Homomorphism of complex neutrosophic set extended to cubic Q neutrosophic set concept via subbisemiring of bisemirings

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Abstract

We introduce the concept of complex cubic Q neutrosophic subbisemiring (CCQNSBS) is a new extension of cubic Q neutrosophic subbisemiring. We examine the characteristics and homomorphic features of CCQNSBS. We communicate the CCQNSBS level sets for bisemirings. A cubic complex Q neutrosophic subset Γ of bisemiring $\mathscr{S}$ if and only if each non-empty level set $R^{(\ell,\flat)}$, where $R = (\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}} \cdot e^{i\tau\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}}, \overline{\Theta^{\mathscr{I}}_{\mathcal{R}}} \cdot e^{i\tau\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}}, \overline{\Theta^{\mathscr{F}}_{\mathcal{R}}} \cdot e^{i\tau\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}}, \Theta^{\mathscr{T}}_{\mathcal{R}} \cdot e^{i\tau\Theta^{\mathscr{T}}_{\mathcal{I}}}, \Theta^{\mathscr{I}}_{\mathcal{R}} \cdot e^{i\tau\Theta^{\mathscr{I}}_{\mathcal{I}}}, \Theta^{\mathscr{F}}_{\mathcal{R}} \cdot e^{i\tau\Theta^{\mathscr{F}}_{\mathcal{I}}})$ is a CCQNSBS of $\mathscr{S}$. We show that the intersection of all CCQNSBSs yields a CCQNSBS of $\mathscr{S}$. If $\Theta_1, \Theta_2, ..., \Theta_n$ be the finite collection of CCQNSBSs of $\mathscr{S}_1, \mathscr{S}_2, ..., \mathscr{S}_n$ respectively. Then $\Theta_1 \times \Theta_2 \times ... \times \Theta_n$ is a CCQNSBS of $\mathscr{S}_1 \times \mathscr{S}_2 \times ... \times \mathscr{S}_n$. If $\mathcal{F} : \mathscr{S}_1 \to \mathscr{S}_2$ is a homomorphism,then $\mathcal{F}(\overline{\Theta_{(\ell,\flat)}})$ is a subbisemiring of CCQNSBS $\overline{\Omega}$ of $\mathscr{S}_2$. Examples are provided to show how our findings are used.

Keywords: CCQNSBS; CCNQNSBS; SBS; Homomorphism

1 Introduction

Algebraic structures are used extensively in theoretical physics, computer science, control engineering, information sciences, coding theory, topological spaces, combinatorics, functional analysis, graph theory, Euclidean geometry, probability theory, commutative and non-commutative ring theory, optimization theory, discrete event dynamical systems, automata theory, mathematical modeling of quantum physics, parallel computation systems, and other fields. The complexity of systems grows by the day, making it more difficult for decision-makers to make the proper option. Achieving a single goal is challenging, but it is doable. Many firms struggled to motivate employees, define goals, and form opinions. As a result, whether individual or committee decisions are made, a variety of objectives must be addressed concurrently. Based on this assessment, it appears that the criteria are answered flexibly, making it difficult for each decision maker to find an optimal solution in each of the criterion concerned. To find the optimal choice, decision-makers must establish trustworthy and acceptable techniques. Historically, crisp procedures have been ineffective when dealing with ambiguity and uncertainty in decision making. Zadeh[1] developed fuzzy set (FS) theory, which excels at addressing ambiguity and uncertainty. An element in an FS is considered a member if it includes a single value from the interval. However, because resistance might exist in real-world settings, the degree of non-membership does not always equal one minus the degree of membership. As FS theory improves, an increasing number of hybrid fuzzy models are generated. Uncertainty has led to the development of several theories, such as FS,[1] intuitionistic FS (IFS),[2] Pythagorean FS (PFS),[3] and spherical FS (SFS).[4] Sets with grades ranging from 0 to 1, known as

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MG, make up an FS. IFS is classified as MG, notwithstanding Atanassov's[2] statement that non-membership grades (NMG) can only have a value of 1. There is a potential that during the decision-making process, the sum of MGs and NMGs will occasionally exceed 1. Yager[3] used PFS logic to create the generalized MG and NMG logic, which is determined by the square of the MGs and NMGs and has a value of no more than 1. These views fail to describe the neutral condition, which is neither positive nor negative. Cuong[5] shared the image with associates. FS used three grading points: positive, neutral, and negative. These grades combined could not be greater than 1. For some uses, it outperforms PFS and IFS. It is an independent generalization of three models that address the truth, indeterminacy, and falsity of FS and IFS.
Smarandache[6] invented the neutrosophic set (NS) to handle ambiguous and contradictory data. This reasoning determines the degree to which a statement is true, ambiguous, or false. Ramot et al.[7] introduce the concept of a complex fuzzy set (CFS). The membership functions of CFSs transactions can take many different values. The unit circle of the complex plane is extended to $[0, 1]$, whereas the unit circle of a fuzzy membership function remains fixed. The CFS X is distinguished by a membership function $\mu_X(x)$ that extends to the unit circle in the complex plane rather than only $[0, 1]$. $\mu_X(x)$ is a complex-valued function that awards a grade of membership of the type $\eta_X(x) \cdot e^{i\tau_X(x)}$, where $i = \sqrt{-1}$, to any element x of the discourse universe. The value of $\mu_X(x)$ is determined by two real-valued variables, $\eta_X(x)$ and $\tau_X(x)$, where $\eta_X(x), \tau_X(x) \in [0, 1]$. Golan[8] pioneered the use of semiring logic. Hussian and colleagues[9] explored the concept and applications of bisemirings. Lee[10] discusses bipolar-valued FSs and their related techniques. Ahsan et al. studied fuzzy semirings in.[11] Sen et al.[12] introduced bisemirings. Palanikumar et al.[13] proposed an intuitionistic fuzzy normal subbisemiring for bisemiring. Palanikumar et al.[14] proposed bisemiring using bipolar-valued neutrosophic normal sets. Several authors have lately published on innovative ideas such as the fuzzy extension set, neutrosopic set, and specific fuzzy set[15]-.[26] He studied their properties on the parellel lines to set theory. In 1971, Rosenfeld[27] defined fuzzy subgroups and gave some of its properties. Kuroki[28] introduced fuzzy semigroups as a generalized of classical semigroups. Mordeson[29] obtained some characterization of fuzzy semigroups.
We will study several aspects of the concepts of SBS and CCQNSBS and draw some conclusions. The following five sections comprise the article. Section 1 describes semirings and SBS. Section 2 discusses semiring and SBS preparation. The characteristics of CCQNSBS are listed in Section 3. Section 4 describes the homomorphism of a complex cubic Q neutrosophic subbisemiring. It is recommended to evaluate CCQNSBS with numerical examples.

