Characterization of bipolar neutrosophic sets to novel concept of complex Q bipolar neutrosophic sets using bisemirings

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Abstract

The notion of the complex Q bipolar neutrosophic subbisemiring (CQBNSBS) is a fundamental notion to be considered for tackling tricky and intricate information. Here, in this study, we want to expand the notion of CQBNSBS by giving a general algebraic structure for tackling bipolar complex fuzzy (BCF) data by fusing the conception of CQBNSBS and subbisemiring. Keeping in view the importance of fuzzy algebraic structures, in this manuscript, we develop the concept of CQBNSBS. We analyze the important properties and homomorphic aspects of CQBNSBS. For bisemirings, we propose the CQBNSBS level sets. We also develop the notions of homomorphic images of all CQBNSBSs is also CQBNSBS and homomorphic pre-images of all CQBNSBSs is also CQBNSBS. Examples are provided to demonstrate our findings.

Keywords: CQBNSBS; SBS; Homomorphism

1 Introduction

Every day, systems become more complex, which makes it harder for decision-makers to select the best option. A single goal is hard to achieve, but you can do it. For many businesses, it was challenging to inspire people, create objectives, and form attitudes. Therefore, many goals must be taken into account at the same time when making decisions, whether they are made by an individual or a group. After giving it some thought, it appears that the criteria are solved flexibly, which makes it challenging for any decision-maker to come up with the best answer possible for each of the relevant criteria. Reliable and adequate methodologies should be developed by decision-makers in order to identify the optimal solution. Generally, when dealing with ambiguity and uncertainty in decision-making, crisp methods don't work. Human life is becoming more and more ambiguous and uncertain as time goes on, and an expert or decision-analyst cannot handle these kinds of ambiguities and uncertainties by using the theory of crisp set. Fuzzy set (FS) theory was developed by Zadeh,[1] and it is the best at dealing with ambiguity and uncertainty. If an element in an FS has a single value inside the interval, it is regarded as a member. The degree of non-membership does not always equal one minus the degree of membership, though, as resistance can occur in real-world circumstances. An increasing number of hybrid fuzzy models are being created as FS theory develops swiftly. The uncertainties have contributed to the development of a number of uncertain theories, such as FS,[1] intuitionistic FS (IFS),[2] Pythagorean FS (PFS),[3] and spherical FS (SFS).[4] MG sets, or sets with grades between 0 and 1, make up an FS. Although an assertion made by Atanassov[2] that non-membership grades (NMG) can only have a value of 1, IFS is categorized as MG. The total of MGs and NMGs may occasionally exceed 1 throughout a decision-making process. Yager[3]

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used PFS to develop the generalized MG and NMG, which has a value not exceeding 1 and is determined by the square of the MGs and NMGs. As the neutral state is neither positive nor negative, these theories are unable to describe it. Cuong[5] talked to colleagues about the picture. FS used three grading points: positive, neutral, and negative. The sum of these grades could not be greater than 1. It also outperforms PFS and IFS in several situations. It addresses the truth, indeterminacy, and falsity of FS and IFS and is an autonomous generalization of three models. The conception of bipolar fuzzy set (BFS) is one of the generalizations of FS, as FS is unable to cover the negative opinion or negative supportive grade of human beings. Thus, Zhang[6] initiated the BFS to cover both positive and negative opinions of human beings by enlarging the range of FS $[0, 1]$ to the BFS $[0, 1], [-1, 0]$. The BFS holds a positive supportive grade (PSG) which contains in $[0, 1]$ and negative supportive grade (NSG) which contains in $[-1, 0]$.
Lee[7] was presented the idea of bipolar fuzzy sets. The membership degree range in conventional fuzzy sets is $[0, 1]$. Fuzzy sets that have their membership degree range expanded from the interval $[0, 1]$ to the interval $[-1, 1]$ are called BFSs. The membership degrees on $(0, 1]$ indicate that elements somewhat satisfy the property, the membership degrees on $[-1, 0]$ indicate that elements somewhat satisfy the implicit counter property, and the membership degree 0 in a bipolar fuzzy set indicates that elements are irrelevant to the corresponding property. To handle conflicting and unclear data, Smarandache[8] developed the neutrosophic set (NS). The degree to which a proposition is true, ambiguous, or false is established using this logic. Ramot et al.[9] introduce the concept of the complex fuzzy set (CFS). The membership functions of CFS's transactions can have a very broad range of values. While the unit circle of a fuzzy membership function remains fixed, the unit circle of the complex plane is expanded to $[0, 1]$. Instead of extending purely to $[0, 1]$, the membership function $\mu_X(x)$ of the CFS X extends to the unit circle in the complex plane. Hence, $\mu_X(x)$ is a complex-valued function that, for any element x in the discourse universe, provides a grade of membership of the type $\eta_X(x) \cdot e^{i\tau_X(x)}$, where $i = \sqrt{-1}$. The two real-valued variables, $\eta_X(x)$ and $\tau_X(x)$, where $\eta_X(x) \in [0, 1]$, define the value of $\mu_X(x)$. Golan[10] established the concept of semiring logic and its applications. Hussian et al.[11] discussed the concept and use of bisemirings. Lee[7] discusses bipolar-valued FSs and associated techniques. Fuzzy semirings were studied by Ahsan et al.[12] Sen et al.[13] introduced the concept of bisemirings. Palanikumar et al. (2019) introduced an intuitionistic fuzzy normal SBS of bisemiring.[14] Palanikumar et al.[15] introduced the concept of bisemiring by using bipolar valued neutrosophic normal sets. Numerous writers have recently written on novel ideas such the fuzzy extension set, neutrosopic set, and SFS[16].[27] The study has benefited from the following contributions:

1. The intersection of a every CQBNSBSs is again a CQBNSBS of bisemiring $\mathscr{S}$.
2. Let Γ be a CQBNSBS of $\mathscr{S}$ and Υ be a strongest complex cubic anti neutrosophic relation of $\mathscr{S}$. Then Γ is a CQBNSBS of bisemiring $\mathscr{S}$ if and only if Υ is a CQBNSBS of $\mathscr{S} \times \mathscr{S}$.
3. The homomorphic image of every CQBNSBS is a CQBNSBS and homomorphic preimage of every CQBNSBS is a CQBNSBS.

Aspects of the SBS and CQBNSBS concepts will be examined, and conclusions drawn. The following five sections make up the article. We introduce semirings and SBS in Section 1. Information on semiring and SBS preparation is provided in Section 2. The properties of CQBNSBS are listed in Section 3. It is advised that numerical examples be used for evaluating CQBNSBS. The conclusion and future course are indicated in Section 4.

2 Preliminaries

Definition 2.1. A bipolar fuzzy set A in a universe U is an object having the
form $A = \{\langle x, \mu_A^p(x), \mu_A^n(x)\rangle : x \in X\}$, where $\mu_A^p : X \to [0, 1]$ and $\mu_A^n : X \to [-1, 0]$.
Here $\mu_A^p(x)$ represents the degree of satisfaction of the element x to the property of A and $\mu_A^n(x)$ represents the degree of satisfaction of x to some implicit counter property of A. For simplicity the symbol $\langle \mu_A^p, \mu_A^n\rangle$ is used for the bipolar fuzzy set $A = \{\langle x, \mu_A^p(x), \mu_A^n(x)\rangle : x \in X\}$.

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Definition 2.2. For two bipolar fuzzy subsets $\mu = (\mu^p, \mu^n)$ and $\lambda = (\lambda^p, \lambda^n)$ of S, the product of μ and λ is denoted by $\mu \circ \lambda$ and is defined as

$$(\mu^p \circ \lambda^p)(x) = \begin{cases} \sup\limits_{(s,t) \in A_x} \Big\{\mu^p(s) \wedge \lambda^p(t)\Big\} \; if \;\; A_x \neq 0 \\ 0 \quad if \qquad\qquad\qquad A_x = 0 \end{cases}$$

$$(\mu^n \circ \lambda^n)(x) = \begin{cases} \inf\limits_{(s,t) \in A_x} \Big\{\lambda^n(s) \vee \lambda^n(t)\Big\} \; if \; A_x \neq 0 \\ -1 \quad if \qquad\qquad A_x = 0 \end{cases}$$

Definition 2.3. [8] A NS v in the universe $\mathcal{U}$ is $v = \{x, u_v^T(x), u_v^I(x), u_v^F(x) | x \in \mathcal{U}\}$,where $u_v^T(x)$,$u_v^I(x)$ $u_v^F(x)$ represents the TD,ID and FD of v respectively. Consider the mapping $u_v^T : \mathcal{U} \to [0, 1], u_v^I : \mathcal{U} \to [0, 1], u_v^F : \mathcal{U} \to [0, 1]$ and $0 \leq u_v^T(x) + u_v^I(x) + u_v^F(x) \leq 3$.

Definition 2.4. [8] Let $\psi_1 = \langle u_{\psi_1}^T, u_{\psi_1}^I, u_{\psi_1}^F\rangle$,$\psi_2 = \langle u_{\psi_2}^T, u_{\psi_2}^I, u_{\psi_2}^F\rangle$ and $\psi_3 = \langle u_{\psi_3}^T, u_{\psi_3}^I, u_{\psi_3}^F\rangle$ be the three neutrosophic numbers over $\mathcal{U}$. Then

1. $\psi_2 \ominus \psi_3 = \Big\langle \max(\chi_{\psi_2}^T, u_{\psi_3}^T), \min(\chi_{\psi_2}^I, u_{\psi_3}^I), \min(\chi_{\psi_2}^F, u_{\psi_3}^F)\Big\rangle$,
2. $\psi_2 \uplus \psi_3 = \Big\langle \min(\chi_{\psi_2}^T, u_{\psi_3}^T), \max(\chi_{\psi_2}^I, u_{\psi_3}^I), \max(\chi_{\psi_2}^F, u_{\psi_3}^F)\Big\rangle$,
3. $\psi_2 \geq \psi_3$ iff $u_{\psi_2}^T \geq u_{\psi_3}^T$ and $u_{\psi_2}^I \leq u_{\psi_3}^I$ and $u_{\psi_2}^F \leq u_{\psi_3}^F$,
4. $\psi_2 = \psi_3$ iff $u_{\psi_2}^T = u_{\psi_3}^T$ and $u_{\psi_2}^I = u_{\psi_3}^I$ and $u_{\psi_2}^F = u_{\psi_3}^F$.

Definition 2.5. [8] For any NS $\psi = \{x, \chi_v^T(x), \chi_v^I(x), \chi_v^F(x)\}$ of $\mathcal{U}$. Then (τ, β)-cut is defined as $\{x \in U | \chi_v^T(x) \geq \tau, \chi_v^I(x) \geq \tau, \chi_v^F(x) \leq \beta\}$.

Definition 2.6. [8] Let V and Y be two NSs of S. Then Cartesian product of V and Y is defined as $V \times Y = \{\chi_{V\times Y}^T(\chi, \psi), \chi_{V\times Y}^I(\chi, \psi), \chi_{V\times Y}^F(\chi, \psi)| \; for \; all \; \chi, \psi \in S\}$, where
$\chi_{V\times Y}^T(\chi, \psi) = \min\{\chi_V^T(x), \chi_Y^T(\psi)\}, \chi_{V\times Y}^I(\chi, \psi) = \frac{\chi_V^I(x) + \chi_Y^I(\psi)}{2}, \chi_{V\times Y}^F(\chi, \psi) = \max\{\chi_V^F(x), \chi_Y^F(\psi)\}$.

Definition 2.7. A fuzzy subset v of a bisemiring $(\mathscr{S}, \bigcirc_1, \bigcirc_2, \bigcirc_3)$ is represents a fuzzy SBS of $\mathscr{S}$ if $\chi_v(\chi \bigcirc_1 \varepsilon) \geq \min\{\chi_v(x), \chi_v(\varepsilon)\}, \chi_v(\chi \bigcirc_2 \varepsilon) \geq \min\{\chi_v(x), \chi_v(\varepsilon)\}, \chi_v(\chi \bigcirc_3 \varepsilon) \geq \min\{\chi_v(x), \chi_v(\varepsilon)\}$,for all $\chi, \varepsilon \in \mathscr{S}$.

3 Complex Q bipolar neutrosophic subbisemiring

Here $\mathscr{S}$ denotes bisemiring unless other stated, $\Re$ stands for real part and $\Im$ stands for imaginary part and $i2\pi = x$.

Definition 3.1. The non-empty set Q is a complex Q bipolar neutrosophic set(CQBNS) ℓ in universal set U, $\ell = \{\tau, \Re_\ell^{T^n}(\tau^q) \cdot e^{x\Im_\ell^{T^n(\tau^q)}}, \Re_\ell^{I^n}(\tau^q) \cdot e^{x\Im_\ell^{I^n(\tau^q)}}, \Re_\ell^{F^n}(\tau^q) \cdot e^{x\Im_\ell^{F^n(\tau^q)}}, \Re_\ell^{T^p}(\tau^q) \cdot e^{x\Im_\ell^{T^p(\tau^q)}}, \Re_\ell^{I^p}(\tau^q) \cdot e^{x\Im_\ell^{I^p(\tau^q)}}, \Re_\ell^{F^p}(\tau^q) \cdot e^{x\Im_\ell^{F^p(\tau^q)}} : \tau \in U\}$, where $\Re_\ell^{T^n}(\tau^q) \cdot e^{x\Im_\ell^{T^n(\tau^q)}}, \Re_\ell^{I^n}(\tau^q) \cdot e^{x\Im_\ell^{I^n(\tau^q)}}, \Re_\ell^{F^n}(\tau^q) \cdot e^{x\Im_\ell^{F^n(\tau^q)}} : U \to [-1, 0] \times [0, 1]$ and $\Re_\ell^{T}(\tau^q) \cdot e^{x\Im_\ell^{T(\tau^q)}}, \Re_\ell^{I}(\tau^q) \cdot e^{x\Im_\ell^{I(\tau^q)}}, \Re_\ell^{F}(\tau^q) \cdot e^{x\Im_\ell^{F(\tau^q)}} : U \to [-1, 0] \times [0, 1]$ represents the truth degree, indeterminacy degree and false degree respectively.
For simplicity the symbol $\Re_\ell^{T^n}, \Re_\ell^{I^n}, \Re_\ell^{F^n}$ is CQBNS $\ell = \{\tau, \Re_\ell^{T^n}(\tau^q) \cdot e^{x\Im_\ell^{T^n(\tau^q)}}, \Re_\ell^{I^n}(\tau^q) \cdot e^{x\Im_\ell^{I^n(\tau^q)}}, \Re_\ell^{F^n}(\tau^q) \cdot e^{x\Im_\ell^{F^n(\tau^q)}}, \Re_\ell^{T^p}(\tau^q) \cdot e^{x\Im_\ell^{T^p(\tau^q)}}, \Re_\ell^{I^p}(\tau^q) \cdot e^{x\Im_\ell^{I^p(\tau^q)}}, \Re_\ell^{F^p}(\tau^q) \cdot e^{x\Im_\ell^{F^p(\tau^q)}} : \tau \in U\}$ and $q \in Q$.

Definition 3.2. Let $\ell = \Big\{\tau, \Re_\ell^{T^n}(\tau^q) \cdot e^{x\Im_\ell^{T^n(\tau^q)}}, \Re_\ell^{I^n}(\tau^q) \cdot e^{x\Im_\ell^{I^n(\tau^q)}}, \Re_\ell^{F^n}(\tau^q) \cdot e^{x\Im_\ell^{F^n(\tau^q)}}, \Re_\ell^{T^p}(\tau^q) \cdot e^{x\Im_\ell^{T^p(\tau^q)}}, \Re_\ell^{I^p}(\tau^q) \cdot e^{x\Im_\ell^{I^p(\tau^q)}}, \Re_\ell^{F^p}(\tau^q) \cdot e^{x\Im_\ell^{F^p(\tau^q)}}\Big\}$ and
$\Game = \Big\{\tau, \Re_\Game^{T^n}(\tau^q) \cdot e^{x\Im_\Game^{T^n(\tau^q)}}, \Re_\Game^{I^n}(\tau^q) \cdot e^{x\Im_\Game^{I^n(\tau^q)}}, \Re_\Game^{F^n}(\tau^q) \cdot e^{x\Im_\Game^{F^n(\tau^q)}}, \Re_\Game^{T^p}(\tau^q) \cdot e^{x\Im_\Game^{T^p(\tau^q)}}, \Re_\Game^{I^p}(\tau^q) \cdot e^{x\Im_\Game^{I^p(\tau^q)}}, \Re_\Game^{F^p}(\tau^q) \cdot e^{x\Im_\Game^{F^p(\tau^q)}}\Big\}$ be two CQBNSs of U.

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Then we define the intersection and union operation is defined as

(i) $\ell \cap \eth = \Big\{\Big(\tau, \max\{\Re_{\ell}^{T^n}(\tau^q)\cdot e^{x\Im_{\ell}^{T^n(\tau^q)}}, \Re_{\eth}^{T^n}(\tau^q)\cdot e^{x\Im_{\eth}^{T^n(\tau^q)}}\}, \max\{\Re_{\ell}^{I^n}(\tau^q)\cdot e^{x\Im_{\ell}^{I^n(\tau^q)}}, \Re_{\eth}^{I^n}(\tau^q)\cdot e^{x\Im_{\eth}^{I^n(\tau^q)}}\}, \min\{\Re_{\ell}^{F^n}(\tau^q)\cdot e^{x\Im_{\ell}^{F^n(\tau^q)}}, \Re_{\eth}^{F^n}(\tau^q)\cdot e^{x\Im_{\eth}^{F^n(\tau^q)}}\},$
$\min\{\Re_{\ell}^{T^p}(\tau^q)\cdot e^{x\Im_{\ell}^{T^p(\tau^q)}}, \Re_{\eth}^{T^p}(\tau^q)\cdot e^{x\Im_{\eth}^{T^p(\tau^q)}}\}, \min\{\Re_{\ell}^{I^p}(\tau^q)\cdot e^{x\Im_{\ell}^{I^p(\tau^q)}}, \Re_{\eth}^{I^p}(\tau^q)\cdot e^{x\Im_{\eth}^{I^p(\tau^q)}}\},$
$\max\{\Re_{\ell}^{F^p}(\tau^q)\cdot e^{x\Im_{\ell}^{F^p(\tau^q)}}, \Re_{\eth}^{F^p}(\tau^q)\cdot e^{x\Im_{\eth}^{F^p(\tau^q)}}\}\Big)\Big|\tau \in U\Big\}.$

(ii) $\ell \cup \eth = \Big\{\Big(\tau, \min\{\Re_{\ell}^{T^n}(\tau^q)\cdot e^{x\Im_{\ell}^{T^n(\tau^q)}}, \Re_{\eth}^{T^n}(\tau^q)\cdot e^{x\Im_{\eth}^{T^n(\tau^q)}}\}, \min\{\Re_{\ell}^{I^n}(\tau^q)\cdot e^{x\Im_{\ell}^{I^n(\tau^q)}}, \Re_{\eth}^{I^n}(\tau^q)\cdot e^{x\Im_{\eth}^{I^n(\tau^q)}}\}, \max\{\Re_{\ell}^{F^n}(\tau^q)\cdot e^{x\Im_{\ell}^{F^n(\tau^q)}}, \Re_{\eth}^{F^n}(\tau^q)\cdot e^{x\Im_{\eth}^{F^n(\tau^q)}}\},$
$\max\{\Re_{\ell}^{T^p}(\tau^q)\cdot e^{x\Im_{\ell}^{T^p(\tau^q)}}, \Re_{\eth}^{T^p}(\tau^q)\cdot e^{x\Im_{\eth}^{T^p(\tau^q)}}\}, \max\{\Re_{\ell}^{I^p}(\tau^q)\cdot e^{x\Im_{\ell}^{I^p(\tau^q)}}, \Re_{\eth}^{I^p}(\tau^q)\cdot e^{x\Im_{\eth}^{I^p(\tau^q)}}\},$
$\min\{\Re_{\ell}^{F^p}(\tau^q)\cdot e^{x\Im_{\ell}^{F^p(\tau^q)}}, \Re_{\eth}^{F^p}(\tau^q)\cdot e^{x\Im_{\eth}^{F^p(\tau^q)}}\}\Big)\Big|\tau \in U\Big\}.$

Definition 3.3. For any CQBNS $\ell = \Big\{\tau, \Re_{\ell}^{T^n}(\tau^q)\cdot e^{x\Im_{\ell}^{T^n(\tau^q)}}, \Re_{\ell}^{I^n}(\tau^q)\cdot e^{x\Im_{\ell}^{I^n(\tau^q)}}, \Re_{\ell}^{F^n}(\tau^q)\cdot e^{x\Im_{\ell}^{F^n(\tau^q)}}, \Re_{\ell}^{T^p}(\tau^q)\cdot e^{x\Im_{\ell}^{T^p(\tau^q)}}, \Re_{\ell}^{I^p}(\tau^q)\cdot e^{x\Im_{\ell}^{I^p(\tau^q)}}, \Re_{\ell}^{F^p}(\tau^q)\cdot e^{x\Im_{\ell}^{F^p(\tau^q)}}\Big\}$ of a universal set U. Then $(\wp_1, \wp_2)$-cut is defined as $\Big\{\tau \in U | \Re_{\ell}^{T^n}(\tau^q)\cdot e^{x\Im_{\ell}^{T^n(\tau^q)}} \leq \wp_1, \Re_{\ell}^{I^n}(\tau^q)\cdot e^{x\Im_{\ell}^{I^n(\tau^q)}} \leq \wp_1, \Re_{\ell}^{F^n}(\tau^q)\cdot e^{x\Im_{\ell}^{F^n(\tau^q)}} \geq \wp_2, \Re_{\ell}^{T^p}(\tau^q)\cdot e^{x\Im_{\ell}^{T^p(\tau^q)}} \geq \wp_1, \Re_{\ell}^{I^p}(\tau^q)\cdot e^{x\Im_{\ell}^{I^p(\tau^q)}} \geq \wp_1, \Re_{\ell}^{F^p}(\tau^q)\cdot e^{x\Im_{\ell}^{F^p(\tau^q)}} \leq \wp_2\Big\}$.

Definition 3.4. The Cartesian product of ℓ and $\eth$ is defined as
$\ell \times \eth = \Big\{\Re_{\ell\times\eth}^{T^n}((\tau,\lambda)^q)\cdot e^{x\Im_{\ell\times\eth}^{T^n((\tau,\lambda)^q)}}, \Re_{\ell\times\eth}^{I^n}((\tau,\lambda)^q)\cdot e^{x\Im_{\ell\times\eth}^{I^n((\tau,\lambda)^q)}}, \Re_{\ell\times\eth}^{F^n}(\tau,\lambda)\cdot e^{x\Im_{\ell\times\eth}^{F^n((\tau,\lambda)^q)}},$
$\Re_{\ell\times\eth}^{T^p}((\tau,\lambda)^q)\cdot e^{x\Im_{\ell\times\eth}^{T^p((\tau,\lambda)^q)}}, \Re_{\ell\times\eth}^{I^p}((\tau,\lambda)^q)\cdot e^{x\Im_{\ell\times\eth}^{I^p((\tau,\lambda)^q)}}, \Re_{\ell\times\eth}^{F^p}(\tau,\lambda)\cdot e^{x\Im_{\ell\times\eth}^{F^p((\tau,\lambda)^q)}} \mid for\ all\ \tau,\lambda \in S\Big\},$
where ℓ and $\eth$ be the CQBNS of U

$$\left\{\begin{array}{l} \Re_{\ell\times\eth}^{T^n}((\tau,\lambda)^q)\cdot e^{x\Im_{\ell\times\eth}^{T^n((\tau,\lambda)^q)}} = \max\Big\{\Re_{\ell}^{T^n}(\tau^q)\cdot e^{x\Im_{\ell}^{T^n(\tau^q)}}, \Re_{\eth}^{T^n}(\lambda^q)\cdot e^{x\Im_{\eth}^{T^n(\lambda^q)}}\Big\} \\ \Re_{\ell\times\eth}^{I^n}((\tau,\lambda)^q)\cdot e^{x\Im_{\ell\times\eth}^{I^n((\tau,\lambda)^q)}} = \frac{\Re_{\ell}^{I^n}(\tau^q)\cdot e^{x\Im_{\ell}^{I^n(\tau^q)}} + \Re_{\eth}^{I^n}(\lambda^q)\cdot e^{x\Im_{\eth}^{I^n(\lambda^q)}}}{2} \\ \Re_{\ell\times\eth}^{F^n}((\tau,\lambda)^q)\cdot e^{x\Im_{\ell\times\eth}^{F^n((\tau,\lambda)^q)}} = \min\Big\{\Re_{\ell}^{F^n}(\tau^q)\cdot e^{x\Im_{\ell}^{F^p(\tau^q)}}, \Re_{\eth}^{F^n}(\lambda^q)\cdot e^{x\Im_{\eth}^{F^n(\lambda^q)}}\Big\} \end{array}\right\}$$

$$\left\{\begin{array}{l} \Re_{\ell\times\eth}^{T^p}((\tau,\lambda)^q)\cdot e^{x\Im_{\ell\times\eth}^{T^p((\tau,\lambda)^q)}} = \min\Big\{\Re_{\ell}^{T^p}(\tau^q)\cdot e^{x\Im_{\ell}^{T^p(\tau^q)}}, \Re_{\eth}^{T^p}(\lambda^q)\cdot e^{x\Im_{\eth}^{T^p(\lambda^q)}}\Big\} \\ \Re_{\ell\times\eth}^{I^p}((\tau,\lambda)^q)\cdot e^{x\Im_{\ell\times\eth}^{I^p((\tau,\lambda)^q)}} = \frac{\Re_{\ell}^{I^p}(\tau^q)\cdot e^{x\Im_{\ell}^{I^p(\tau^q)}} + \Re_{\eth}^{I^p}(\lambda^q)\cdot e^{x\Im_{\eth}^{I^p(\lambda^q)}}}{2} \\ \Re_{\ell\times\eth}^{F^p}((\tau,\lambda)^q)\cdot e^{x\Im_{\ell\times\eth}^{F^p((\tau,\lambda)^q)}} = \max\Big\{\Re_{\ell}^{F^p}(\tau^q)\cdot e^{x\Im_{\ell}^{F^p(\tau^q)}}, \Re_{\eth}^{F^p}(\lambda^q)\cdot e^{x\Im_{\eth}^{F^p(\lambda^q)}}\Big\} \end{array}\right\}$$

