t$  is a subsemigroup of  $\mathcal{Z}$  for each  $t \in [0, 1]$ . Let  $t = \min\{\Delta(\varrho_1), \Delta(\varrho_2)\}$ . Then  $\varrho_1, \varrho_2 \in \Delta_t$  for each  $\varrho_1, \varrho_2 \in \mathcal{Z}$ . Thus  $\max\{\Delta(\varrho_1\tau\varrho_2), b\} \geq t = \min\{\Delta(\varrho_1), \Delta(\varrho_2), \ell\}$ . Let  $t = \min\{\Psi(\varrho_1), \Psi(\varrho_2)\}$ . Then  $\varrho_1, \varrho_2 \in \Psi_t$  for each  $\varrho_1, \varrho_2 \in \mathcal{Z}$ . Thus  $\max\{\Psi(\varrho_1\tau\varrho_2), b\} \geq t = \min\{\Psi(\varrho_1), \Psi(\varrho_2), \ell\}$ . Let  $t = \max\{\mathcal{U}(\varrho_1), \mathcal{U}(\varrho_2)\}$ . Then  $\varrho_1, \varrho_2 \in \mathcal{U}_t$  for each  $\varrho_1, \varrho_2 \in \mathcal{Z}$ . Thus  $\min\{\mathcal{U}(\varrho_1\tau\varrho_2), b\} \leq t = \max\{\mathcal{U}(\varrho_1), \mathcal{U}(\varrho_2), \ell\}$ . This shows that  $\xi^t$  is NSS of  $\mathcal{Z}$ .

Conversely, assume that  $\xi^t$  is a NSS of  $\mathcal{Z}$ . For each  $t \in [0, 1]$  and  $\varrho_1, \varrho_2 \in \Delta_t$ . We have  $\Delta(\varrho_1) \geq t, \Delta(\varrho_2) \geq t$ . Since  $\Delta$  is a subsemigroup of  $\mathcal{Z}$ ,  $\max\{\Delta(\varrho_1\tau\varrho_2), b\} \geq \min\{\Delta(\varrho_1), \Delta(\varrho_2), \ell\} \geq t$ . This implies that  $\varrho_1\tau\varrho_2 \in \Delta_t$ . We have  $\Psi(\varrho_1) \geq t, \Psi(\varrho_2) \geq t$ . Since  $\Psi$  is a subsemigroup of  $\mathcal{Z}$ ,  $\max\{\Psi(\varrho_1\tau\varrho_2), b\} \geq \min\{\Psi(\varrho_1), \Psi(\varrho_2), \ell\} \geq t$ . This implies that  $\varrho_1\tau\varrho_2 \in \Psi_t$ . We have  $\mathcal{U}(\varrho_1) \leq t, \mathcal{U}(\varrho_2) \leq t$ . Since  $\mathcal{U}$  is a subsemigroup of  $\mathcal{Z}$ ,  $\min\{\mathcal{U}(\varrho_1\tau\varrho_2), b\} \leq \max\{\mathcal{U}(\varrho_1), \mathcal{U}(\varrho_2), \ell\} \leq t$ . This implies that  $\varrho_1\tau\varrho_2 \in \mathcal{U}_t$ . Therefore  $\xi_t$  is a subsemigroup of  $\mathcal{Z}$  for each  $t \in (b, \ell]$ . Similar proofs holds.

**Example 3.8.** Every NSS  $\xi$  of  $\mathcal{Z}$  is a  $(b, \ell)$ -NSS of  $\mathcal{Z}$ , but converse is not true.

For the Example 3.2, we define subset  $\xi$  by

$$\Delta(\epsilon) = \begin{cases} 0.80 & \text{if } \epsilon = \epsilon_1 \\ 0.73 & \text{if } \epsilon = \epsilon_2 \\ 0.63 & \text{if } \epsilon = \epsilon_3 \\ 0.68 & \text{if } \epsilon = \epsilon_4 \end{cases} \quad \Psi(\epsilon) = \begin{cases} 0.45 & \text{if } \epsilon = \epsilon_1 \\ 0.38 & \text{if } \epsilon = \epsilon_2 \\ 0.28 & \text{if } \epsilon = \epsilon_3 \\ 0.33 & \text{if } \epsilon = \epsilon_4 \end{cases} \quad \mathcal{U}(\epsilon) = \begin{cases} 0.65 & \text{if } \epsilon = \epsilon_1 \\ 0.70 & \text{if } \epsilon = \epsilon_2 \\ 0.80 & \text{if } \epsilon = \epsilon_3 \\ 0.75 & \text{if } \epsilon = \epsilon_4 \end{cases}$$

Then  $\xi$  is a  $(0.71, 0.85)$ -NSS of  $\mathcal{Z}$ , but not a NSS. Since  $\Delta(\epsilon_4\tau\epsilon_4) = 0.63 \not\geq \min\{\Delta(\epsilon_4, q), \Delta(\epsilon_4, q)\} = 0.68$ ,  $\Psi(\epsilon_4\tau\epsilon_4) = 0.28 \not\geq \min\{\Delta(\epsilon_4, q), \Delta(\epsilon_4, q)\} = 0.33$  and  $\mathcal{U}(\epsilon_4\tau\epsilon_4) = 0.80 \not\leq \max\{\mathcal{U}(\epsilon_4, q), \mathcal{U}(\epsilon_4, q)\} = 0.75$ .

**Example 3.9.** Every NBI  $\xi = [\Delta, \Psi, \mathcal{U}]$  of  $\mathcal{Z}$  is a  $(b, \ell)$ -NBI of  $\mathcal{Z}$ , but converse need not be true by the following Example.

For the Example 3.4, we define subset  $\xi$  by,

$$\Delta(\epsilon) = \begin{cases} 0.94 & \text{if } \epsilon = \epsilon_1 \\ 0.75 & \text{if } \epsilon = \epsilon_2 \\ 0.47 & \text{if } \epsilon = \epsilon_3 \\ 0.56 & \text{if } \epsilon = \epsilon_4 \end{cases} \quad \Psi(\epsilon) = \begin{cases} 0.92 & \text{if } \epsilon = \epsilon_1 \\ 0.73 & \text{if } \epsilon = \epsilon_2 \\ 0.45 & \text{if } \epsilon = \epsilon_3 \\ 0.54 & \text{if } \epsilon = \epsilon_4 \end{cases} \quad \mathcal{U}(\epsilon) = \begin{cases} 0.60 & \text{if } \epsilon = \epsilon_1 \\ 0.75 & \text{if } \epsilon = \epsilon_2 \\ 0.85 & \text{if } \epsilon = \epsilon_3 \\ 0.80 & \text{if } \epsilon = \epsilon_4 \end{cases}$$

Then  $\xi$  is a  $(0.70, 0.95)$ NBI, but not a NBI. Since  $\Delta(\epsilon_4\tau_1\epsilon_4\tau_2\epsilon_4) = \Delta(\epsilon_3) = 0.47 \not\geq \min\{\Delta(\epsilon_4), \Delta(\epsilon_4)\} = 0.56$ ,  $\Psi(\epsilon_4\tau_1\epsilon_4\tau_2\epsilon_4) = \Psi(\epsilon_3) = 0.45 \not\geq \min\{\Delta(\epsilon_4), \Delta(\epsilon_4)\} = 0.54$  and  $\mathcal{U}(\epsilon_4\tau_1\epsilon_4\tau_2\epsilon_4) = \mathcal{U}(\epsilon_3) = 0.85 \not\leq \max\{\Delta(\epsilon_4), \Delta(\epsilon_4)\} = 0.80$ .