2 Preliminaries

Definition 2.1. [12] An algebraic structure $(\mathscr{S}, \uplus, \ominus, \odot)$ is a bisemiring,if $(\mathscr{S}, \uplus, \ominus)$ and $(\mathscr{S}, \ominus, \odot)$ are semirings, ie.,$(\mathscr{S}, \uplus)$, $(\mathscr{S}, \ominus)$ and $(\mathscr{S}, \odot)$ are semigroups and

1. $\wp_v \ominus (\wp_\zeta \uplus \wp_\eta) = (\wp_v \ominus \wp_\zeta) \uplus (\wp_v \ominus \wp_\eta)$,
2. $(\wp_\zeta \uplus \wp_\eta) \ominus \wp_v = (\wp_\zeta \ominus \wp_v) \uplus (\wp_\eta \ominus \wp_v)$,
3. $\wp_v \odot (\wp_\zeta \ominus \wp_\eta) = (\wp_v \odot \wp_\zeta) \ominus (\wp_v \odot \wp_\eta)$,
4. $(\wp_\zeta \ominus \wp_\eta) \odot \wp_v = (\wp_\zeta \odot \wp_v) \ominus (\wp_\eta \odot \wp_v), \ \forall \ \wp_v, \wp_\zeta, \wp_\eta \in \mathscr{S}$.

Definition 2.2. [6] A NS v in the universe U is $v = \{x, u_v^{\mathscr{T}}(x), u_v^{\mathscr{I}}(x), u_v^{\mathscr{F}}(x) | x \in U\}$,where $u_v^{\mathscr{T}}(x)$,$u_v^{\mathscr{I}}(x)$ $u_v^{\mathscr{F}}(x)$ represents the TD,ID and FD of v respectively. Consider the mapping $u_v^{\mathscr{T}} : U \to [0, 1], u_v^{\mathscr{I}} : U \to [0, 1], u_v^{\mathscr{F}} : U \to [0, 1]$ and $0 \leq u_v^{\mathscr{T}}(x) + u_v^{\mathscr{I}}(x) + u_v^{\mathscr{F}}(x) \leq 3$.

Definition 2.3. [6] Let $\psi_1 = \langle u_{\psi_1}^{\mathscr{T}}, u_{\psi_1}^{\mathscr{I}}, u_{\psi_1}^{\mathscr{F}} \rangle$,$\psi_2 = \langle u_{\psi_2}^{\mathscr{T}}, u_{\psi_2}^{\mathscr{I}}, u_{\psi_2}^{\mathscr{F}} \rangle$ and $\psi_3 = \langle u_{\psi_3}^{\mathscr{T}}, u_{\psi_3}^{\mathscr{I}}, u_{\psi_3}^{\mathscr{F}} \rangle$ be the three neutrosophic numbers over U. Then

1. $\psi_2 \Cup \psi_3 = \Big\langle \max(\chi_{\psi_2}^{\mathscr{T}}, u_{\psi_3}^{\mathscr{T}}), \min(\chi_{\psi_2}^{\mathscr{I}}, u_{\psi_3}^{\mathscr{I}}), \min(\chi_{\psi_2}^{\mathscr{F}}, u_{\psi_3}^{\mathscr{F}}) \Big\rangle$,
2. $\psi_2 \Cap \psi_3 = \Big\langle \min(\chi_{\psi_2}^{\mathscr{T}}, u_{\psi_3}^{\mathscr{T}}), \max(\chi_{\psi_2}^{\mathscr{I}}, u_{\psi_3}^{\mathscr{I}}), \max(\chi_{\psi_2}^{\mathscr{F}}, u_{\psi_3}^{\mathscr{F}}) \Big\rangle$,
3. $\psi_2 \geq \psi_3$ iff $u_{\psi_2}^{\mathscr{T}} \geq u_{\psi_3}^{\mathscr{T}}$ and $u_{\psi_2}^{\mathscr{I}} \leq u_{\psi_3}^{\mathscr{I}}$ and $u_{\psi_2}^{\mathscr{F}} \leq u_{\psi_3}^{\mathscr{F}}$,
4. $\psi_2 = \psi_3$ iff $u_{\psi_2}^{\mathscr{T}} = u_{\psi_3}^{\mathscr{T}}$ and $u_{\psi_2}^{\mathscr{I}} = u_{\psi_3}^{\mathscr{I}}$ and $u_{\psi_2}^{\mathscr{F}} = u_{\psi_3}^{\mathscr{F}}$.

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Definition 2.4. [6] For any NS $\psi = \{x, \chi_v^{\mathscr{T}}(x), \chi_v^{\mathscr{I}}(x), \chi_v^{\mathscr{F}}(x)\}$ of U. Then (ζ, τ)-cut is defined as $\{x \in U | \chi_v^{\mathscr{T}}(x) \geq \zeta, \chi_v^{\mathscr{I}}(x) \geq \zeta, \chi_v^{\mathscr{F}}(x) \leq \tau\}$.

Definition 2.5. [6] Let V and Y be two NSs of $\mathscr{S}$. Then Cartesian product of V and Y is defined as $V \times Y = \{\chi_{V\times Y}^{\mathscr{T}}(\chi, \psi), \chi_{V\times Y}^{\mathscr{I}}(\chi, \psi), \chi_{V\times Y}^{\mathscr{F}}(\chi, \psi) |\, for\ all\ \chi, \psi \in \mathscr{S}\}$, where
$\chi_{V\times Y}^{\mathscr{T}}(\chi, \psi) = \min\{\chi_V^{\mathscr{T}}(x), \chi_Y^{\mathscr{T}}(\psi)\}, \chi_{V\times Y}^{\mathscr{I}}(\chi, \psi) = \frac{\chi_V^{\mathscr{I}}(x)+\chi_Y^{\mathscr{I}}(\psi)}{2}, \chi_{V\times Y}^{\mathscr{F}}(\chi, \psi) = \max\{\chi_V^{\mathscr{F}}(x), \chi_Y^{\mathscr{F}}(\psi)\}$.

Definition 2.6. A fuzzy subset v of a bisemiring $(\mathscr{S}, \bigcirc_1, \bigcirc_2, \bigcirc_3)$ is represents a fuzzy SBS of $\mathscr{S}$ if $\chi_v(\chi \bigcirc_1 \varepsilon) \geq \min\{\chi_v(x), \chi_v(\varepsilon)\}, \chi_v(\chi \bigcirc_2 \varepsilon) \geq \min\{\chi_v(x), \chi_v(\varepsilon)\}, \chi_v(\chi \bigcirc_3 \varepsilon) \geq \min\{\chi_v(x), \chi_v(\varepsilon)\}$,for all $\chi, \varepsilon \in \mathscr{S}$.

3 Complex cubic Q neutrosophic SBS

Here follows that $\mathscr{S}$ denotes bisemiring unless other stated, $\mathcal{R}$ denotes for real part and $\mathcal{I}$ denotes for imaginary part and $e^{i\tau 2\pi} = M$.