Definition 3.5. For any CQBNS ℓ of $\mathscr{S}$ is said to be a Q-complex bipolar neutrosophic SBS (CQBNSBS) of $\mathscr{S}$ if

$$\left\{\begin{array}{l} \Re_{\ell}^{T^n}((\tau\Diamond_1\lambda)^q)\cdot e^{x\Im_{\ell}^{T^n}((\tau\Diamond_1\lambda)^q)} \leq \max\{\Re_{\ell}^{T^n}(\tau^q)\cdot e^{x\Im_{\ell}^{T^n}((\tau^q)}, \Re_{\ell}^{T^n}(\lambda^q)\cdot e^{x\Im_{\ell}^{T^n}((\lambda^q)}\} \\ \Re_{\ell}^{T^n}((\tau\Diamond_2\lambda)^q)\cdot e^{x\Im_{\ell}^{T^n}((\tau\Diamond_2\lambda)^q)} \leq \max\{\Re_{\ell}^{T^n}(\tau^q)\cdot e^{x\Im_{\ell}^{T^n}((\tau^q)}, \Re_{\ell}^{T^n}(\lambda^q)\cdot e^{x\Im_{\ell}^{T^n}((\lambda^q)}\} \\ \Re_{\ell}^{T^n}((\tau\Diamond_3\lambda)^q)\cdot e^{x\Im_{\ell}^{T^n}((\tau\Diamond_3\lambda)^q)} \leq \max\{\Re_{\ell}^{T^n}(\tau^q)\cdot e^{x\Im_{\ell}^{T^n}((\tau^q)}, \Re_{\ell}^{T^n}(\lambda^q)\cdot e^{x\Im_{\ell}^{T^n}((\lambda^q)}\} \end{array}\right\}$$

$$\left\{\begin{array}{l} \Re_{\ell}^{I^n}((\tau\Diamond_1\lambda)^q)\cdot e^{x\Im_{\ell}^{I^n}((\tau\Diamond_1\lambda)^q)} \leq \frac{\Re_{\ell}^{I^n}(\tau^q)\cdot e^{x\Im_{\ell}^{I^n}((\tau^q)} + \Re_{\ell}^{I^n}(\lambda^q)\cdot e^{x\Im_{\ell}^{I^n}((\lambda^q)}}{2} \\ \qquad\qquad \text{OR} \\ \Re_{\ell}^{I^n}((\tau\Diamond_2\lambda)^q)\cdot e^{x\Im_{\ell}^{T^n}((\tau\Diamond_2\lambda)^q)} \leq \frac{\Re_{\ell}^{I^n}(\tau^q)\cdot e^{x\Im_{\ell}^{I^n}((\tau^q)} + \Re_{\ell}^{I^n}(\lambda^q)\cdot e^{x\Im_{\ell}^{I^n}((\lambda^q)}}{2} \\ \qquad\qquad \text{OR} \\ \Re_{\ell}^{I^n}((\tau\Diamond_3\lambda)^q)\cdot e^{x\Im_{\ell}^{I^n}((\tau\Diamond_3\lambda)^q)} \leq \frac{\Re_{\ell}^{I^n}(\tau^q)\cdot e^{x\Im_{\ell}^{I^n}((\tau^q)} + \Re_{\ell}^{I^n}(\lambda^q)\cdot e^{x\Im_{\ell}^{T^n}((\lambda^q)}}{2} \end{array}\right\}$$

$$\left\{\begin{array}{l} \Re_{\ell}^{F^n}((\tau\Diamond_1\lambda)^q)\cdot e^{x\Im_{\ell}^{F^n}((\tau\Diamond_1\lambda)^q)} \geq \min\{\Re_{\ell}^{F^n}(\tau^q)\cdot e^{x\Im_{\ell}^{F^n}((\tau^q)}, \Re_{\ell}^{F^n}(\lambda^q)\cdot e^{x\Im_{\ell}^{F^n}((\lambda^q)}\} \\ \Re_{\ell}^{F^n}((\tau\Diamond_2\lambda)^q)\cdot e^{x\Im_{\ell}^{F^n}((\tau\Diamond_2\lambda)^q)} \geq \min\{\Re_{\ell}^{F^n}(\tau^q)\cdot e^{x\Im_{\ell}^{F^n}((\tau^q)}, \Re_{\ell}^{F^n}(\lambda^q)\cdot e^{x\Im_{\ell}^{F^n}((\lambda^q)}\} \\ \Re_{\ell}^{F^n}((\tau\Diamond_3\lambda)^q)\cdot e^{x\Im_{\ell}^{F^n}((\tau\Diamond_3\lambda)^q)} \geq \min\{\Re_{\ell}^{F^n}(\tau^q)\cdot e^{x\Im_{\ell}^{F^n}((\tau^q)}, \Re_{\ell}^{F^n}(\lambda^q)\cdot e^{x\Im_{\ell}^{F^n}((\lambda^q)}\} \end{array}\right\}$$

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$$\left\{\begin{array}{l} \Re_\ell^{T^p}((\tau\Diamond_1\lambda)^q)\cdot e^{x\Im_\ell^{T^p}((\tau\Diamond_1\lambda)^q)} \geq \min\{\Re_\ell^{T^p}(\tau^q)\cdot e^{x\Im_\ell^{T^p}((\tau^q)}, \Re_\ell^{T^p}(\lambda^q)\cdot e^{x\Im_\ell^{T^p}((\lambda^q)}\} \\ \Re_\ell^{T^p}((\tau\Diamond_2\lambda)^q)\cdot e^{x\Im_\ell^{T^p}((\tau\Diamond_2\lambda)^q)} \geq \min\{\Re_\ell^{T^p}(\tau^q)\cdot e^{x\Im_\ell^{T^p}((\tau^q)}, \Re_\ell^{T^p}(\lambda^q)\cdot e^{x\Im_\ell^{T^p}((\lambda^q)}\} \\ \Re_\ell^{T^p}((\tau\Diamond_3\lambda)^q)\cdot e^{x\Im_\ell^{T^p}((\tau\Diamond_3\lambda)^q)} \geq \min\{\Re_\ell^{T^p}(\tau^q)\cdot e^{x\Im_\ell^{T^p}((\tau^q)}, \Re_\ell^{T^p}(\lambda^q)\cdot e^{x\Im_\ell^{T^p}((\lambda^q)}\} \end{array}\right\}$$

$$\left\{\begin{array}{l} \Re_\ell^{I^p}((\tau\Diamond_1\lambda)^q)\cdot e^{x\Im_\ell^{I^p}((\tau\Diamond_1\lambda)^q)} \geq \frac{\Re_\ell^{I^p}(\tau^q)\cdot e^{x\Im_\ell^{I^p}((\tau^q)} + \Re_\ell^{I^p}(\lambda^q)\cdot e^{x\Im_\ell^{I^p}((\lambda^q)}}{2} \\ \qquad\qquad \text{OR} \\ \Re_\ell^{I^p}((\tau\Diamond_2\lambda)^q)\cdot e^{x\Im_\ell^{T^p}((\tau\Diamond_2\lambda)^q)} \geq \frac{\Re_\ell^{I^p}(\tau^q)\cdot e^{x\Im_\ell^{I^p}((\tau^q)} + \Re_\ell^{I^p}(\lambda^q)\cdot e^{x\Im_\ell^{I^p}((\lambda^q)}}{2} \\ \qquad\qquad \text{OR} \\ \Re_\ell^{I^p}((\tau\Diamond_3\lambda)^q)\cdot e^{x\Im_\ell^{I^p}((\tau\Diamond_3\lambda)^q)} \geq \frac{\Re_\ell^{I^p}(\tau^q)\cdot e^{x\Im_\ell^{I^p}((\tau^q)} + \Re_\ell^{I^p}(\lambda^q)\cdot e^{x\Im_\ell^{T^p}((\lambda^q)}}{2} \end{array}\right\}$$

$$\left\{\begin{array}{l} \Re_\ell^{F^p}((\tau\Diamond_1\lambda)^q)\cdot e^{x\Im_\ell^{F^p}((\tau\Diamond_1\lambda)^q)} \leq \max\{\Re_\ell^{F^p}(\tau^q)\cdot e^{x\Im_\ell^{F^p}((\tau^q)}, \Re_\ell^{F^p}(\lambda^q)\cdot e^{x\Im_\ell^{F^p}((\lambda^q)}\} \\ \Re_\ell^{F^p}((\tau\Diamond_2\lambda)^q)\cdot e^{x\Im_\ell^{F^p}((\tau\Diamond_2\lambda)^q)} \leq \max\{\Re_\ell^{F^p}(\tau^q)\cdot e^{x\Im_\ell^{F^p}((\tau^q)}, \Re_\ell^{F^p}(\lambda^q)\cdot e^{x\Im_\ell^{F^p}((\lambda^q)}\} \\ \Re_\ell^{F^p}((\tau\Diamond_3\lambda)^q)\cdot e^{x\Im_\ell^{F^p}((\tau\Diamond_3\lambda)^q)} \leq \max\{\Re_\ell^{F^p}(\tau^q)\cdot e^{x\Im_\ell^{F^p}((\tau^q)}, \Re_\ell^{F^p}(\lambda^q)\cdot e^{x\Im_\ell^{F^p}((\lambda^q)}\} \end{array}\right\}$$

for all $\tau, \lambda \in \mathscr{S}$.

Example 3.6. Consider the bisemiring $\mathscr{S} = \{\eta_1, \eta_2, \eta_3, \eta_4\}$ with the Cayley table:

$\Diamond_1$	η_1	η_2	η_3	η_4
η_1	η_1	η_1	η_1	η_1
η_2	η_1	η_2	η_1	η_2
η_3	η_1	η_1	η_3	η_3
η_4	η_1	η_2	η_3	η_4

$\Diamond_2$	η_1	η_2	η_3	η_4
η_1	η_1	η_2	η_3	η_4
η_2	η_2	η_2	η_4	η_4
η_3	η_3	η_4	η_3	η_4
η_4	η_4	η_4	η_4	η_4

$\Diamond_3$	η_1	η_2	η_3	η_4
η_1	η_1	η_1	η_1	η_1
η_2	η_1	η_2	η_3	η_4
η_3	η_4	η_4	η_4	η_4
η_4	η_4	η_4	η_4	η_4

	$(\flat) = \eta_1$	$(\flat) = \eta_2$	$(\flat) = \eta_3$	$(\flat) = \eta_4$
$(\Re_\ell^{T^n}, \Im_\ell^{T^n})(\flat)$	$-0.75e^{i2\pi(-0.60)}$	$-0.7e^{i2\pi(-0.55)}$	$-0.6e^{i2\pi(-0.45)}$	$-0.65e^{i2\pi(0-.50)}$
$(\Re_\ell^{I^n}, \Im_\ell^{I^n})(\flat)$	$-0.85e^{i2\pi(-0.70)}$	$-0.8e^{i2\pi(-0.65)}$	$-0.7e^{i2\pi(-0.55)}$	$-0.75e^{i2\pi(-0.60)}$
$(\Re_\ell^{F^n}, \Im_\ell^{F^n})(\flat)$	$-0.65e^{i2\pi(-0.50)}$	$-0.75e^{i2\pi(-0.6)}$	$-0.85e^{i2\pi(-0.7)}$	$-0.8e^{i2\pi(-0.65)}$

	$(\flat) = \eta_1$	$(\flat) = \eta_2$	$(\flat) = \eta_3$	$(\flat) = \eta_4$
$(\Re_\ell^{T^p}, \Im_\ell^{T^p})(\flat)$	$0.65e^{i2\pi(0.55)}$	$0.55e^{i2\pi(0.45)}$	$0.35e^{i2\pi(0.25)}$	$0.45e^{i2\pi(0.35)}$
$(\Re_\ell^{I^p}, \Im_\ell^{I^p})(\flat)$	$0.85e^{i2\pi(0.75)}$	$0.75e^{i2\pi(0.65)}$	$0.45e^{i2\pi(0.35)}$	$0.55e^{i2\pi(0.45)}$
$(\Re_\ell^{F^p}, \Im_\ell^{F^p})(\flat)$	$0.55e^{i2\pi(0.45)}$	$0.65e^{i2\pi(0.55)}$	$0.85e^{i2\pi(0.75)}$	$0.75e^{i2\pi(0.65)}$

Hence, ℓ is a CQBNSBS of $\mathscr{S}$.

Theorem 3.7. *The intersection of a every CQBNSBSs is again a CQBNSBS of $\mathscr{S}$.*

Proof. Let $\{\varpi_i : i \in I\}$ be the family of CQBNSBSs of $\mathscr{S}$ and $\ell = \bigwedge_{i\in I}\varpi_i$. Let $\tau, \lambda \in \mathscr{S}$.

Now,

$$\begin{aligned} \Re_\ell^{T^n}((\tau\Diamond_1\lambda)^q)\cdot e^{x\Im_\ell^{T^n}((\tau\Diamond_1\lambda)^q)} &= \bigvee_{i\in I} \Re_{\varpi_i}^{T^n}((\tau\Diamond_1\lambda)^q)\cdot e^{x\Im_\ell^{T^n}((\tau\Diamond_1\lambda)^q)} \\ &\leq \bigvee_{i\in I} \max\{\Re_{\varpi_i}^{T^n}(\tau^q)\cdot e^{x\Im_{\varpi_i}^{T^n}(\tau^q)}, \Re_{\varpi_i}^{T^n}(\lambda^q)\cdot e^{x\Im_{\varpi_i}^{T^n}(\lambda^q)}\} \\ &= \max\Big\{\bigvee_{i\in I} \Re_{\varpi_i}^{T^n}(\tau^q)\cdot e^{x\Im_{\varpi_i}^{T^n}(\tau^q)}, \bigvee_{i\in I} \Re_{\varpi_i}^{T^n}(\lambda^q)\cdot e^{x\Im_{\varpi_i}^{T^n}(\lambda^q)}\Big\} \\ &= \max\{\Re_\ell^{T^n}(\tau^q)\cdot e^{x\Im_\ell^{T^n}(\tau^q)}, \Re_\ell^{T^n}(\lambda^q)\cdot e^{x\Im_\ell^{T^n}(\lambda^q)}\} \end{aligned}$$

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Similarly,
$\Re_{\ell}^{T^n}((\tau\diamondsuit_2\lambda)^q)\cdot e^{x\Im_{\ell}^{T^n}((\tau\diamondsuit_2\lambda)^q)} \leq \max\{\Re_{\ell}^{T^n}(\tau^q)\cdot e^{x\Im_{\ell}^{T^n}(\tau^q)}, \Re_{\ell}^{T^n}(\lambda^q)\cdot e^{x\Im_{\ell}^{T^n}(\lambda^q)}\}$,
$\Re_{\ell}^{T^n}((\tau\diamondsuit_3\lambda)^q)\cdot e^{x\Im_{\ell}^{T^n}((\tau\diamondsuit_3\lambda)^q)} \leq \max\{\Re_{\ell}^{T^n}(\tau^q)\cdot e^{x\Im_{\ell}^{T^n}(\tau^q)}, \Re_{\ell}^{T^n}(\lambda^q)\cdot e^{x\Im_{\ell}^{T^n}(\lambda^q)}\}$.
Now,

$$\begin{aligned}
\Re_{\ell}^{I^n}((\tau\diamondsuit_1\lambda)^q)\cdot e^{x\Im_{\ell}^{I^n}((\tau\diamondsuit_1\lambda)^q)} &= \bigvee_{i\in I} \Re_{\varpi_i}^{I^n}((\tau\diamondsuit_1\lambda)^q)\cdot e^{x\Im_{\ell}^{I^n}((\tau\diamondsuit_1\lambda)^q)} \\
&\leq \bigvee_{i\in I} \frac{\Re_{\varpi_i}^{I^n}(\tau^q)\cdot e^{x\Im_{\varpi_i}^{I^n}(\tau^q)} + \Re_{\varpi_i}^{I^n}(\lambda^q)\cdot e^{x\Im_{\varpi_i}^{I^n}(\lambda^q)}}{2} \\
&= \frac{\bigvee_{i\in I} \Re_{\varpi_i}^{I^n}(\tau^q)\cdot e^{x\Im_{\varpi_i}^{I^n}(\tau^q)} + \bigvee_{i\in I} \Re_{\varpi_i}^{I^n}(\lambda^q)\cdot e^{x\Im_{\varpi_i}^{I^n}(\lambda^q)}}{2} \\
&= \frac{\Re_{\ell}^{I^n}(\tau^q)\cdot e^{x\Im_{\ell}^{I^n}(\tau^q)} + \Re_{\ell}^{I^n}(\lambda^q)\cdot e^{x\Im_{\ell}^{I^n}(\lambda^q)}}{2}
\end{aligned}$$

Similarly,
$\Re_{\ell}^{I^n}((\tau\diamondsuit_2\lambda)^q)\cdot e^{x\Im_{\ell}^{I^n}((\tau\diamondsuit_2\lambda)^q)} \leq \frac{\Re_{\ell}^{I^n}(\tau^q)\cdot e^{x\Im_{\ell}^{I^n}(\tau^q)}+\Re_{\ell}^{I^n}(\lambda^q)\cdot e^{x\Im_{\ell}^{I^n}(\lambda^q)}}{2}$ and
$\Re_{\ell}^{I^n}((\tau\diamondsuit_3\lambda)^q)\cdot e^{x\Im_{\ell}^{I^n}((\tau\diamondsuit_3\lambda)^q)} \leq \frac{\Re_{\ell}^{I^n}(\tau^q)\cdot e^{x\Im_{\ell}^{I^n}(\tau^q)}+\Re_{\ell}^{I^n}(\lambda^q)\cdot e^{x\Im_{\ell}^{I^n}(\lambda^q)}}{2}$.
Now,

$$\begin{aligned}
\Re_{\ell}^{F^n}((\tau\diamondsuit_1\lambda)^q)\cdot e^{x\Im_{\ell}^{F^n}((\tau\diamondsuit_1\lambda)^q)} &= \bigwedge_{i\in I} \Re_{\varpi_i}^{F^n}((\tau\diamondsuit_1\lambda)^q)\cdot e^{x\Im_{\ell}^{F^n}((\tau\diamondsuit_1\lambda)^q)} \\
&\geq \bigwedge_{i\in I} \min\{\Re_{\varpi_i}^{F^n}(\tau^q)\cdot e^{x\Im_{\varpi_i}^{F^n}(\tau^q)}, \Re_{\varpi_i}^{F^n}(\lambda^q)\cdot e^{x\Im_{\varpi_i}^{F^n}(\lambda^q)}\} \\
&= \min\left\{\bigwedge_{i\in I} \Re_{\varpi_i}^{F^n}(\tau^q)\cdot e^{x\Im_{\varpi_i}^{F^n}(\tau^q)}, \bigwedge_{i\in I} \Re_{\varpi_i}^{F^n}(\lambda^q)\cdot e^{x\Im_{\varpi_i}^{F^n}(\lambda^q)}\right\} \\
&= \min\{\Re_{\ell}^{F^n}(\tau^q)\cdot e^{x\Im_{\ell}^{F^n}(\tau^q)}, \Re_{\ell}^{F^n}(\lambda^q)\cdot e^{x\Im_{\ell}^{F^n}(\lambda^q)}\}
\end{aligned}$$

Similarly,
$\Re_{\ell}^{F^n}((\tau\diamondsuit_2\lambda)^q)\cdot e^{x\Im_{\ell}^{F^n}((\tau\diamondsuit_2\lambda)^q)} \geq \min\{\Re_{\ell}^{F^n}(\tau^q)\cdot e^{x\Im_{\ell}^{F^n}(\tau^q)}, \Re_{\ell}^{F^n}(\lambda^q)\cdot e^{x\Im_{\ell}^{F^n}(\lambda^q)}\}$ and
$\Re_{\ell}^{F^n}((\tau\diamondsuit_3\lambda)^q)\cdot e^{x\Im_{\ell}^{F^n}((\tau\diamondsuit_3\lambda)^q)} \geq \min\{\Re_{\ell}^{F^n}(\tau^q)\cdot e^{x\Im_{\ell}^{F^n}(\tau^q)}, \Re_{\ell}^{F^n}(\lambda^q)\cdot e^{x\Im_{\ell}^{F^n}(\lambda^q)}\}$.
Let $\{\varpi_i : i \in I\}$ be the family of CQBNSBSs of $\mathscr{S}$ and $\ell = \bigwedge_{i\in I}\varpi_i$. Let $\tau, \lambda \in \mathscr{S}$.

Now,

$$\begin{aligned}
\Re_{\ell}^{T^p}((\tau\diamondsuit_1\lambda)^q)\cdot e^{x\Im_{\ell}^{T^p}((\tau\diamondsuit_1\lambda)^q)} &= \bigwedge_{i\in I} \Re_{\varpi_i}^{T^p}((\tau\diamondsuit_1\lambda)^q)\cdot e^{x\Im_{\ell}^{T^p}((\tau\diamondsuit_1\lambda)^q)} \\
&\geq \bigwedge_{i\in I} \min\{\Re_{\varpi_i}^{T^p}(\tau^q)\cdot e^{x\Im_{\varpi_i}^{T^p}(\tau^q)}, \Re_{\varpi_i}^{T^p}(\lambda^q)\cdot e^{x\Im_{\varpi_i}^{T^p}(\lambda^q)}\} \\
&= \min\left\{\bigwedge_{i\in I} \Re_{\varpi_i}^{T^p}(\tau^q)\cdot e^{x\Im_{\varpi_i}^{T^p}(\tau^q)}, \bigwedge_{i\in I} \Re_{\varpi_i}^{T^p}(\lambda^q)\cdot e^{x\Im_{\varpi_i}^{T^p}(\lambda^q)}\right\} \\
&= \min\{\Re_{\ell}^{T^p}(\tau^q)\cdot e^{x\Im_{\ell}^{T^p}(\tau^q)}, \Re_{\ell}^{T^p}(\lambda^q)\cdot e^{x\Im_{\ell}^{T^p}(\lambda^q)}\}
\end{aligned}$$

Similarly,
$\Re_{\ell}^{T^p}((\tau\diamondsuit_2\lambda)^q)\cdot e^{x\Im_{\ell}^{T^p}((\tau\diamondsuit_2\lambda)^q)} \geq \min\{\Re_{\ell}^{T^p}(\tau^q)\cdot e^{x\Im_{\ell}^{T^p}(\tau^q)}, \Re_{\ell}^{T^p}(\lambda^q)\cdot e^{x\Im_{\ell}^{T^p}(\lambda^q)}\}$,
$\Re_{\ell}^{T^p}((\tau\diamondsuit_3\lambda)^q)\cdot e^{x\Im_{\ell}^{T^p}((\tau\diamondsuit_3\lambda)^q)} \geq \min\{\Re_{\ell}^{T^p}(\tau^q)\cdot e^{x\Im_{\ell}^{T^p}(\tau^q)}, \Re_{\ell}^{T^p}(\lambda^q)\cdot e^{x\Im_{\ell}^{T^p}(\lambda^q)}\}$.
Now,

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$$\begin{aligned}
\Re_{\ell}^{I^p}((\tau\Diamond_1\lambda)^q)\cdot e^{x\Im_{\ell}^{I^p}((\tau\Diamond_1\lambda)^q)} &= \bigwedge_{i\in I}\ \Re_{\varpi_i}^{I^p}((\tau\Diamond_1\lambda)^q)\cdot e^{x\Im_{\ell}^{I^p}((\tau\Diamond_1\lambda)^q)}\\
&\geq \bigwedge_{i\in I}\ \frac{\Re_{\varpi_i}^{I^p}(\tau^q)\cdot e^{x\Im_{\varpi_i}^{I^p}(\tau^q)}+\Re_{\varpi_i}^{I^p}(\lambda^q)\cdot e^{x\Im_{\varpi_i}^{I^p}(\lambda^q)}}{2}\\
&= \frac{\bigwedge_{i\in I}\ \Re_{\varpi_i}^{I^p}(\tau^q)\cdot e^{x\Im_{\varpi_i}^{I^p}(\tau^q)}+\bigwedge_{i\in I}\ \Re_{\varpi_i}^{I^p}(\lambda^q)\cdot e^{x\Im_{\varpi_i}^{I^p}(\lambda^q)}}{2}\\
&= \frac{\Re_{\ell}^{I^p}(\tau^q)\cdot e^{x\Im_{\ell}^{I^p}(\tau^q)}+\Re_{\ell}^{I^p}(\lambda^q)\cdot e^{x\Im_{\ell}^{I^p}(\lambda^q)}}{2}
\end{aligned}$$

Similarly,
$\Re_{\ell}^{I^p}((\tau\Diamond_2\lambda)^q)\cdot e^{x\Im_{\ell}^{I^p}((\tau\Diamond_2\lambda)^q)} \geq \frac{\Re_{\ell}^{I^p}(\tau^q)\cdot e^{x\Im_{\ell}^{I^p}(\tau^q)}+\Re_{\ell}^{I^p}(\lambda^q)\cdot e^{x\Im_{\ell}^{I^p}(\lambda^q)}}{2}$ and
$\Re_{\ell}^{I^p}((\tau\Diamond_3\lambda)^q)\cdot e^{x\Im_{\ell}^{I^p}((\tau\Diamond_3\lambda)^q)} \geq \frac{\Re_{\ell}^{I^p}(\tau^q)\cdot e^{x\Im_{\ell}^{I^p}(\tau^q)}+\Re_{\ell}^{I^p}(\lambda^q)\cdot e^{x\Im_{\ell}^{I^p}(\lambda^q)}}{2}$.
Now,

$$\begin{aligned}
\Re_{\ell}^{F^p}((\tau\Diamond_1\lambda)^q)\cdot e^{x\Im_{\ell}^{F^p}((\tau\Diamond_1\lambda)^q)} &= \bigvee_{i\in I}\ \Re_{\varpi_i}^{F^p}((\tau\Diamond_1\lambda)^q)\cdot e^{x\Im_{\ell}^{F^p}((\tau\Diamond_1\lambda)^q)}\\
&\leq \bigvee_{i\in I}\ \max\{\Re_{\varpi_i}^{F^p}(\tau^q)\cdot e^{x\Im_{\varpi_i}^{F^p}(\tau^q)},\Re_{\varpi_i}^{F^p}(\lambda^q)\cdot e^{x\Im_{\varpi_i}^{F^p}(\lambda^q)}\}\\
&= \max\left\{\bigvee_{i\in I}\ \Re_{\varpi_i}^{F^p}(\tau^q)\cdot e^{x\Im_{\varpi_i}^{F^p}(\tau^q)},\bigvee_{i\in I}\ \Re_{\varpi_i}^{F^p}(\lambda^q)\cdot e^{x\Im_{\varpi_i}^{F^p}(\lambda^q)}\right\}\\
&= \max\{\Re_{\ell}^{F^p}(\tau^q)\cdot e^{x\Im_{\ell}^{F^p}(\tau^q)},\Re_{\ell}^{F^p}(\lambda^q)\cdot e^{x\Im_{\ell}^{F^p}(\lambda^q)}\}
\end{aligned}$$

Similarly,
$\Re_{\ell}^{F^p}((\tau\Diamond_2\lambda)^q)\cdot e^{x\Im_{\ell}^{F^p}((\tau\Diamond_2\lambda)^q)} \leq \max\{\Re_{\ell}^{F^p}(\tau^q)\cdot e^{x\Im_{\ell}^{F^p}(\tau^q)},\Re_{\ell}^{F^p}(\lambda^q)\cdot e^{x\Im_{\ell}^{F^p}(\lambda^q)}\}$ and
$\Re_{\ell}^{F^p}((\tau\Diamond_3\lambda)^q)\cdot e^{x\Im_{\ell}^{F^p}((\tau\Diamond_3\lambda)^q)} \leq \max\{\Re_{\ell}^{F^p}(\tau^q)\cdot e^{x\Im_{\ell}^{F^p}(\tau^q)},\Re_{\ell}^{F^p}(\lambda^q)\cdot e^{x\Im_{\ell}^{F^p}(\lambda^q)}\}$.
Thus,ℓ is a CQBNSBS of $\mathscr{S}$.

Theorem 3.8. *If ℓ and $\eth$ be the CQBNSBSs of $\mathscr{S}_1$ and $\mathscr{S}_2$ respectively, then $\ell\times\eth$is a CQBNSBS of $\mathscr{S}_1\times\mathscr{S}_2$.*

Proof. Let $\tau_1,\tau_2\in\mathscr{S}_1$ and $\lambda_1,\lambda_2\in\mathscr{S}_2$. Then (τ_1,λ_1) and (τ_2,λ_2) are in $\mathscr{S}_1\times\mathscr{S}_2$. Now

$$\begin{aligned}
&\Re_{\ell\times\eth}^{T^n}[((\tau_1,\lambda_1)\Diamond_1(\tau_2,\lambda_2)^q)]\cdot e^{x\Im_{\ell\times\eth}^{T^n}[((\tau_1,\lambda_1)\Diamond_1(\tau_2,\lambda_2)^q)]}\\
=\ &\Re_{\ell\times\eth}^{T^n}((\tau_1\Diamond_1\tau_2,\lambda_1\Diamond_1\lambda_2)^q)\cdot e^{x\Im_{\ell\times\eth}^{T^n}((\tau_1\Diamond_1\tau_2,\lambda_1\Diamond_1\lambda_2)^q)}\\
=\ &\max\{\Re_{\ell}^{T^n}((\tau_1\Diamond_1\tau_2)^q)\cdot e^{x\Im_{\ell}^{T^n}((\tau_1\Diamond_1\tau_2)^q)},\Re_{\eth}^{T^n}((\lambda_1\Diamond_1\lambda_2)^q)\cdot e^{x\Im_{\eth}^{T^n}((\lambda_1\Diamond_1\lambda_2)^q)}\}\\
\leq\ &\max\{\max\{\Re_{\ell}^{T^n}(\tau_1)\cdot e^{x\Im_{\ell}^{T^n}(\tau_1)},\Re_{\ell}^{T^n}(\tau_2)\cdot e^{x\Im_{\ell}^{T^n}(\tau_2)}\},\\
&\max\{\Re_{\eth}^{T^n}(\lambda_1)\cdot e^{x\Im_{\ell}^{T^n}(\lambda_1)},\Re_{\eth}^{T^n}(\lambda_2)\cdot e^{x\Im_{\ell}^{T^n}(\lambda_2)}\}\}\\
=\ &\max\{\max\{\Re_{\ell}^{T^n}(\tau_1)\cdot e^{x\Im_{\ell}^{T^n}(\tau_1)},\Re_{\eth}^{T^n}(\lambda_1)\cdot e^{x\Im_{\ell}^{T^n}(\lambda_1)}\},\\
&\max\{\Re_{\ell}^{T^n}(\tau_2)\cdot e^{x\Im_{\ell}^{T^n}(\tau_2)},\Re_{\eth}^{T^n}(\lambda_2)\cdot e^{x\Im_{\ell}^{T^n}(\lambda_2)}\}\}\\
=\ &\max\{\Re_{\ell\times\eth}^{T^n}((\tau_1,\lambda_1)^q)\cdot e^{x\Im_{\ell\times\eth}^{T^n}((\tau_1,\lambda_1)^q)},\Re_{\ell\times\eth}^{T^n}((\tau_2,\lambda_2)^q)\cdot e^{x\Im_{\ell\times\eth}^{T^n}((\tau_2,\lambda_2)^q)}\}
\end{aligned}$$

Also $\Re_{\ell\times\eth}^{T^n}[((\tau_1,\lambda_1)\Diamond_2(\tau_2,\lambda_2)^q)]\cdot e^{x\Im_{\ell\times\eth}^{T^n}[((\tau_1,\lambda_1)\Diamond_2(\tau_2,\lambda_2)^q)]}$
$\leq\max\{\Re_{\ell\times\eth}^{T^n}((\tau_1,\lambda_1)^q)\cdot e^{x\Im_{\ell\times\eth}^{T^n}((\tau_1,\lambda_1)^q)},\Re_{\ell\times\eth}^{T^n}((\tau_2,\lambda_2)^q)e^{x\Im_{\ell\times\eth}^{T^n}((\tau_2,\lambda_2)^q)}\}$

and $\Re_{\ell\times\eth}^{T^n}[((\tau_1,\lambda_1)\Diamond_3(\tau_2,\lambda_2)^q)]\cdot e^{x\Im_{\ell\times\eth}^{T^n}[((\tau_1,\lambda_1)\Diamond_3(\tau_2,\lambda_2)^q)]}$
$\leq\max\{\Re_{\ell\times\eth}^{T^n}((\tau_1,\lambda_1)^q)\cdot e^{x\Im_{\ell\times\eth}^{T^n}((\tau_1,\lambda_1)^q)},\Re_{\ell\times\eth}^{T^n}((\tau_2,\lambda_2)^q)\cdot e^{x\Im_{\ell\times\eth}^{T^n}((\tau_2,\lambda_2)^q)}\}$.