**Definition 3.10.** If  $\chi_M$  is the characteristic function of  $M$ , then  $(\chi_M)_b^\ell$  is defined as

$$(\chi_M^T)_b^\ell(\varrho) = \begin{cases} \ell & \text{if } \varrho \in (M) \\ b & \text{if } \varrho \notin (M) \end{cases} \quad (\chi_M^I)_b^\ell(\varrho) = \begin{cases} \ell & \text{if } \varrho \in (M) \\ b & \text{if } \varrho \notin (M) \end{cases} \quad (\chi_M^F)_b^\ell(\varrho) = \begin{cases} b & \text{if } \varrho \in (M) \\ \ell & \text{if } \varrho \notin (M) \end{cases}$$

**Theorem 3.11.** A non empty subset  $M$  of  $\mathcal{Z}$  is a subsemigroup  $[LI, RI, BI]$  of  $\mathcal{Z}$  if and only if subset  $\chi_{(M)}$  is a  $(b, \ell)$ -NSS $[NLI, NRI, NBI]$  of  $\mathcal{Z}$ .

**Proof.** Assume that  $M$  is a subsemigroup of  $\mathcal{Z}$ . Then  $\chi_{(M)}$  is a NSS of  $\mathcal{Z}$  and hence  $\chi_{(M)}$  is an  $(b, \ell)$ -NSS of  $\mathcal{Z}$ .

Conversely, Let  $\chi_{(M)}$  is an  $(b, \ell)$ -NSS of  $\mathcal{Z}$ . Let  $\varrho, \sigma \in \mathcal{Z}$  be such that  $\varrho, \sigma \in (M)$ . Then  $\chi_{(M)}^T(\varrho) = \ell = \chi_{(M)}^T(\sigma) = \ell$  Since  $\chi_{(M)}^T$  is a  $(b, \ell)$ NSS. Consider

$$\begin{aligned} \max\{\chi_{(M)}^T(\varrho\tau\sigma), b\} &\geq \min\{\chi_{(M)}^T(\varrho), \chi_{(M)}^T(\sigma), \ell\} \\ &= \min\{\ell, \ell, \ell\} \\ &= \ell \end{aligned}$$

as  $b < \ell$ , this implies that  $\chi_{(M)}^T(\varrho\tau\sigma) \geq \ell$ . Thus  $\varrho\tau\sigma \in (M)$ . Thus  $\varrho\tau\sigma \in (M)$ .

Let  $\varrho, \sigma \in \mathcal{Z}$  be such that  $\varrho, \sigma \in (M)$ . Then  $\chi_{(M)}^I(\varrho) = \ell = \chi_{(M)}^I(\sigma) = \ell$ . Since  $\chi_{(M)}^I$  is a  $(b, \ell)$ NSS. Consider

$$\begin{aligned} \max\{\chi_{(M)}^I(\varrho\tau\sigma), b\} &\geq \min\{\chi_{(M)}^I(\varrho), \chi_{(M)}^I(\sigma), \ell\} \\ &= \min\{\ell, \ell, \ell\} \\ &= \ell \end{aligned}$$

as  $b < \ell$ , this implies that  $\chi_{(M)}^I(\varrho\tau\sigma) \geq \ell$ . Thus  $\varrho\tau\sigma \in (M)$ . Thus  $\varrho\tau\sigma \in (M)$ .

Let  $\varrho, \sigma \in \mathcal{Z}$  be such that  $\varrho, \sigma \in (M)$ . Then  $\chi_{(M)}^F(\varrho) = b = \chi_{(M)}^F(\sigma) = b$ . Since  $\chi_{(M)}^F$  is a  $(b, \ell)$ NSS. Consider

$$\begin{aligned} \min\{\chi_{(M)}^F(\varrho\tau\sigma), b\} &\leq \max\{\chi_{(M)}^F(\varrho), \chi_{(M)}^F(\sigma), \ell\} \\ &= \max\{b, b, \ell\} \\ &= \ell \end{aligned}$$

as  $b < \ell$ , this implies that  $\chi_{(M)}^F(\varrho\tau\sigma) \leq b$ . Thus  $\varrho\tau\sigma \in (M)$ . Thus  $\varrho\tau\sigma \in (M)$ .

Therefore  $M$  is a subsemigroup of  $\mathcal{Z}$ .

Let  $\varrho, \sigma \in \mathcal{Z}$  be such that  $\varrho, \sigma \notin (M)$ . Then  $\chi_{(M)}^T(\varrho) = b = \chi_{(M)}^T(\sigma) = b$ . Since  $\chi_{(M)}^T$  is a  $(b, \ell)$ NSS.

$$\begin{aligned} \max\{\chi_{(M)}^T(\varrho\tau\sigma), b\} &\geq \min\{\chi_{(M)}^T(\varrho), \chi_{(M)}^T(\sigma), \ell\} \\ &= \min\{b, b, \ell\} \\ &= b \end{aligned}$$

as  $b < \ell$ , this implies that  $\chi_{(M)}^T(\varrho\tau\sigma) \geq b$ . Thus  $\varrho\tau\sigma \notin (M)$ .

Let  $\varrho, \sigma \in \mathcal{Z}$  be such that  $\varrho, \sigma \notin (M)$ . Then  $\chi_{(M)}^I(\varrho) = b = \chi_{(M)}^I(\sigma) = b$ . Since  $\tilde{\chi}_{(M)}^I$  is a  $(b, \ell)$ NSS.

$$\begin{aligned} \max\{\chi_{(M)}^I(\varrho\tau\sigma), b\} &\geq \min\{\chi_{(M)}^I(\varrho), \chi_{(M)}^I(\sigma), \ell\} \\ &= \min\{b, b, \ell\} \\ &= b \end{aligned}$$

as  $b < \ell$ , this implies that  $\chi_{(M)}^I(\varrho\tau\sigma) \geq b$ . Thus  $\varrho\tau\sigma \notin (M)$ .

Let  $\varrho, \sigma \in \mathcal{Z}$  be such that  $\varrho, \sigma \notin (M)$ . Then  $\chi_{(M)}^F(\varrho) = \ell = \chi_{(M)}^F(\sigma) = \ell$ . Since  $\tilde{\chi}_{(M)}^F$  is a  $(b, \ell)$ NSS.

$$\begin{aligned} \min\{\chi_{(M)}^F(\varrho\tau\sigma), b\} &\leq \max\{\chi_{(M)}^F(\varrho), \chi_{(M)}^F(\sigma), \ell\} \\ &= \max\{\ell, \ell, \ell\} \\ &= \ell \end{aligned}$$

as  $b < \ell$ , this implies that  $\chi_{(M)}^F(\varrho\tau\sigma) \leq \ell$ . Thus  $\varrho\tau\sigma \notin (M)$ . Therefore  $M$  is a subsemigroup of  $\mathcal{Z}$ . Similar to proof holds.