Definition 3.1. The complex cubic Q NS (CCQNS) Θ in universal set Ψ,
$\Theta = \{\alpha, \overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}(\alpha, \varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}(\alpha,\varrho)}}}, \overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}(\alpha, \varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}}}, \overline{\Theta_{\mathcal{R}}^{\mathscr{F}}}(\alpha, \varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{F}(\alpha,\varrho)}}}, \Theta_{\mathcal{R}}^{\mathscr{T}}(\alpha, \varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}(\alpha,\varrho)}}, \Theta_{\mathcal{R}}^{\mathscr{I}}(\alpha, \varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}}, \Theta_{\mathcal{R}}^{\mathscr{F}}(\alpha, \varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}(\alpha,\varrho)}} : \alpha \in \Psi\}$,
where $\overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}(\alpha, \varrho) = [\Theta_{\mathcal{R}}^{\mathscr{T}L}, \Theta_{\mathcal{R}}^{\mathscr{T}U}], \overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}(\alpha, \varrho) = [\Theta_{\mathcal{R}}^{\mathscr{I}L}, \Theta_{\mathcal{R}}^{\mathscr{I}U}], \overline{\Theta_{\mathcal{R}}^{\mathscr{F}}}(\alpha, \varrho) = [\Theta_{\mathcal{R}}^{\mathscr{F}L}, \Theta_{\mathcal{R}}^{\mathscr{F}U}]$ and $\overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}(\alpha, \varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}(\alpha,\varrho)}}}, \overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}(\alpha, \varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}}}, \overline{\Theta_{\mathcal{R}}^{\mathscr{F}}}(\alpha, \varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{F}(\alpha,\varrho)}}} : \Psi \to D[0,1]$ and $\Theta_{\mathcal{R}}^{\mathscr{T}}(\alpha, \varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}(\alpha,\varrho)}}, \Theta_{\mathcal{R}}^{\mathscr{I}}(\alpha, \varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}}, \Theta_{\mathcal{R}}^{\mathscr{F}}(\alpha, \varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}(\alpha,\varrho)}} : \Psi \to [0,1]$ represents the truth degree,indeterminacy degree and false degree respectively.
For $\overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}, \overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}, \overline{\Theta_{\mathcal{R}}^{\mathscr{F}}}$ is CCQNS $\Theta = \{\alpha, \overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}(\alpha, \varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}(\alpha,\varrho)}}}, \overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}(\alpha, \varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}}}, \overline{\Theta_{\mathcal{R}}^{\mathscr{F}}}(\alpha, \varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{F}(\alpha,\varrho)}}}, \Theta_{\mathcal{R}}^{\mathscr{T}}(\alpha, \varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}(\alpha,\varrho)}}, \Theta_{\mathcal{R}}^{\mathscr{I}}(\alpha, \varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}}, \Theta_{\mathcal{R}}^{\mathscr{F}}(\alpha, \varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}(\alpha,\varrho)}} : \alpha \in \Psi\}$ and $\varrho \in Q$, Q is a non-empty set.

Definition 3.2. Let $\Theta = \Big\{\alpha, \overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}(\alpha, \varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}(\alpha,\varrho)}}}, \overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}(\alpha, \varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}}}, \overline{\Theta_{\mathcal{R}}^{\mathscr{F}}}(\alpha, \varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{F}(\alpha,\varrho)}}}, \Theta_{\mathcal{R}}^{\mathscr{T}}(\alpha, \varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}(\alpha,\varrho)}}, \Theta_{\mathcal{R}}^{\mathscr{I}}(\alpha, \varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}}, \Theta_{\mathcal{R}}^{\mathscr{F}}(\alpha, \varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}(\alpha,\varrho)}}\Big\}$ and $\Xi = \Big\{\alpha, \overline{\Xi_{\mathcal{R}}^{\mathscr{T}}}(\alpha, \varrho)\cdot M^{\overline{\Xi_{\mathcal{I}}^{\mathscr{T}(\alpha,\varrho)}}}, \overline{\Xi_{\mathcal{R}}^{\mathscr{I}}}(\alpha, \varrho)\cdot M^{\overline{\Xi_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}}}, \overline{\Xi_{\mathcal{R}}^{\mathscr{F}}}(\alpha, \varrho)\cdot M^{\overline{\Xi_{\mathcal{I}}^{\mathscr{F}(\alpha,\varrho)}}}, \Xi_{\mathcal{R}}^{\mathscr{T}}(\alpha, \varrho)\cdot M^{\Xi_{\mathcal{I}}^{\mathscr{T}(\alpha,\varrho)}}, \Xi_{\mathcal{R}}^{\mathscr{I}}(\alpha, \varrho)\cdot M^{\Xi_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}}, \Xi_{\mathcal{R}}^{\mathscr{F}}(\alpha, \varrho)\cdot M^{\Xi_{\mathcal{I}}^{\mathscr{F}(\alpha,\varrho)}}\Big\}$ be two CCQNSs of Ψ. Then
(i) $\Theta \cap \Xi = \Big\{\Big(\alpha, \min\{\overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}(\alpha, \varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}(\alpha,\varrho)}}}, \overline{\Xi_{\mathcal{R}}^{\mathscr{T}}}(\alpha, \varrho)\cdot M^{\overline{\Xi_{\mathcal{I}}^{\mathscr{T}(\alpha,\varrho)}}}\}, \min\{\overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}(\alpha, \varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}}}, \overline{\Xi_{\mathcal{R}}^{\mathscr{I}}}(\alpha, \varrho)\cdot M^{\overline{\Xi_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}}}\}, \max\{\overline{\Theta_{\mathcal{R}}^{\mathscr{F}}}(\alpha, \varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{F}(\alpha,\varrho)}}}, \overline{\Xi_{\mathcal{R}}^{\mathscr{F}}}(\alpha, \varrho)\cdot M^{\overline{\Xi_{\mathcal{I}}^{\mathscr{F}(\alpha,\varrho)}}}\},$
$\min\{\Theta_{\mathcal{R}}^{\mathscr{T}}(\alpha, \varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}(\alpha,\varrho)}}, \Xi_{\mathcal{R}}^{\mathscr{T}}(\alpha, \varrho)\cdot M^{\Xi_{\mathcal{I}}^{\mathscr{T}(\alpha,\varrho)}}\}, \min\{\Theta_{\mathcal{R}}^{\mathscr{I}}(\alpha, \varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}}, \Xi_{\mathcal{R}}^{\mathscr{I}}(\alpha, \varrho)\cdot M^{\Xi_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}}\},$
$\max\{\Theta_{\mathcal{R}}^{\mathscr{F}}(\alpha, \varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}(\alpha,\varrho)}}, \Xi_{\mathcal{R}}^{\mathscr{F}}(\alpha, \varrho)\cdot M^{\Xi_{\mathcal{I}}^{\mathscr{F}(\alpha,\varrho)}}\}\Big)\Big|\alpha \in \Psi\Big\}$.
(ii) $\Theta \cup \Xi = \Big\{\Big(\alpha, \max\{\overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}(\alpha, \varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}(\alpha,\varrho)}}}, \overline{\Xi_{\mathcal{R}}^{\mathscr{T}}}(\alpha, \varrho)\cdot M^{\overline{\Xi_{\mathcal{I}}^{\mathscr{T}(\alpha,\varrho)}}}\}, \max\{\overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}(\alpha, \varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}}}, \overline{\Xi_{\mathcal{R}}^{\mathscr{I}}}(\alpha, \varrho)\cdot M^{\overline{\Xi_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}}}\}, \min\{\overline{\Theta_{\mathcal{R}}^{\mathscr{F}}}(\alpha, \varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{F}(\alpha,\varrho)}}}, \overline{\Xi_{\mathcal{R}}^{\mathscr{F}}}(\alpha, \varrho)\cdot M^{\overline{\Xi_{\mathcal{I}}^{\mathscr{F}(\alpha,\varrho)}}}\},$
$\max\{\Theta_{\mathcal{R}}^{\mathscr{T}}(\alpha, \varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}(\alpha,\varrho)}}, \Xi_{\mathcal{R}}^{\mathscr{T}}(\alpha, \varrho)\cdot M^{\Xi_{\mathcal{I}}^{\mathscr{T}(\alpha,\varrho)}}\}, \max\{\Theta_{\mathcal{R}}^{\mathscr{I}}(\alpha, \varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}}, \Xi_{\mathcal{R}}^{\mathscr{I}}(\alpha, \varrho)\cdot M^{\Xi_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}}\},$
$\min\{\Theta_{\mathcal{R}}^{\mathscr{F}}(\alpha, \varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}(\alpha,\varrho)}}, \Xi_{\mathcal{R}}^{\mathscr{F}}(\alpha, \varrho)\cdot M^{\Xi_{\mathcal{I}}^{\mathscr{F}(\alpha,\varrho)}}\}\Big)\Big|\alpha \in \Psi\Big\}$ and $\varrho \in Q$, Q is a non-empty set.