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Now,

$$\begin{aligned}
& \Re^{I^n}_{\ell\times\eth}[((\tau_1,\lambda_1)\diamondsuit_1(\tau_2,\lambda_2)^q)]\cdot e^{x\Im^{I^n}_{\ell\times\eth}[((\tau_1,\lambda_1)\diamondsuit_1(\tau_2,\lambda_2)^q)]}\\
= & \Re^{I^n}_{\ell\times\eth}((\tau_1\diamondsuit_1\tau_2,\lambda_1\diamondsuit_1\lambda_2)^q)\cdot e^{x\Im^{I^n}_{\ell\times\eth}((\tau_1\diamondsuit_1\tau_2,\lambda_1\diamondsuit_1\lambda_2)^q)}\\
= & \frac{\Re^{I^n}_{\ell}((\tau_1\diamondsuit_1\tau_2)^q)\cdot e^{x\Im^{I^n}_{\ell\times\eth}((\tau_1\diamondsuit_1\tau_2)^q)}+\Re^{I^n}_{\eth}((\lambda_1\diamondsuit_1\lambda_2)^q)\cdot e^{x\Im^{T^n}_{\ell\times\eth}((\lambda_1\diamondsuit_1\lambda_2)^q)}}{2}\\
\le & \frac{1}{2}\left[\frac{\Re^{I^n}_{\ell}(\tau_1)\cdot e^{x\Im^{I^n}_{\ell}(\tau_1)}+\Re^{I^n}_{\ell}(\tau_2)\cdot e^{x\Im^{I^n}_{\ell}(\tau_2)}}{2}+\frac{\Re^{I^n}_{\eth}(\lambda_1)\cdot e^{x\Im^{I^n}_{\eth}(\lambda_1)}+\Re^{I^n}_{\eth}(\lambda_2)\cdot e^{x\Im^{I^n}_{\eth}(\lambda_2)}}{2}\right]\\
= & \frac{1}{2}\left[\frac{\Re^{I^n}_{\ell}(\tau_1)\cdot e^{x\Im^{I^n}_{\ell}(\tau_1)}+\Re^{I^n}_{\eth}(\lambda_1)\cdot e^{x\Im^{I^n}_{\eth}(\lambda_1)}}{2}+\frac{\Re^{I^n}_{\ell}(\tau_2)\cdot e^{x\Im^{I^n}_{\ell}(\tau_2)}+\Re^{I^n}_{\eth}(\lambda_2)\cdot e^{x\Im^{I^n}_{\eth}(\lambda_2)}}{2}\right]\\
= & \frac{1}{2}\Big[\Re^{I^n}_{\ell\times\eth}((\tau_1,\lambda_1)^q)\cdot e^{x\Im^{T^n}_{\ell\times\eth}((\tau_1,\lambda_1)^q)}+\Re^{I^n}_{\ell\times\eth}((\tau_2,\lambda_2)^q)\cdot e^{x\Im^{T^n}_{\ell\times\eth}((\tau_2,\lambda_2)^q)}\Big]
\end{aligned}$$

Also $\Re^{I^n}_{\ell\times\eth}[((\tau_1,\lambda_1)\diamondsuit_2(\tau_2,\lambda_2)^q)]\cdot e^{x\Im^{I^n}_{\ell\times\eth}[((\tau_1,\lambda_1)\diamondsuit_2(\tau_2,\lambda_2)^q)]}\le$ $\frac{1}{2}\Big[\Re^{I^n}_{\ell\times\eth}((\tau_1,\lambda_1)^q)\cdot e^{x\Im^{I^n}_{\ell\times\eth}((\tau_1,\lambda_1)^q)}+\Re^{I^n}_{\ell\times\eth}((\tau_2,\lambda_2)^q)\cdot e^{x\Im^{I^n}_{\ell\times\eth}((\tau_2,\lambda_2)^q)}\Big]$ and $\Re^{I^n}_{\ell\times\eth}[((\tau_1,\lambda_1)\diamondsuit_3(\tau_2,\lambda_2)^q)]\cdot e^{x\Im^{I^n}_{\ell\times\eth}[((\tau_1,\lambda_1)\diamondsuit_3(\tau_2,\lambda_2)^q)]}\le$ $\frac{1}{2}\Big[\Re^{I^n}_{\ell\times\eth}((\tau_1,\lambda_1)^q)\cdot e^{x\Im^{I^n}_{\ell\times\eth}((\tau_1,\lambda_1)^q)}+\Re^{I^n}_{\ell\times\eth}((\tau_2,\lambda_2)^q)\cdot e^{x\Im^{I^n}_{\ell\times\eth}((\tau_2,\lambda_2)^q)}\Big]$.
Now,

$$\begin{aligned}
& \Re^{F^n}_{\ell\times\eth}[((\tau_1,\lambda_1)\diamondsuit_1(\tau_2,\lambda_2)^q)]\cdot e^{x\Im^{F^n}_{\ell\times\eth}[((\tau_1,\lambda_1)\diamondsuit_1(\tau_2,\lambda_2)^q)]}\\
= & \Re^{F^n}_{\ell\times\eth}((\tau_1\diamondsuit_1\tau_2,\lambda_1\diamondsuit_1\lambda_2)^q)\cdot e^{x\Im^{F^n}_{\ell\times\eth}((\tau_1\diamondsuit_1\tau_2,\lambda_1\diamondsuit_1\lambda_2)^q)}\\
= & \min\{\Re^{F^n}_{\ell}((\tau_1\diamondsuit_1\tau_2)^q)\cdot e^{x\Im^{F^n}_{\ell\times\eth}((\tau_1\diamondsuit_1\tau_2)^q)},\Re^{F^n}_{\eth}((\lambda_1\diamondsuit_1\lambda_2)^q)\cdot e^{x\Im^{F^n}_{\ell\times\eth}((\lambda_1\diamondsuit_1\lambda_2)^q)}\}\\
\ge & \min\{\min\{\Re^{F^n}_{\ell}(\tau_1)\cdot e^{x\Im^{F^n}_{\ell}(\tau_1)},\Re^{F^n}_{\ell}(\tau_2)\cdot e^{x\Im^{F^n}_{\ell}(\tau_2)}\},\\
& \min\{\Re^{F^n}_{\eth}(\lambda_1)\cdot e^{x\Im^{F^n}_{\eth}(\lambda_1)},\Re^{F^n}_{\eth}(\lambda_2)\cdot e^{x\Im^{F^n}_{\eth}(\lambda_2)}\}\}\\
= & \min\{\min\{\Re^{F^n}_{\ell}(\tau_1)\cdot e^{x\Im^{F^n}_{\ell}(\tau_1)},\Re^{F^n}_{\eth}(\lambda_1)\cdot e^{x\Im^{F^n}_{\eth}(\lambda_1)}\},\\
& \min\{\Re^{F^n}_{\ell}(\tau_2)\cdot e^{x\Im^{F^n}_{\ell}(\tau_2)},\Re^{F^n}_{\eth}(\lambda_2)\cdot e^{x\Im^{F^n}_{\eth}(\lambda_2)}\}\}\\
= & \min\{\Re^{F^n}_{\ell\times\eth}((\tau_1,\lambda_1)^q)\cdot e^{x\Im^{T^n}_{\ell\times\eth}((\tau_1,\lambda_1)^q)},\Re^{F^n}_{\ell\times\eth}((\tau_2,\lambda_2)^q)\cdot e^{x\Im^{T^n}_{\ell\times\eth}((\tau_2,\lambda_2)^q)}\}
\end{aligned}$$

Also $\Re^{F^n}_{\ell\times\eth}[((\tau_1,\lambda_1)\diamondsuit_2(\tau_2,\lambda_2)^q)]\cdot e^{x\Im^{T^n}_{\ell\times\eth}[((\tau_1,\lambda_1)\diamondsuit_2(\tau_2,\lambda_2)^q)]}$
$\ge\min\{\Re^{F^n}_{\ell\times\eth}((\tau_1,\lambda_1)^q)\cdot e^{x\Im^{T^n}_{\ell\times\eth}((\tau_1,\lambda_1)^q)},\Re^{F^n}_{\ell\times\eth}((\tau_2,\lambda_2)^q)\cdot e^{x\Im^{T^n}_{\ell\times\eth}}((\tau_2,\lambda_2)^q)\}$,
$\Re^{F^n}_{\ell\times\eth}[((\tau_1,\lambda_1)\diamondsuit_3(\tau_2,\lambda_2)^q)]\cdot e^{x\Im^{T^n}_{\ell\times\eth}[((\tau_1,\lambda_1)\diamondsuit_3(\tau_2,\lambda_2)^q)]}$
$\ge\min\{\Re^{F^n}_{\ell\times\eth}((\tau_1,\lambda_1)^q)\cdot e^{x\Im^{T^n}_{\ell\times\eth}((\tau_1,\lambda_1)^q)},\Re^{F^n}_{\ell\times\eth}((\tau_2,\lambda_2)^q)\cdot e^{x\Im^{T^n}_{\ell\times\eth}((\tau_2,\lambda_2)^q)}\}$.
Let $\tau_1,\tau_2\in\mathscr{S}_1$ and $\lambda_1,\lambda_2\in\mathscr{S}_2$. Then (τ_1,λ_1) and (τ_2,λ_2) are in $\mathscr{S}_1\times\mathscr{S}_2$. Now

$$\begin{aligned}
& \Re^{T^p}_{\ell\times\eth}[((\tau_1,\lambda_1)\diamondsuit_1(\tau_2,\lambda_2)^q)]\cdot e^{x\Im^{T^p}_{\ell\times\eth}[((\tau_1,\lambda_1)\diamondsuit_1(\tau_2,\lambda_2)^q)]}\\
= & \Re^{T^p}_{\ell\times\eth}((\tau_1\diamondsuit_1\tau_2,\lambda_1\diamondsuit_1\lambda_2)^q)\cdot e^{x\Im^{T^p}_{\ell\times\eth}((\tau_1\diamondsuit_1\tau_2,\lambda_1\diamondsuit_1\lambda_2)^q)}\\
= & \min\{\Re^{T^p}_{\ell}((\tau_1\diamondsuit_1\tau_2)^q)\cdot e^{x\Im^{T^p}_{\ell}((\tau_1\diamondsuit_1\tau_2)^q)},\Re^{T^p}_{\eth}((\lambda_1\diamondsuit_1\lambda_2)^q)\cdot e^{x\Im^{T^p}_{\eth}((\lambda_1\diamondsuit_1\lambda_2)^q)}\}\\
\ge & \min\{\min\{\Re^{T^p}_{\ell}(\tau_1)\cdot e^{x\Im^{T^p}_{\ell}(\tau_1)},\Re^{T^p}_{\ell}(\tau_2)\cdot e^{x\Im^{T^p}_{\ell}(\tau_2)}\},\min\{\Re^{T^p}_{\eth}(\lambda_1)\cdot e^{x\Im^{T^p}_{\ell}(\lambda_1)},\Re^{T^p}_{\eth}(\lambda_2)\cdot e^{x\Im^{T^p}_{\ell}(\lambda_2)}\}\}\\
= & \min\{\min\{\Re^{T^p}_{\ell}(\tau_1)\cdot e^{x\Im^{T^p}_{\ell}(\tau_1)},\Re^{T^p}_{\eth}(\lambda_1)\cdot e^{x\Im^{T^p}_{\ell}(\lambda_1)}\},\min\{\Re^{T^p}_{\ell}(\tau_2)\cdot e^{x\Im^{T^p}_{\ell}(\tau_2)},\Re^{T^p}_{\eth}(\lambda_2)\cdot e^{x\Im^{T^p}_{\ell}(\lambda_2)}\}\}\\
= & \min\{\Re^{T^p}_{\ell\times\eth}((\tau_1,\lambda_1)^q)\cdot e^{x\Im^{T^p}_{\ell\times\eth}((\tau_1,\lambda_1)^q)},\Re^{T^p}_{\ell\times\eth}((\tau_2,\lambda_2)^q)\cdot e^{x\Im^{T^p}_{\ell\times\eth}((\tau_2,\lambda_2)^q)}\}
\end{aligned}$$

Also $\Re^{T^p}_{\ell\times\eth}[((\tau_1,\lambda_1)\diamondsuit_2(\tau_2,\lambda_2)^q)]\cdot e^{x\Im^{T^p}_{\ell\times\eth}[((\tau_1,\lambda_1)\diamondsuit_2(\tau_2,\lambda_2)^q)]}$
$\ge\min\{\Re^{T^p}_{\ell\times\eth}((\tau_1,\lambda_1)^q)\cdot e^{x\Im^{T^p}_{\ell\times\eth}((\tau_1,\lambda_1)^q)},\Re^{T^p}_{\ell\times\eth}((\tau_2,\lambda_2)^q)e^{x\Im^{T^p}_{\ell\times\eth}((\tau_2,\lambda_2)^q)}\}$

and $\Re^{T^p}_{\ell\times\eth}[((\tau_1,\lambda_1)\diamondsuit_3(\tau_2,\lambda_2)^q)]\cdot e^{x\Im^{T^p}_{\ell\times\eth}[((\tau_1,\lambda_1)\diamondsuit_3(\tau_2,\lambda_2)^q)]}$
$\ge\min\{\Re^{T^p}_{\ell\times\eth}((\tau_1,\lambda_1)^q)\cdot e^{x\Im^{T^p}_{\ell\times\eth}((\tau_1,\lambda_1)^q)},\Re^{T^p}_{\ell\times\eth}((\tau_2,\lambda_2)^q)\cdot e^{x\Im^{T^p}_{\ell\times\eth}((\tau_2,\lambda_2)^q)}\}$.

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Now,

$$\begin{aligned}
& \Re^{I^p}_{\ell\times\partial}[((\tau_1,\lambda_1)\lozenge_1(\tau_2,\lambda_2)^q)]\cdot e^{x\Im^{I^p}_{\ell\times\partial}[((\tau_1,\lambda_1)\lozenge_1(\tau_2,\lambda_2)^q)]} \\
= \; & \Re^{I^p}_{\ell\times\partial}((\tau_1\lozenge_1\tau_2,\lambda_1\lozenge_1\lambda_2)^q)\cdot e^{x\Im^{I^p}_{\ell\times\partial}((\tau_1\lozenge_1\tau_2,\lambda_1\lozenge_1\lambda_2)^q)} \\
= \; & \frac{\Re^{I^p}_{\ell}((\tau_1\lozenge_1\tau_2)^q)\cdot e^{x\Im^{I^p}_{\ell\times\partial}((\tau_1\lozenge_1\tau_2)^q)}+\Re^{I^p}_{\partial}((\lambda_1\lozenge_1\lambda_2)^q)\cdot e^{x\Im^{T^p}_{\ell\times\partial}((\lambda_1\lozenge_1\lambda_2)^q)}}{2} \\
\geq \; & \frac{1}{2}\left[\frac{\Re^{I^p}_{\ell}(\tau_1)\cdot e^{x\Im^{I^p}_{\ell}(\tau_1)}+\Re^{I^p}_{\ell}(\tau_2)\cdot e^{x\Im^{I^p}_{\ell}(\tau_2)}}{2}+\frac{\Re^{I^p}_{\partial}(\lambda_1)\cdot e^{x\Im^{I^p}_{\partial}(\lambda_1)}+\Re^{I^p}_{\partial}(\lambda_2)\cdot e^{x\Im^{I^p}_{\partial}(\lambda_2)}}{2}\right] \\
= \; & \frac{1}{2}\left[\frac{\Re^{I^p}_{\ell}(\tau_1)\cdot e^{x\Im^{I^p}_{\ell}(\tau_1)}+\Re^{I^p}_{\partial}(\lambda_1)\cdot e^{x\Im^{I^p}_{\partial}(\lambda_1)}}{2}+\frac{\Re^{I^p}_{\ell}(\tau_2)\cdot e^{x\Im^{I^p}_{\ell}(\tau_2)}+\Re^{I^p}_{\partial}(\lambda_2)\cdot e^{x\Im^{I^p}_{\partial}(\lambda_2)}}{2}\right] \\
= \; & \frac{1}{2}\left[\Re^{I^p}_{\ell\times\partial}((\tau_1,\lambda_1)^q)\cdot e^{x\Im^{T^p}_{\ell\times\partial}((\tau_1,\lambda_1)^q)}+\Re^{I^p}_{\ell\times\partial}((\tau_2,\lambda_2)^q)\cdot e^{x\Im^{T^p}_{\ell\times\partial}((\tau_2,\lambda_2)^q)}\right]
\end{aligned}$$

Also $\Re^{I^p}_{\ell\times\partial}[((\tau_1,\lambda_1)\lozenge_2(\tau_2,\lambda_2)^q)]\cdot e^{x\Im^{I^p}_{\ell\times\partial}[((\tau_1,\lambda_1)\lozenge_2(\tau_2,\lambda_2)^q)]}\geq$
$\frac{1}{2}\left[\Re^{I^p}_{\ell\times\partial}((\tau_1,\lambda_1)^q)\cdot e^{x\Im^{I^p}_{\ell\times\partial}((\tau_1,\lambda_1)^q)}+\Re^{I^p}_{\ell\times\partial}((\tau_2,\lambda_2)^q)\cdot e^{x\Im^{I^p}_{\ell\times\partial}((\tau_2,\lambda_2)^q)}\right]$
and $\Re^{I^p}_{\ell\times\partial}[((\tau_1,\lambda_1)\lozenge_3(\tau_2,\lambda_2)^q)]\cdot e^{x\Im^{I^p}_{\ell\times\partial}[((\tau_1,\lambda_1)\lozenge_3(\tau_2,\lambda_2)^q)]}\geq$
$\frac{1}{2}\left[\Re^{I^p}_{\ell\times\partial}((\tau_1,\lambda_1)^q)\cdot e^{x\Im^{I^p}_{\ell\times\partial}((\tau_1,\lambda_1)^q)}+\Re^{I^p}_{\ell\times\partial}((\tau_2,\lambda_2)^q)\cdot e^{x\Im^{I^p}_{\ell\times\partial}((\tau_2,\lambda_2)^q)}\right]$.
Now,

$$\begin{aligned}
& \Re^{F^p}_{\ell\times\partial}[((\tau_1,\lambda_1)\lozenge_1(\tau_2,\lambda_2)^q)]\cdot e^{x\Im^{F^p}_{\ell\times\partial}[((\tau_1,\lambda_1)\lozenge_1(\tau_2,\lambda_2)^q)]} \\
= \; & \Re^{F^p}_{\ell\times\partial}((\tau_1\lozenge_1\tau_2,\lambda_1\lozenge_1\lambda_2)^q)\cdot e^{x\Im^{F^p}_{\ell\times\partial}((\tau_1\lozenge_1\tau_2,\lambda_1\lozenge_1\lambda_2)^q)} \\
= \; & \max\{\Re^{F^p}_{\ell}((\tau_1\lozenge_1\tau_2)^q)\cdot e^{x\Im^{F^p}_{\ell\times\partial}((\tau_1\lozenge_1\tau_2)^q)},\Re^{F^p}_{\partial}((\lambda_1\lozenge_1\lambda_2)^q)\cdot e^{x\Im^{F^p}_{\ell\times\partial}((\lambda_1\lozenge_1\lambda_2)^q)}\} \\
\leq \; & \max\{\max\{\Re^{F^p}_{\ell}(\tau_1)\cdot e^{x\Im^{F^p}_{\ell}(\tau_1)},\Re^{F^p}_{\ell}(\tau_2)\cdot e^{x\Im^{F^p}_{\ell}(\tau_2)}\}, \\
& \max\{\Re^{F^p}_{\partial}(\lambda_1)\cdot e^{x\Im^{F^p}_{\partial}(\lambda_1)},\Re^{F^p}_{\partial}(\lambda_2)\cdot e^{x\Im^{F^p}_{\partial}(\lambda_2)}\}\} \\
= \; & \max\{\max\{\Re^{F^p}_{\ell}(\tau_1)\cdot e^{x\Im^{F^p}_{\ell}(\tau_1)},\Re^{F^p}_{\partial}(\lambda_1)\cdot e^{x\Im^{F^p}_{\partial}(\lambda_1)}\}, \\
& \max\{\Re^{F^p}_{\ell}(\tau_2)\cdot e^{x\Im^{F^p}_{\ell}(\tau_2)},\Re^{F^p}_{\partial}(\lambda_2)\cdot e^{x\Im^{F^p}_{\partial}(\lambda_2)}\}\} \\
= \; & \max\{\Re^{F^p}_{\ell\times\partial}((\tau_1,\lambda_1)^q)\cdot e^{x\Im^{T^p}_{\ell\times\partial}((\tau_1,\lambda_1)^q)},\Re^{F^p}_{\ell\times\partial}((\tau_2,\lambda_2)^q)\cdot e^{x\Im^{T^p}_{\ell\times\partial}((\tau_2,\lambda_2)^q)}\}
\end{aligned}$$

Also $\Re^{F^p}_{\ell\times\partial}[((\tau_1,\lambda_1)\lozenge_2(\tau_2,\lambda_2)^q)]\cdot e^{x\Im^{T^p}_{\ell\times\partial}[((\tau_1,\lambda_1)\lozenge_2(\tau_2,\lambda_2)^q)]}$
$\leq \max\{\Re^{F^p}_{\ell\times\partial}((\tau_1,\lambda_1)^q)\cdot e^{x\Im^{T^p}_{\ell\times\partial}((\tau_1,\lambda_1)^q)}$,
$\Re^{F^p}_{\ell\times\partial}((\tau_2,\lambda_2)^q)\cdot e^{x\Im^{T^p}_{\ell\times\partial}}((\tau_2,\lambda_2)^q)\}$,
$\Re^{F^p}_{\ell\times\partial}[((\tau_1,\lambda_1)\lozenge_3(\tau_2,\lambda_2)^q)]\cdot e^{x\Im^{T^p}_{\ell\times\partial}[((\tau_1,\lambda_1)\lozenge_3(\tau_2,\lambda_2)^q)]}$
$\leq \max\{\Re^{F^p}_{\ell\times\partial}((\tau_1,\lambda_1)^q)\cdot e^{x\Im^{T^p}_{\ell\times\partial}((\tau_1,\lambda_1)^q)},\Re^{F^p}_{\ell\times\partial}((\tau_2,\lambda_2)^q)\cdot e^{x\Im^{T^p}_{\ell\times\partial}((\tau_2,\lambda_2)^q)}\}$.
Thus,$\ell\times\partial$ is a CQBNSBS of $\mathscr{S}$.