**Definition 3.12.** For two neutrosophic subsets  $\xi$  and  $\lambda$  of  $\mathcal{Z}$ , their product  $\xi \circ \lambda$  is defined as

$$(\xi^T \circ \lambda^T)(\varrho) = \begin{cases} \sup_{(s,t) \in M_\varrho} \{\xi^T(s) \wedge \lambda^T(t)\} & \text{if } M_\varrho \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$(\xi^I \circ \lambda^I)(\varrho) = \begin{cases} \sup_{(s,t) \in M_\varrho} \{\xi^I(s) \wedge \lambda^I(t)\} & \text{if } M_\varrho \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$(\xi^F \circ \lambda^F)(\varrho) = \begin{cases} \inf_{(s,t) \in M_\varrho} \{\xi^F(s) \vee \lambda^F(t)\} & \text{if } M_\varrho \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

**Definition 3.13.** Let  $\xi$  be subset of  $\mathcal{Z}$ , we define the subset  $(\Delta)_b^\ell(\varrho) = \{\Delta(\varrho) \wedge \ell\} \vee b$ ,  $(\Psi)_b^\ell(\varrho) = \{\Psi(\varrho) \wedge \ell\} \vee b$  and  $(\cup)_b^\ell(\varrho) = \{\cup(\varrho) \vee \ell\} \wedge b$ , for all  $\varrho \in \mathcal{Z}$ .

**Lemma 3.14.** Let  $M$  and  $N$  be non-empty subsets of  $\mathcal{Z}$ . Then the following hold:

1.  $(\chi_{(M)} \wedge_b^\ell \chi_{(N)}) = (\chi_{(M \cap N)})_b^\ell$ ,
2.  $(\chi_{(M)} \vee_b^\ell \chi_{(N)}) = (\chi_{(M \cup N)})_b^\ell$ ,

$$3. (\chi_{(M)} \circ_{\Gamma}^{\ell} \chi_{(N)}) = (\chi_{(M\Gamma N)})_{\mathfrak{b}}^{\ell}.$$

**Proof.** (i) and (ii) Straightforward.

(iii) Let  $\varrho \in \mathcal{Z}$ . If  $\varrho \in (M\Gamma N)$ , then  $(\chi_{(M\Gamma N)})(\varrho) = \ell$ .

Since  $\varrho \leq a\tau b$  for some  $a \in (M)$ ,  $b \in (N)$  and  $\tau \in \Gamma$ , we have  $(a, b) \in M_{\varrho}$  and  $M_{\varrho} \neq \emptyset$ .

$$\begin{aligned} (\chi_{(M)}^T \circ_{\Gamma} \chi_{(N)}^T)(\varrho) &= \sup_{\varrho=y\tau z} \min\{\chi_{(M)}^T(y), \chi_{(N)}^T(z)\} \\ &\geq \min\{\chi_{(M)}^T(a), \chi_{(N)}^T(b)\} \\ &= \ell \end{aligned}$$

$$\begin{aligned} (\chi_{(M)}^I \circ_{\Gamma} \chi_{(N)}^I)(\varrho) &= \sup_{\varrho=y\tau z} \min\{\chi_{(M)}^I(y), \chi_{(N)}^I(z)\} \\ &\geq \min\{\chi_{(M)}^I(a), \chi_{(N)}^I(b)\} \\ &= \ell \end{aligned}$$

$$\begin{aligned} (\chi_{(M)}^F \circ_{\Gamma} \chi_{(N)}^F)(\varrho) &= \inf_{\varrho=y\tau z} \max\{\chi_{(M)}^F(y), \chi_{(N)}^F(z)\} \\ &\leq \max\{\chi_{(M)}^F(a), \chi_{(N)}^F(b)\} \\ &= \mathfrak{b} \end{aligned}$$

Therefore  $(\chi_{(M)} \circ_{\Gamma} \chi_{(N)})(\varrho) = (\chi_{(M\Gamma N)})(\varrho)$ .

If  $\varrho \notin (M\Gamma N)$  then  $(\chi_{(M\Gamma N)}^T)(\varrho) = \mathfrak{b}$ ,  $(\chi_{(M\Gamma N)}^I)(\varrho) = \mathfrak{b}$  and  $(\chi_{(M\Gamma N)}^F)(\varrho) = \ell$ . Since  $\varrho \leq a\tau b$  for some  $a \notin (M)$ ,  $b \notin (N)$  and  $\tau \in \Gamma$ . We have

$$\begin{aligned} (\chi_{(M)}^T \circ_{\Gamma} \chi_{(N)}^T)(\varrho) &= \sup_{\varrho=y\tau z} \min\{\chi_{(M)}^T(y), \chi_{(N)}^T(z)\} \\ &\geq \min\{\chi_{(M)}^T(a), \chi_{(N)}^T(b)\} \\ &= \mathfrak{b} \end{aligned}$$

$$\begin{aligned} (\chi_{(M)}^I \circ_{\Gamma} \chi_{(N)}^I)(\varrho) &= \sup_{\varrho=y\tau z} \min\{\chi_{(M)}^I(y), \chi_{(N)}^I(z)\} \\ &\geq \min\{\chi_{(M)}^I(a), \chi_{(N)}^I(b)\} \\ &= \mathfrak{b} \end{aligned}$$

$$\begin{aligned} (\chi_{(M)}^F \circ_{\Gamma} \chi_{(N)}^F)(\varrho) &= \inf_{\varrho=y\tau z} \max\{\chi_{(M)}^F(y), \chi_{(N)}^F(z)\} \\ &\leq \max\{\chi_{(M)}^F(a), \chi_{(N)}^F(b)\} \\ &= \ell \end{aligned}$$

Hence  $(\chi_{(M)} \circ_{\Gamma} \chi_{(N)})(\varrho) = (\chi_{(M\Gamma N)})(\varrho)$ .

**Theorem 3.15.** For  $M, N \subseteq \mathcal{Z}$  and  $\{M_i | i \in I\}$  be a family of subsets of  $\mathcal{Z}$  then

(i)  $(M) \subseteq (N)$  if and only if  $(\chi_{(M)})_{\mathfrak{b}}^{\ell} \leq (\chi_{(N)})_{\mathfrak{b}}^{\ell}$ .

(ii)  $(\bigcap_{i \in I} \chi_{(M_i)})_{\mathfrak{b}}^{\ell} = (\chi_{\bigcap_{i \in I} (M_i)})_{\mathfrak{b}}^{\ell}$ .

(iii)  $(\bigcup_{i \in I} \chi_{(M_i)})_{\mathfrak{b}}^{\ell} = (\chi_{\bigcup_{i \in I} (M_i)})_{\mathfrak{b}}^{\ell}$ .

**Theorem 3.16.** If  $M$  is a  $(\mathfrak{b}, \ell)$ -NLI[NSS, NRI] of  $\mathcal{Z}$ , then  $(M)_{\mathfrak{b}}^{\ell}$  is a NLI[NSS, NRI] of  $\mathcal{Z}$ .