Definition 3.3. For any CCQNS $\Theta = \Big\{\alpha, \overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}(\alpha, \varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}(\alpha,\varrho)}}}, \overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}(\alpha, \varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}}}, \overline{\Theta_{\mathcal{R}}^{\mathscr{F}}}(\alpha, \varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{F}(\alpha,\varrho)}}}, \Theta_{\mathcal{R}}^{\mathscr{T}}(\alpha, \varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}(\alpha,\varrho)}}, \Theta_{\mathcal{R}}^{\mathscr{I}}(\alpha, \varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}},$
$\Theta_{\mathcal{R}}^{\mathscr{F}}(\alpha, \varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}(\alpha,\varrho)}}\Big\}$ of a universal set Ψ. Then $(\ell, \flat)$-cut is defined as $\Big\{\alpha \in \Psi | \overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}(\alpha, \varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}(\alpha,\varrho)}}} \succeq \ell, \overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}(\alpha, \varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}}} \succeq \ell, \overline{\Theta_{\mathcal{R}}^{\mathscr{F}}}(\alpha, \varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{F}(\alpha,\varrho)}}} \preceq \flat, \Theta_{\mathcal{R}}^{\mathscr{T}}(\alpha, \varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}(\alpha,\varrho)}} \succeq \ell, \Theta_{\mathcal{R}}^{\mathscr{I}}(\alpha, \varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}} \succeq \ell, \Theta_{\mathcal{R}}^{\mathscr{F}}(\alpha, \varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}(\alpha,\varrho)}} \preceq \flat\Big\}$.

Definition 3.4. The Cartesian product of Θ and Ξ is defined as
$\Theta \times \Xi = \Big\{\overline{\Theta \times \Xi_{\mathcal{R}}^{\mathscr{T}}}((\alpha, \varpi), \varrho)\cdot M^{\overline{\Theta\times\Xi_{\mathcal{I}}^{\mathscr{T}((\alpha,\varpi),\varrho)}}}, \overline{\Theta \times \Xi_{\mathcal{R}}^{\mathscr{I}}}((\alpha, \varpi), \varrho)\cdot M^{\overline{\Theta\times\Xi_{\mathcal{I}}^{\mathscr{I}((\alpha,\varpi),\varrho)}}},$

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$\overline{\Theta\times\Xi_{\mathcal{R}}^{\mathscr{F}}}(\alpha,\varpi)\cdot M^{\overline{\Theta\times\Xi_{\mathcal{I}}^{\mathscr{F}((\alpha,\varpi),\varrho)}}}, \Theta\times\Xi_{\mathcal{R}}^{\mathscr{T}}((\alpha,\varpi),\varrho)\cdot M^{\Theta\times\Xi_{\mathcal{I}}^{\mathscr{T}((\alpha,\varpi),\varrho)}}, \Theta\times\Xi_{\mathcal{R}}^{\mathscr{I}}((\alpha,\varpi),\varrho)\cdot M^{\Theta\times\Xi_{\mathcal{I}}^{\mathscr{I}((\alpha,\varpi),\varrho)}}, \Theta\times\Xi_{\mathcal{R}}^{\mathscr{F}}(\alpha,\varpi)\cdot$ $M^{\Theta\times\Xi_{\mathcal{I}}^{\mathscr{F}((\alpha,\varpi),\varrho)}} \mid for\ all\ \alpha,\varpi\in S\Big\}$,
where Θ and Ξ be the CCQNS of Ψ, where

$$\left\{\begin{array}{l}
\overline{\Theta\times\Xi_{\mathcal{R}}^{\mathscr{T}}}((\alpha,\varpi),\varrho)\cdot M^{\overline{\Theta\times\Xi_{\mathcal{I}}^{\mathscr{T}((\alpha,\varpi),\varrho)}}} = \min\left\{\overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}(\alpha,\varrho)\cdot M^{\overline{\mathcal{I}_{\Theta}^{\mathscr{T}(\alpha,\varrho)}}}, \overline{\Xi_{\mathcal{R}}^{\mathscr{T}}}(\varpi,\varrho)\cdot M^{\overline{\Xi_{\mathcal{I}}^{\mathscr{T}(\varpi,\varrho)}}}\right\} \\
\overline{\Theta\times\Xi_{\mathcal{R}}^{\mathscr{I}}}((\alpha,\varpi),\varrho)\cdot M^{\overline{\Theta\times\Xi_{\mathcal{I}}^{\mathscr{I}((\alpha,\varpi),\varrho)}}} = \frac{\overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}(\alpha,\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}}}+\overline{\Xi_{\mathcal{R}}^{\mathscr{I}}}(\varpi,\varrho)\cdot M^{\overline{\Xi_{\mathcal{I}}^{\mathscr{I}(\varpi,\varrho)}}}}{2} \\
\overline{\Theta\times\Xi_{\mathcal{R}}^{\mathscr{F}}}((\alpha,\varpi),\varrho)\cdot M^{\overline{\Theta\times\Xi_{\mathcal{I}}^{\mathscr{F}((\alpha,\varpi),\varrho)}}} = \max\left\{\overline{\Theta_{\mathcal{R}}^{\mathscr{F}}}(\alpha,\varrho)\cdot M^{\overline{\mathcal{I}_{\Theta}^{\mathscr{F}(\alpha,\varrho)}}}, \overline{\Xi_{\mathcal{R}}^{\mathscr{F}}}(\varpi,\varrho)\cdot M^{\overline{\Xi_{\mathcal{I}}^{\mathscr{F}(\varpi,\varrho)}}}\right\}
\end{array}\right\}$$