Corollary 3.9. *If $\ell_1,\ell_2,...,\ell_n$ be the finite collection of CQBNSBSs of $\mathscr{S}_1,\mathscr{S}_2,...,\mathscr{S}_n$ respectively. Then $\ell_1\times\ell_2\times...\times\ell_n$ is a CQBNSBS of $\mathscr{S}_1\times\mathscr{S}_2\times...\times\mathscr{S}_n$.*

Definition 3.10. Let $\ell\subseteq\mathscr{S}$,the strongest CQBN relation on $\mathscr{S}$ is

$$\left\{\begin{array}{l}
\Re^{T^n}_{\varpi}((\tau,\lambda)^q)\cdot e^{x\Im^{T^n}_{\varpi}((\tau,\lambda)^q)}=min\{\Re^{T^n}_{\ell}(\tau^q)\cdot e^{x\Im^{T^n}_{\ell}(\tau^q)},\Re^{T^n}_{\ell}(\lambda^q)\cdot e^{x\Im^{T^n}_{\ell}(\lambda^q)}\} \\
\Re^{I^n}_{\varpi}((\tau,\lambda)^q)\cdot e^{x\Im^{I^n}_{\varpi}((\tau,\lambda)^q)}=\frac{\Re^{I^n}_{\ell}(\tau^q)\cdot e^{x\Im^{I^n}_{\ell}(\tau^q)}+\Re^{I^n}_{\ell}(\lambda^q)\cdot e^{x\Im^{I^n}_{\ell}(\lambda^q)}}{2} \\
\Re^{F^n}_{\varpi}((\tau,\lambda)^q)\cdot e^{x\Im^{F^n}_{\varpi}((\tau,\lambda)^q)}=max\{\Re^{F^n}_{\ell}(\tau^q)\cdot e^{x\Im^{F^n}_{\ell}(\tau^q)},\Re^{F^n}_{\ell}(\lambda^q)\cdot e^{x\Im^{F^n}_{\ell}(\lambda^q)}\}
\end{array}\right\}$$

$$\left\{\begin{array}{l}
\Re^{T^p}_{\varpi}((\tau,\lambda)^q)\cdot e^{x\Im^{T^p}_{\varpi}((\tau,\lambda)^q)}=\max\{\Re^{T^p}_{\ell}(\tau^q)\cdot e^{x\Im^{T^p}_{\ell}(\tau^q)},\Re^{T^p}_{\ell}(\lambda^q)\cdot e^{x\Im^{T^p}_{\ell}(\lambda^q)}\} \\
\Re^{I^p}_{\varpi}((\tau,\lambda)^q)\cdot e^{x\Im^{I^p}_{\varpi}((\tau,\lambda)^q)}=\frac{\Re^{I^p}_{\ell}(\tau^q)\cdot e^{x\Im^{I^p}_{\ell}(\tau^q)}+\Re^{I^p}_{\ell}(\lambda^q)\cdot e^{x\Im^{I^p}_{\ell}(\lambda^q)}}{2} \\
\Re^{F^p}_{\varpi}((\tau,\lambda)^q)\cdot e^{x\Im^{F^p}_{\varpi}((\tau,\lambda)^q)}=\min\{\Re^{F^p}_{\ell}(\tau^q)\cdot e^{x\Im^{F^p}_{\ell}(\tau^q)},\Re^{F^p}_{\ell}(\lambda^q)\cdot e^{x\Im^{F^p}_{\ell}(\lambda^q)}\}
\end{array}\right\}$$

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Theorem 3.11. *Let ℓ be a CQBNSBS of $\mathscr{S}$ and ϖ be a strongest complex bipolar neutrosophic relation of $\mathscr{S}$. Then ℓ is a CQBNSBS of $\mathscr{S} \times \mathscr{S}$ if and only if ϖ is a CQBNSBS of $\mathscr{S} \times \mathscr{S}$.*

Proof. Suppose ℓ is a CQBNSBS of $\mathscr{S} \times \mathscr{S}$ and ϖ be the strongest complex bipolar neutrosophic relation of $\mathscr{S}$.
For any $\tau = (\tau_1, \tau_2), \lambda = (\lambda_1, \lambda_2) \in S \times \mathscr{S}$. Now,

$$\begin{aligned}
& \Re_{\varpi}^{T^n}((\tau\Diamond_1\lambda)^q)\cdot e^{x\Im_{\varpi}^{T^n}((\tau\Diamond_1\lambda)^q)} \\
= \;& \Re_{\varpi}^{T^n}[((((\tau_1,\tau_2)^q)\Diamond_1((\lambda_1,\lambda_2)^q)]\cdot e^{x\Im_{\varpi}^{T^n}[((((\tau_1,\tau_2)^q)\Diamond_1((\lambda_1,\lambda_2)^q)]} \\
= \;& \Re_{\varpi}^{T^n}(\tau_1\Diamond_1\lambda_1,\tau_2\Diamond_1\lambda_2)\cdot e^{x\Im_{\varpi}^{T^n}(\tau_1\Diamond_1\lambda_1,\tau_2\Diamond_1\lambda_2)} \\
= \;& \max\{\Re_{\ell}^{T^n}((\tau_1\Diamond_1\lambda_1)^q)\cdot e^{x\Im_{\ell}^{T^n}((\tau_1\Diamond_1\lambda_1)^q)}, \Re_{\ell}^{T^n}((\tau_2\Diamond_1\lambda_2)^q)\cdot e^{x\Im_{\ell}^{T^n}((\tau_2\Diamond_1\lambda_2)^q)}\} \\
\leq \;& \max\{\max\{\Re_{\ell}^{T^n}(\tau_1)\cdot e^{x\Im_{\ell}^{T^n}(\tau_1)}, \Re_{\ell}^{T^n}(\lambda_1)\cdot e^{x\Im_{\ell}^{T^n}(\lambda_1)}\}, \\
& \max\{\Re_{\ell}^{T^n}(\tau_2)\cdot e^{x\Im_{\ell}^{T^n}(\tau_2)}, \Re_{\ell}^{T^n}(\lambda_2)\cdot e^{x\Im_{\ell}^{T^n}(\lambda_2)}\}\} \\
= \;& \max\{\max\{\Re_{\ell}^{T^n}(\tau_1)\cdot e^{x\Im_{\ell}^{T^n}(\tau_1)}, \Re_{\ell}^{T^n}(\tau_2)\cdot e^{x\Im_{\ell}^{T^n}(\tau_2)}\}, \\
& \max\{\Re_{\ell}^{T^n}(\lambda_1)\cdot e^{x\Im_{\ell}^{T^n}(\lambda_1)}, \Re_{\ell}^{T^n}(\lambda_2)\cdot e^{x\Im_{\ell}^{T^n}(\lambda_2)}\}\} \\
= \;& \max\{\Re_{\varpi}^{T^n}((\tau_1,\tau_2)^q)\cdot e^{x\Im_{\varpi}^{T^n}((\tau_1,\tau_2)^q)}, \Re_{\varpi}^{T^n}((\lambda_1,\lambda_2)^q)\cdot e^{x\Im_{\varpi}^{T^n}((\lambda_1,\lambda_2)^q)}\} \\
= \;& \max\{\Re_{\varpi}^{T^n}(\tau^q)\cdot e^{x\Im_{\varpi}^{T^n}(\tau^q)}, \Re_{\varpi}^{T^n}(\lambda^q)\cdot e^{x\Im_{\varpi}^{T^n}(\lambda^q)}\}
\end{aligned}$$

Also $\Re_{\varpi}^{T^n}((\tau\Diamond_2\lambda)^q)\cdot e^{x\Im_{\varpi}^{T^n}((\tau\Diamond_2\lambda)^q)} \leq \max\{\Re_{\varpi}^{T^n}(\tau^q)\cdot e^{x\Im_{\varpi}^{T^n}(\tau^q)}, \Re_{\varpi}^{T^n}(\lambda^q)\cdot e^{x\Im_{\varpi}^{T^n}(\lambda^q)}\}$,
$\Re_{\varpi}^{T^n}((\tau\Diamond_3\lambda)^q)\cdot e^{x\Im_{\varpi}^{T^n}((\tau\Diamond_3\lambda)^q)} \leq \max\{\Re_{\varpi}^{T^n}(\tau^q)\cdot e^{x\Im_{\varpi}^{T^n}(\tau^q)}, \Re_{\varpi}^{T^n}(\lambda^q)\cdot e^{x\Im_{\varpi}^{T^n}(\lambda^q)}\}$.
Now, $\Re_{\varpi}^{I^n}(\tau\Diamond_1\lambda)\cdot e^{x\Im_{\varpi}^{I^n}(\tau\Diamond_1\lambda)}$

$$\begin{aligned}
&= \Re_{\varpi}^{I^n}[((((\tau_1,\tau_2)^q)\Diamond_1(\lambda_1,\lambda_2)^q)]\cdot e^{x\Im_{\varpi}^{I^n}[((((\tau_1,\tau_2)^q)\Diamond_1((\lambda_1,\lambda_2)^q))]} \\
&= \Re_{\varpi}^{I^n}((\tau_1\Diamond_1\lambda_1,\tau_2\Diamond_1\lambda_2)^q)\cdot e^{x\Im_{\varpi}^{I^n}((\tau_1\Diamond_1\lambda_1,\tau_2\Diamond_1\lambda_2)^q)} \\
&= \frac{\Re_{\ell}^{I^n}((\tau_1\Diamond_1\lambda_1)^q)\cdot e^{x\Im_{\ell}^{I^n}((\tau_1\Diamond_1\lambda_1)^q)} + \Re_{\ell}^{I^n}((\tau_2\Diamond_1\lambda_2)^q)\cdot e^{x\Im_{\ell}^{I^n}}((\tau_2\Diamond_1\lambda_2)^q)}{2} \\
&\leq \frac{1}{2}\left[\frac{\Re_{\ell}^{I^n}(\tau_1)\cdot e^{x\Im_{\ell}^{I^n}(\tau_1)} + \Re_{\ell}^{I^n}(\lambda_1)\cdot e^{x\Im_{\ell}^{I^n}(\lambda_1)}}{2} + \frac{\Re_{\ell}^{I^n}(\tau_2)\cdot e^{x\Im_{\ell}^{I^n}(\tau_2)} + \Re_{\ell}^{I^n}(\lambda_2)\cdot e^{x\Im_{\ell}^{I^n}(\lambda_2)}}{2}\right] \\
&= \frac{1}{2}\left[\frac{\Re_{\ell}^{I^n}(\tau_1)\cdot e^{x\Im_{\ell}^{I^n}(\tau_1)} + \Re_{\ell}^{I^n}(\tau_2)\cdot e^{x\Im_{\ell}^{I^n}(\tau_2)}}{2} + \frac{\Re_{\ell}^{I^n}(\lambda_1)\cdot e^{x\Im_{\ell}^{I^n}(\lambda_1)} + \Re_{\ell}^{I^n}(\lambda_2)\cdot e^{x\Im_{\ell}^{I^n}(\lambda_2)}}{2}\right] \\
&= \frac{\Re_{\varpi}^{I^n}((\tau_1,\tau_2)^q)\cdot e^{x\Im_{\varpi}^{I^n}((\tau_1,\tau_2)^q)} + \Re_{\varpi}^{I^n}((\lambda_1,\lambda_2)^q)\cdot e^{x\Im_{\varpi}^{I^n}((\lambda_1,\lambda_2)^q)}}{2} \\
&= \frac{\Re_{\varpi}^{I^n}(\tau^q)\cdot e^{x\Im_{\varpi}^{I^n}(\tau^q)} + \Re_{\varpi}^{I^n}(\lambda^q)\cdot e^{x\Im_{\varpi}^{I^n}(\tau^q)}}{2}
\end{aligned}$$

Also $\Re_{\varpi}^{I^n}((\tau\Diamond_2\lambda)^q)\cdot e^{x\Im_{\varpi}^{I^n}((\tau\Diamond_2\lambda)^q)} \leq \frac{\Re_{\varpi}^{I^n}(\tau^q)\cdot e^{x\Im_{\varpi}^{I^n}(\tau^q)} + \Re_{\varpi}^{I^n}(\lambda^q)\cdot e^{x\Im_{\varpi}^{I^n}(\lambda^q)}}{2}$
and $\Re_{\varpi}^{I^n}((\tau\Diamond_3\lambda)^q)\cdot e^{x\Im_{\varpi}^{I^n}((\tau\Diamond_3\lambda)^q)} \leq \frac{\Re_{\varpi}^{I^n}(\tau^q)\cdot e^{x\Im_{\varpi}^{I^n}(\tau^q)} + \Re_{\varpi}^{I^n}(\lambda^q)\cdot e^{x\Im_{\varpi}^{I^n}(\lambda^q)}}{2}$.
Similarly,$\Re_{\varpi}^{F^n}((\tau\Diamond_1\lambda)^q)\cdot e^{x\Im_{\varpi}^{F^n}((\tau\Diamond_1\lambda)^q)} \geq \min\{\Re_{\varpi}^{F^n}(\tau^q)\cdot e^{x\Im_{\varpi}^{F^n}(\tau^q)\cdot}, \Re_{\varpi}^{F^n}(\lambda^q)\cdot e^{x\Im_{\varpi}^{F^n}(\lambda^q)}\}$,
$\Re_{\varpi}^{F^n}((\tau\Diamond_2\lambda)^q)\cdot e^{x\Im_{\varpi}^{F^n}((\tau\Diamond_2\lambda)^q)} \geq \min\{\Re_{\varpi}^{F^n}(\tau^q)\cdot e^{x\Im_{\varpi}^{F^n}(\tau^q)}, \Re_{\varpi}^{F^n}(\lambda^q)\cdot e^{x\Im_{\varpi}^{F^n}(\lambda^q)}\}$ and
$\Re_{\varpi}^{F^n}((\tau\Diamond_3\lambda)^q)\cdot e^{x\Im_{\varpi}^{F^n}((\tau\Diamond_3\lambda)^q)} \geq \min\{\Re_{\varpi}^{F^n}(\tau^q)\cdot e^{x\Im_{\varpi}^{F^n}(\tau^q)}, \Re_{\varpi}^{F^n}(\lambda^q)\cdot e^{x\Im_{\varpi}^{I^n}(\lambda}\}$.

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For any $\tau = (\tau_1, \tau_2), \lambda = (\lambda_1, \lambda_2) \in \mathscr{S} \times \mathscr{S}$. Now,

$$\begin{aligned}
& \Re_{\varpi}^{T^p}((\tau\Diamond_1\lambda)^q) \cdot e^{x\Im_{\varpi}^{T^p}((\tau\Diamond_1\lambda)^q)} \\
= \; & \Re_{\varpi}^{T^p}[((((\tau_1,\tau_2)^q)\Diamond_1((\lambda_1,\lambda_2)^q)] \cdot e^{x\Im_{\varpi}^{T^p}[((((\tau_1,\tau_2)^q)\Diamond_1((\lambda_1,\lambda_2)^q)]} \\
= \; & \Re_{\varpi}^{T^p}(\tau_1\Diamond_1\lambda_1, \tau_2\Diamond_1\lambda_2) \cdot e^{x\Im_{\varpi}^{T^p}(\tau_1\Diamond_1\lambda_1, \tau_2\Diamond_1\lambda_2)} \\
= \; & \min\{\Re_{\ell}^{T^p}((\tau_1\Diamond_1\lambda_1)^q) \cdot e^{x\Im_{\ell}^{T^p}((\tau_1\Diamond_1\lambda_1)^q)}, \Re_{\ell}^{T^p}((\tau_2\Diamond_1\lambda_2)^q) \cdot e^{x\Im_{\ell}^{T^p}((\tau_2\Diamond_1\lambda_2)^q)}\} \\
\geq \; & \min\{\min\{\Re_{\ell}^{T^p}(\tau_1) \cdot e^{x\Im_{\ell}^{T^p}(\tau_1)}, \Re_{\ell}^{T^p}(\lambda_1) \cdot e^{x\Im_{\ell}^{T^p}(\lambda_1)}\}, \\
& \min\{\Re_{\ell}^{T^p}(\tau_2) \cdot e^{x\Im_{\ell}^{T^p}(\tau_2)}, \Re_{\ell}^{T^p}(\lambda_2) \cdot e^{x\Im_{\ell}^{T^p}(\lambda_2)}\}\} \\
= \; & \min\{\min\{\Re_{\ell}^{T^p}(\tau_1) \cdot e^{x\Im_{\ell}^{T^p}(\tau_1)}, \Re_{\ell}^{T^p}(\tau_2) \cdot e^{x\Im_{\ell}^{T^p}(\tau_2)}\}, \\
& \min\{\Re_{\ell}^{T^p}(\lambda_1) \cdot e^{x\Im_{\ell}^{T^p}(\lambda_1)}, \Re_{\ell}^{T^p}(\lambda_2) \cdot e^{x\Im_{\ell}^{T^p}(\lambda_2)}\}\} \\
= \; & \min\{\Re_{\varpi}^{T^p}((\tau_1,\tau_2)^q) \cdot e^{x\Im_{\varpi}^{T^p}((\tau_1,\tau_2)^q)}, \Re_{\varpi}^{T^p}((\lambda_1,\lambda_2)^q) \cdot e^{x\Im_{\varpi}^{T^p}((\lambda_1,\lambda_2)^q)}\} \\
= \; & \min\{\Re_{\varpi}^{T^p}(\tau^q) \cdot e^{x\Im_{\varpi}^{T^p}(\tau^q)}, \Re_{\varpi}^{T^p}(\lambda^q) \cdot e^{x\Im_{\varpi}^{T^p}(\lambda^q)}\}
\end{aligned}$$

Also $\Re_{\varpi}^{T^p}((\tau\Diamond_2\lambda)^q) \cdot e^{x\Im_{\varpi}^{T^p}((\tau\Diamond_2\lambda)^q)} \geq \min\{\Re_{\varpi}^{T^p}(\tau^q) \cdot e^{x\Im_{\varpi}^{T^p}(\tau^q)}, \Re_{\varpi}^{T^p}(\lambda^q) \cdot e^{x\Im_{\varpi}^{T^p}(\lambda^q)}\}$,
$\Re_{\varpi}^{T^p}((\tau\Diamond_3\lambda)^q) \cdot e^{x\Im_{\varpi}^{T^p}((\tau\Diamond_3\lambda)^q)} \geq \min\{\Re_{\varpi}^{T^p}(\tau^q) \cdot e^{x\Im_{\varpi}^{T^p}(\tau^q)}, \Re_{\varpi}^{T^p}(\lambda^q) \cdot e^{x\Im_{\varpi}^{T^p}(\lambda^q)}\}$.
Now, $\Re_{\varpi}^{I^p}(\tau\Diamond_1\lambda) \cdot e^{x\Im_{\varpi}^{I^p}(\tau\Diamond_1\lambda)}$

$$\begin{aligned}
& = \Re_{\varpi}^{I^p}[((((\tau_1,\tau_2)^q)\Diamond_1(\lambda_1,\lambda_2)^q)] \cdot e^{x\Im_{\varpi}^{I^p}[((((\tau_1,\tau_2)^q)\Diamond_1((\lambda_1,\lambda_2)^q))]} \\
& = \Re_{\varpi}^{I^p}((\tau_1\Diamond_1\lambda_1, \tau_2\Diamond_1\lambda_2)^q) \cdot e^{x\Im_{\varpi}^{I^p}((\tau_1\Diamond_1\lambda_1, \tau_2\Diamond_1\lambda_2)^q)} \\
& = \frac{\Re_{\ell}^{I^p}((\tau_1\Diamond_1\lambda_1)^q) \cdot e^{x\Im_{\ell}^{I^p}((\tau_1\Diamond_1\lambda_1)^q)} + \Re_{\ell}^{I^p}((\tau_2\Diamond_1\lambda_2)^q) \cdot e^{x\Im_{\ell}^{I^p}}((\tau_2\Diamond_1\lambda_2)^q)}{2} \\
& \geq \frac{1}{2}\left[\frac{\Re_{\ell}^{I^p}(\tau_1) \cdot e^{x\Im_{\ell}^{I^p}(\tau_1)} + \Re_{\ell}^{I^p}(\lambda_1) \cdot e^{x\Im_{\ell}^{I^p}(\lambda_1)}}{2} + \frac{\Re_{\ell}^{I^p}(\tau_2) \cdot e^{x\Im_{\ell}^{I^p}(\tau_2)} + \Re_{\ell}^{I^p}(\lambda_2) \cdot e^{x\Im_{\ell}^{I^p}(\lambda_2)}}{2}\right] \\
& = \frac{1}{2}\left[\frac{\Re_{\ell}^{I^p}(\tau_1) \cdot e^{x\Im_{\ell}^{I^p}(\tau_1)} + \Re_{\ell}^{I^p}(\tau_2) \cdot e^{x\Im_{\ell}^{I^p}(\tau_2)}}{2} + \frac{\Re_{\ell}^{I^p}(\lambda_1) \cdot e^{x\Im_{\ell}^{I^p}(\lambda_1)} + \Re_{\ell}^{I^p}(\lambda_2) \cdot e^{x\Im_{\ell}^{I^p}(\lambda_2)}}{2}\right] \\
& = \frac{\Re_{\varpi}^{I^p}((\tau_1,\tau_2)^q) \cdot e^{x\Im_{\varpi}^{I^p}((\tau_1,\tau_2)^q)} + \Re_{\varpi}^{I^p}((\lambda_1,\lambda_2)^q) \cdot e^{x\Im_{\varpi}^{I^p}((\lambda_1,\lambda_2)^q)}}{2} \\
& = \frac{\Re_{\varpi}^{I^p}(\tau^q) \cdot e^{x\Im_{\varpi}^{I^p}(\tau^q)} + \Re_{\varpi}^{I^p}(\lambda^q) \cdot e^{x\Im_{\varpi}^{I^p}(\tau^q)}}{2}
\end{aligned}$$

Also $\Re_{\varpi}^{I^p}((\tau\Diamond_2\lambda)^q) \cdot e^{x\Im_{\varpi}^{I^p}((\tau\Diamond_2\lambda)^q)} \geq \frac{\Re_{\varpi}^{I^p}(\tau^q) \cdot e^{x\Im_{\varpi}^{I^p}(\tau^q)} + \Re_{\varpi}^{I^p}(\lambda^q) \cdot e^{x\Im_{\varpi}^{I^p}(\lambda^q)}}{2}$
and $\Re_{\varpi}^{I^p}((\tau\Diamond_3\lambda)^q) \cdot e^{x\Im_{\varpi}^{I^p}((\tau\Diamond_3\lambda)^q)} \geq \frac{\Re_{\varpi}^{I^p}(\tau^q) \cdot e^{x\Im_{\varpi}^{I^p}(\tau^q)} + \Re_{\varpi}^{I^p}(\lambda^q) \cdot e^{x\Im_{\varpi}^{I^p}(\lambda^q)}}{2}$.
Similarly, $\Re_{\varpi}^{F^p}((\tau\Diamond_1\lambda)^q) \cdot e^{x\Im_{\varpi}^{F^p}((\tau\Diamond_1\lambda)^q)} \leq \max\{\Re_{\varpi}^{F^p}(\tau^q) \cdot e^{x\Im_{\varpi}^{F^p}(\tau^q) \cdot}, \Re_{\varpi}^{F^p}(\lambda^q) \cdot e^{x\Im_{\varpi}^{F^p}(\lambda^q)}\}$,
$\Re_{\varpi}^{F^p}((\tau\Diamond_2\lambda)^q) \cdot e^{x\Im_{\varpi}^{F^p}((\tau\Diamond_2\lambda)^q)} \leq \max\{\Re_{\varpi}^{F^p}(\tau^q) \cdot e^{x\Im_{\varpi}^{F^p}(\tau^q)}, \Re_{\varpi}^{F^p}(\lambda^q) \cdot e^{x\Im_{\varpi}^{F^p}(\lambda^q)}\}$ and
$\Re_{\varpi}^{F^p}((\tau\Diamond_3\lambda)^q) \cdot e^{x\Im_{\varpi}^{F^p}((\tau\Diamond_3\lambda)^q)} \leq \max\{\Re_{\varpi}^{F^p}(\tau^q) \cdot e^{x\Im_{\varpi}^{F^p}(\tau^q)}, \Re_{\varpi}^{F^p}(\lambda^q) \cdot e^{x\Im_{\varpi}^{I^p}(\lambda}\}$.
Therefore, ϖ is a CQBNSBS of $\mathscr{S} \times \mathscr{S}$.

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Conversely, suppose that ϖ is a CQBNSBS of $\mathscr{S}\times\mathscr{S}$. Let $\tau=((\tau_1,\tau_2)^q),\lambda=((\lambda_1,\lambda_2)^q)\in\mathscr{S}\times\mathscr{S}$. Now,

$$\begin{aligned}
&\max\{\Re_\ell^{T^n}((\tau_1\diamondsuit_1\lambda_1)^q)\cdot e^{x\Im_\ell^{T^n}((\tau_1\diamondsuit_1\lambda_1)^q)},\Re_\ell^{T^n}((\tau_2\diamondsuit_1\lambda_2)^q)\cdot e^{x\Im_\ell^{T^n}((\tau_2\diamondsuit_1\lambda_2)^q)}\}\\
&=\Re_\varpi^{T^n}(\tau_1\diamondsuit_1\lambda_1,\tau_2\diamondsuit_1\lambda_2)\cdot e^{x\Im_\varpi^{T^n}(\tau_1\diamondsuit_1\lambda_1,\tau_2\diamondsuit_1\lambda_2)}\\
&=\Re_\varpi^{T^n}[((\tau_1,\tau_2)^q)\diamondsuit_1((\lambda_1,\lambda_2)^q)]\cdot e^{x\Im_v^{T^p}[((\tau_1,\tau_2)^q)\diamondsuit_1((\lambda_1,\lambda_2)^q)]}\\
&=\Re_\varpi^{T^n}(\tau\diamondsuit_1\lambda)\cdot e^{x\Im_\varpi^{T^n}(\tau\diamondsuit_1\lambda)}\\
&\le\max\{\Re_\varpi^{T^n}(\tau^q)\cdot e^{x\Im_\varpi^{T^n}(\tau^q)},\Re_\varpi^{T^n}(\lambda^q)\cdot e^{x\Im_\varpi^{T^n}(\lambda^q)}\}\\
&=\max\{\Re_\varpi^{T^n}((\tau_1,\tau_2)^q)\}\cdot e^{x\Im_\varpi^{T^n}((\tau_1,\tau_2)^q)\}},\Re_\varpi^{T^n}((\lambda_1,\lambda_2)^q)\cdot e^{x\Im_\varpi^{T^n}((\lambda_1,\lambda_2)^q)}\}\\
&=\max\{\max\{\Re_\ell^{T^n}(\tau_1)\cdot e^{x\Im_\ell^{T^n}(\tau_1)},\Re_\ell^{T^n}(\tau_2)\cdot e^{x\Im_\ell^{T^n}(\tau_2)}\},\max\{\Re_\ell^{T^n}(\lambda_1)\cdot e^{x\Im_\ell^{T^n}(\lambda_1)},\Re_\ell^{T^n}(\lambda_2)\cdot e^{x\Im_\ell^{T^n}(\lambda_2)}\}\}
\end{aligned}$$

If $\Re_\ell^{T^n}((\tau_1\diamondsuit_1\lambda_1)^q)\cdot e^{x\Im_\ell^{T^n}((\tau_1\diamondsuit_1\lambda_1)^q)}\ge\Re_\ell^{T^n}((\tau_2\diamondsuit_1\lambda_2)^q)\cdot e^{x\Im_\ell^{T^n}((\tau_2\diamondsuit_1\lambda_2)^q)}$,
then $\Re_\ell^{T^n}(\tau_1)\cdot e^{x\Im_\ell^{T^n}(\tau_1)}\ge\Re_\ell^{T^n}(\tau_2)\cdot e^{x\Im_\ell^{T^n}(\tau_2)}$ and $\Re_\ell^{T^n}(\lambda_1)\cdot e^{x\Im_\ell^{T^n}(\lambda_1)}\ge\Re_\ell^{T^n}(\lambda_2)\cdot e^{x\Im_\ell^{T^n}(\lambda_2)}$.
We get $\Re_\ell^{T^n}((\tau_1\diamondsuit_1\lambda_1)^q)\cdot e^{x\Im_\ell^{T^n}((\tau_1\diamondsuit_1\lambda_1)^q)}\le\max\{\Re_\ell^{T^n}(\tau_1)\cdot e^{x\Im_\ell^{T^n}(\tau_1)},\Re_\ell^{T^n}(\lambda_1)\cdot e^{x\Im_\ell^{T^n}(\lambda_1)}\}$ for all $\tau_1,\lambda_1\in\mathscr{S}$,and
$\max\{\Re_\ell^{T^n}((\tau_1\diamondsuit_2\lambda_1)^q)\cdot e^{x\Im_\ell^{T^n}((\tau_1\diamondsuit_2\lambda_1)^q)},\Re_\ell^{T^n}((\tau_2\diamondsuit_2\lambda_2)^q)\cdot e^{x\Im_\ell^{T^n}((\tau_2\diamondsuit_2\lambda_2)^q)}\}\le\max\{\max\{\Re_\ell^{T^n}(\tau_1)\cdot e^{x\Im_\ell^{T^n}(\tau_1)},\Re_\ell^{T^n}(\tau_2)\cdot e^{x\Im_\ell^{T^n}(\tau_2)}\},\max\{\Re_\ell^{T^n}(\lambda_1)\cdot e^{x\Im_\ell^{T^n}(\lambda_1)},\Re_\ell^{T^n}(\lambda_2)\cdot e^{x\Im_\ell^{T^n}(\lambda_2)}\}\}$
If $\Re_\ell^{T^n}((\tau_1\diamondsuit_2\lambda_1)^q)\cdot e^{x\Im_\ell^{T^n}((\tau_1\diamondsuit_2\lambda_1)^q)}\ge\Re_\ell^{T^n}((\tau_2\diamondsuit_2\lambda_2)^q)\cdot e^{x\Im_\ell^{T^n}((\tau_2\diamondsuit_2\lambda_2)^q)}$,then $\Re_\ell^{T^n}((\tau_1\diamondsuit_2\lambda_1)^q)\cdot e^{x\Im_\ell^{T^n}((\tau_1\diamondsuit_2\lambda_1)^q)}\le\max\{\Re_\ell^{T^n}(\tau_1)\cdot e^{x\Im_\ell^{T^n}(\tau_1)},\Re_\ell^{T^n}(\lambda_1)\cdot e^{x\Im_\ell^{T^n}(\lambda_1)}\}$.
$\max\{\Re_\ell^{T^n}((\tau_1\diamondsuit_3\lambda_1)^q)\cdot e^{x\Im_\ell^{T^n}((\tau_1\diamondsuit_3\lambda_1)^q)},\Re_\ell^{T^n}((\tau_2\diamondsuit_3\lambda_2)^q)\cdot e^{x\Im_\ell^{T^n}((\tau_2\diamondsuit_3\lambda_2)^q)}\}\le\max\{\max\{\Re_\ell^{T^n}(\tau_1)\cdot e^{x\Im_\ell^{T^n}(\tau_1)},\Re_\ell^{T^n}(\tau_2)\cdot e^{x\Im_\ell^{T^n}(\tau_2)}\},\max\{\Re_\ell^{T^n}(\lambda_1)\cdot e^{x\Im_\ell^{T^n}(\lambda_1)},\Re_\ell^{T^n}(\lambda_2)\cdot e^{x\Im_\ell^{T^n}(\lambda_2)}\}\}$
If $\Re_\ell^{T^n}((\tau_1\diamondsuit_3\lambda_1)^q)\cdot e^{x\Im_\ell^{T^n}((\tau_1\diamondsuit_3\lambda_1)^q)}\ge\Re_\ell^{T^n}((\tau_2\diamondsuit_3\lambda_2)^q)\cdot e^{x\Im_\ell^{T^n}((\tau_2\diamondsuit_3\lambda_2)^q)}$,then $\Re_\ell^{T^n}((\tau_1\diamondsuit_3\lambda_1)^q)\cdot e^{x\Im_\ell^{T^n}((\tau_1\diamondsuit_3\lambda_1)^q)}\le\max\{\Re_\ell^{T^n}(\tau_1)\cdot e^{x\Im_\ell^{T^n}(\tau_1)},\Re_\ell^{T^n}(\lambda_1)\cdot e^{x\Im_\ell^{T^n}(\lambda_1)}\}$.
Now,

$$\begin{aligned}
&\frac{1}{2}\Big[\Re_\ell^{I^n}((\tau_1\diamondsuit_1\lambda_1)^q)\cdot e^{x\Im_\ell^{I^n}((\tau_1\diamondsuit_1\lambda_1)^q)}+\Re_\ell^{I^n}((\tau_2\diamondsuit_1\lambda_2)^q)\cdot e^{x\Im_\ell^{I^n((\tau_2\diamondsuit_1\lambda_2)^q)}}\Big]\\
&=\Re_\varpi^{I^n}(\tau_1\diamondsuit_1\lambda_1,\tau_2\diamondsuit_1\lambda_2)\cdot e^{x\Im_\varpi^{I^n}(\tau_1\diamondsuit_1\lambda_1,\tau_2\diamondsuit_1\lambda_2)}\\
&=\Re_\varpi^{I^n}[((\tau_1,\tau_2)^q)\diamondsuit_1((\lambda_1,\lambda_2)^q)]\cdot e^{x\Im_\varpi^{I^n}[((\tau_1,\tau_2)^q)\diamondsuit_1((\lambda_1,\lambda_2)^q)]}\\
&=\Re_\varpi^{I^n}((\tau\diamondsuit_1\lambda)^q)\cdot e^{x\Im_\varpi^{I^n}((\tau\diamondsuit_1\lambda)^q)}\\
&\le\frac{\Re_\varpi^{I^n}(\tau^q)\cdot e^{x\Im_\varpi^{I^n(\tau^q)}}+\Re_\varpi^{I^n}(\lambda^q)\cdot e^{x\Im_\varpi^{I^n}(\lambda^q)}}{2}\\
&=\frac{\Re_\varpi^{I^n}((\tau_1,\tau_2)^q)\cdot e^{x\Im_\varpi^{I^n}((\tau_1,\tau_2)^q)}+\Re_\varpi^{I^n}((\lambda_1,\lambda_2)^q)\cdot e^{x\Im_\varpi^{I^n}}}{2}\\
&=\frac{1}{2}\left[\frac{\Re_\ell^{I^n}(\tau_1)\cdot e^{x\Im_\ell^{I^n}(\tau_1)}+\Re_\ell^{I^n}(\tau_2)\cdot e^{x\Im_\ell^{I^n}(\tau_2)}}{2}+\frac{\Re_\ell^{I^n}(\lambda_1)\cdot e^{x\Im_\ell^{I^n}(\lambda_1)}+\Re_\ell^{I^n}(\lambda_2)\cdot e^{x\Im_\ell^{I^n}(\lambda_2)}}{2}\right]
\end{aligned}$$