**Proof.** Assume that  $M$  is a  $(\mathfrak{b}, \ell)$ NLI of  $\mathcal{Z}$ . If there exist  $\varrho, \sigma \in \mathcal{Z}$  and  $\tau \in \Gamma$ . Now

$$\begin{aligned} \max\{(\Delta)_{\mathfrak{b}}^{\ell}(\varrho\tau\sigma), \mathfrak{b}\} &= \max\{(\{\Delta(\varrho\tau\sigma) \wedge \ell\} \vee \mathfrak{b}), \mathfrak{b}\} \\ &= \{\Delta(\varrho\tau\sigma) \wedge \ell\} \vee \mathfrak{b} \\ &= \{\Delta(\varrho\tau\sigma) \vee \mathfrak{b}\} \wedge \{\ell \vee \mathfrak{b}\} \\ &= \{(\Delta(\varrho\tau\sigma) \vee \mathfrak{b}) \vee \mathfrak{b}\} \wedge \ell \\ &\geq \{(\Delta(\sigma) \wedge \ell) \vee \mathfrak{b}\} \wedge \ell \\ &= \{(\Delta(\sigma) \wedge \ell) \wedge \ell\} \vee (\mathfrak{b} \wedge \ell) \\ &= \{(\Delta(\sigma) \wedge \ell) \wedge \ell\} \vee \mathfrak{b} \\ &\geq (\Delta)_{\mathfrak{b}}^{\ell}(\sigma) \wedge \ell. \end{aligned}$$

$$\begin{aligned}
 \max\{(\Psi)_b^\ell(\varrho\tau\sigma), b\} &= \max\{(\{\Psi(\varrho\tau\sigma) \wedge \ell\} \vee b), b\} \\
 &= \{\Psi(\varrho\tau\sigma) \wedge \ell\} \vee b \\
 &= \{\Psi(\varrho\tau\sigma) \vee b\} \wedge \{\ell \vee b\} \\
 &= \{(\Psi(\varrho\tau\sigma) \vee b) \vee b\} \wedge \ell \\
 &\geq \{(\Psi(\sigma) \wedge \ell) \vee b\} \wedge \ell \\
 &= \{(\Psi(\sigma) \wedge \ell) \wedge \ell\} \vee (b \wedge \ell) \\
 &= \{(\Psi(\sigma) \wedge \ell) \wedge \ell\} \vee b \\
 &\geq (\Psi)_b^\ell(\sigma) \wedge \ell.
 \end{aligned}$$

$$\begin{aligned}
 \min\{(\mathcal{U})_b^\ell(\varrho\tau\sigma), b\} &= \min\{(\{\mathcal{U}(\varrho\tau\sigma) \vee \ell\} \wedge b), b\} \\
 &= \{\mathcal{U}(\varrho\tau\sigma) \vee \ell\} \wedge b \\
 &= \{\mathcal{U}(\varrho\tau\sigma) \wedge b\} \vee \{\ell \wedge b\} \\
 &= \{(\mathcal{U}(\varrho\tau\sigma) \wedge b) \wedge b\} \vee b \\
 &\leq \{(\mathcal{U}(\sigma) \vee \ell) \wedge b\} \vee b \\
 &= \{(\mathcal{U}(\sigma) \vee \ell) \vee \ell\} \wedge b \\
 &\leq (\mathcal{U})_b^\ell(\sigma) \vee \ell
 \end{aligned}$$

Hence  $M = [(\Delta)_b^\ell, (\Psi)_b^\ell, (\mathcal{U})_b^\ell]$  is a NLI of  $\mathcal{Z}$ .

**Theorem 3.17.** Let  $M$  be an  $(b, \ell)$ NRI and  $N$  be an  $(b, \ell)$ NLI of  $\mathcal{Z}$  then  $((M \circ_{\Gamma} N))_b^\ell \subseteq (M \cap_b^\ell N)$ .

**Proof.** Let  $M = [\Delta_M, \Psi_M, \mathcal{U}_M]$  be an  $(b, \ell)$ NRI and  $N = [\Delta_N, \Psi_N, \mathcal{U}_N]$  be an  $(b, \ell)$ NLI of  $\mathcal{Z}$ . Let  $(\varrho, \sigma) \in I_{\varpi}$ . If  $I_{\varpi} \neq \emptyset$ , then  $\varpi \leq \varrho\tau\sigma$ . Thus  $\Delta_M(\varpi) \geq \Delta_M(\varrho\tau\sigma) \geq \Delta_M(\varrho)$ ,  $\Psi_M(\varpi) \geq \Psi_M(\varrho\tau\sigma) \geq \Psi_M(\varrho)$  and  $\mathcal{U}_M(\varpi) \leq \mathcal{U}_M(\varrho\tau\sigma) \leq \mathcal{U}_M(\varrho)$ . Similarly  $\Delta_N(\varpi) \geq \Delta_N(\varrho\tau\sigma) \geq \Delta_N(\varrho)$ ,  $\Psi_N(\varpi) \geq \Psi_N(\varrho\tau\sigma) \geq \Psi_N(\varrho)$  and  $\mathcal{U}_N(\varpi) \leq \mathcal{U}_N(\varrho\tau\sigma) \leq \mathcal{U}_N(\varrho)$ . We have

$$\begin{aligned}
 (\Delta_{(M \circ_{\Gamma} N)})_b^\ell(\varpi) &= (\Delta_{(M \circ_{\Gamma} N)}(\varpi) \wedge \ell) \vee b \\
 &= \left[ \sup_{\varpi \leq \varrho\tau\sigma} \{\Delta_M(\varrho) \wedge \Delta_N(\sigma)\} \wedge \ell \right] \vee b \\
 &= \left[ \sup_{\varpi \leq \varrho\tau\sigma} \{\Delta_M(\varrho) \wedge \Delta_N(\sigma)\} \wedge \ell \wedge \ell \right] \vee b \\
 &= \left[ \sup_{\varpi \leq \varrho\tau\sigma} \{(\Delta_M(\varrho) \wedge \ell) \wedge (\Delta_N(\sigma) \wedge \ell)\} \wedge \ell \right] \vee b \\
 &\leq (\{(\Delta_M(\varpi) \vee b) \wedge (\Delta_N(\varpi) \vee b)\} \wedge \ell) \vee b \\
 &= \{((\Delta_M(\varpi) \wedge \Delta_N(\varpi)) \vee b) \wedge \ell\} \vee b \\
 &= \{((\Delta_M \wedge \Delta_N)(\varpi) \wedge \ell)\} \vee b \\
 &= (\Delta_{M \cap_b^\ell N})(\varpi)
 \end{aligned}$$

$$\begin{aligned}
 (\Psi_{(M \circ_{\Gamma} N)})_b^\ell(\varpi) &= (\Psi_{(M \circ_{\Gamma} N)}(\varpi) \wedge \ell) \vee b \\
 &= \left[ \sup_{\varpi \leq \varrho\tau\sigma} \{\Psi_M(\varrho) \wedge \Psi_N(\sigma)\} \wedge \ell \right] \vee b \\
 &= \left[ \sup_{\varpi \leq \varrho\tau\sigma} \{\Psi_M(\varrho) \wedge \Psi_N(\sigma)\} \wedge \ell \wedge \ell \right] \vee b \\
 &= \left[ \sup_{\varpi \leq \varrho\tau\sigma} \{(\Psi_M(\varrho) \wedge \ell) \wedge (\Psi_N(\sigma) \wedge \ell)\} \wedge \ell \right] \vee b \\
 &\leq (\{\Psi_M(\varpi) \vee b\} \wedge \{\Psi_N(\varpi) \vee b\}) \wedge \ell \vee b \\
 &= \{((\Psi_M(\varpi) \wedge \Psi_N(\varpi)) \vee b) \wedge \ell\} \vee b \\
 &= \{((\Psi_M \wedge \Psi_N)(\varpi) \wedge \ell)\} \vee b \\
 &= (\Psi_{M \cap_b^\ell N})(\varpi)
 \end{aligned}$$