$$\left\{\begin{array}{l}
\Theta\times\Xi_{\mathcal{R}}^{\mathscr{T}}((\alpha,\varpi),\varrho)\cdot M^{\Theta\times\Xi_{\mathcal{I}}^{\mathscr{T}((\alpha,\varpi),\varrho)}} = \min\left\{\Theta_{\mathcal{R}}^{\mathscr{T}}(\alpha,\varrho)\cdot M^{\mathcal{I}_{\Theta}^{\mathscr{T}(\alpha,\varrho)}}, \Xi_{\mathcal{R}}^{\mathscr{T}}(\varpi,\varrho)\cdot M^{\Xi_{\mathcal{I}}^{\mathscr{T}(\varpi,\varrho)}}\right\} \\
\Theta\times\Xi_{\mathcal{R}}^{\mathscr{I}}((\alpha,\varpi),\varrho)\cdot M^{\Theta\times\Xi_{\mathcal{I}}^{\mathscr{I}((\alpha,\varpi),\varrho)}} = \frac{\Theta_{\mathcal{R}}^{\mathscr{I}}(\alpha,\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}}+\Xi_{\mathcal{R}}^{\mathscr{I}}(\varpi,\varrho)\cdot M^{\Xi_{\mathcal{I}}^{\mathscr{I}(\varpi,\varrho)}}}{2} \\
\Theta\times\Xi_{\mathcal{R}}^{\mathscr{F}}((\alpha,\varpi),\varrho)\cdot M^{\Theta\times\Xi_{\mathcal{I}}^{\mathscr{F}((\alpha,\varpi),\varrho)}} = \max\left\{\Theta_{\mathcal{R}}^{\mathscr{F}}(\alpha,\varrho)\cdot M^{\mathcal{I}_{\Theta}^{\mathscr{F}(\alpha,\varrho)}}, \Xi_{\mathcal{R}}^{\mathscr{F}}(\varpi,\varrho)\cdot M^{\Xi_{\mathcal{I}}^{\mathscr{F}(\varpi,\varrho)}}\right\}
\end{array}\right\}$$

Definition 3.5. For any CCQNS Θ of $\mathscr{S}$ is said to be a CCQNSBS of $\mathscr{S}$ if

$$\left\{\begin{array}{l}
\overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}((\alpha\oslash_1\varpi),\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}}}((\alpha\oslash_1\varpi),\varrho)} \succeq \min\{\overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}(\alpha,\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}}}((\alpha,\varrho)}, \overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}(\varpi,\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}}}((\varpi,\varrho)}\} \\
\overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}((\alpha\oslash_2\varpi),\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}}}((\alpha\oslash_2\varpi),\varrho)} \succeq \min\{\overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}(\alpha,\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}}}((\alpha,\varrho)}, \overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}(\varpi,\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}}}((\varpi,\varrho)}\} \\
\overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}((\alpha\oslash_3\varpi),\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}}}((\alpha\oslash_3\varpi),\varrho)} \succeq \min\{\overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}(\alpha,\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}}}((\alpha,\varrho)}, \overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}(\varpi,\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}}}((\varpi,\varrho)}\}
\end{array}\right\}$$

$$\left\{\begin{array}{l}
\overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}((\alpha\oslash_1\varpi),\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{I}}}((\alpha\oslash_1\varpi),\varrho)} \succeq \frac{\overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}(\alpha,\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{I}}}((\alpha,\varrho)}+\overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}(\varpi,\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{I}}}((\varpi,\varrho)}}{2} \\
\qquad\qquad \text{OR} \\
\overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}((\alpha\oslash_2\varpi),\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}}}((\alpha\oslash_2\varpi),\varrho)} \succeq \frac{\overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}(\alpha,\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{I}}}((\alpha,\varrho)}+\overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}(\varpi,\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{I}}}((\varpi,\varrho)}}{2} \\
\qquad\qquad \text{OR} \\
\overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}((\alpha\oslash_3\varpi),\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{I}}}((\alpha\oslash_3\varpi),\varrho)} \succeq \frac{\overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}(\alpha,\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{I}}}((\alpha,\varrho)}+\overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}(\varpi,\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}}}((\varpi,\varrho)}}{2}
\end{array}\right\}$$

$$\left\{\begin{array}{l}
\overline{\Theta_{\mathcal{R}}^{\mathscr{F}}}((\alpha\oslash_1\varpi),\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{F}}}((\alpha\oslash_1\varpi),\varrho)} \preceq \max\{\overline{\Theta_{\mathcal{R}}^{\mathscr{F}}}(\alpha,\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{F}}}((\alpha,\varrho)}, \overline{\Theta_{\mathcal{R}}^{\mathscr{F}}}(\varpi,\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{F}}}((\varpi,\varrho)}\} \\
\overline{\Theta_{\mathcal{R}}^{\mathscr{F}}}((\alpha\oslash_2\varpi),\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{F}}}((\alpha\oslash_2\varpi),\varrho)} \preceq \max\{\overline{\Theta_{\mathcal{R}}^{\mathscr{F}}}(\alpha,\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{F}}}((\alpha,\varrho)}, \overline{\Theta_{\mathcal{R}}^{\mathscr{F}}}(\varpi,\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{F}}}((\varpi,\varrho)}\} \\
\overline{\Theta_{\mathcal{R}}^{\mathscr{F}}}((\alpha\oslash_3\varpi),\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{F}}}((\alpha\oslash_3\varpi),\varrho)} \preceq \max\{\overline{\Theta_{\mathcal{R}}^{\mathscr{F}}}(\alpha,\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{F}}}((\alpha,\varrho)}, \overline{\Theta_{\mathcal{R}}^{\mathscr{F}}}(\varpi,\varrho)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{F}}}((\varpi,\varrho)}\}
\end{array}\right\}$$

$$\left\{\begin{array}{l}
\Theta_{\mathcal{R}}^{\mathscr{T}}((\alpha\oslash_1\varpi),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\alpha\oslash_1\varpi),\varrho)} \succeq \min\{\Theta_{\mathcal{R}}^{\mathscr{T}}(\alpha,\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\alpha,\varrho)}, \Theta_{\mathcal{R}}^{\mathscr{T}}(\varpi,\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\varpi,\varrho)}\} \\
\Theta_{\mathcal{R}}^{\mathscr{T}}((\alpha\oslash_2\varpi),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\alpha\oslash_2\varpi),\varrho)} \succeq \min\{\Theta_{\mathcal{R}}^{\mathscr{T}}(\alpha,\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\alpha,\varrho)}, \Theta_{\mathcal{R}}^{\mathscr{T}}(\varpi,\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\varpi,\varrho)}\} \\
\Theta_{\mathcal{R}}^{\mathscr{T}}((\alpha\oslash_3\varpi),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\alpha\oslash_3\varpi),\varrho)} \succeq \min\{\Theta_{\mathcal{R}}^{\mathscr{T}}(\alpha,\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\alpha,\varrho)}, \Theta_{\mathcal{R}}^{\mathscr{T}}(\varpi,\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\varpi,\varrho)}\}
\end{array}\right\}$$