If $\Re_\ell^{I^n}((\tau_1\diamondsuit_1\lambda_1)^q)\cdot e^{x\Im_\ell^{I^n}((\tau_1\diamondsuit_1\lambda_1)^q)}\ge\Re_\ell^{I^n}((\tau_2\diamondsuit_1\lambda_2)^q)\cdot e^{x\Im_\ell^{I^n}((\tau_2\diamondsuit_1\lambda_2)^q)}$, then $\Re_\ell^{I^n}(\tau_1)\cdot e^{x\Im_\ell^{I^n}(\tau_1)}\ge\Re_\ell^{I^n}(\tau_2)\cdot e^{x\Im_\ell^{I^n}(\tau_2)}$ and $\Re_\ell^{I^n}(\lambda_1)\cdot e^{x\Im_\ell^{I^n}(\lambda_1)}\ge\Re_\ell^{I^n}(\lambda_2)\cdot e^{x\Im_\ell^{I^n}(\lambda_2)}$.
We get $\Re_\ell^{I^n}((\tau_1\diamondsuit_1\lambda_1)^q)\cdot e^{x\Im_\ell^{T^n}((\tau_1\diamondsuit_1\lambda_1)^q)}\le\frac{\Re_\ell^{I^n}(\tau_1)\cdot e^{x\Im_\ell^{I^n}(\tau_1)}+\Re_\ell^{I^n}(\lambda_1)\cdot e^{x\Im_\ell^{T^n}(\lambda_1)}}{2}$.
Similarly,$\Re_\ell^{I^n}((\tau_1\diamondsuit_2\lambda_1)^q)\cdot e^{x\Im_\ell^{I^n}((\tau_1\diamondsuit_2\lambda_1)^q)}\le\frac{\Re_\ell^{I^n}(\tau_1)\cdot e^{x\Im_\ell^{I^n}(\tau_1)}+\Re_\ell^{I^n}(\lambda_1)\cdot e^{x\Im_\ell^{I^n}(\lambda_1)}}{2}$
and $\Re_\ell^{I^n}((\tau_1\diamondsuit_3\lambda_1)^q)\cdot e^{x\Im_\ell^{I^n}((\tau_1\diamondsuit_3\lambda_1)^q)}\le\frac{\Re_\ell^{I^n}(\tau_1)\cdot e^{x\Im_\ell^{I^n}(\tau_1)}+\Re_\ell^{I^n}(\lambda_1)\cdot e^{x\Im_\ell^{I^n}(\lambda_1)}}{2}$.
Similarly to prove that
$\min\{\Re_\ell^{F^n}((\tau_1\diamondsuit_1\lambda_1)^q)\cdot e^{x\Im_\ell^{F^n}((\tau_1\diamondsuit_1\lambda_1)^q)},\Re_\ell^{F^n}((\tau_2\diamondsuit_1\lambda_2)^q)\cdot e^{x\Im_\ell^{F^n}((\tau_2\diamondsuit_1\lambda_2)^q)}\}\ge\min\{\min\{\Re_\ell^{F^n}(\tau_1)\cdot e^{x\Im_\ell^{F^n}(\tau_1)},\Re_\ell^{F^n}(\tau_2)\cdot e^{x\Im_\ell^{F^n}(\tau_2)}\},\min\{\Re_\ell^{F^n}(\lambda_1)\cdot e^{x\Im_\ell^{F^n}(\lambda_1)},\Re_\ell^{F^n}(\lambda_2)\cdot e^{x\Im_\ell^{F^n}(\lambda_2)}\}\}$
If $\Re_\ell^{F^n}((\tau_1\diamondsuit_1\lambda_1)^q)\cdot e^{x\Im_\ell^{F^n}((\tau_1\diamondsuit_1\lambda_1)^q)}\le\Re_\ell^{F^n}((\tau_2\diamondsuit_1\lambda_2)^q)\cdot e^{x\Im_\ell^{F^n}((\tau_2\diamondsuit_1\lambda_2)^q)}$,then $\Re_\ell^{F^n}(\tau_1)\cdot e^{x\Im_\ell^{F^n}(\tau_1)}\le\Re_\ell^{F^n}(\tau_2)\cdot e^{x\Im_\ell^{F^n}(\tau_2)}$ and $\Re_\ell^{F^n}(\lambda_1)\cdot e^{x\Im_\ell^{F^n}(\lambda_1)}\le\Re_\ell^{F^n}(\lambda_2)\cdot e^{x\Im_\ell^{F^n}(\lambda_2)}$.

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We get $\Re_\ell^{F^n}((\tau_1\diamondsuit_1\lambda_1)^q)\cdot e^{x\Im_\ell^{F^n}((\tau_1\diamondsuit_1\lambda_1)^q)} \geq \min\{\Re_\ell^{F^n}(\tau_1)\cdot e^{x\Im_\ell^{F^n}(\tau_1)}, \Re_\ell^{F^n}(\lambda_1)\cdot e^{x\Im_\ell^{F^n}(\lambda_1)}\}$.
$\min\{\Re_\ell^{F^n}((\tau_1\diamondsuit_2\lambda_1)^q)\cdot e^{x\Im_\ell^{F^n}((\tau_1\diamondsuit_2\lambda_1)^q)}, \Re_\ell^{F^n}((\tau_2\diamondsuit_2\lambda_2)^q)\cdot e^{x\Im_\ell^{F^n}((\tau_2\diamondsuit_2\lambda_2)^q)}\}$
$\geq \min\{\min\{\Re_\ell^{F^n}(\tau_1)\cdot e^{x\Im_\ell^{F^n}(\tau_1)}, \Re_\ell^{F^n}(\tau_2)\cdot e^{x\Im_\ell^{F^n}(\tau_2)}\}, \min\{\Re_\ell^{F^n}(\lambda_1)\cdot e^{x\Im_\ell^{F^n}(\lambda_1)}, \Re_\ell^{F^n}(\lambda_2)\cdot e^{x\Im_\ell^{F^n}}\}\}$
If $\Re_\ell^{F^n}((\tau_1\diamondsuit_2\lambda_1)^q)\cdot e^{x\Im_\ell^{F^n}((\tau_1\diamondsuit_2\lambda_1)^q)} \leq \Re_\ell^{F^n}((\tau_2\diamondsuit_2\lambda_2)^q)\cdot e^{x\Im_\ell^{F^n}((\tau_2\diamondsuit_2\lambda_2)^q)}$,then $\Re_\ell^{F^n}((\tau_1\diamondsuit_2\lambda_1)^q)\cdot e^{x\Im_\ell^{F^n}((\tau_1\diamondsuit_2\lambda_1)^q)} \geq \min\{\Re_\ell^{F^n}(\tau_1)\cdot e^{x\Im_\ell^{F^n}(\tau_1)}, \Re_\ell^{F^n}(\lambda_1)\cdot e^{x\Im_\ell^{F^n}(\lambda_1)}\}$.
$\min\{\Re_\ell^{F^n}((\tau_1\diamondsuit_3\lambda_1)^q)\cdot e^{x\Im_\ell^{F^n}((\tau_1\diamondsuit_3\lambda_1)^q)}, \Re_\ell^{F^n}((\tau_2\diamondsuit_3\lambda_2)^q)\cdot e^{x\Im_\ell^{F^n}((\tau_2\diamondsuit_3\lambda_2)^q)}\} \geq \min\{\min\{\Re_\ell^{F^n}(\tau_1)\cdot e^{x\Im_\ell^{F^n}(\tau_1)}, \Re_\ell^{F^n}(\tau_2)\cdot e^{x\Im_\ell^{F^n}(\tau_2)}\}, \min\{\Re_\ell^{F^n}(\lambda_1)\cdot e^{x\Im_\ell^{F^n}(\lambda_1)}, \Re_\ell^{F^n}(\lambda_2)\cdot e^{x\Im_\ell^{F^n}(\lambda_2)}\}\}$
If $\Re_\ell^{F^n}((\tau_1\diamondsuit_3\lambda_1)^q)\cdot e^{x\Im_\ell^{T^n}((\tau_1\diamondsuit_3\lambda_1)^q)} \leq \Re_\ell^{F^n}((\tau_2\diamondsuit_3\lambda_2)^q)\cdot e^{x\Im_\ell^{T^n}((\tau_2\diamondsuit_3\lambda_2)^q)}$,then $\Re_\ell^{F^n}((\tau_1\diamondsuit_3\lambda_1)^q)\cdot e^{x\Im_\ell^{T^n}((\tau_1\diamondsuit_3\lambda_1)^q)} \geq \min\{\Re_\ell^{F^n}(\tau_1)\cdot e^{x\Im_\ell^{F^n}(\tau_1)}, \Re_\ell^{F^n}(\lambda_1)\cdot e^{x\Im_\ell^{T^n}(\lambda_1)}\}$.

Let $\tau = ((\tau_1,\tau_2)^q), \lambda = ((\lambda_1,\lambda_2)^q) \in \mathscr{S}\times\mathscr{S}$. Now,

$$\begin{aligned}
&\min\{\Re_\ell^{T^p}((\tau_1\diamondsuit_1\lambda_1)^q)\cdot e^{x\Im_\ell^{T^p}((\tau_1\diamondsuit_1\lambda_1)^q)}, \Re_\ell^{T^p}((\tau_2\diamondsuit_1\lambda_2)^q)\cdot e^{x\Im_\ell^{T^p}((\tau_2\diamondsuit_1\lambda_2)^q)}\}\\
&= \Re_\varpi^{T^p}(\tau_1\diamondsuit_1\lambda_1,\tau_2\diamondsuit_1\lambda_2)\cdot e^{x\Im_\varpi^{T^p}(\tau_1\diamondsuit_1\lambda_1,\tau_2\diamondsuit_1\lambda_2)}\\
&= \Re_\varpi^{T^p}[((\tau_1,\tau_2)^q)\diamondsuit_1((\lambda_1,\lambda_2)^q)]\cdot e^{x\Im_v^{T^p}[((\tau_1,\tau_2)^q)\diamondsuit_1((\lambda_1,\lambda_2)^q)]}\\
&= \Re_\varpi^{T^p}(\tau\diamondsuit_1\lambda)\cdot e^{x\Im_\varpi^{T^p}(\tau\diamondsuit_1\lambda)}\\
&\geq \min\{\Re_\varpi^{T^p}(\tau^q)\cdot e^{x\Im_\varpi^{T^p}(\tau^q)}, \Re_\varpi^{T^p}(\lambda^q)\cdot e^{x\Im_\varpi^{T^p}(\lambda^q)}\}\\
&= \min\{\Re_\varpi^{T^p}((\tau_1,\tau_2)^q)\}\cdot e^{x\Im_\varpi^{T^p}((\tau_1,\tau_2)^q)}\}, \Re_\varpi^{T^p}((\lambda_1,\lambda_2)^q)\cdot e^{x\Im_\varpi^{T^p}((\lambda_1,\lambda_2)^q)}\}\\
&= \min\{\min\{\Re_\ell^{T^p}(\tau_1)\cdot e^{x\Im_\ell^{T^p}(\tau_1)}, \Re_\ell^{T^p}(\tau_2)\cdot e^{x\Im_\ell^{T^p}(\tau_2)}\}, \min\{\Re_\ell^{T^p}(\lambda_1)\cdot e^{x\Im_\ell^{T^p}(\lambda_1)}, \Re_\ell^{T^p}(\lambda_2)\cdot e^{x\Im_\ell^{T^p}(\lambda_2)}\}\}
\end{aligned}$$

If $\Re_\ell^{T^p}((\tau_1\diamondsuit_1\lambda_1)^q)\cdot e^{x\Im_\ell^{T^p}((\tau_1\diamondsuit_1\lambda_1)^q)} \leq \Re_\ell^{T^p}((\tau_2\diamondsuit_1\lambda_2)^q)\cdot e^{x\Im_\ell^{T^p}((\tau_2\diamondsuit_1\lambda_2)^q)}$, then $\Re_\ell^{T^p}(\tau_1)\cdot e^{x\Im_\ell^{T^p}(\tau_1)} \leq \Re_\ell^{T^p}(\tau_2)\cdot e^{x\Im_\ell^{T^p}(\tau_2)}$ and $\Re_\ell^{T^p}(\lambda_1)\cdot e^{x\Im_\ell^{T^p}(\lambda_1)} \leq \Re_\ell^{T^p}(\lambda_2)\cdot e^{x\Im_\ell^{T^p}(\lambda_2)}$.
We get $\Re_\ell^{T^p}((\tau_1\diamondsuit_1\lambda_1)^q)\cdot e^{x\Im_\ell^{T^p}((\tau_1\diamondsuit_1\lambda_1)^q)} \geq \min\{\Re_\ell^{T^p}(\tau_1)\cdot e^{x\Im_\ell^{T^p}(\tau_1)}, \Re_\ell^{T^p}(\lambda_1)\cdot e^{x\Im_\ell^{T^p}(\lambda_1)}\}$ for all $\tau_1,\lambda_1 \in \mathscr{S}$,and
$\min\{\Re_\ell^{T^p}((\tau_1\diamondsuit_2\lambda_1)^q)\cdot e^{x\Im_\ell^{T^p}((\tau_1\diamondsuit_2\lambda_1)^q)}, \Re_\ell^{T^p}((\tau_2\diamondsuit_2\lambda_2)^q)\cdot e^{x\Im_\ell^{T^p}((\tau_2\diamondsuit_2\lambda_2)^q)}\} \geq \min\{\min\{\Re_\ell^{T^p}(\tau_1)\cdot e^{x\Im_\ell^{T^p}(\tau_1)}, \Re_\ell^{T^p}(\tau_2)\cdot e^{x\Im_\ell^{T^p}(\tau_2)}\}, \min\{\Re_\ell^{T^p}(\lambda_1)\cdot e^{x\Im_\ell^{T^p}(\lambda_1)}, \Re_\ell^{T^p}(\lambda_2)\cdot e^{x\Im_\ell^{T^p}(\lambda_2)}\}\}$
If $\Re_\ell^{T^p}((\tau_1\diamondsuit_2\lambda_1)^q)\cdot e^{x\Im_\ell^{T^p}((\tau_1\diamondsuit_2\lambda_1)^q)} \leq \Re_\ell^{T^p}((\tau_2\diamondsuit_2\lambda_2)^q)\cdot e^{x\Im_\ell^{T^p}((\tau_2\diamondsuit_2\lambda_2)^q)}$,then $\Re_\ell^{T^p}((\tau_1\diamondsuit_2\lambda_1)^q)\cdot e^{x\Im_\ell^{T^p}((\tau_1\diamondsuit_2\lambda_1)^q)} \geq \min\{\Re_\ell^{T^p}(\tau_1)\cdot e^{x\Im_\ell^{T^p}(\tau_1)}, \Re_\ell^{T^p}(\lambda_1)\cdot e^{x\Im_\ell^{T^p}(\lambda_1)}\}$.
$\min\{\Re_\ell^{T^p}((\tau_1\diamondsuit_3\lambda_1)^q)\cdot e^{x\Im_\ell^{T^p}((\tau_1\diamondsuit_3\lambda_1)^q)}, \Re_\ell^{T^p}((\tau_2\diamondsuit_3\lambda_2)^q)\cdot e^{x\Im_\ell^{T^p}((\tau_2\diamondsuit_3\lambda_2)^q)}\} \geq \min\{\min\{\Re_\ell^{T^p}(\tau_1)\cdot e^{x\Im_\ell^{T^p}(\tau_1)}, \Re_\ell^{T^p}(\tau_2)\cdot e^{x\Im_\ell^{T^p}(\tau_2)}\}, \min\{\Re_\ell^{T^p}(\lambda_1)\cdot e^{x\Im_\ell^{T^p}(\lambda_1)}, \Re_\ell^{T^p}(\lambda_2)\cdot e^{x\Im_\ell^{T^p}(\lambda_2)}\}\}$
If $\Re_\ell^{T^p}((\tau_1\diamondsuit_3\lambda_1)^q)\cdot e^{x\Im_\ell^{T^p}((\tau_1\diamondsuit_3\lambda_1)^q)} \leq \Re_\ell^{T^p}((\tau_2\diamondsuit_3\lambda_2)^q)\cdot e^{x\Im_\ell^{T^p}((\tau_2\diamondsuit_3\lambda_2)^q)}$,then $\Re_\ell^{T^p}((\tau_1\diamondsuit_3\lambda_1)^q)\cdot e^{x\Im_\ell^{T^p}((\tau_1\diamondsuit_3\lambda_1)^q)} \geq \min\{\Re_\ell^{T^p}(\tau_1)\cdot e^{x\Im_\ell^{T^p}(\tau_1)}, \Re_\ell^{T^p}(\lambda_1)\cdot e^{x\Im_\ell^{T^p}(\lambda_1)}\}$.
Now,

$$\begin{aligned}
&\frac{1}{2}\Big[\Re_\ell^{I^p}((\tau_1\diamondsuit_1\lambda_1)^q)\cdot e^{x\Im_\ell^{I^p}((\tau_1\diamondsuit_1\lambda_1)^q)} + \Re_\ell^{I^p}((\tau_2\diamondsuit_1\lambda_2)^q)\cdot e^{x\Im_\ell^{I^p((\tau_2\diamondsuit_1\lambda_2)^q)}}\Big]\\
&= \Re_\varpi^{I^p}(\tau_1\diamondsuit_1\lambda_1,\tau_2\diamondsuit_1\lambda_2)\cdot e^{x\Im_\varpi^{I^p}(\tau_1\diamondsuit_1\lambda_1,\tau_2\diamondsuit_1\lambda_2)}\\
&= \Re_\varpi^{I^p}[((\tau_1,\tau_2)^q)\diamondsuit_1((\lambda_1,\lambda_2)^q)]\cdot e^{x\Im_\varpi^{I^p}[((\tau_1,\tau_2)^q)\diamondsuit_1((\lambda_1,\lambda_2)^q)]}\\
&= \Re_\varpi^{I^p}((\tau\diamondsuit_1\lambda)^q)\cdot e^{x\Im_\varpi^{I^p}((\tau\diamondsuit_1\lambda)^q)}\\
&\geq \frac{\Re_\varpi^{I^p}(\tau^q)\cdot e^{x\Im_\varpi^{I^p(\tau^q)}} + \Re_\varpi^{I^p}(\lambda^q)\cdot e^{x\Im_\varpi^{I^p}(\lambda^q)}}{2}\\
&= \frac{\Re_\varpi^{I^p}((\tau_1,\tau_2)^q)\cdot e^{x\Im_\varpi^{I^p}((\tau_1,\tau_2)^q)} + \Re_\varpi^{I^p}((\lambda_1,\lambda_2)^q)\cdot e^{x\Im_\varpi^{I^p}}}{2}\\
&= \frac{1}{2}\left[\frac{\Re_\ell^{I^p}(\tau_1)\cdot e^{x\Im_\ell^{I^p}(\tau_1)} + \Re_\ell^{I^p}(\tau_2)\cdot e^{x\Im_\ell^{I^p}(\tau_2)}}{2} + \frac{\Re_\ell^{I^p}(\lambda_1)\cdot e^{x\Im_\ell^{I^p}(\lambda_1)} + \Re_\ell^{I^p}(\lambda_2)\cdot e^{x\Im_\ell^{I^p}(\lambda_2)}}{2}\right]
\end{aligned}$$

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If $\Re_\ell^{I^p}((\tau_1\diamondsuit_1\lambda_1)^q)\cdot e^{x\Im_\ell^{I^p}((\tau_1\diamondsuit_1\lambda_1)^q)} \leq \Re_\ell^{I^p}((\tau_2\diamondsuit_1\lambda_2)^q)\cdot e^{x\Im_\ell^{I^p}((\tau_2\diamondsuit_1\lambda_2)^q)}$, then $\Re_\ell^{I^p}(\tau_1)\cdot e^{x\Im_\ell^{I^p}(\tau_1)} \leq \Re_\ell^{I^p}(\tau_2)\cdot e^{x\Im_\ell^{I^p}(\tau_2)}$ and $\Re_\ell^{I^p}(\lambda_1)\cdot e^{x\Im_\ell^{I^p}(\lambda_1)} \leq \Re_\ell^{I^p}(\lambda_2)\cdot e^{x\Im_\ell^{I^p}(\lambda_2)}$.
We get $\Re_\ell^{I^p}((\tau_1\diamondsuit_1\lambda_1)^q)\cdot e^{x\Im_\ell^{T^p}((\tau_1\diamondsuit_1\lambda_1)^q)} \geq \frac{\Re_\ell^{I^p}(\tau_1)\cdot e^{x\Im_\ell^{I^p}(\tau_1)}+\Re_\ell^{I^p}(\lambda_1)\cdot e^{x\Im_\ell^{T^p}(\lambda_1)}}{2}$.
Similarly,$\Re_\ell^{I^p}((\tau_1\diamondsuit_2\lambda_1)^q)\cdot e^{x\Im_\ell^{I^p}((\tau_1\diamondsuit_2\lambda_1)^q)} \geq \frac{\Re_\ell^{I^p}(\tau_1)\cdot e^{x\Im_\ell^{I^p}(\tau_1)}+\Re_\ell^{I^p}(\lambda_1)\cdot e^{x\Im_\ell^{I^p}(\lambda_1)}}{2}$
and $\Re_\ell^{I^p}((\tau_1\diamondsuit_3\lambda_1)^q)\cdot e^{x\Im_\ell^{I^p}((\tau_1\diamondsuit_3\lambda_1)^q)} \geq \frac{\Re_\ell^{I^p}(\tau_1)\cdot e^{x\Im_\ell^{I^p}(\tau_1)}+\Re_\ell^{I^p}(\lambda_1)\cdot e^{x\Im_\ell^{I^p}(\lambda_1)}}{2}$.
Similarly to prove that
$\max\{\Re_\ell^{F^p}((\tau_1\diamondsuit_1\lambda_1)^q)\cdot e^{x\Im_\ell^{F^p}((\tau_1\diamondsuit_1\lambda_1)^q)}, \Re_\ell^{F^p}((\tau_2\diamondsuit_1\lambda_2)^q)\cdot e^{x\Im_\ell^{F^p}((\tau_2\diamondsuit_1\lambda_2)^q)}\}$
$\leq \max\{\max\{\Re_\ell^{F^p}(\tau_1)\cdot e^{x\Im_\ell^{F^p}(\tau_1)}, \Re_\ell^{F^p}(\tau_2)\cdot e^{x\Im_\ell^{F^p}(\tau_2)}\}, \max\{\Re_\ell^{F^p}(\lambda_1)\cdot e^{x\Im_\ell^{F^p}(\lambda_1)}, \Re_\ell^{F^p}(\lambda_2)\cdot e^{x\Im_\ell^{F^p}(\lambda_2)}\}\}$
If $\Re_\ell^{F^p}((\tau_1\diamondsuit_1\lambda_1)^q)\cdot e^{x\Im_\ell^{F^p}((\tau_1\diamondsuit_1\lambda_1)^q)} \geq \Re_\ell^{F^p}((\tau_2\diamondsuit_1\lambda_2)^q)\cdot e^{x\Im_\ell^{F^p}((\tau_2\diamondsuit_1\lambda_2)^q)}$,then $\Re_\ell^{F^p}(\tau_1)\cdot e^{x\Im_\ell^{F^p}(\tau_1)} \geq \Re_\ell^{F^p}(\tau_2)\cdot e^{x\Im_\ell^{F^p}(\tau_2)}$ and $\Re_\ell^{F^p}(\lambda_1)\cdot e^{x\Im_\ell^{F^p}(\lambda_1)} \geq \Re_\ell^{F^p}(\lambda_2)\cdot e^{x\Im_\ell^{F^p}(\lambda_2)}$.
We get $\Re_\ell^{F^p}((\tau_1\diamondsuit_1\lambda_1)^q)\cdot e^{x\Im_\ell^{F^p}((\tau_1\diamondsuit_1\lambda_1)^q)} \leq \max\{\Re_\ell^{F^p}(\tau_1)\cdot e^{x\Im_\ell^{F^p}(\tau_1)}, \Re_\ell^{F^p}(\lambda_1)\cdot e^{x\Im_\ell^{F^p}(\lambda_1)}\}$.
$\max\{\Re_\ell^{F^p}((\tau_1\diamondsuit_2\lambda_1)^q)\cdot e^{x\Im_\ell^{F^p}((\tau_1\diamondsuit_2\lambda_1)^q)}, \Re_\ell^{F^p}((\tau_2\diamondsuit_2\lambda_2)^q)\cdot e^{x\Im_\ell^{F^p}((\tau_2\diamondsuit_2\lambda_2)^q)}\}$
$\leq \max\{\max\{\Re_\ell^{F^p}(\tau_1)\cdot e^{x\Im_\ell^{F^p}(\tau_1)}, \Re_\ell^{F^p}(\tau_2)\cdot e^{x\Im_\ell^{F^p}(\tau_2)}\}, \max\{\Re_\ell^{F^p}(\lambda_1)\cdot e^{x\Im_\ell^{F^p}(\lambda_1)}, \Re_\ell^{F^p}(\lambda_2)\cdot e^{x\Im_\ell^{F^p}}\}\}$
If $\Re_\ell^{F^p}((\tau_1\diamondsuit_2\lambda_1)^q)\cdot e^{x\Im_\ell^{F^p}((\tau_1\diamondsuit_2\lambda_1)^q)} \geq \Re_\ell^{F^p}((\tau_2\diamondsuit_2\lambda_2)^q)\cdot e^{x\Im_\ell^{F^p}((\tau_2\diamondsuit_2\lambda_2)^q)}$,then $\Re_\ell^{F^p}((\tau_1\diamondsuit_2\lambda_1)^q)\cdot e^{x\Im_\ell^{F^p}((\tau_1\diamondsuit_2\lambda_1)^q)} \leq \max\{\Re_\ell^{F^p}(\tau_1)\cdot e^{x\Im_\ell^{F^p}(\tau_1)}, \Re_\ell^{F^p}(\lambda_1)\cdot e^{x\Im_\ell^{F^p}(\lambda_1)}\}$.
$\max\{\Re_\ell^{F^p}((\tau_1\diamondsuit_3\lambda_1)^q)\cdot e^{x\Im_\ell^{F^p}((\tau_1\diamondsuit_3\lambda_1)^q)}, \Re_\ell^{F^p}((\tau_2\diamondsuit_3\lambda_2)^q)\cdot e^{x\Im_\ell^{F^p}((\tau_2\diamondsuit_3\lambda_2)^q)}\}$
$\leq \max\{\max\{\Re_\ell^{F^p}(\tau_1)\cdot e^{x\Im_\ell^{F^p}(\tau_1)}, \Re_\ell^{F^p}(\tau_2)\cdot e^{x\Im_\ell^{F^p}(\tau_2)}\}, \max\{\Re_\ell^{F^p}(\lambda_1)\cdot e^{x\Im_\ell^{F^p}(\lambda_1)}, \Re_\ell^{F^p}(\lambda_2)\cdot e^{x\Im_\ell^{F^p}(\lambda_2)}\}\}$
If $\Re_\ell^{F^p}((\tau_1\diamondsuit_3\lambda_1)^q)\cdot e^{x\Im_\ell^{T^p}((\tau_1\diamondsuit_3\lambda_1)^q)} \geq \Re_\ell^{F^p}((\tau_2\diamondsuit_3\lambda_2)^q)\cdot e^{x\Im_\ell^{T^p}((\tau_2\diamondsuit_3\lambda_2)^q)}$,then $\Re_\ell^{F^p}((\tau_1\diamondsuit_3\lambda_1)^q)\cdot e^{x\Im_\ell^{T^p}((\tau_1\diamondsuit_3\lambda_1)^q)} \leq \max\{\Re_\ell^{F^p}(\tau_1)\cdot e^{x\Im_\ell^{F^p}(\tau_1)}, \Re_\ell^{F^p}(\lambda_1)\cdot e^{x\Im_\ell^{T^p}(\lambda_1)}\}$.
Therefore, ℓ is a CQBNSBS of $\mathscr{S}$.