$$\begin{aligned}
 (\mathcal{U}_{(M \circ_{\Gamma} N]})_b^{\ell}(\varpi) &= (\mathcal{U}_{(M \circ_{\Gamma} N]}(\varpi) \vee \ell) \wedge b \\
 &= \left[ \inf_{\varpi \leq \varrho\tau\sigma} \{ \mathcal{U}_M(\varrho) \vee \mathcal{U}_N(\sigma) \} \vee \ell \right] \wedge b \\
 &= \left[ \inf_{\varpi \leq \varrho\tau\sigma} \{ \mathcal{U}_M(\varrho) \vee \mathcal{U}_N(\sigma) \} \vee \ell \vee \ell \right] \wedge b \\
 &= \left[ \inf_{\varpi \leq \varrho\tau\sigma} \{ (\mathcal{U}_M(\varrho) \vee \ell) \vee (\mathcal{U}_N(\sigma) \vee \ell) \} \vee \ell \right] \wedge b \\
 &\geq \{ (\mathcal{U}_M(\varpi) \wedge b) \vee (\mathcal{U}_N(\varpi) \wedge b) \} \vee \ell \wedge b \\
 &= \{ (\mathcal{U}_M(\varpi) \vee \mathcal{U}_N(\varpi)) \wedge b \} \vee \ell \wedge b \\
 &= \{ (\mathcal{U}_M \vee \mathcal{U}_N)(\varpi) \vee \ell \} \wedge b \\
 &= (\mathcal{U}_{M \cup N^{\ell}})(\varpi)
 \end{aligned}$$

Let  $\varrho, \sigma \notin I_{\varpi}$ . If  $I_{\varpi} = \emptyset$ , then  $(\Delta_M \circ_{\Gamma} \Delta_N)(\varpi) = 0$ ,  $(\Psi_M \circ_{\Gamma} \Psi_N)(\varpi) = 0$  and  $(\mathcal{U}_M \circ_{\Gamma} \mathcal{U}_N)(\varpi) = 1$  and  $\tau \in \Gamma$  such that  $\varpi \leq \varrho\tau\sigma$ .

$$\begin{aligned}
 (\Delta_{(M \circ_{\Gamma} N]})_b^{\ell}(\varpi) &= (\Delta_{(M \circ_{\Gamma} N]}(\varpi) \wedge \ell) \vee b \\
 &= 0 \vee b \\
 &\leq (\Delta_{M \cap N}(\varpi) \wedge \ell) \vee b \\
 &= (\Delta_{M \cap N}(\varpi) \wedge \ell)
 \end{aligned}$$

$$\begin{aligned}
 (\Psi_{(M \circ_{\Gamma} N]})_b^{\ell}(\varpi) &= (\Psi_{(M \circ_{\Gamma} N]}(\varpi) \wedge \ell) \vee b \\
 &= 0 \vee b \\
 &\leq (\Psi_{M \cap N}(\varpi) \wedge \ell) \vee b \\
 &= (\Psi_{M \cap N}(\varpi) \wedge \ell)
 \end{aligned}$$

$$\begin{aligned}
 (\mathcal{U}_{(M \circ_{\Gamma} N]})_b^{\ell}(\varpi) &= (\mathcal{U}_{(M \circ_{\Gamma} N]}(\varpi) \vee \ell) \wedge b \\
 &= 1 \wedge b \\
 &= b \\
 &\geq (\mathcal{U}_{M \cup N}(\varpi) \vee \ell) \wedge b \\
 &= (\mathcal{U}_{M \cup N}(\varpi) \vee \ell)
 \end{aligned}$$

Therefore  $((M \circ_{\Gamma} N]_b^{\ell}) \subseteq (M \cap_b^{\ell} N]$ .

**Theorem 3.18.** An ordered  $\Gamma$ -semigroup  $\mathcal{Z}$  is regular,  $M$  be an  $(b, \ell)$ NRI and  $N$  be an  $(b, \ell)$ NLI of  $\mathcal{Z}$  if and only if  $((M \circ_{\Gamma} N]_b^{\ell}) = (M \cap_b^{\ell} N]$ .

**Proof.** Let  $\mathcal{Z}$  be an ordered  $\Gamma$ -regular semigroup and  $M$  be an  $(b, \ell)$ NRI and  $N$  be an  $(b, \ell)$ NLI of  $\mathcal{Z}$ . Let  $(\varrho, \sigma) \in I_{\varpi}$ . If  $I_{\varpi} \neq \emptyset$ , then  $\varpi \leq \varrho\tau\sigma$ . Thus  $\Delta_M(\varpi) \geq \Delta_M(\varrho\tau\sigma) \geq \Delta_M(\varrho)$ ,  $\Psi_M(\varpi) \geq \Psi_M(\varrho\tau\sigma) \geq \Psi_M(\varrho)$  and  $\mathcal{U}_M(\varpi) \leq \mathcal{U}_M(\varrho\tau\sigma) \leq \mathcal{U}_M(\varrho)$ . Similarly  $\Delta_N(\varpi) \geq \Delta_N(\varrho\tau\sigma) \geq \Delta_N(\varrho)$ ,  $\Psi_N(\varpi) \geq \Psi_N(\varrho\tau\sigma) \geq \Psi_N(\varrho)$  and  $\mathcal{U}_N(\varpi) \leq \mathcal{U}_N(\varrho\tau\sigma) \leq \mathcal{U}_N(\varrho)$ . For  $\varpi \in \mathcal{Z}$ , there exists  $x \in \mathcal{Z}$  such that  $\varpi \leq (\varpi\tau x)\beta\varpi$ . Then  $(\varpi\tau x), \varpi \in I_{\varpi}$ . We have

$$\begin{aligned}
 (\Delta_{(M \circ_{\Gamma} N]})_b^{\ell}(\varpi) &= (\Delta_{(M \circ_{\Gamma} N]}(\varpi) \wedge \ell) \vee b \\
 &= \left[ \sup_{\varpi \leq \varrho\tau\sigma} \{ \Delta_M(\varrho) \wedge \Delta_N(\sigma) \} \wedge \ell \right] \vee b \\
 &= \left[ \sup_{\varpi \leq \varrho\tau\sigma} \{ \Delta_M(\varrho) \wedge \Delta_N(\sigma) \} \wedge \ell \wedge \ell \right] \vee b \\
 &= \left[ \sup_{\varpi \leq \varrho\tau\sigma} \{ (\Delta_M(\varrho) \wedge \ell) \wedge (\Delta_N(\sigma) \wedge \ell) \} \wedge \ell \right] \vee b \\
 &\geq \{ (\Delta_M(\varpi\tau x) \vee b) \wedge (\Delta_N(\varpi) \vee b) \} \wedge \ell \vee b \\
 &\geq ((\Delta_M(\varpi) \vee b) \wedge (\Delta_N(\varpi) \vee b) \wedge \ell) \vee b \\
 &= \{ ((\Delta_M(\varpi) \wedge \Delta_N(\varpi)) \vee b) \wedge \ell \} \vee b \\
 &= \{ ((\Delta_M \wedge \Delta_N)(\varpi) \wedge \ell) \} \vee b \\
 &= (\Delta_{M \cap_b^{\ell} N})(\varpi)
 \end{aligned}$$