$$\left\{\begin{array}{l}
\Theta_{\mathcal{R}}^{\mathscr{I}}((\alpha\oslash_1\varpi),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}((\alpha\oslash_1\varpi),\varrho)} \succeq \frac{\Theta_{\mathcal{R}}^{\mathscr{I}}(\alpha,\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}((\alpha,\varrho)}+\Theta_{\mathcal{R}}^{\mathscr{I}}(\varpi,\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}((\varpi,\varrho)}}{2} \\
\qquad\qquad \text{OR} \\
\Theta_{\mathcal{R}}^{\mathscr{I}}((\alpha\oslash_2\varpi),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\alpha\oslash_2\varpi),\varrho)} \succeq \frac{\Theta_{\mathcal{R}}^{\mathscr{I}}(\alpha,\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}((\alpha,\varrho)}+\Theta_{\mathcal{R}}^{\mathscr{I}}(\varpi,\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}((\varpi,\varrho)}}{2} \\
\qquad\qquad \text{OR} \\
\Theta_{\mathcal{R}}^{\mathscr{I}}((\alpha\oslash_3\varpi),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}((\alpha\oslash_3\varpi),\varrho)} \succeq \frac{\Theta_{\mathcal{R}}^{\mathscr{I}}(\alpha,\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}((\alpha,\varrho)}+\Theta_{\mathcal{R}}^{\mathscr{I}}(\varpi,\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\varpi,\varrho)}}{2}
\end{array}\right\}$$

$$\left\{\begin{array}{l}
\Theta_{\mathcal{R}}^{\mathscr{F}}((\alpha\oslash_1\varpi),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}((\alpha\oslash_1\varpi),\varrho)} \preceq \max\{\Theta_{\mathcal{R}}^{\mathscr{F}}(\alpha,\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}((\alpha,\varrho)}, \Theta_{\mathcal{R}}^{\mathscr{F}}(\varpi,\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}((\varpi,\varrho)}\} \\
\Theta_{\mathcal{R}}^{\mathscr{F}}((\alpha\oslash_2\varpi),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}((\alpha\oslash_2\varpi),\varrho)} \preceq \max\{\Theta_{\mathcal{R}}^{\mathscr{F}}(\alpha,\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}((\alpha,\varrho)}, \Theta_{\mathcal{R}}^{\mathscr{F}}(\varpi,\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}((\varpi,\varrho)}\} \\
\Theta_{\mathcal{R}}^{\mathscr{F}}((\alpha\oslash_3\varpi),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}((\alpha\oslash_3\varpi),\varrho)} \preceq \max\{\Theta_{\mathcal{R}}^{\mathscr{F}}(\alpha,\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}((\alpha,\varrho)}, \Theta_{\mathcal{R}}^{\mathscr{F}}(\varpi,\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}((\varpi,\varrho)}\}
\end{array}\right\}$$

for all $\alpha,\varpi\in\mathscr{S}$.

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Example 3.6. Consider the bisemiring $\mathscr{S} = \{\mu_1, \mu_2, \mu_3, \mu_4\}$ with the Cayley table:

$\oslash_1$	μ_1	μ_2	μ_3	μ_4
μ_1	μ_1	μ_1	μ_1	μ_1
μ_2	μ_1	μ_2	μ_1	μ_2
μ_3	μ_1	μ_1	μ_3	μ_3
μ_4	μ_1	μ_2	μ_3	μ_4

$\oslash_2$	μ_1	μ_2	μ_3	μ_4
μ_1	μ_1	μ_2	μ_3	μ_4
μ_2	μ_2	μ_2	μ_4	μ_4
μ_3	μ_3	μ_4	μ_3	μ_4
μ_4	μ_4	μ_4	μ_4	μ_4

$\oslash_3$	μ_1	μ_2	μ_3	μ_4
μ_1	μ_1	μ_1	μ_1	μ_1
μ_2	μ_1	μ_2	μ_3	μ_4
μ_3	μ_4	μ_4	μ_4	μ_4
μ_4	μ_4	μ_4	μ_4	μ_4

	$(w) = \mu_1$	$(w) = \mu_2$
$(\overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}, \overline{\Theta_{\mathcal{I}}^{\mathscr{T}}})(w)$	$[0.85e^{i2\pi(0.7)}, 0.95e^{i2\pi(0.8)}]$	$[0.75e^{i2\pi(0.6)}, 0.85e^{i2\pi(0.7)}]$
$(\overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}, \overline{\Theta_{\mathcal{I}}^{\mathscr{I}}})(w)$	$[0.55e^{i2\pi(0.4)}, 0.65e^{i2\pi(0.5)}]$	$[0.45e^{i2\pi(0.3)}, 0.55e^{i2\pi(0.4)}]$
$(\overline{\Theta_{\mathcal{R}}^{\mathscr{F}}}, \overline{\Theta_{\mathcal{I}}^{\mathscr{F}}})(w)$	$[0.65e^{i2\pi(0.5)}, 0.75e^{i2\pi(0.6)}]$	$[0.75e^{i2\pi(0.6)}, 0.85e^{i2\pi(0.65)}]$

	$(w) = \mu_3$	$(w) = \mu_4$
$(\overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}, \overline{\Theta_{\mathcal{I}}^{\mathscr{T}}})(w)$	$[0.45e^{i2\pi(0.3)}, 0.55e^{i2\pi(0.4)}]$	$[0.65e^{i2\pi(0.5)}, 0.75e^{i2\pi(0.6)}]$
$(\overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}, \overline{\Theta_{\mathcal{I}}^{\mathscr{I}}})(w)$	$[0.25e^{i2\pi(0.2)}, 0.35e^{i2\pi(0.3)}]$	$[0.35e^{i2\pi(0.2)}, 0.45e^{i2\pi(0.3)}]$
$(\overline{\Theta_{\mathcal{R}}^{\mathscr{F}}}, \overline{\Theta_{\mathcal{I}}^{\mathscr{F}}})(w)$	$[0.95e^{i2\pi(0.8)}, 0.85e^{i2\pi(0.7)}]$	$[0.9e^{i2\pi(0.75)}, 0.95e^{i2\pi(0.8)}]$

	$(w) = \mu_1$	$(w) = \mu_2$
$(\Theta_{\mathcal{R}}^{\mathscr{T}}, \Theta_{\mathcal{I}}^{\mathscr{T}})(w)$	$0.8e^{i2\pi(0.65)}$	$0.7e^{i2\pi(0.55)}$
$(\Theta_{\mathcal{R}}^{\mathscr{I}}, \Theta_{\mathcal{I}}^{\mathscr{I}})(w)$	$0.5e^{i2\pi(0.35)}$	$0.4e^{i2\pi(0.25)}$
$(\Theta_{\mathcal{R}}^{\mathscr{F}}, \Theta_{\mathcal{I}}^{\mathscr{F}})(w)$	$0.6e^{i2\pi(0.45)}$	$0.7e^{i2\pi(0.55)}$

	$(w) = \mu_3$	$(w) = \mu_4$
$(\Theta_{\mathcal{R}}^{\mathscr{T}}, \Theta_{\mathcal{I}}^{\mathscr{T}})(w)$	$0.4e^{i2\pi(0.25)}$	$0.6e^{i2\pi(0.45)}$
$(\Theta_{\mathcal{R}}^{\mathscr{I}}, \Theta_{\mathcal{I}}^{\mathscr{I}})(w)$	$0.2e^{i2\pi(0.15)}$	$0.3e^{i2\pi(0.15)}$
$(\Theta_{\mathcal{R}}^{\mathscr{F}}, \Theta_{\mathcal{I}}^{\mathscr{F}})(w)$	$0.9e^{i2\pi(0.75)}$	$0.85e^{i2\pi(0.7)}$

Hence, Θ is a CCQNSBS of $\mathscr{S}$.