Theorem 3.12. *Suppose that ℓ is a subset of $\mathscr{S}$. Then $R = (\Re_\ell^{T^n}\cdot e^{x\Im_\ell^{T^n}}, \Re_\ell^{I^n}\cdot e^{x\Im_\ell^{I^n}}, \Re_\ell^{F^n}\cdot e^{x\Im_\ell^{F^n}}, \Re_\ell^{T^p}\cdot e^{x\Im_\ell^{T^p}}, \Re_\ell^{I^p}\cdot e^{x\Im_\ell^{I^p}}, \Re_\ell^{F^p}\cdot e^{x\Im_\ell^{F^p}})$ is a CQBNSBS of $\mathscr{S}$ if and only if $\Re^{(\wp_1,\wp_2)}$ is a SBS of $\mathscr{S}$ for all $\wp_1,\wp_2\in D[0,1]$.*

Proof. Assume that $\Re$ is a CQBNSBS of $\mathscr{S}$. For each $\wp_1,\wp_2\in D[0,1]$ and $\tau_1,\tau_2\in\Re^{(\wp_1,\wp_2)}$. Now,$\Re_\ell^{T^n}(\tau_1)\cdot e^{x\Im_\ell^{T^n}}(\tau_1)\leq\wp_1, \Re_\ell^{T^n}(\tau_2)\cdot e^{x\Im_\ell^{T^n}}(\tau_2)\leq\wp_1$ and $\Re_\ell^{I^n}(\tau_1)\cdot e^{x\Im_\ell^{I^n}}(\tau_1)\leq\wp_1, \Re_\ell^{I^n}(\tau_2)\cdot e^{x\Im_\ell^{I^n}}(\tau_2)\leq\wp_1$ and $\Re_\ell^{F^n}(\tau_1)\cdot e^{x\Im_\ell^{F^n}}(\tau_1)\geq\wp_2, \Re_\ell^{F^n}(\tau_2)\cdot e^{x\Im_\ell^{F^n}}(\tau_2)\geq\wp_2$. Now,$\Re_\ell^{T^n}((\tau_1\diamondsuit_1\tau_2)^q)\cdot e^{x\Im_\ell^{T^n}}((\tau_1\diamondsuit_1\tau_2)^q)\leq \max\{\Re_\ell^{T^n}(\tau_1)\cdot e^{x\Im_\ell^{T^n}}(\tau_1), \Re_\ell^{T^n}(\tau_2)\cdot e^{x\Im_\ell^{T^n}}(\tau_2)\}\leq\wp_1$ and
$\Re_\ell^{I^n}((\tau_1\diamondsuit_1\tau_2)^q)\cdot e^{x\Im_\ell^{I^n}}((\tau_1\diamondsuit_1\tau_2)^q)\leq\frac{\Re_\ell^{I^n}(\tau_1)\cdot e^{x\Im_\ell^{I^n}}(\tau_1)+\Re_\ell^{I^n}(\tau_2)\cdot e^{x\Im_\ell^{I^n}}(\tau_2)}{2}\leq\frac{\wp_1+\wp_1}{2}=\wp_1$ and
$\Re_\ell^{F^n}((\tau_1\diamondsuit_1\tau_2)^q)\cdot e^{x\Im_\ell^{F^n}}((\tau_1\diamondsuit_1\tau_2)^q)\geq\min$
$\{\Re_\ell^{F^n}(\tau_1)\cdot e^{x\Im_\ell^{F^n}}(\tau_1), \Re_\ell^{F^n}(\tau_2)\cdot e^{x\Im_\ell^{F^n}}(\tau_2)\}\geq\wp_2$. This implies that $\tau_1\diamondsuit_1\tau_2\in\Re^{(\wp_1,\wp_2)}$. Similarly,$\tau_1\diamondsuit_2\tau_2\in\Re^{(\wp_1,\wp_2)}$ and $\tau_1\diamondsuit_3\tau_2\in\Re^{(\wp_1,\wp_2)}$. Hence,$\Re^{(\wp_1,\wp_2)}$ is a SBS of $\mathscr{S}$,for all $\wp_1,\wp_2\in D[0,1]$.
For each $\wp_1,\wp_2\in[0,1]$ and $\tau_1,\tau_2\in\Re^{(\wp_1,\wp_2)}$. Now,$\Re_\ell^{T^p}(\tau_1)\cdot e^{x\Im_\ell^{T^p}}(\tau_1)\geq\wp_1, \Re_\ell^{T^p}(\tau_2)\cdot e^{x\Im_\ell^{T^p}}(\tau_2)\geq\wp_1$ and $\Re_\ell^{I^p}(\tau_1)\cdot e^{x\Im_\ell^{I^p}}(\tau_1)\geq\wp_1, \Re_\ell^{I^p}(\tau_2)\cdot e^{x\Im_\ell^{I^p}}(\tau_2)\geq\wp_1$ and $\Re_\ell^{F^p}(\tau_1)\cdot e^{x\Im_\ell^{F^p}}(\tau_1)\leq\wp_2, \Re_\ell^{F^p}(\tau_2)\cdot e^{x\Im_\ell^{F^p}}(\tau_2)\leq\wp_2$. Now,$\Re_\ell^{T^p}((\tau_1\diamondsuit_1\tau_2)^q)\cdot e^{x\Im_\ell^{T^p}}((\tau_1\diamondsuit_1\tau_2)^q)\geq\min\{\Re_\ell^{T^p}(\tau_1)\cdot e^{x\Im_\ell^{T^p}}(\tau_1), \Re_\ell^{T^p}(\tau_2)\cdot e^{x\Im_\ell^{T^p}}(\tau_2)\}\geq\wp_1$ and $\Re_\ell^{I^p}((\tau_1\diamondsuit_1\tau_2)^q)\cdot e^{x\Im_\ell^{I^p}}((\tau_1\diamondsuit_1\tau_2)^q)\geq\frac{\Re_\ell^{I^p}(\tau_1)\cdot e^{x\Im_\ell^{I^p}}(\tau_1)+\Re_\ell^{I^p}(\tau_2)\cdot e^{x\Im_\ell^{I^p}}(\tau_2)}{2}\geq\frac{\wp_1+\wp_1}{2}=\wp_1$ and $\Re_\ell^{F^p}((\tau_1\diamondsuit_1\tau_2)^q)\cdot e^{x\Im_\ell^{F^p}}((\tau_1\diamondsuit_1\tau_2)^q)\leq\max\{\Re_\ell^{F^p}(\tau_1)\cdot e^{x\Im_\ell^{F^p}}(\tau_1), \Re_\ell^{F^p}(\tau_2)\cdot e^{x\Im_\ell^{F^p}}(\tau_2)\}\leq\wp_2$. This implies that $\tau_1\diamondsuit_1\tau_2\in\Re^{(\wp_1,\wp_2)}$. Similarly,$\tau_1\diamondsuit_2\tau_2\in\Re^{(\wp_1,\wp_2)}$ and $\tau_1\diamondsuit_3\tau_2\in\Re^{(\wp_1,\wp_2)}$. Hence,$\Re^{(\wp_1,\wp_2)}$ is a SBS of $\mathscr{S}$,for all $\wp_1,\wp_2\in D[0,1]$.
Conversely, assume that $\Re^{(\wp_1,\wp_2)}$ is a SBS of $\mathscr{S}$ and $\wp_1,\wp_2\in D[0,1]$. Suppose if there exist $\tau_1,\tau_2\in\mathscr{S}$ such that $\Re_\ell^{T^n}((\tau_1\diamondsuit_1\tau_2)^q)\cdot e^{x\Im_\ell^{T^n}}((\tau_1\diamondsuit_1\tau_2)^q)>\max\{\Re_\ell^{T^n}(\tau_1)\cdot e^{x\Im_\ell^{T^n}}(\tau_1), \Re_\ell^{T^n}(\tau_2)\cdot e^{x\Im_\ell^{T^n}}(\tau_2)\}$,
$\Re_\ell^{I^n}((\tau_1\diamondsuit_1\tau_2)^q)\cdot e^{x\Im_\ell^{I^n}}((\tau_1\diamondsuit_1\tau_2)^q)>\frac{\Re_\ell^{I^n}(\tau_1)\cdot e^{x\Im_\ell^{I^n}}(\tau_1)+\Re_\ell^{I^n}(\tau_2)\cdot e^{x\Im_\ell^{I^n}}(\tau_2)}{2}$ and
$\Re_\ell^{F^n}((\tau_1\diamondsuit_1\tau_2)^q)\cdot e^{x\Im_\ell^{F^n}}((\tau_1\diamondsuit_1\tau_2)^q)<\min\{\Re_\ell^{F^n}(\tau_1)\cdot e^{x\Im_\ell^{F^n}}(\tau_1), \Re_\ell^{F^n}(\tau_2)\cdot e^{x\Im_\ell^{F^n}}(\tau_2)\}$. For $\wp_1,\wp_2\in D[0,1]$ such that $\Re_\ell^{T^n}((\tau_1\diamondsuit_1\tau_2)^q)\cdot e^{x\Im_\ell^{T^n}}((\tau_1\diamondsuit_1\tau_2)^q)>\wp_1\geq\max\{\Re_\ell^{T^n}(\tau_1)\cdot e^{x\Im_\ell^{T^n}}(\tau_1), \Re_\ell^{T^n}(\tau_2)\cdot e^{x\Im_\ell^{T^n}}(\tau_2)\}$ and $\Re_\ell^{I^n}((\tau_1\diamondsuit_1\tau_2)^q)\cdot e^{x\Im_\ell^{I^n}}((\tau_1\diamondsuit_1\tau_2)^q)>\wp_1\geq\frac{\Re_\ell^{I^n}(\tau_1)\cdot e^{x\Im_\ell^{I^n}}(\tau_1)+\Re_\ell^{I^n}(\tau_2)\cdot e^{x\Im_\ell^{I^n}}(\tau_2)}{2}$ and $\Re_\ell^{F^n}((\tau_1\diamondsuit_1\tau_2)^q)\cdot e^{x\Im_\ell^{F^n}}((\tau_1\diamondsuit_1\tau_2)^q)<\wp_2\leq\min\{\Re_\ell^{F^n}(\tau_1)\cdot e^{x\Im_\ell^{F^n}}(\tau_1), \Re_\ell^{F^n}(\tau_2)\cdot e^{x\Im_\ell^{F^n}}(\tau_2)\}$. Thus,$\tau_1,\tau_2\in\Re^{(\wp_1,\wp_2)}$,but $\tau_1\diamondsuit_1\tau_2\notin\Re^{(\wp_1,\wp_2)}$. This contradicts,$\Re^{(\wp_1,\wp_2)}$ is a SBS of $\mathscr{S}$. Therefore $\Re_\ell^{T^n}((\tau_1\diamondsuit_1\tau_2)^q)\cdot e^{x\Im_\ell^{T^n}}((\tau_1\diamondsuit_1\tau_2)^q)\leq$

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$\max\{\Re_{\ell}^{T^n}(\tau_1)\cdot e^{x\Im_{\ell}^{T^n}}(\tau_1),\Re_{\ell}^{T^n}(\tau_2)\cdot e^{x\Im_{\ell}^{T^n}}(\tau_2)\}$, $\Re_{\ell}^{I^n}((\tau_1\Diamond_1\tau_2)^q)\cdot e^{x\Im_{\ell}^{I^n}}((\tau_1\Diamond_1\tau_2)^q)\leq \frac{\Re_{\ell}^{I^n}(\tau_1)\cdot e^{x\Im_{\ell}^{I^n}}(\tau_1)+\Re_{\ell}^{I^n}(\tau_2)\cdot e^{x\Im_{\ell}^{I^n}}(\tau_2)}{2}$ and
$\Re_{\ell}^{F^n}((\tau_1\Diamond_1\tau_2)^q)\cdot e^{x\Im_{\ell}^{F^n}}((\tau_1\Diamond_1\tau_2)^q)\geq \min\{\Re_{\ell}^{F^n}(\tau_1)\cdot e^{x\Im_{\ell}^{F^n}}(\tau_1),\Re_{\ell}^{F^n}(\tau_2)\cdot e^{x\Im_{\ell}^{F^n}}(\tau_2)\}$.
Similarly, $\Diamond_2$ and $\Diamond_3$ cases.
Hence $\Re=(\Re_{\ell}^{T^n}\cdot e^{x\Im_{\ell}^{T^n}},\Re_{\ell}^{I^n}\cdot e^{x\Im_{\ell}^{I^n}},\Re_{\ell}^{F^n}\cdot e^{x\Im_{\ell}^{F^n}})$ is a CQBNSBS of $\mathscr{S}$.
Let us assume that $\Re^{(\wp_1,\wp_2)}$ is a SBS of $\mathscr{S}$ and $\wp_1,\wp_2\in[0,1]$. Suppose if there exist $\tau_1,\tau_2\in\mathscr{S}$ such that $\Re_{\ell}^{T^p}((\tau_1\Diamond_1\tau_2)^q)\cdot e^{x\Im_{\ell}^{T^p}}((\tau_1\Diamond_1\tau_2)^q)>\min\{\Re_{\ell}^{T^p}(\tau_1)\cdot e^{x\Im_{\ell}^{T^p}}(\tau_1),\Re_{\ell}^{T^p}(\tau_2)\cdot e^{x\Im_{\ell}^{T^p}}(\tau_2)\}$,$\Re_{\ell}^{I^p}((\tau_1\Diamond_1\tau_2)^q)\cdot e^{x\Im_{\ell}^{I^p}}((\tau_1\Diamond_1\tau_2)^q)>\frac{\Re_{\ell}^{I^p}(\tau_1)\cdot e^{x\Im_{\ell}^{I^p}}(\tau_1)+\Re_{\ell}^{I^p}(\tau_2)\cdot e^{x\Im_{\ell}^{I^p}}(\tau_2)}{2}$ and $\Re_{\ell}^{F^p}((\tau_1\Diamond_1\tau_2)^q)\cdot e^{x\Im_{\ell}^{F^p}}((\tau_1\Diamond_1\tau_2)^q)<\max\{\Re_{\ell}^{F^p}(\tau_1)\cdot e^{x\Im_{\ell}^{F^p}}(\tau_1),\Re_{\ell}^{F^p}(\tau_2)\cdot e^{x\Im_{\ell}^{F^p}}(\tau_2)\}$. For $\wp_1,\wp_2\in D[0,1]$ such that $\Re_{\ell}^{T^p}((\tau_1\Diamond_1\tau_2)^q)\cdot e^{x\Im_{\ell}^{T^p}}((\tau_1\Diamond_1\tau_2)^q)>\wp_1\leq\min\{\Re_{\ell}^{T^p}(\tau_1)\cdot e^{x\Im_{\ell}^{T^p}}(\tau_1),\Re_{\ell}^{T^p}(\tau_2)\cdot e^{x\Im_{\ell}^{T^p}}(\tau_2)\}$ and $\Re_{\ell}^{I^p}((\tau_1\Diamond_1\tau_2)^q)\cdot e^{x\Im_{\ell}^{I^p}}((\tau_1\Diamond_1\tau_2)^q)>\wp_1\leq \frac{\Re_{\ell}^{I^p}(\tau_1)\cdot e^{x\Im_{\ell}^{I^p}}(\tau_1)+\Re_{\ell}^{I^p}(\tau_2)\cdot e^{x\Im_{\ell}^{I^p}}(\tau_2)}{2}$ and
$\Re_{\ell}^{F^p}((\tau_1\Diamond_1\tau_2)^q)\cdot e^{x\Im_{\ell}^{F^p}}((\tau_1\Diamond_1\tau_2)^q)<\wp_2\geq\max\{\Re_{\ell}^{F^p}(\tau_1)\cdot e^{x\Im_{\ell}^{F^p}}(\tau_1),\Re_{\ell}^{F^p}(\tau_2)\cdot e^{x\Im_{\ell}^{F^p}}(\tau_2)\}$. Thus,$\tau_1,\tau_2\in\Re^{(\wp_1,\wp_2)}$,but $\tau_1\Diamond_1\tau_2\notin\Re^{(\wp_1,\wp_2)}$. This contradicts,$\Re^{(\wp_1,\wp_2)}$ is a SBS of $\mathscr{S}$. Therefore $\Re_{\ell}^{T^p}((\tau_1\Diamond_1\tau_2)^q)\cdot e^{x\Im_{\ell}^{T^p}}((\tau_1\Diamond_1\tau_2)^q)\geq\min\{\Re_{\ell}^{T^p}(\tau_1)\cdot e^{x\Im_{\ell}^{T^p}}(\tau_1),\Re_{\ell}^{T^p}(\tau_2)\cdot e^{x\Im_{\ell}^{T^p}}(\tau_2)\}$, $\Re_{\ell}^{I^p}((\tau_1\Diamond_1\tau_2)^q)\cdot e^{x\Im_{\ell}^{I^p}}((\tau_1\Diamond_1\tau_2)^q)\geq \frac{\Re_{\ell}^{I^p}(\tau_1)\cdot e^{x\Im_{\ell}^{I^p}}(\tau_1)+\Re_{\ell}^{I^p}(\tau_2)\cdot e^{x\Im_{\ell}^{I^p}}(\tau_2)}{2}$ and $\Re_{\ell}^{F^p}((\tau_1\Diamond_1\tau_2)^q)\cdot e^{x\Im_{\ell}^{F^p}}((\tau_1\Diamond_1\tau_2)^q)\leq\max\{\Re_{\ell}^{F^p}(\tau_1)\cdot e^{x\Im_{\ell}^{F^p}}(\tau_1),\Re_{\ell}^{F^p}(\tau_2)\cdot e^{x\Im_{\ell}^{F^p}}(\tau_2)\}$. Similarly, $\Diamond_2$ and $\Diamond_3$ cases.
Hence $\Re=(\Re_{\ell}^{T^p}\cdot e^{x\Im_{\ell}^{T^p}},\Re_{\ell}^{I^p}\cdot e^{x\Im_{\ell}^{I^p}},\Re_{\ell}^{F^p}\cdot e^{x\Im_{\ell}^{F^p}})$ is a CQBNSBS of $\mathscr{S}$.

Definition 3.13. Let $(\mathscr{S}_1,\ominus_1,\ominus_2,\ominus_3)$ and $(\mathscr{S}_2,\uplus_1,\uplus_2,\uplus_3)$ be any two bisemirings. The mapping $\Bbbk:\mathscr{S}_1\to\mathscr{S}_2$ and ℓ be any CQBNSBS in $\mathscr{S}_1$, ϖ be any CQBNSBS in $\Bbbk(\mathscr{S}_1)=\mathscr{S}_2$. If $\Re_{\ell}\cdot e^{x\Im_{\ell}}=[\Re_{\ell}^{T^n}\cdot e^{x\Im_{\ell}^{T^n}},\Re_{\ell}^{I^n}\cdot e^{x\Im_{\ell}^{I^n}},\Re_{\ell}^{F^n}\cdot e^{x\Im_{\ell}^{F^n}},\Re_{\ell}\cdot e^{x\Im_{\ell}},\Re_{\ell}^{T^p}\cdot e^{x\Im_{\ell}^{T^p}},\Re_{\ell}^{I^p}\cdot e^{x\Im_{\ell}^{I^p}},\Re_{\ell}^{F^p}\cdot e^{x\Im_{\ell}^{F^p}}$ is a CQBNS in $\mathscr{S}_1$, then $\Re_{\varpi}$ is a CQBNS in $\mathscr{S}_2$,defined by

$$\Re_{\varpi}^{T^n}(\lambda^q)\cdot e^{x\Im_{\ell}^{T^n}}(\lambda^q)=\begin{cases}\bigwedge\Re_{\ell}^{T^n}(\tau^q)\cdot e^{x\Im_{\ell}^{T^n}}(\tau^q) & if\ \ \tau\in\Bbbk^{-1}\lambda\\ 0 & otherwise\end{cases}$$

$$\Re_{\varpi}^{I^n}(\lambda^q)\cdot e^{x\Im_{\ell}^{I^n}}(\lambda^q)=\begin{cases}\bigwedge\Re_{\ell}^{I^n}(\tau^q)\cdot e^{x\Im_{\ell}^{I^n}}(\tau^q) & if\ \ \tau\in\Bbbk^{-1}\lambda\\ 0 & otherwise\end{cases}$$

$$\Re_{\varpi}^{F^n}(\lambda^q)\cdot e^{x\Im_{\ell}^{F^n}}(\lambda^q)=\begin{cases}\bigvee\Re_{\ell}^{F^n}(\tau^q)\cdot e^{x\Im_{\ell}^{F^n}}(\tau^q) & if\ \ \tau\in\Bbbk^{-1}\lambda\\ -1 & otherwise\end{cases}$$

$$\Re_{\varpi}^{T^p}(\lambda^q)\cdot e^{x\Im_{\ell}^{T^p}}(\lambda^q)=\begin{cases}\bigvee\Re_{\ell}^{T^p}(\tau^q)\cdot e^{x\Im_{\ell}^{T^p}}(\tau^q) & if\ \ \tau\in\Bbbk^{-1}\lambda\\ 0 & otherwise\end{cases}$$

$$\Re_{\varpi}^{I^p}(\lambda^q)\cdot e^{x\Im_{\ell}^{I^p}}(\lambda^q)=\begin{cases}\bigvee\Re_{\ell}^{I^p}(\tau^q)\cdot e^{x\Im_{\ell}^{I^p}}(\tau^q) & if\ \ \tau\in\Bbbk^{-1}\lambda\\ 0 & otherwise\end{cases}$$

$$\Re_{\varpi}^{F^p}(\lambda^q)\cdot e^{x\Im_{\ell}^{F^p}}(\lambda^q)=\begin{cases}\bigwedge\Re_{\ell}^{F^p}(\tau^q)\cdot e^{x\Im_{\ell}^{F^p}}(\tau^q) & if\ \ \tau\in\Bbbk^{-1}\lambda\\ 1 & otherwise\end{cases}$$

for all $\tau\in\mathscr{S}_1$ and $\lambda\in\mathscr{S}_2$ is represents the image of R_{ℓ} under $\Bbbk$.
Similarly,If $\Re_{\varpi}\cdot e^{x\Im_{\varpi}}=[\Re_{\varpi}^{T^n}\cdot e^{x\Im_{\ell}^{T^n}},\Re_{\varpi}^{I^n}\cdot e^{x\Im_{\varpi}^{I^n}},\Re_{\varpi}^{F^n}\cdot e^{x\Im_{\varpi}^{F^n}}],\Re_{\varpi}\cdot e^{x\Im_{\varpi}},\Re_{\varpi}^{T^p}\cdot e^{x\Im_{\ell}^{T^p}},\Re_{\varpi}^{I^p}\cdot e^{x\Im_{\varpi}^{I^p}},\Re_{\varpi}^{F^p}\cdot e^{x\Im_{\varpi}^{F^p}}$ is a CQBNS in $\mathscr{S}_2$,then CQBNS $\Re_{\ell}=\Bbbk\circ\Re_{\varpi}$ in $\mathscr{S}_1$ ie,the CQBNS defined by $\Re_{\ell}(\tau^q)\cdot e^{x\Im_{\ell}(\tau^q)},\Re_{\varpi}(\Bbbk(\tau^q))\cdot e^{x\Im_{\ell}(\Bbbk(\tau^q))}$, $\Re_{\ell}=\Bbbk\circ\Re_{\varpi}$ in $\mathscr{S}_1$ [ie,the CQBNS defined by $\Re_{\ell}(\tau^q)\cdot e^{x\Im_{\ell}(\tau^q)}=\Re_{\varpi}(\Bbbk(\tau^q))\cdot e^{x\Im_{\ell}(\Bbbk(\tau^q))}$ is represents the preimage of $\Re_{\varpi}$ under $\Bbbk$.