$$\begin{aligned}
 (\Psi_{(M \circ_{\Gamma} N)})_b^{\ell}(\varpi) &= (\Psi_{(M \circ_{\Gamma} N)}(\varpi) \wedge \ell) \vee b \\
 &= \left[ \sup_{\varpi \leq \varrho \tau \sigma} \{ \Psi_M(\varrho) \wedge \Psi_N(\sigma) \} \wedge \ell \right] \vee b \\
 &= \left[ \sup_{\varpi \leq \varrho \tau \sigma} \{ \Psi_M(\varrho) \wedge \Psi_N(\sigma) \} \wedge \ell \wedge \ell \right] \vee b \\
 &= \left[ \sup_{\varpi \leq \varrho \tau \sigma} \{ (\Psi_M(\varrho) \wedge \ell) \wedge (\Psi_N(\sigma) \wedge \ell) \} \wedge \ell \right] \vee b \\
 &\geq (\{ \Psi_M(\varpi \tau x) \vee b \} \wedge \{ \Psi_N(\varpi) \vee b \}) \wedge \ell \vee b \\
 &\geq ((\Psi_M(\varpi) \vee b) \wedge (\Psi_N(\varpi) \vee b) \wedge \ell) \vee b \\
 &= \{ (\Psi_M(\varpi) \wedge \Psi_N(\varpi)) \vee b \} \wedge \ell \vee b \\
 &= \{ (\Psi_M \wedge \Psi_N)(\varpi) \wedge \ell \} \vee b \\
 &= (\Psi_{M \cap_b^{\ell} N})(\varpi) \\
 (\mathcal{U}_{(M \circ_{\Gamma} N)})_b^{\ell}(\varpi) &= (\mathcal{U}_{(M \circ_{\Gamma} N)}(\varpi) \vee \ell) \wedge b \\
 &= \left[ \inf_{\varpi \leq \varrho \tau \sigma} \{ \mathcal{U}_M(\varrho) \vee \mathcal{U}_N(\sigma) \} \vee \ell \right] \wedge b \\
 &= \left[ \inf_{\varpi \leq \varrho \tau \sigma} \{ \mathcal{U}_M(\varrho) \vee \mathcal{U}_N(\sigma) \} \vee \ell \vee \ell \right] \wedge b \\
 &= \left[ \inf_{\varpi \leq \varrho \tau \sigma} \{ (\mathcal{U}_M(\varrho) \vee \ell) \vee (\mathcal{U}_N(\sigma) \vee \ell) \} \vee \ell \right] \wedge b \\
 &\leq (\{ \mathcal{U}_M(\varpi \tau x) \wedge b \} \vee \{ \mathcal{U}_N(\varpi) \wedge b \}) \vee \ell \wedge b \\
 &\leq ((\mathcal{U}_M(\varpi) \wedge b) \vee (\mathcal{U}_N(\varpi) \wedge b) \vee \ell) \wedge b \\
 &= \{ (\mathcal{U}_M(\varpi) \vee \mathcal{U}_N(\varpi)) \wedge b \} \vee \ell \wedge b \\
 &= \{ (\mathcal{U}_M \vee \mathcal{U}_N)(\varpi) \vee \ell \} \wedge b \\
 &= (\mathcal{U}_{M \cap_b^{\ell} N})(\varpi)
 \end{aligned}$$

Thus  $((M \circ_{\Gamma} N))_b^{\ell} \supseteq (M \cap_b^{\ell} N)$  and by Theorem 3.17 and hence  $((M \circ_{\Gamma} N))_b^{\ell} = (M \cap_b^{\ell} N)$ . Conversely assume that  $((M \circ_{\Gamma} N))_b^{\ell} = M \cap_b^{\ell} N$ . Let  $M = (\Delta_M, \Psi_M, \mathcal{U}_M)$  be an  $(b, \ell)$ NRI and  $N = (\Delta_N, \Psi_N, \mathcal{U}_N)$  be an  $(b, \ell)$ NLI of  $\mathcal{Z}$ . Then by Theorem 3.11,  $\chi_M$  is a  $(b, \ell)$ NRI and  $\chi_N$  be a  $(b, \ell)$ NLI of  $\mathcal{Z}$ . By Lemma 3.14 and Theorem 3.15,  $(\chi_{(M \cap_b^{\ell} N)})_b^{\ell} = (\chi_M \cap_b^{\ell} \chi_N) = (\chi_M \circ_{\Gamma} \chi_N)_b^{\ell} = (\chi_{(M \circ_{\Gamma} N)})_b^{\ell}$ . This implies  $(M \cap_b^{\ell} N) = ((M \circ_{\Gamma} N))_b^{\ell}$ . Hence by Corollary 2.9,  $\mathcal{Z}$  is regular.

**Theorem 3.19.** An ordered  $\Gamma$ -semigroup  $\mathcal{Z}$  is regular,  $M$  be an  $(b, \ell)$ NBI and  $N$  be an  $(b, \ell)$ NLI if and only if  $((M \circ_{\Gamma} N))_b^{\ell} = (M \cap_b^{\ell} N)$ .

**Proof.** Let  $\mathcal{Z}$  be an ordered  $\Gamma$ -regular semigroup and  $M$  be an  $(b, \ell)$ NBI and  $N$  be an  $(b, \ell)$ NLI of  $\mathcal{Z}$ . Let  $(\varrho, \sigma) \in I_{\varpi}$ . If  $I_{\varpi} \neq \emptyset$ , then  $\varpi \leq \varrho \tau \sigma$ . Thus  $\Delta_M(\varpi) \geq \Delta_M(\varrho \tau \sigma) \geq \Delta_M(\varrho)$ ,  $\Psi_M(\varpi) \geq \Psi_M(\varrho \tau \sigma) \geq \Psi_M(\varrho)$  and  $\mathcal{U}_M(\varpi) \leq \mathcal{U}_M(\varrho \tau \sigma) \leq \mathcal{U}_M(\varrho)$ . Similarly  $\Delta_N(\varpi) \geq \Delta_N(\varrho \tau \sigma) \geq \Delta_N(\varrho)$ ,  $\Psi_N(\varpi) \geq \Psi_N(\varrho \tau \sigma) \geq \Psi_N(\varrho)$  and  $\mathcal{U}_N(\varpi) \leq \mathcal{U}_N(\varrho \tau \sigma) \leq \mathcal{U}_N(\varrho)$ . For  $\varpi \in \mathcal{Z}$ , there exists  $x \in \mathcal{Z}$  such that  $\varpi \leq \varpi \tau_1 x \tau_2 \varpi = \varpi \tau_1 (x \tau_2 \varpi) \leq (\varpi \tau_1 x \tau_2 \varpi) \tau_1 (x \tau_2 \varpi)$ . Then  $(\varpi \tau_1 x \tau_2 \varpi), (x \tau_2 \varpi) \in I_{\varpi}$ . We have