Theorem 3.7. *The intersection of all CCQNSBSs is a CCQNSBS of* $\mathscr{S}$*.*

Proof. Let $\{\overline{\Omega_i} : i \in I\}$ be the collection of CCQNSBSs of $\mathscr{S}$ and $\Theta = \bigcap_{i\in I} \overline{\Omega_i}$. Let $\alpha, \varpi \in \mathscr{S}$.

Now,

$$\begin{aligned}
\overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}((\alpha \oslash_1 \varpi), \varrho) \cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}}}((\alpha\oslash_1\varpi),\varrho)} &= \bigwedge_{i\in I} \overline{\Omega_{i\mathcal{R}}^{\mathscr{T}}}((\alpha \oslash_1 \varpi), \varrho) \cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}}}((\alpha\oslash_1\varpi),\varrho)} \\
&\succeq \bigwedge_{i\in I} \min\{\overline{\Omega_{i\mathcal{R}}^{\mathscr{T}}}(\alpha, \varrho) \cdot M^{\overline{\Omega_{i\mathcal{I}}^{\mathscr{T}}}(\alpha,\varrho)}, \overline{\Omega_{i\mathcal{R}}^{\mathscr{T}}}(\varpi, \varrho) \cdot M^{\overline{\Omega_{i\mathcal{I}}^{\mathscr{T}}}(\varpi,\varrho)}\} \\
&= \min\left\{\bigwedge_{i\in I} \overline{\Omega_{i\mathcal{R}}^{\mathscr{T}}}(\alpha, \varrho) \cdot M^{\overline{\Omega_{i\mathcal{I}}^{\mathscr{T}}}(\alpha,\varrho)}, \bigwedge_{i\in I} \overline{\Omega_{i\mathcal{R}}^{\mathscr{T}}}(\varpi, \varrho) \cdot M^{\overline{\Omega_{i\mathcal{I}}^{\mathscr{T}}}(\varpi,\varrho)}\right\} \\
&= \min\{\overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}(\alpha, \varrho) \cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}}}(\alpha,\varrho)}, \overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}(\varpi, \varrho) \cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}}}(\varpi,\varrho)}\}
\end{aligned}$$

Similarly,

$$\overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}((\alpha \oslash_2 \varpi), \varrho) \cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}}}((\alpha\oslash_2\varpi),\varrho)} \succeq \min\{\overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}(\alpha, \varrho) \cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}}}(\alpha,\varrho)}, \overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}(\varpi, \varrho) \cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}}}(\varpi,\varrho)}\},$$

$$\overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}((\alpha \oslash_3 \varpi), \varrho) \cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}}}((\alpha\oslash_3\varpi),\varrho)} \succeq \min\{\overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}(\alpha, \varrho) \cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}}}(\alpha,\varrho)}, \overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}(\varpi, \varrho) \cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}}}(\varpi,\varrho)}\}.$$

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Now,

$$\begin{aligned}\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}((\alpha\oslash_1\varpi),\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}((\alpha\oslash_1\varpi),\varrho)} &= \bigwedge_{i\in I}\ \overline{\Omega_{i\mathcal{R}}^{\mathscr{I}}}((\alpha\oslash_1\varpi),\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}((\alpha\oslash_1\varpi),\varrho)}\\ &\succeq \bigwedge_{i\in I}\ \frac{\overline{\Omega_{i\mathcal{R}}^{\mathscr{I}}}(\alpha,\varrho)\cdot M^{\overline{\Omega_{i\mathcal{I}}^{\mathscr{I}}}(\alpha,\varrho)}+\overline{\Omega_{i\mathcal{R}}^{\mathscr{I}}}(\varpi,\varrho)\cdot M^{\overline{\Omega_{i\mathcal{I}}^{\mathscr{I}}}(\varpi,\varrho)}}{2}\\ &= \frac{\bigwedge_{i\in I}\ \overline{\Omega_{i\mathcal{R}}^{\mathscr{I}}}(\alpha,\varrho)\cdot M^{\overline{\Omega_{i\mathcal{I}}^{\mathscr{I}}}(\alpha,\varrho)}+\bigwedge_{i\in I}\ \overline{\Omega_{i\mathcal{R}}^{\mathscr{I}}}(\varpi,\varrho)\cdot M^{\overline{\Omega_{i\mathcal{I}}^{\mathscr{I}}}(\varpi,\varrho)}}{2}\\ &= \frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha,\varrho)}+\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\varpi,\varrho)}}{2}\end{aligned}$$

Similarly,
$\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}((\alpha\oslash_2\varpi),\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}((\alpha\oslash_2\varpi),\varrho)} \succeq \frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha,\varrho)}+\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\varpi,\varrho)}}{2}$ and
$\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}((\alpha\oslash_3\varpi),\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}((\alpha\oslash_3\varpi),\varrho)} \succeq \frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha,\varrho)}+\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\varpi,\varrho)}}{2}$.
Now,

$$\begin{aligned}\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha\oslash_1\varpi),\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}((\alpha\oslash_1\varpi),\varrho)} &= \bigvee_{i\in I}\ \overline{\Omega_{i\mathcal{R}}^{\mathscr{F}}}((\alpha\oslash_1\varpi),\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}((\alpha\oslash_1\varpi),\varrho)}\\ &\preceq \bigvee_{i\in I}\ \max\{\overline{\Omega_{i\mathcal{R}}^{\mathscr{F}}}(\alpha,\varrho)\cdot M^{\overline{\Omega_{i\mathcal{I}}^{\mathscr{F}}}(\alpha,\varrho)},\overline{\Omega_{i\mathcal{R}}^{\mathscr{F}}}(\varpi,\varrho)\cdot M^{\overline{\Omega_{i\mathcal{I}}^{\mathscr{F}}}(\varpi,\varrho)}\}\\ &= \max\left\{\bigvee_{i\in I}\ \overline{\Omega_{i\mathcal{R}}^{\mathscr{F}}}(\alpha,\varrho)\cdot M^{\overline{\Omega_{i\mathcal{I}}^{\mathscr{F}}}(\alpha,\varrho)},\bigvee_{i\in I}\ \overline{\Omega_{i\mathcal{R}}^{\mathscr{F}}}(\varpi,\varrho)\cdot M^{\overline{\Omega_{i\mathcal{I}}^{\mathscr{F}}}(\varpi,\varrho)}\right\}\\ &= \max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha,\varrho)},\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\varpi,\varrho)}\}\end{aligned}$$

Similarly,
$\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha\oslash_2\varpi),\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}((\alpha\oslash_2\varpi),\varrho)} \preceq \max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha,\varrho)},\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\varpi,\varrho)}\}$ and
$\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha\oslash_3\varpi),\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}((\alpha\oslash_3\varpi),\varrho)} \preceq \max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha,\varrho)},\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\varpi,\varrho)}\}$.
Let $\{\Omega_i : i\in I\}$ be the family of CCQNSBSs of $\mathscr{S}$ and $\Theta=\bigcap_{i\in I}\Omega_i$. Let $\alpha,\varpi\in\mathscr{S}$.