Theorem 3.14. *The homomorphic image of every CQBNSBS is a CQBNSBS.*

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Proof. The mapping $\Bbbk : \mathscr{S}_1 \to \mathscr{S}_2$ be any homomorphism. Now, $\Bbbk((\tau \ominus_1 \lambda)^q) = \Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q), \Bbbk((\tau \ominus_2 \lambda)^q) = \Bbbk(\tau^q) \biguplus_2 \Bbbk(\lambda^q)$ and $\Bbbk((\tau \ominus_3 \lambda)^q) = \Bbbk(\tau^q) \biguplus_3 \Bbbk(\lambda^q)$ for all $\tau, \lambda \in \mathscr{S}_1$. Let $\varpi = \Bbbk(\ell)$,ℓ is any CQBNSBS of $\mathscr{S}_1$. Let $\Bbbk(\tau^q), \Bbbk(\lambda^q) \in \mathscr{S}_2$. Let $\tau \in u\Bbbk^{-1}(\Bbbk(\tau^q))$ and $\lambda \in \Bbbk^{-1}(\Bbbk(\lambda^q))$ be such that $\Re_\ell^{T^n}(\tau^q) \cdot e^{x\Im_\ell^{T^n}(\tau^q)} = \bigwedge_{\tau \in \Bbbk^{-1}(\Bbbk(\tau^q))} \Re_\ell^{T^n}(\tau^q) \cdot e^{x\Im_\ell^{T^n}(\tau^q)}$ and $\Re_\ell^{T^n}(\lambda^q) \cdot e^{x\Im_\ell^{T^n}(\lambda^q)} = \bigwedge_{\tau \in \Bbbk^{-1}(\Bbbk(\lambda^q))} \Re_\ell^{T^n}(\tau^q) \cdot e^{x\Im_\ell^{T^n}(\tau^q)}$.
Now, $\Re_\varpi^{T^n}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q))) \cdot e^{x\Im_\ell^{T^n}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q)))}$

$$\begin{aligned}
&= \bigwedge_{(\tau') \in \Bbbk^{-1}(\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q))} \Re_\ell^{T^n}(\tau') \cdot e^{x\Im_\ell^{T^n}(\tau')} \\
&= \bigwedge_{(\tau') \in \Bbbk^{-1}(\Bbbk((\tau \ominus_1 \lambda)^q)} \Re_\ell^{T^n}(\tau') \cdot e^{x\Im_\ell^{T^n}(\tau')} \\
&= \Re_\ell^{T^n}((\tau \ominus_1 \lambda)^q) \cdot e^{x\Im_\ell^{T^n}((\tau \ominus_1 \lambda)^q)} \\
&\leq \max\{\Re_\ell^{T^n}(\tau^q) \cdot e^{x\Im_\ell^{T^n}(\tau^q)}, \Re_\ell^{T^n}(\lambda^q) \cdot e^{x\Im_\ell^{T^n}(\lambda^q)}\} \\
&= \max\{\Re_\varpi^{T^n}\Bbbk(\tau^q) \cdot e^{x\Im_\varpi^{T^n}\Bbbk(\tau^q)}, \Re_\varpi^{T^n}\Bbbk(\lambda^q) \cdot e^{x\Im_\varpi^{T^n}\Bbbk(\lambda^q)}\}.
\end{aligned}$$

Thus,$\Re_\varpi^{T^n}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q))) \cdot e^{x\Im_\varpi^{T^n}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q)))} \leq \max\{\Re_\varpi^{T^n}\Bbbk(\tau^q) \cdot e^{x\Im_\varpi^{T^n}\Bbbk(\tau^q)}, \Re_\varpi^{T^n}\Bbbk(\lambda^q) \cdot e^{x\Im_\varpi^{T^n}\Bbbk(\lambda^q)}\}$.
Similarly,$\Re_\varpi^{T^n}((\Bbbk(\tau^q) \biguplus_2 \Bbbk(\lambda^q))) \cdot e^{x\Im_\varpi^{T^n}((\Bbbk(\tau^q) \biguplus_2 \Bbbk(\lambda^q)))} \leq \max\{\Re_\varpi^{T^n}\Bbbk(\tau^q) \cdot e^{x\Im_\varpi^{T^n}\Bbbk(\tau^q)}, \Re_\varpi^{T^n}\Bbbk(\lambda^q) \cdot e^{x\Im_\varpi^{T^n}\Bbbk(\lambda^q)}\}$
and
$\Re_\varpi^{T^n}((\Bbbk(\tau^q) \biguplus_3 \Bbbk(\lambda^q))) \cdot e^{x\Im_\varpi^{T^n}((\Bbbk(\tau^q) \biguplus_3 \Bbbk(\lambda^q)))} \leq \max\{\Re_\varpi^{T^n}\Bbbk(\tau^q) \cdot e^{x\Im_\varpi^{T^n}\Bbbk(\tau^q)}, \Re_\varpi^{T^n}\Bbbk(\lambda^q) \cdot e^{x\Im_\varpi^{T^n}\Bbbk(\lambda^q)}\}$.
Let $\tau \in \Bbbk^{-1}(\Bbbk(\tau^q))$ and $\lambda \in \Bbbk^{-1}(\Bbbk(\lambda^q))$ be such that $\Re_\ell^{I^n}(\tau^q) \cdot e^{x\Im_\varpi^{T^n}(\tau^q)} = \bigwedge_{\tau \in \Bbbk^{-1}(\Bbbk(\tau^q))} \Re_\ell^{I^n}(\tau^q) \cdot e^{x\Im_\varpi^{T^n}(\tau^q)}$ and $\Re_\ell^{I^n}(\lambda^q) \cdot e^{x\Im_\varpi^{T^n}(\lambda^q)} = \bigwedge_{\tau \in \Bbbk^{-1}(\Bbbk(\lambda^q))} \Re_\ell^{I^n}(\tau^q) \cdot e^{x\Im_\varpi^{T^n}\Bbbk(\tau^q)}$.
Now, $\Re_\varpi^{I^n}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q))) \cdot e^{x\Im_\varpi^{I^n}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q)))}$

$$\begin{aligned}
&= \bigwedge_{(\tau') \in \Bbbk^{-1}(\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q))} \Re_\ell^{I^n}(\tau') \cdot e^{x\Im_\ell^{I^n}(\tau')} \\
&= \bigwedge_{(\tau') \in \Bbbk^{-1}(\Bbbk((\tau \ominus_1 \lambda)^q)} \Re_\ell^{I^n}(\tau') \cdot e^{x\Im_\ell^{I^n}(\tau')} \\
&= \Re_\ell^{I^n}((\tau \ominus_1 \lambda)^q) \cdot e^{x\Im_\ell^{I^n}((\tau \ominus_1 \lambda)^q)} \\
&\leq \frac{\Re_\ell^{I^n}(\tau^q) \cdot e^{x\Im_\ell^{I^n}(\tau^q)} + \Re_\ell^{I^n}(\lambda^q) \cdot e^{x\Im_\ell^{I^n}(\lambda^q)}}{2} \\
&= \frac{\Re_\varpi^{I^n}\Bbbk(\tau^q) \cdot e^{x\Im_\varpi^{I^n}\Bbbk(\tau^q)} + \Re_\varpi^{I^n}\Bbbk(\lambda^q) \cdot e^{x\Im_\varpi^{I^n}\Bbbk(\lambda^q)}}{2}.
\end{aligned}$$

Thus,
$\Re_\varpi^{I^n}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q))) \cdot e^{x\Im_\varpi^{I^n}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q)))} \leq \frac{\Re_\varpi^{I^n}\Bbbk(\tau^q) \cdot e^{x\Im_\varpi^{I^n}\Bbbk(\tau^q)} + \Re_\varpi^{I^n}\Bbbk(\lambda^q) \cdot e^{x\Im_\varpi^{I^n}\Bbbk(\lambda^q)}}{2}$.
Similarly,
$\Re_\varpi^{I^n}((\Bbbk(\tau^q) \biguplus_2 \Bbbk(\lambda^q))) \cdot e^{x\Im_\varpi^{I^n}((\Bbbk(\tau^q) \biguplus_2 \Bbbk(\lambda^q)))} \leq \frac{\Re_\varpi^{I^n}\Bbbk(\tau^q) \cdot e^{x\Im_\varpi^{I^n}\Bbbk(\tau^q)} + \Re_\varpi^{I^n}\Bbbk(\lambda^q) \cdot e^{x\Im_\varpi^{I^n}\Bbbk(\lambda^q)}}{2}$ and
$\Re_\varpi^{I^n}((\Bbbk(\tau^q) \biguplus_3 \Bbbk(\lambda^q))) \cdot e^{x\Im_\varpi^{I^n}((\Bbbk(\tau^q) \biguplus_3 \Bbbk(\lambda^q)))} \leq \frac{\Re_\varpi^{I^n}\Bbbk(\tau^q) \cdot e^{x\Im_\varpi^{I^n}\Bbbk(\tau^q)} + \Re_\varpi^{I^n}\Bbbk(\lambda^q) \cdot e^{x\Im_\varpi^{I^n}\Bbbk(\lambda^q)}}{2}$.
Let $\Bbbk(\tau^q), \Bbbk(\lambda^q) \in \mathscr{S}_2$. Let $\tau \in \Bbbk^{-1}(\Bbbk(\tau^q))$ and $\lambda \in \Bbbk^{-1}(\Bbbk(\lambda^q))$ be such that $\Re_\ell^{F^n}(\tau^q) \cdot e^{x\Im_\ell^{F^n}(\tau^q)} = \bigvee_{\tau \in \Bbbk^{-1}(\Bbbk(\tau^q))} \Re_\ell^{F^n}(\tau^q) \cdot e^{x\Im_\ell^{F^n}(\tau^q)}$ and $\Re_\ell^{F^n}(\lambda^q) \cdot e^{x\Im_\ell^{F^n}(\lambda^q)} = \bigvee_{\tau \in \Bbbk^{-1}(\Bbbk(\lambda^q))} \Re_\ell^{F^n}(\tau^q) \cdot e^{x\Im_\ell^{F^n}(\tau^q)}$.

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Now, $\Re_{\varpi}^{F^n}((\Bbbk(\tau^q)\biguplus_1 \Bbbk(\lambda^q)))\cdot e^{x\Im_{\varpi}^{F^n}((\Bbbk(\tau^q)\biguplus_1 \Bbbk(\lambda^q)))}$

$$\begin{aligned}
&= \bigvee_{(\tau')\in\Bbbk^{-1}(\Bbbk(\tau^q)\biguplus_1 \Bbbk(\lambda^q))} \Re_{\ell}^{F^n}(\tau')\cdot e^{x\Im_{\ell}^{F^n}(\tau')}\\
&= \bigvee_{(\tau')\in\Bbbk^{-1}(\Bbbk((\tau\ominus_1\lambda)^q)} \Re_{\ell}^{F^n}(\tau')\cdot e^{x\Im_{\ell}^{F^n}(\tau')}\\
&= \Re_{\ell}^{F^n}((\tau\ominus_1\lambda)^q)\cdot e^{x\Im_{\ell}^{F^n}((\tau\ominus_1\lambda)^q)}\\
&\geq \min\{\Re_{\ell}^{F^n}(\tau^q)\cdot e^{x\Im_{\ell}^{F^n}(\tau^q)}, \Re_{\ell}^{F^n}(\lambda^q)\cdot e^{x\Im_{\ell}^{F^n}(\lambda^q)}\}\\
&= \min\{\Re_{\varpi}^{F^n}\Bbbk(\tau^q)\cdot e^{x\Im_{\varpi}^{F^n}\Bbbk(\tau^q)}, \Re_{\varpi}^{F^n}\Bbbk(\lambda^q)\cdot e^{x\Im_{\varpi}^{F^n}\Bbbk(\lambda^q)}\}.
\end{aligned}$$

Thus,$\Re_{\varpi}^{F^n}((\Bbbk(\tau^q)\biguplus_1 \Bbbk(\lambda^q)))\cdot e^{x\Im_{\varpi}^{F^n}((\Bbbk(\tau^q)\biguplus_1 \Bbbk(\lambda^q)))} \geq \min\{\Re_{\varpi}^{F^n}\Bbbk(\tau^q)\cdot e^{x\Im_{\varpi}^{F^n}\Bbbk(\tau^q)}, \Re_{\varpi}^{F^n}\Bbbk(\lambda^q)\cdot e^{x\Im_{\varpi}^{F^n}\Bbbk(\lambda^q)}\}$.
Similarly,
$\Re_{\varpi}^{F^n}((\Bbbk(\tau^q)\biguplus_2 \Bbbk(\lambda^q)))\cdot e^{x\Im_{\varpi}^{F^n}((\Bbbk(\tau^q)\biguplus_2 \Bbbk(\lambda^q)))} \geq \min\{\Re_{\varpi}^{F^n}\Bbbk(\tau^q)\cdot e^{x\Im_{\varpi}^{F^n}\Bbbk(\tau^q)}, \Re_{\varpi}^{F^n}\Bbbk(\lambda^q)\cdot e^{x\Im_{\varpi}^{F^n}\Bbbk(\lambda^q)}\}$ and
$\Re_{\varpi}^{F^n}((\Bbbk(\tau^q)\biguplus_3 \Bbbk(\lambda^q)))\cdot e^{x\Im_{\varpi}^{F^n}((\Bbbk(\tau^q)\biguplus_1 \Bbbk(\lambda^q)))} \geq \min\{\Re_{\varpi}^{F^n}\Bbbk(\tau^q)\cdot e^{x\Im_{\varpi}^{F^n}\Bbbk(\lambda^q)}, \Re_{\varpi}^{F^n}\Bbbk(\lambda^q)\cdot e^{x\Im_{\varpi}^{F^n}\Bbbk(\lambda^q)}\}$.

The mapping $\Bbbk:\mathscr{S}_1\to\mathscr{S}_2$ be any homomorphism. Now, $\Bbbk((\tau\ominus_1\lambda)^q)=\Bbbk(\tau^q)\biguplus_1\Bbbk(\lambda^q), \Bbbk((\tau\ominus_2\lambda)^q)=\Bbbk(\tau^q)\biguplus_2\Bbbk(\lambda^q)$ and $\Bbbk((\tau\ominus_3\lambda)^q)=\Bbbk(\tau^q)\biguplus_3\Bbbk(\lambda^q)$ for all $\tau,\lambda\in\mathscr{S}_1$. Let $\varpi=\Bbbk(\ell)$,ℓ is any CQB-NSBS of $\mathscr{S}_1$. Let $\Bbbk(\tau^q),\Bbbk(\lambda^q)\in\mathscr{S}_2$. Let $\tau\in\Bbbk^{-1}(\Bbbk(\tau^q))$ and $\lambda\in\Bbbk^{-1}(\Bbbk(\lambda^q))$ be such that $\Re_{\ell}^{T^p}(\tau^q)\cdot e^{x\Im_{\ell}^{T^p}(\tau^q)} = \bigvee_{\tau\in\Bbbk^{-1}(\Bbbk(\tau^q))}\Re_{\ell}^{T^p}(\tau^q)\cdot e^{x\Im_{\ell}^{T^p}(\tau^q)}$ and $\Re_{\ell}^{T^p}(\lambda^q)\cdot e^{x\Im_{\ell}^{T^p}(\lambda^q)} = \bigvee_{\tau\in\Bbbk^{-1}(\Bbbk(\lambda^q))}\Re_{\ell}^{T^p}(\tau^q)\cdot e^{x\Im_{\ell}^{T^p}(\tau^q)}$.
Now, $\Re_{\varpi}^{T^p}((\Bbbk(\tau^q)\biguplus_1 \Bbbk(\lambda^q)))\cdot e^{x\Im_{\ell}^{T^p}((\Bbbk(\tau^q)\biguplus_1 \Bbbk(\lambda^q)))}$

$$\begin{aligned}
&= \bigvee_{(\tau')\in\Bbbk^{-1}(\Bbbk(\tau^q)\biguplus_1 \Bbbk(\lambda^q))} \Re_{\ell}^{T^p}(\tau')\cdot e^{x\Im_{\ell}^{T^p}(\tau')}\\
&= \bigvee_{(\tau')\in\Bbbk^{-1}(\Bbbk((\tau\ominus_1\lambda)^q)} \Re_{\ell}^{T^p}(\tau')\cdot e^{x\Im_{\ell}^{T^p}(\tau')}\\
&= \Re_{\ell}^{T^p}((\tau\ominus_1\lambda)^q)\cdot e^{x\Im_{\ell}^{T^p}((\tau\ominus_1\lambda)^q)}\\
&\geq \min\{\Re_{\ell}^{T^p}(\tau^q)\cdot e^{x\Im_{\ell}^{T^p}(\tau^q)}, \Re_{\ell}^{T^p}(\lambda^q)\cdot e^{x\Im_{\ell}^{T^p}(\lambda^q)}\}\\
&= \min\{\Re_{\varpi}^{T^p}\Bbbk(\tau^q)\cdot e^{x\Im_{\varpi}^{T^p}\Bbbk(\tau^q)}, \Re_{\varpi}^{T^p}\Bbbk(\lambda^q)\cdot e^{x\Im_{\varpi}^{T^p}\Bbbk(\lambda^q)}\}.
\end{aligned}$$

Thus,$\Re_{\varpi}^{T^p}((\Bbbk(\tau^q)\biguplus_1 \Bbbk(\lambda^q)))\cdot e^{x\Im_{\varpi}^{T^p}((\Bbbk(\tau^q)\biguplus_1 \Bbbk(\lambda^q)))} \geq \min\{\Re_{\varpi}^{T^p}\Bbbk(\tau^q)\cdot e^{x\Im_{\varpi}^{T^p}\Bbbk(\tau^q)}, \Re_{\varpi}^{T^p}\Bbbk(\lambda^q)\cdot e^{x\Im_{\varpi}^{T^p}\Bbbk(\lambda^q)}\}$.
Similarly,$\Re_{\varpi}^{T^p}((\Bbbk(\tau^q)\biguplus_2 \Bbbk(\lambda^q)))\cdot e^{x\Im_{\varpi}^{T^p}((\Bbbk(\tau^q)\biguplus_2 \Bbbk(\lambda^q)))} \geq \min\{\Re_{\varpi}^{T^p}\Bbbk(\tau^q)\cdot e^{x\Im_{\varpi}^{T^p}\Bbbk(\tau^q)}, \Re_{\varpi}^{T^p}\Bbbk(\lambda^q)\cdot e^{x\Im_{\varpi}^{T^p}\Bbbk(\lambda^q)}\}$
and
$\Re_{\varpi}^{T^p}((\Bbbk(\tau^q)\biguplus_3 \Bbbk(\lambda^q)))\cdot e^{x\Im_{\varpi}^{T^p}((\Bbbk(\tau^q)\biguplus_3 \Bbbk(\lambda^q)))} \geq \min\{\Re_{\varpi}^{T^p}\Bbbk(\tau^q)\cdot e^{x\Im_{\varpi}^{T^p}\Bbbk(\tau^q)}, \Re_{\varpi}^{T^p}\Bbbk(\lambda^q)\cdot e^{x\Im_{\varpi}^{T^p}\Bbbk(\lambda^q)}\}$.
Let $\tau\in\Bbbk^{-1}(\Bbbk(\tau^q))$ and $\lambda\in\Bbbk^{-1}(\Bbbk(\lambda^q))$ be such that $\Re_{\ell}^{I^p}(\tau^q)\cdot e^{x\Im_{\varpi}^{T^p}(\tau^q)} = \bigvee_{\tau\in\Bbbk^{-1}(\Bbbk(\tau^q))}\Re_{\ell}^{I^p}(\tau^q)\cdot e^{x\Im_{\varpi}^{T^p}(\tau^q)}$ and $\Re_{\ell}^{I^p}(\lambda^q)\cdot e^{x\Im_{\varpi}^{T^p}(\lambda^q)} = \bigvee_{\tau\in\Bbbk^{-1}(\Bbbk(\lambda^q))}\Re_{\ell}^{I^p}(\tau^q)\cdot e^{x\Im_{\varpi}^{T^p}\Bbbk(\tau^q)}$.
Now, $\Re_{\varpi}^{I^p}((\Bbbk(\tau^q)\biguplus_1 \Bbbk(\lambda^q)))\cdot e^{x\Im_{\varpi}^{I^p}((\Bbbk(\tau^q)\biguplus_1 \Bbbk(\lambda^q)))}$

$$\begin{aligned}
&= \bigvee_{(\tau')\in\Bbbk^{-1}(\Bbbk(\tau^q)\biguplus_1 \Bbbk(\lambda^q))} \Re_{\ell}^{I^p}(\tau')\cdot e^{x\Im_{\ell}^{I^p}(\tau')}\\
&= \bigvee_{(\tau')\in\Bbbk^{-1}(\Bbbk((\tau\ominus_1\lambda)^q)} \Re_{\ell}^{I^p}(\tau')\cdot e^{x\Im_{\ell}^{I^p}(\tau')}\\
&= \Re_{\ell}^{I^p}((\tau\ominus_1\lambda)^q)\cdot e^{x\Im_{\ell}^{I^p}((\tau\ominus_1\lambda)^q)}\\
&\geq \frac{\Re_{\ell}^{I^p}(\tau^q)\cdot e^{x\Im_{\ell}^{I^p}(\tau^q)} + \Re_{\ell}^{I^p}(\lambda^q)\cdot e^{x\Im_{\ell}^{I^p}(\lambda^q)}}{2}\\
&= \frac{\Re_{\varpi}^{I^p}\Bbbk(\tau^q)\cdot e^{x\Im_{\varpi}^{I^p}\Bbbk(\tau^q)} + \Re_{\varpi}^{I^p}\Bbbk(\lambda^q)\cdot e^{x\Im_{\varpi}^{I^p}\Bbbk(\lambda^q)}}{2}.
\end{aligned}$$

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Thus,$\Re^{I^p}_{\varpi}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q))) \cdot e^{x\Im^{I^p}_{\varpi}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q)))} \geq \frac{\Re^{I^p}_{\varpi}\Bbbk(\tau^q)\cdot e^{x\Im^{I^p}_{\varpi}\Bbbk(\tau^q)}+\Re^{I^p}_{\varpi}\Bbbk(\lambda^q)\cdot e^{x\Im^{I^p}_{\varpi}\Bbbk(\lambda^q)}}{2}$. Similarly, $\Re^{I^p}_{\varpi}((\Bbbk(\tau^q) \biguplus_2 \Bbbk(\lambda^q))) \cdot e^{x\Im^{I^p}_{\varpi}((\Bbbk(\tau^q) \biguplus_2 \Bbbk(\lambda^q)))} \geq \frac{\Re^{I^p}_{\varpi}\Bbbk(\tau^q)\cdot e^{x\Im^{I^p}_{\varpi}\Bbbk(\tau^q)}+\Re^{I^p}_{\varpi}\Bbbk(\lambda^q)\cdot e^{x\Im^{I^p}_{\varpi}\Bbbk(\lambda^q)}}{2}$ and $\Re^{I^p}_{\varpi}((\Bbbk(\tau^q) \biguplus_3 \Bbbk(\lambda^q))) \cdot e^{x\Im^{I^p}_{\varpi}((\Bbbk(\tau^q) \biguplus_3 \Bbbk(\lambda^q)))} \geq \frac{\Re^{I^p}_{\varpi}\Bbbk(\tau^q)\cdot e^{x\Im^{I^p}_{\varpi}\Bbbk(\tau^q)}+\Re^{I^p}_{\varpi}\Bbbk(\lambda^q)\cdot e^{x\Im^{I^p}_{\varpi}\Bbbk(\lambda^q)}}{2}$.
Let $\Bbbk(\tau^q), \Bbbk(\lambda^q) \in \mathscr{S}_2$. Let $\tau \in \Bbbk^{-1}(\Bbbk(\tau^q))$ and $\lambda \in \Bbbk^{-1}(\Bbbk(\lambda^q))$ be such that $\Re^{F^p}_{\ell}(\tau^q) \cdot e^{x\Im^{F^p}_{\ell}(\tau^q)} = \bigwedge_{\tau\in\Bbbk^{-1}(\Bbbk(\tau^q))} \Re^{F^p}_{\ell}(\tau^q) \cdot e^{x\Im^{F^p}_{\ell}(\tau^q)}$ and $\Re^{F^p}_{\ell}(\lambda^q) \cdot e^{x\Im^{F^p}_{\ell}(\lambda^q)} = \bigwedge_{\tau\in\Bbbk^{-1}(\Bbbk(\lambda^q))} \Re^{F^p}_{\ell}(\tau^q) \cdot e^{x\Im^{F^p}_{\ell}(\tau^q)}$.
Now, $\Re^{F^p}_{\varpi}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q))) \cdot e^{x\Im^{F^p}_{\varpi}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q)))}$

$$\begin{aligned}
&= \bigwedge_{(\tau')\in\Bbbk^{-1}(\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q))} \Re^{F^p}_{\ell}(\tau') \cdot e^{x\Im^{F^p}_{\ell}(\tau')} \\
&= \bigwedge_{(\tau')\in\Bbbk^{-1}(\Bbbk((\tau\ominus_1\lambda)^q)} \Re^{F^p}_{\ell}(\tau') \cdot e^{x\Im^{F^p}_{\ell}(\tau')} \\
&= \Re^{F^p}_{\ell}((\tau \ominus_1 \lambda)^q) \cdot e^{x\Im^{F^p}_{\ell}((\tau\ominus_1\lambda)^q)} \\
&\leq \max\{\Re^{F^p}_{\ell}(\tau^q) \cdot e^{x\Im^{F^p}_{\ell}(\tau^q)}, \Re^{F^p}_{\ell}(\lambda^q) \cdot e^{x\Im^{F^p}_{\ell}(\lambda^q)}\} \\
&= \max\{\Re^{F^p}_{\varpi}\Bbbk(\tau^q) \cdot e^{x\Im^{F^p}_{\varpi}\Bbbk(\tau^q)}, \Re^{F^p}_{\varpi}\Bbbk(\lambda^q) \cdot e^{x\Im^{F^p}_{\varpi}\Bbbk(\lambda^q)}\}.
\end{aligned}$$

Thus,$\Re^{F^p}_{\varpi}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q)))\cdot e^{x\Im^{F^p}_{\varpi}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q)))} \leq \max\{\Re^{F^p}_{\varpi}\Bbbk(\tau^q)\cdot e^{x\Im^{F^p}_{\varpi}\Bbbk(\tau^q)}, \Re^{F^p}_{\varpi}\Bbbk(\lambda^q)\cdot e^{x\Im^{F^p}_{\varpi}\Bbbk(\lambda^q)}\}$.
Similarly,$\Re^{F^p}_{\varpi}((\Bbbk(\tau^q) \biguplus_2 \Bbbk(\lambda^q)))\cdot e^{x\Im^{F^p}_{\varpi}((\Bbbk(\tau^q) \biguplus_2 \Bbbk(\lambda^q)))} \leq \max\{\Re^{F^p}_{\varpi}\Bbbk(\tau^q)\cdot e^{x\Im^{F^p}_{\varpi}\Bbbk(\tau^q)}, \Re^{F^p}_{\varpi}\Bbbk(\lambda^q)\cdot e^{x\Im^{F^p}_{\varpi}\Bbbk(\lambda^q)}\}$ and
$\Re^{F^p}_{\varpi}((\Bbbk(\tau^q) \biguplus_3 \Bbbk(\lambda^q))) \cdot e^{x\Im^{F^p}_{\varpi}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q)))} \leq \max\{\Re^{F^p}_{\varpi}\Bbbk(\tau^q) \cdot e^{x\Im^{F^p}_{\varpi}\Bbbk(\lambda^q)}, \Re^{F^p}_{\varpi}\Bbbk(\lambda^q) \cdot e^{x\Im^{F^p}_{\varpi}\Bbbk(\lambda^q)}\}$.
Thus,ϖ is a CQBNSBS of $\mathscr{S}_2$.

Theorem 3.15. *The homomorphic preimage of every CQBNSBS is a CQBNSBS.*

Proof. The mapping $\Bbbk : \mathscr{S}_1 \to \mathscr{S}_2$ be a homomorphism. Now,$\Bbbk((\tau\ominus_1\lambda)^q) = \Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q), \Bbbk((\tau\ominus_2\lambda)^q) = \Bbbk(\tau^q) \biguplus_2 \Bbbk(\lambda^q)$ and $\Bbbk((\tau \ominus_3 \lambda)^q) = \Bbbk(\tau^q) \biguplus_3 \Bbbk(\lambda^q)$ for all $\tau, \lambda \in \mathscr{S}_1$. Let $\varpi = \Bbbk(\ell)$,ϖ is a CQBNSBS of $\mathscr{S}_2$. Let $\tau, \lambda \in \mathscr{S}_1$. Now,$\Re^{T^n}_{\ell}((\tau \ominus_1 \lambda)^q) \cdot e^{x\Im^{T^n}_{\ell}((\tau\ominus_1\lambda)^q)} = \Re^{T^n}_{\varpi}(\Bbbk((\tau \ominus_1 \lambda)^q)) \cdot e^{x\Im^{T^n}_{\varpi}(\Bbbk((\tau\ominus_1\lambda)^q))} = \Re^{T^n}_{\varpi}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q))) \cdot e^{x\Im^{T^n}_{\varpi}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q)))} \leq \max\{\Re^{T^n}_{\varpi}\Bbbk(\tau^q) \cdot e^{x\Im^{T^n}_{\varpi}\Bbbk(\tau^q)}, \Re^{T^n}_{\varpi}\Bbbk(\lambda^q) \cdot e^{x\Im^{T^n}_{\varpi}\Bbbk(\lambda^q)}\} = \max\{\Re^{T^n}_{\ell}(\tau^q) \cdot e^{x\Im^{T^n}_{\ell}(\tau^q)}, \Re^{T^n}_{\ell}(\lambda^q) \cdot e^{x\Im^{T^n}_{\ell}(\lambda^q)}\}$. Thus,$\Re^{T^n}_{\ell}((\tau \ominus_1 \lambda)^q) \cdot e^{x\Im^{T^n}_{\ell}((\tau\ominus_1\lambda)^q)} \leq \max\{\Re^{T^n}_{\ell}(\tau^q) \cdot e^{x\Im^{T^n}_{\ell}(\tau)}, \Re^{T^n}_{\ell}(\lambda^q) \cdot e^{x\Im^{T^n}_{\ell}(\lambda^q)}\}$.
Now,$\Re^{I^n}_{\ell}((\tau \ominus_1 \lambda)^q) \cdot e^{x\Im^{I^n}_{\ell}((\tau\ominus_1\lambda)^q)} = \Re^{I^n}_{\varpi}(\Bbbk((\tau \ominus_1 \lambda)^q)) \cdot e^{x\Im^{I^n}_{\varpi}((\tau\ominus_1\lambda)^q)} = \Re^{I^n}_{\varpi}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q))) \cdot e^{x\Im^{I^n}_{\varpi}((\tau\ominus_1\lambda)^q)} \leq \frac{\Re^{I^n}_{\varpi}\Bbbk(\tau^q)\cdot e^{x\Im^{I^n}_{\varpi}\Bbbk(\tau^q)}+\Re^{I^n}_{\varpi}\Bbbk(\lambda^q)\cdot e^{x\Im^{I^n}_{\varpi}\Bbbk(\lambda^q)}}{2} = \frac{\Re^{I^n}_{\ell}(\tau^q)\cdot e^{x\Im^{I^n}_{\ell}(\tau^q)}+\Re^{I^n}_{\ell}(\lambda^q)\cdot e^{x\Im^{I^n}_{\ell}(\lambda^q)}}{2}$.
Thus,$\Re^{I^n}_{\ell}((\tau \ominus_1 \lambda)^q) \cdot e^{x\Im^{I^n}_{\ell}((\tau\ominus_1\lambda)^q)} \leq \frac{\Re^{I^n}_{\ell}(\tau^q)\cdot e^{x\Im^{I^n}_{\ell}(\tau^q)}+\Re^{I^n}_{\ell}(\lambda^q)\cdot e^{x\Im^{I^n}_{\ell}(\lambda^q)}}{2}$.
Now, $\Re^{F^n}_{\ell}((\tau \ominus_1 \lambda)^q) \cdot e^{x\Im^{F^n}_{\ell}((\tau\ominus_1\lambda)^q)} = \Re^{F^n}_{\varpi}(\Bbbk((\tau \ominus_1 \lambda)^q)) \cdot e^{x\Im^{F^n}_{\varpi}(\Bbbk((\tau\ominus_1\lambda)^q))} = \Re^{F^n}_{\varpi}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q)))\cdot e^{x\Im^{F^n}_{\varpi}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q)))} \geq \min\{\Re^{F^n}_{\varpi}\Bbbk(\tau^q)\cdot e^{x\Im^{F^n}_{\varpi}\Bbbk(\tau^q)}, \Re^{F^n}_{\varpi}\Bbbk(\lambda^q)\cdot e^{x\Im^{F^n}_{\varpi}\Bbbk(\lambda^q)}\} = \min\{\Re^{F^n}_{\ell}(\tau^q)\cdot e^{x\Im^{F^n}_{\ell}(\tau^q)}, \Re^{F^n}_{\ell}(\lambda^q)\cdot e^{x\Im^{F^n}_{\ell}(\lambda^q)}\}$.
Thus,$\Re^{F^n}_{\ell}((\tau \ominus_1 \lambda)^q) \cdot e^{x\Im^{F^n}_{\ell}((\tau\ominus_1\lambda)^q)} \geq \min\{\Re^{F^n}_{\ell}(\tau^q) \cdot e^{x\Im^{F^n}_{\ell}(\tau^q)}, \Re^{F^n}_{\ell}(\lambda^q) \cdot e^{x\Im^{F^n}_{\ell}(\lambda^q)}\}$.