$$\begin{aligned}
 (\Delta_{M \circ_{\Gamma} N})_b^{\ell}(\varpi) &= (\Delta_{M \circ_{\Gamma} N}(\varpi) \wedge \ell) \vee b \\
 &= \left[ \sup_{\varpi \leq a_1 \tau a_2} \{ \Delta_M(a_1) \wedge \Delta_N(a_2) \} \wedge \ell \right] \vee b \\
 &= \left[ \sup_{\varpi \leq a_1 \tau a_2} \{ \Delta_M(a_1) \wedge \Delta_N(a_2) \} \wedge \ell \wedge \ell \right] \vee b \\
 &= \left[ \sup_{\varpi \leq a_1 \tau a_2} \{ (\Delta_M(a_1) \wedge \ell) \wedge (\Delta_N(a_2) \wedge \ell) \} \wedge \ell \right] \vee b \\
 &\geq (\{ \Delta_M(\varpi \tau_1 x \tau_2 \varpi) \vee b \} \wedge \{ \Delta_N(x \tau_2 \varpi) \vee b \}) \wedge \ell \vee b \\
 &\geq ((\Delta_M(\varpi) \vee b) \wedge (\Delta_N(\varpi) \vee b) \wedge \ell) \vee b \\
 &= \{ (\Delta_M(\varpi) \wedge \Delta_N(\varpi)) \vee b \} \wedge \ell \vee b \\
 &= \{ (\Delta_M \wedge \Delta_N)(\varpi) \wedge \ell \} \vee b \\
 &= (\Delta_{M \cap_b^{\ell} N})(\varpi)
 \end{aligned}$$

$$\begin{aligned}
 (\Psi_{M \circ_{\Gamma} N})_b^\ell(\varpi) &= (\Psi_{M \circ_{\Gamma} N}(\varpi) \wedge \ell) \vee b \\
 &= \left[ \sup_{\varpi \leq a_1 \tau a_2} \{ \Psi_M(a_1) \wedge \Psi_N(a_2) \} \wedge \ell \right] \vee b \\
 &= \left[ \sup_{\varpi \leq a_1 \tau a_2} \{ \Psi_M(a_1) \wedge \Psi_N(a_2) \} \wedge \ell \wedge \ell \right] \vee b \\
 &= \left[ \sup_{\varpi \leq a_1 \tau a_2} \{ (\Psi_M(a_1) \wedge \ell) \wedge (\Psi_N(a_2) \wedge \ell) \} \wedge \ell \right] \vee b \\
 &\geq (\{ \Psi_M(\varpi \tau_1 x \tau_2 \varpi) \vee b \} \wedge \{ \Psi_N(x \tau_2 \varpi) \vee b \}) \wedge \ell \vee b \\
 &\geq ((\Psi_M(\varpi) \vee b) \wedge (\Psi_N(\varpi) \vee b) \wedge \ell) \vee b \\
 &= \{ (\Psi_M(\varpi) \wedge \Psi_N(\varpi)) \vee b \} \wedge \ell \vee b \\
 &= \{ (\Psi_M \wedge \Psi_N)(\varpi) \wedge \ell \} \vee b \\
 &= (\Psi_{M \cap_b^\ell N})(\varpi) \\
 (\mathcal{U}_{M \circ_{\Gamma} N})_b^\ell(\varpi) &= (\mathcal{U}_{M \circ_{\Gamma} N}(\varpi) \vee \ell) \wedge b \\
 &= \left[ \inf_{\varpi \leq a_1 \tau a_2} \{ \mathcal{U}_M(a_1) \vee \mathcal{U}_N(a_2) \} \vee \ell \right] \wedge b \\
 &= \left[ \inf_{\varpi \leq a_1 \tau a_2} \{ \mathcal{U}_M(a_1) \vee \mathcal{U}_N(a_2) \} \vee \ell \vee \ell \right] \wedge b \\
 &= \left[ \inf_{\varpi \leq a_1 \tau a_2} \{ (\mathcal{U}_M(a_1) \vee \ell) \vee (\mathcal{U}_N(a_2) \vee \ell) \} \vee \ell \right] \wedge b \\
 &\leq (\{ \mathcal{U}_M(\varpi \tau_1 x \tau_2 \varpi) \wedge b \} \vee \{ \mathcal{U}_N(x \tau_2 \varpi) \wedge b \}) \vee \ell \wedge b \\
 &\leq ((\mathcal{U}_M(\varpi) \wedge b) \vee (\mathcal{U}_N(\varpi) \wedge b) \vee \ell) \wedge b \\
 &= \{ (\mathcal{U}_M(\varpi) \vee \mathcal{U}_N(\varpi)) \wedge b \} \vee \ell \wedge b \\
 &= \{ (\mathcal{U}_M \vee \mathcal{U}_N)(\varpi) \vee \ell \} \wedge b \\
 &= (\mathcal{U}_{M \cap_b^\ell N})(\varpi_3)
 \end{aligned}$$

Thus  $((M \circ_{\Gamma} N))_b^\ell \supseteq (M \cap_b^\ell N)$  and by Theorem 3.17 and hence  $((M \circ_{\Gamma} N))_b^\ell = (M \cap_b^\ell N)$ . Conversely assume that  $((M \circ_{\Gamma} N))_b^\ell \supseteq (M \cap_b^\ell N)$ . Let  $M$  be an  $(b, \ell)$ NBI and  $N$  be an  $(b, \ell)$ NLI of  $\mathcal{Z}$ . Since every  $(b, \ell)$ NRI of  $\mathcal{Z}$  is a  $(b, \ell)$ NBI of  $\mathcal{Z}$ . By Theorem 3.18,  $\mathcal{Z}$  is regular.

**Theorem 3.20.** Let  $M$  be an  $(b, \ell)$ NRI and  $N$  be an  $(b, \ell)$ NLI of  $\mathcal{Z}$  is weakly regular if and only if  $((M \circ_{\Gamma} N))_b^\ell \supseteq (M \cap_b^\ell N)$ .

**Proof.** Straightforward.

**Theorem 3.21.** Let  $M$  be an  $(b, \ell)$ NBI and  $N$  be an  $(b, \ell)$ NLI of  $\mathcal{Z}$  is weakly regular if and only if  $((M \circ_{\Gamma} N))_b^\ell \supseteq (M \cap_b^\ell N)$ .

**Proof.** Straightforward.

**4  $(\epsilon, \epsilon \vee q)$  Fuzzy ideals**

**Corollary 4.1.** A non-empty subset  $M$  of  $\mathcal{Z}$  is an subsemigroup  $[LI, RI, BI, (1, 2) - ideal]$  of  $\mathcal{Z}$  if and only if the  $\xi$  of  $\mathcal{Z}$  defined as

$$\Delta(\varrho) = \begin{cases} \geq 0.5 \text{ for all } \varrho \in (M) \\ 0 \text{ for all } \varrho \notin (M) \end{cases} \quad \Psi(\varrho) = \begin{cases} \geq 0.5 \text{ for all } \varrho \in (M) \\ 0 \text{ for all } \varrho \notin (M) \end{cases} \quad \mathcal{U}(\varrho) = \begin{cases} \leq 0.5 \text{ for all } \varrho \in (M) \\ 0 \text{ for all } \varrho \notin (M) \end{cases}$$

is a  $(\epsilon, \epsilon \vee q)$ NSS[NLI,NRI,NBI,(1, 2)-ideal]

**Proof.** The proof follows from Theorem 3.6, taking  $b = 0, \ell = 0.5$ .

**Corollary 4.2.** Every  $(\epsilon, \epsilon \vee q)$ NSS of  $\mathcal{Z}$  is a  $(b, \ell)$ NSS of  $\mathcal{Z}$ , but converse need not be true by the following Example.