The mapping $\Bbbk : \mathscr{S}_1 \to \mathscr{S}_2$ be a homomorphism. Now,$\Bbbk((\tau \ominus_1 \lambda)^q) = \Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q), \Bbbk((\tau \ominus_2 \lambda)^q) = \Bbbk(\tau^q) \biguplus_2 \Bbbk(\lambda^q)$ and $\Bbbk((\tau \ominus_3 \lambda)^q) = \Bbbk(\tau^q) \biguplus_3 \Bbbk(\lambda^q)$ for all $\tau, \lambda \in \mathscr{S}_1$. Let $\varpi = \Bbbk(\ell)$,ϖ is a CQBNSBS of $\mathscr{S}_2$. Let $\tau, \lambda \in \mathscr{S}_1$. Now,$\Re^{T^p}_{\ell}((\tau \ominus_1 \lambda)^q) \cdot e^{x\Im^{T^p}_{\ell}((\tau\ominus_1\lambda)^q)} = \Re^{T^p}_{\varpi}(\Bbbk((\tau \ominus_1 \lambda)^q)) \cdot e^{x\Im^{T^p}_{\varpi}(\Bbbk((\tau\ominus_1\lambda)^q))} = \Re^{T^p}_{\varpi}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q))) \cdot e^{x\Im^{T^p}_{\varpi}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q)))} \geq \min\{\Re^{T^p}_{\varpi}\Bbbk(\tau^q) \cdot e^{x\Im^{T^p}_{\varpi}\Bbbk(\tau^q)}, \Re^{T^p}_{\varpi}\Bbbk(\lambda^q) \cdot e^{x\Im^{T^p}_{\varpi}\Bbbk(\lambda^q)}\} = \min\{\Re^{T^p}_{\ell}(\tau^q) \cdot e^{x\Im^{T^p}_{\ell}(\tau^q)}, \Re^{T^p}_{\ell}(\lambda^q) \cdot e^{x\Im^{T^p}_{\ell}(\lambda^q)}\}$. Thus,$\Re^{T^p}_{\ell}((\tau \ominus_1 \lambda)^q) \cdot e^{x\Im^{T^p}_{\ell}((\tau\ominus_1\lambda)^q)} \geq \min\{\Re^{T^p}_{\ell}(\tau^q) \cdot e^{x\Im^{T^p}_{\ell}(\tau)}, \Re^{T^p}_{\ell}(\lambda^q) \cdot e^{x\Im^{T^p}_{\ell}(\lambda^q)}\}$.
Now,$\Re^{I^p}_{\ell}((\tau \ominus_1 \lambda)^q) \cdot e^{x\Im^{I^p}_{\ell}((\tau\ominus_1\lambda)^q)} = \Re^{I^p}_{\varpi}(\Bbbk((\tau \ominus_1 \lambda)^q)) \cdot e^{x\Im^{I^p}_{\varpi}((\tau\ominus_1\lambda)^q)} = \Re^{I^p}_{\varpi}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q))) \cdot e^{x\Im^{I^p}_{\varpi}((\tau\ominus_1\lambda)^q)} \geq \frac{\Re^{I^p}_{\varpi}\Bbbk(\tau^q)\cdot e^{x\Im^{I^p}_{\varpi}\Bbbk(\tau^q)}+\Re^{I^p}_{\varpi}\Bbbk(\lambda^q)\cdot e^{x\Im^{I^p}_{\varpi}\Bbbk(\lambda^q)}}{2} = \frac{\Re^{I^p}_{\ell}(\tau^q)\cdot e^{x\Im^{I^p}_{\ell}(\tau^q)}+\Re^{I^p}_{\ell}(\lambda^q)\cdot e^{x\Im^{I^p}_{\ell}(\lambda^q)}}{2}$.
Thus, $\Re^{I^p}_{\ell}((\tau \ominus_1 \lambda)^q) \cdot e^{x\Im^{I^p}_{\ell}((\tau\ominus_1\lambda)^q)} \geq \frac{\Re^{I^p}_{\ell}(\tau^q)\cdot e^{x\Im^{I^p}_{\ell}(\tau^q)}+\Re^{I^p}_{\ell}(\lambda^q)\cdot e^{x\Im^{I^p}_{\ell}(\lambda^q)}}{2}$.

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Now,$\Re_{\ell}^{F^p}((\tau \ominus_1 \lambda)^q) \cdot e^{x\Im_{\ell}^{F^p}((\tau\ominus_1\lambda)^q)} = \Re_{\varpi}^{F^p}(\Bbbk((\tau \ominus_1 \lambda)^q)) \cdot e^{x\Im_{\varpi}^{F^p}(\Bbbk((\tau\ominus_1\lambda)^q))} = \Re_{\varpi}^{F^p}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q))) \cdot e^{x\Im_{\varpi}^{F^p}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q)))} \leq \max\{\Re_{\varpi}^{F^p}\Bbbk(\tau^q) \cdot e^{x\Im_{\varpi}^{F^p}\Bbbk(\tau^q)}, \Re_{\varpi}^{F^p}\Bbbk(\lambda^q) \cdot e^{x\Im_{\varpi}^{F^p}\Bbbk(\lambda^q)}\} = \max\{\Re_{\ell}^{F^p}(\tau^q) \cdot e^{x\Im_{\ell}^{F^p}(\tau^q)}, \Re_{\ell}^{F^p}(\lambda^q) \cdot e^{x\Im_{\ell}^{F^p}(\lambda^q)}\}$. Thus,$\Re_{\ell}^{F^p}((\tau \ominus_1 \lambda)^q) \cdot e^{x\Im_{\ell}^{F^p}((\tau\ominus_1\lambda)^q)} \leq \max\{\Re_{\ell}^{F^p}(\tau^q) \cdot e^{x\Im_{\ell}^{F^p}(\tau^q)}, \Re_{\ell}^{F^p}(\lambda^q) \cdot e^{x\Im_{\ell}^{F^p}(\lambda^q)}\}$.

Theorem 3.16. *If $\Bbbk : \mathscr{S}_1 \to \mathscr{S}_2$ is a homomorphism,then $\Bbbk(\ell_{(\wp_1,\wp_2)})$ is a SBS of CQBNSBS ϖ of $\mathscr{S}_2$.*

Proof. The mapping $\Bbbk : \mathscr{S}_1 \to \mathscr{S}_2$ be a homomorphism. Now,$\Bbbk((\tau\ominus_1\lambda)^q) = \Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q), \Bbbk((\tau\ominus_2\lambda)^q) = \Bbbk(\tau^q) \biguplus_2 \Bbbk(\lambda^q)$ and $\Bbbk((\tau \ominus_3 \lambda)^q) = \Bbbk(\tau^q) \biguplus_3 \Bbbk(\lambda^q)$ for all $\tau, \lambda \in \mathscr{S}_1$. Let $\varpi = \Bbbk(\ell)$,ℓ is a CQBNSBS of $\mathscr{S}_1$. By Theorem 3.14, ϖ is a CQBNSBS of $\mathscr{S}_2$. Let $\ell_{(\wp_1,\wp_2)}$ be any SBS of ℓ. Suppose that $\tau, \lambda \in \ell_{(\wp_1,\wp_2)}$. Then $\tau \ominus_1 \lambda, \tau \ominus_2 \lambda$ and $\tau \ominus_3 \lambda \in \ell_{(\wp_1,\wp_2)}$. Now,$\Re_{\varpi}^{T^n}(\Bbbk(\tau^q)) \cdot e^{x\Im_{\varpi}^{T^n}(\Bbbk(\tau^q))} = \Re_{\ell}^{T^n}(\tau^q) \cdot e^{x\Im_{\ell}^{T^n}(\tau^q)} \leq \wp_1, \Re_{\varpi}^{T^n}(\Bbbk(\lambda^q)) \cdot e^{x\Im_{\varpi}^{T^n}(\Bbbk(\lambda^q))} = \Re_{\ell}^{T^n}(\lambda^q) \cdot e^{x\Im_{\ell}^{T^n}(\lambda^q)} \leq \wp_1$. Thus,$\Re_{\varpi}^{T^n}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q))) \cdot e^{x\Im_{\varpi}^{T^n}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q)))} \leq \Re_{\ell}^{T^n}((\tau \ominus_1 \lambda)^q) \cdot e^{x\Im_{\ell}^{T^n}((\tau\ominus_1\lambda)^q)} \leq \wp_1$. Now,$\Re_{\varpi}^{I^n}(\Bbbk(\tau^q)) \cdot e^{x\Im_{\varpi}^{I^n}(\Bbbk(\tau^q))} = \Re_{\ell}^{I^n}(\tau^q) \cdot e^{x\Im_{\ell}^{I^n}(\tau^q)} \leq \wp_1, \Re_{\varpi}^{I^n}(\Bbbk(\lambda^q)) \cdot e^{x\Im_{\varpi}^{I^n}(\Bbbk(\lambda^q))} = \Re_{\ell}^{I^n}(\lambda^q) \cdot e^{x\Im_{\ell}^{I^n}(\lambda^q)} \leq \wp_1$.
Thus,$\Re_{\varpi}^{I^n}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q))) \cdot e^{x\Im_{\varpi}^{I^n}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q)))} \leq \Re_{\ell}^{I^n}((\tau \ominus_1 \lambda)^q) \cdot e^{x\Im_{\ell}^{I^n}((\tau\ominus_1\lambda)^q)} \leq \wp_1$.
Now, $\Re_{\varpi}^{F^n}(\Bbbk(\tau^q)) \cdot e^{x\Im_{\varpi}^{F^n}(\Bbbk(\tau^q))} = \Re_{\ell}^{F^n}(\tau^q) \cdot e^{x\Im_{\ell}^{F^n}(\tau^q)} \geq \wp_2, \Re_{\varpi}^{F^n}(\Bbbk(\lambda^q)) \cdot e^{x\Im_{\varpi}^{F^n}(\Bbbk(\lambda^q))} = \Re_{\ell}^{F^n}(\lambda^q) \cdot e^{x\Im_{\ell}^{F^n}(\lambda^q)} \geq \wp_2$.
Thus,$\Re_{\varpi}^{F^n}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q))) \cdot e^{x\Im_{\varpi}^{F^n}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q)))} \geq \Re_{\ell}^{F^n}((\tau \ominus_1 \lambda)^q) \cdot e^{x\Im_{\ell}^{F^n}((\tau\ominus_1\lambda)^q)} \geq \wp_2$,for all $\Bbbk(\tau^q), \Bbbk(\lambda^q) \in \mathscr{S}_2$. Similarly other operations,$\Bbbk(\ell_{(\wp_1,\wp_2)})$ is a SBS of CQBNSBS ϖ of $\mathscr{S}_2$.
The mapping $\Bbbk : \mathscr{S}_1 \to \mathscr{S}_2$ be a homomorphism. Now,$\Bbbk((\tau \ominus_1 \lambda)^q) = \Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q), \Bbbk((\tau \ominus_2 \lambda)^q) = \Bbbk(\tau^q) \biguplus_2 \Bbbk(\lambda^q)$ and $\Bbbk((\tau \ominus_3 \lambda)^q) = \Bbbk(\tau^q) \biguplus_3 \Bbbk(\lambda^q)$ for all $\tau, \lambda \in \mathscr{S}_1$. Let $\varpi = \Bbbk(\ell)$,ℓ is a CQBNSBS of $\mathscr{S}_1$. By Theorem 3.14, ϖ is a CQBNSBS of $\mathscr{S}_2$. Let $\ell_{(\wp_1,\wp_2)}$ be any SBS of ℓ. Suppose that $\tau, \lambda \in \ell_{(\wp_1,\wp_2)}$. Then $\tau \ominus_1 \lambda, \tau \ominus_2 \lambda$ and $\tau \ominus_3 \lambda \in \ell_{(\wp_1,\wp_2)}$. Now,$\Re_{\varpi}^{T^p}(\Bbbk(\tau^q)) \cdot e^{x\Im_{\varpi}^{T^p}(\Bbbk(\tau^q))} = \Re_{\ell}^{T^p}(\tau^q) \cdot e^{x\Im_{\ell}^{T^p}(\tau^q)} \geq \wp_1, \Re_{\varpi}^{T^p}(\Bbbk(\lambda^q)) \cdot e^{x\Im_{\varpi}^{T^p}(\Bbbk(\lambda^q))} = \Re_{\ell}^{T^p}(\lambda^q) \cdot e^{x\Im_{\ell}^{T^p}(\lambda^q)} \geq \wp_1$.
Thus,$\Re_{\varpi}^{T^p}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q))) \cdot e^{x\Im_{\varpi}^{T^p}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q)))} \geq \Re_{\ell}^{T^p}((\tau \ominus_1 \lambda)^q) \cdot e^{x\Im_{\ell}^{T^p}((\tau\ominus_1\lambda)^q)} \geq \wp_1$.
Now, $\Re_{\varpi}^{I^p}(\Bbbk(\tau^q)) \cdot e^{x\Im_{\varpi}^{I^p}(\Bbbk(\tau^q))} = \Re_{\ell}^{I^p}(\tau^q) \cdot e^{x\Im_{\ell}^{I^p}(\tau^q)} \geq \wp_1, \Re_{\varpi}^{I^p}(\Bbbk(\lambda^q)) \cdot e^{x\Im_{\varpi}^{I^p}(\Bbbk(\lambda^q))} = \Re_{\ell}^{I^p}(\lambda^q) \cdot e^{x\Im_{\ell}^{I^p}(\lambda^q)} \geq \wp_1$.
Thus, $\Re_{\varpi}^{I^p}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q))) \cdot e^{x\Im_{\varpi}^{I^p}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q)))} \geq \Re_{\ell}^{I^p}((\tau \ominus_1 \lambda)^q) \cdot e^{x\Im_{\ell}^{I^p}((\tau\ominus_1\lambda)^q)} \geq \wp_1$.
Now, $\Re_{\varpi}^{F^p}(\Bbbk(\tau^q)) \cdot e^{x\Im_{\varpi}^{F^p}(\Bbbk(\tau^q))} = \Re_{\ell}^{F^p}(\tau^q) \cdot e^{x\Im_{\ell}^{F^p}(\tau^q)} \leq \wp_2, \Re_{\varpi}^{F^p}(\Bbbk(\lambda^q)) \cdot e^{x\Im_{\varpi}^{F^p}(\Bbbk(\lambda^q))} = \Re_{\ell}^{F^p}(\lambda^q) \cdot e^{x\Im_{\ell}^{F^p}(\lambda^q)} \leq \wp_2$.
Thus,$\Re_{\varpi}^{F^p}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q))) \cdot e^{x\Im_{\varpi}^{F^p}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q)))} \leq \Re_{\ell}^{F^p}((\tau \ominus_1 \lambda)^q) \cdot e^{x\Im_{\ell}^{F^p}((\tau\ominus_1\lambda)^q)} \leq \wp_2$,
for all $\Bbbk(\tau^q), \Bbbk(\lambda^q) \in \mathscr{S}_2$. Similarly other operations, $\Bbbk(\ell_{(\wp_1,\wp_2)})$ is a SBS of CQBNSBS ϖ of $\mathscr{S}_2$.

Theorem 3.17. *If $\Bbbk : \mathscr{S}_1 \to \mathscr{S}_2$ is any homomorphism,then $\ell_{(\wp_1,\wp_2)}$ is a SBS of CQBNSBS ℓ of $\mathscr{S}_1$.*

Proof. The mapping $\Bbbk : \mathscr{S}_1 \to \mathscr{S}_2$ be any homomorphism. We have $\Bbbk((\tau\ominus_1\lambda)^q) = \Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q), \Bbbk((\tau\ominus_2 \lambda)^q) = \Bbbk(\tau^q) \biguplus_2 \Bbbk(\lambda^q)$ and $\Bbbk((\tau \ominus_3 \lambda)^q) = \Bbbk(\tau^q) \biguplus_3 \Bbbk(\lambda^q)$ for all $\tau, \lambda \in \mathscr{S}_1$. Let $\varpi = \Bbbk(\ell)$,ϖ is a CQBNSBS of $\mathscr{S}_2$. By Theorem 3.15,ℓ is a CQBNSBS of $\mathscr{S}_1$. Let $\Bbbk(\ell_{(\wp_1,\wp_2)})$ be a SBS of ϖ. Suppose that $\Bbbk(\tau^q), \Bbbk(\lambda^q) \in \Bbbk(\ell_{(\wp_1,\wp_2)})$. Now,$\Bbbk((\tau \ominus_1 \lambda)^q), \Bbbk((\tau \ominus_2 \lambda)^q)$ and $\Bbbk((\tau \ominus_3 \lambda)^q) \in \Bbbk(\ell_{(\wp_1,\wp_2)})$. Now,$\Re_{\ell}^{T^n}(\tau^q) \cdot e^{x\Im_{\ell}^{T^n}(\tau^q)} = \Re_{\varpi}^{T^n}(\Bbbk(\tau^q)) \cdot e^{x\Im_{\varpi}^{T^n}(\Bbbk(\tau^q))} \leq \wp_1, \Re_{\ell}^{T^n}(\lambda^q) \cdot e^{x\Im_{\ell}^{T^n}(\lambda^q)} = \Re_{\varpi}^{T^n}(\Bbbk(\lambda^q)) \cdot e^{x\Im_{\varpi}^{T^n}(\Bbbk(\lambda^q))} \leq \wp_1$. Thus,$\Re_{\ell}^{T^n}((\tau \ominus_1 \lambda)^q) \cdot e^{x\Im_{\ell}^{T^n}((\tau\ominus_1\lambda)^q)} \leq \max\{\Re_{\ell}^{T^n}(\tau^q) \cdot e^{x\Im_{\ell}^{T^n}(\tau^q)}, \Re_{\ell}^{T^n}(\lambda^q) \cdot e^{x\Im_{\ell}^{T^n}(\lambda^q)}\} \leq \wp_1$. Now,$\Re_{\ell}^{I^n}(\tau^q) \cdot e^{x\Im_{\ell}^{I^n}(\tau^q)} = \Re_{\varpi}^{I^n}(\Bbbk(\tau^q)) \cdot e^{x\Im_{\varpi}^{I^n}(\Bbbk(\tau^q))} \leq \wp_1, \Re_{\ell}^{I^n}(\lambda^q) \cdot e^{x\Im_{\ell}^{I^n}(\lambda^q)} = \Re_{\varpi}^{I^n}(\Bbbk(\lambda^q)) \cdot e^{x\Im_{\varpi}^{I^n}(\Bbbk(\lambda^q))} \leq \wp_1$.
Thus,$\Re_{\ell}^{I^n}((\tau\ominus_1\lambda)^q) \cdot e^{x\Im_{\ell}^{I^n}((\tau\ominus_1\lambda)^q)} \leq \frac{\Re_{\ell}^{I^n}(\tau^q) \cdot e^{x\Im_{\ell}^{I^n}(\tau^q)} + \Re_{\ell}^{I^n}(\lambda^q) \cdot e^{x\Im_{\ell}^{I^n}(\lambda^q)}}{2} \leq \wp_1$. Now,$\Re_{\ell}^{F^n}(\tau^q) \cdot e^{x\Im_{\ell}^{F^n}(\tau^q)} = \Re_{\varpi}^{F^n}(\Bbbk(\tau^q)) \cdot e^{x\Im_{\varpi}^{F^n}(\Bbbk(\tau^q))} \geq \wp_2, \Re_{\ell}^{F^n}(\lambda^q) \cdot e^{x\Im_{\ell}^{F^n}(\lambda^q)} = \Re_{\varpi}^{F^n}(\Bbbk(\lambda^q)) \cdot e^{x\Im_{\varpi}^{F^n}(\Bbbk(\lambda^q))} \geq \wp_2$.
Thus,$\Re_{\ell}^{F^n}((\tau \ominus_1 \lambda)^q) \cdot e^{x\Im_{\ell}^{F^n}((\tau\ominus_1\lambda)^q)} = \Re_{\varpi}^{F^n}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q))) \cdot e^{x\Im_{\varpi}^{F^n}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q)))}$
$\geq \min\{\Re_{\ell}^{F^n}(\tau^q) \cdot e^{x\Im_{\ell}^{F^n}(\tau^q)}, \Re_{\ell}^{F^n}(\lambda^q) \cdot e^{x\Im_{\ell}^{F^n}(\lambda^q)}\} \geq \wp_2$,for all $\tau, \lambda \in \mathscr{S}_1$.
Similarly other operations,$\ell_{(\wp_1,\wp_2)}$ is a SBS of CQBNSBS ℓ of $\mathscr{S}_1$.

The mapping $\Bbbk : \mathscr{S}_1 \to \mathscr{S}_2$ be any homomorphism. We have $\Bbbk((\tau\ominus_1\lambda)^q) = \Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q), \Bbbk((\tau\ominus_2\lambda)^q) = \Bbbk(\tau^q) \biguplus_2 \Bbbk(\lambda^q)$ and $\Bbbk((\tau \ominus_3 \lambda)^q) = \Bbbk(\tau^q) \biguplus_3 \Bbbk(\lambda^q)$ for all $\tau, \lambda \in \mathscr{S}_1$. Let $\varpi = \Bbbk(\ell)$,ϖ is a CQBNSBS of

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$\mathscr{S}_2$. By Theorem 3.15,ℓ is a CQBNSBS of $\mathscr{S}_1$. Let $\Bbbk(\ell_{(\wp_1,\wp_2)})$ be a SBS of ϖ. Suppose that $\Bbbk(\tau^q), \Bbbk(\lambda^q) \in \Bbbk(\ell_{(\wp_1,\wp_2)})$. Now,$\Bbbk((\tau \ominus_1 \lambda)^q), \Bbbk((\tau \ominus_2 \lambda)^q)$ and $\Bbbk((\tau \ominus_3 \lambda)^q) \in \Bbbk(\ell_{(\wp_1,\wp_2)})$. Now,$\Re_\ell^{T^p}(\tau^q) \cdot e^{x\Im_\ell^{T^p}(\tau^q)} = \Re_\varpi^{T^p}(\Bbbk(\tau^q)) \cdot e^{x\Im_\varpi^{T^p}(\Bbbk(\tau^q))} \geq \wp_1, \Re_\ell^{T^p}(\lambda^q) \cdot e^{x\Im_\ell^{T^p}(\lambda^q)} = \Re_\varpi^{T^p}(\Bbbk(\lambda^q)) \cdot e^{x\Im_\varpi^{T^p}(\Bbbk(\lambda^q))} \geq \wp_1$.
Thus,$\Re_\ell^{T^p}((\tau\ominus_1\lambda)^q)\cdot e^{x\Im_\ell^{T^p}((\tau\ominus_1\lambda)^q)} \geq \min\{\Re_\ell^{T^p}(\tau^q)\cdot e^{x\Im_\ell^{T^p}(\tau^q)}, \Re_\ell^{T^p}(\lambda^q)\cdot e^{x\Im_\ell^{T^p}(\lambda^q)}\} \geq \wp_1$. Now,$\Re_\ell^{I^p}(\tau^q)\cdot e^{x\Im_\ell^{I^p}(\tau^q)} = \Re_\varpi^{I^p}(\Bbbk(\tau^q)) \cdot e^{x\Im_\varpi^{I^p}(\Bbbk(\tau^q))} \geq \wp_1, \Re_\ell^{I^p}(\lambda^q) \cdot e^{x\Im_\ell^{I^p}(\lambda^q)} = \Re_\varpi^{I^p}(\Bbbk(\lambda^q)) \cdot e^{x\Im_\varpi^{I^p}(\Bbbk(\lambda^q))} \geq \wp_1$.
Thus,$\Re_\ell^{I^p}((\tau\ominus_1\lambda)^q)\cdot e^{x\Im_\ell^{I^p}((\tau\ominus_1\lambda)^q)} \geq \frac{\Re_\ell^{I^p}(\tau^q)\cdot e^{x\Im_\ell^{I^p}(\tau^q)}+\Re_\ell^{I^p}(\lambda^q)\cdot e^{x\Im_\ell^{I^p}(\lambda^q)}}{2} \geq \wp_1$. Now,$\Re_\ell^{F^p}(\tau^q)\cdot e^{x\Im_\ell^{F^p}(\tau^q)} = \Re_\varpi^{F^p}(\Bbbk(\tau^q)) \cdot e^{x\Im_\varpi^{F^p}(\Bbbk(\tau^q))} \leq \wp_2, \Re_\ell^{F^p}(\lambda^q) \cdot e^{x\Im_\ell^{F^p}(\lambda^q)} = \Re_\varpi^{F^p}(\Bbbk(\lambda^q)) \cdot e^{x\Im_\varpi^{F^p}(\Bbbk(\lambda^q))} \leq \wp_2$.
Thus,$\Re_\ell^{F^p}((\tau \ominus_1 \lambda)^q) \cdot e^{x\Im_\ell^{F^p}((\tau\ominus_1\lambda)^q)} = \Re_\varpi^{F^p}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q))) \cdot e^{x\Im_\varpi^{F^p}((\Bbbk(\tau^q) \biguplus_1 \Bbbk(\lambda^q)))} \leq \max\{\Re_\ell^{F^p}(\tau^q) \cdot e^{x\Im_\ell^{F^p}(\tau^q)}, \Re_\ell^{F^p}(\lambda^q) \cdot e^{x\Im_\ell^{F^p}(\lambda^q)}\} \leq \wp_2$,for all $\tau, \lambda \in \mathscr{S}_1$.
Similarly other operations,$\ell_{(\wp_1,\wp_2)}$ is a SBS of CQBNSBS ℓ of $\mathscr{S}_1$.

4 Conclusion

This paper presents a novel form of CQBNSBS. Three grades are expressed in terms of a complex number by the complex bipolar neutrosophic subbisemiring, which approaches the concept of three grades in a novel way. A bipolar neutrosophic SBS carrying complicated interval values was defined. Level sets for CQBNSBS and CQBNNSBS were defined. Applying the set to bisemiring is our aim. Furthermore, an analysis of the properties of various conversions is conducted. As a result, we should think about using soft set CQBNSBS, soft set CQBNNSBS in the future.

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