For the Example 3.2, define the subset  $\xi$  by,

$$\Delta(\epsilon) = \begin{cases} 0.45 & \text{if } \epsilon = \epsilon_1 \\ 0.41 & \text{if } \epsilon = \epsilon_2 \\ 0.29 & \text{if } \epsilon = \epsilon_3 \\ 0.33 & \text{if } \epsilon = \epsilon_4 \end{cases} \quad \Psi(\epsilon) = \begin{cases} 0.44 & \text{if } \epsilon = \epsilon_1 \\ 0.42 & \text{if } \epsilon = \epsilon_2 \\ 0.30 & \text{if } \epsilon = \epsilon_3 \\ 0.34 & \text{if } \epsilon = \epsilon_4 \end{cases} \quad \mathcal{U}(\epsilon) = \begin{cases} 0.30 & \text{if } \epsilon = \epsilon_1 \\ 0.33 & \text{if } \epsilon = \epsilon_2 \\ 0.40 & \text{if } \epsilon = \epsilon_3 \\ 0.36 & \text{if } \epsilon = \epsilon_4 \end{cases}$$

Then  $\xi$  is a  $(0.38, 0.48)$  FSZ, but not a  $(\epsilon, \epsilon \vee q)$  FSZ of  $\mathcal{Z}$ , since  $\Delta(\epsilon_4 \tau \epsilon_4) = \Delta(\epsilon_3, q) = 0.29 \not\geq \min\{\Delta(\epsilon_4, q), \Delta(\epsilon_4, q)\} = 0.33$ ,  $\Psi(\epsilon_4 \tau \epsilon_4) = \Psi(\epsilon_3, q) = 0.30 \not\geq \min\{\Psi(\epsilon_4, q), \Psi(\epsilon_4, q)\} = 0.34$  and  $\mathcal{U}(\epsilon_4 \tau \epsilon_4) = \mathcal{U}(\epsilon_3, q) = 0.40 \not\leq \max\{\mathcal{U}(\epsilon_4, q), \mathcal{U}(\epsilon_4, q)\} = 0.36$ .

**Corollary 4.3.** Every  $(\epsilon, \epsilon \vee q)$ NBI of  $\mathcal{Z}$  is a  $(b, \ell)$ NBI of  $\mathcal{Z}$ , but converse need not be true by the following Example.

For the Example 3.4, we define the subset  $\xi$  by,

$$\Delta(\epsilon) = \begin{cases} 0.46 & \text{if } \epsilon = \epsilon_1 \\ 0.41 & \text{if } \epsilon = \epsilon_2 \\ 0.28 & \text{if } \epsilon = \epsilon_3 \\ 0.33 & \text{if } \epsilon = \epsilon_4 \end{cases} \quad \Psi(\epsilon) = \begin{cases} 0.48 & \text{if } \epsilon = \epsilon_1 \\ 0.43 & \text{if } \epsilon = \epsilon_2 \\ 0.30 & \text{if } \epsilon = \epsilon_3 \\ 0.35 & \text{if } \epsilon = \epsilon_4 \end{cases} \quad \mathcal{U}(\epsilon) = \begin{cases} 0.35 & \text{if } \epsilon = \epsilon_1 \\ 0.37 & \text{if } \epsilon = \epsilon_2 \\ 0.41 & \text{if } \epsilon = \epsilon_3 \\ 0.39 & \text{if } \epsilon = \epsilon_4 \end{cases}$$

Then  $\xi$  is a  $(0.45, 0.49)$  FBI of  $\mathcal{Z}$ , but not a  $(\epsilon, \epsilon \vee q)$  FBI,  $\Delta(\epsilon_4 \tau \epsilon_4 \beta \epsilon_4, q) = \Delta(\epsilon_3, q) = 0.28 \not\geq \min\{\Delta(\epsilon_4, q), \Delta(\epsilon_4, q)\} = 0.33$ ,  $\Psi(\epsilon_4 \tau \epsilon_4 \beta \epsilon_4, q) = \Psi(\epsilon_3, q) = 0.30 \not\geq \min\{\Psi(\epsilon_4, q), \Psi(\epsilon_4, q)\} = 0.35$  and  $\mathcal{U}(\epsilon_4 \tau \epsilon_4 \beta \epsilon_4, q) = \mathcal{U}(\epsilon_3, q) = 0.41 \not\leq \max\{\mathcal{U}(\epsilon_4, q), \mathcal{U}(\epsilon_4, q)\} = 0.39$ .

**Corollary 4.4.** Let  $M$  be an  $(\epsilon, \epsilon \vee q)$ NRI and  $N$  be an  $(\epsilon, \epsilon \vee q)$ NLI of  $\mathcal{Z}$ , then  $((M \circ_{\Gamma} N]) \subseteq (M \cup_b^{\ell} N]$ .

**Proof.** Taking  $b = 0, \ell = 0.5$  in Theorem 3.17.

**Corollary 4.5.** Let  $M$  be an  $(\epsilon, \epsilon \vee q)$ NBI and  $N$  be an  $(\epsilon, \epsilon \vee q)$ NLI of  $\mathcal{Z}$ . Then  $\mathcal{Z}$  is regular if and only if  $(M \circ_{\Gamma} N] \supseteq (M \cap_b^{\ell} N]$ .

**Corollary 4.6.** Let  $M$  be an  $(\epsilon, \epsilon \vee q)$ NRI and  $N$  be an  $(\epsilon, \epsilon \vee q)$ NLI of  $\mathcal{Z}$ . Then  $\mathcal{Z}$  is weakly regular if and only if  $(M \circ_{\Gamma} N] \supseteq (M \cap_b^{\ell} N]$ .

**Corollary 4.7.** Let  $M$  be an  $(\epsilon, \epsilon \vee q)$ NBI and  $N$  be an  $(\epsilon, \epsilon \vee q)$ NLI of  $\mathcal{Z}$ . Then  $\mathcal{Z}$  is weakly regular if and only if  $(M \circ_{\Gamma} N] \supseteq (M \cap_b^{\ell} N]$ .

### 5 Conclusion and future direction

we introduced the notion of  $b, \ell$ -neutrosophic subsemigroup, neutrosophic left ideal, neutrosophic right ideal, neutrosophic ideal, neutrosophic bi-ideal,  $(\epsilon, \epsilon \vee q)$ -neutrosophic ideal, neutrosophic bi-ideal of an ordered  $\Gamma$ -semigroups and discuss some of their properties. Furthermore, an analysis of the properties of various transformations is conducted. We are attempting to handle cubic and interval valued new fuzzy structures. Therefore, in the future, we should think about utilizing complex, soft set.

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