Now,

$$\begin{aligned}\Theta^{\mathscr{T}}_{\mathcal{R}}((\alpha\oslash_1\varpi),\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha\oslash_1\varpi),\varrho)} &= \bigwedge_{i\in I}\ \Omega_{i\mathcal{R}}^{\mathscr{T}}((\alpha\oslash_1\varpi),\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha\oslash_1\varpi),\varrho)}\\ &\succeq \bigwedge_{i\in I}\ \min\{\Omega_{i\mathcal{R}}^{\mathscr{T}}(\alpha,\varrho)\cdot M^{\Omega_{i\mathcal{I}}^{\mathscr{T}}(\alpha,\varrho)},\Omega_{i\mathcal{R}}^{\mathscr{T}}(\varpi,\varrho)\cdot M^{\Omega_{i\mathcal{I}}^{\mathscr{T}}(\varpi,\varrho)}\}\\ &= \min\left\{\bigwedge_{i\in I}\ \Omega_{i\mathcal{R}}^{\mathscr{T}}(\alpha,\varrho)\cdot M^{\Omega_{i\mathcal{I}}^{\mathscr{T}}(\alpha,\varrho)},\bigwedge_{i\in I}\ \Omega_{i\mathcal{R}}^{\mathscr{T}}(\varpi,\varrho)\cdot M^{\Omega_{i\mathcal{I}}^{\mathscr{T}}(\varpi,\varrho)}\right\}\\ &= \min\{\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha,\varrho)},\Theta^{\mathscr{T}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi,\varrho)}\}\end{aligned}$$

Similarly,
$\Theta^{\mathscr{T}}_{\mathcal{R}}((\alpha\oslash_2\varpi),\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha\oslash_2\varpi),\varrho)} \succeq \min\{\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha,\varrho)},\Theta^{\mathscr{T}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi,\varrho)}\}$,
$\Theta^{\mathscr{T}}_{\mathcal{R}}((\alpha\oslash_3\varpi),\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha\oslash_3\varpi),\varrho)} \succeq \min\{\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha,\varrho)},\Theta^{\mathscr{T}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi,\varrho)}\}$.
Now,

$$\begin{aligned}\Theta^{\mathscr{I}}_{\mathcal{R}}((\alpha\oslash_1\varpi),\varrho)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}((\alpha\oslash_1\varpi),\varrho)} &= \bigwedge_{i\in I}\ \Omega_{i\mathcal{R}}^{\mathscr{I}}((\alpha\oslash_1\varpi),\varrho)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}((\alpha\oslash_1\varpi),\varrho)}\\ &\succeq \bigwedge_{i\in I}\ \frac{\Omega_{i\mathcal{R}}^{\mathscr{I}}(\alpha,\varrho)\cdot M^{\Omega_{i\mathcal{I}}^{\mathscr{I}}(\alpha,\varrho)}+\Omega_{i\mathcal{R}}^{\mathscr{I}}(\varpi,\varrho)\cdot M^{\Omega_{i\mathcal{I}}^{\mathscr{I}}(\varpi,\varrho)}}{2}\\ &= \frac{\bigwedge_{i\in I}\ \Omega_{i\mathcal{R}}^{\mathscr{I}}(\alpha,\varrho)\cdot M^{\Omega_{i\mathcal{I}}^{\mathscr{I}}(\alpha,\varrho)}+\bigwedge_{i\in I}\ \Omega_{i\mathcal{R}}^{\mathscr{I}}(\varpi,\varrho)\cdot M^{\Omega_{i\mathcal{I}}^{\mathscr{I}}(\varpi,\varrho)}}{2}\\ &= \frac{\Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha,\varrho)}+\Theta^{\mathscr{I}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\varpi,\varrho)}}{2}\end{aligned}$$

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Similarly,
$\Theta^{\mathscr{I}}_{\mathcal{R}}((\alpha \oslash_2 \varpi),\varrho) \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}((\alpha \oslash_2 \varpi),\varrho)} \succeq \frac{\Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha,\varrho)}+\Theta^{\mathscr{I}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\varpi,\varrho)}}{2}$ and
$\Theta^{\mathscr{I}}_{\mathcal{R}}((\alpha \oslash_3 \varpi),\varrho) \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}((\alpha \oslash_3 \varpi),\varrho)} \succeq \frac{\Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha,\varrho)}+\Theta^{\mathscr{I}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\varpi,\varrho)}}{2}$.
Now,

$$\begin{aligned}
\Theta^{\mathscr{F}}_{\mathcal{R}}((\alpha \oslash_1 \varpi),\varrho) \cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}((\alpha \oslash_1 \varpi),\varrho)} &= \bigvee_{i\in I} \Omega_{i}{}^{\mathscr{F}}_{\mathcal{R}}((\alpha \oslash_1 \varpi),\varrho) \cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}((\alpha \oslash_1 \varpi),\varrho)} \\
&\preceq \bigvee_{i\in I} \max\{\Omega_{i}{}^{\mathscr{F}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Omega_{i}{}^{\mathscr{F}}_{\mathcal{I}}(\alpha,\varrho)}, \Omega_{i}{}^{\mathscr{F}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Omega_{i}{}^{\mathscr{F}}_{\mathcal{I}}(\varpi,\varrho)}\} \\
&= \max\left\{\bigvee_{i\in I} \Omega_{i}{}^{\mathscr{F}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Omega_{i}{}^{\mathscr{F}}_{\mathcal{I}}(\alpha,\varrho)}, \bigvee_{i\in I} \Omega_{i}{}^{\mathscr{F}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Omega_{i}{}^{\mathscr{F}}_{\mathcal{I}}(\varpi,\varrho)}\right\} \\
&= \max\{\Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha,\varrho)}, \Theta^{\mathscr{F}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\varpi,\varrho)}\}
\end{aligned}$$

Similarly,
$\Theta^{\mathscr{F}}_{\mathcal{R}}((\alpha \oslash_2 \varpi),\varrho) \cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}((\alpha \oslash_2 \varpi),\varrho)} \preceq \max\{\Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha,\varrho)}, \Theta^{\mathscr{F}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\varpi,\varrho)}\}$ and
$\Theta^{\mathscr{F}}_{\mathcal{R}}((\alpha \oslash_3 \varpi),\varrho) \cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}((\alpha \oslash_3 \varpi),\varrho)} \preceq \max\{\Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha,\varrho)}, \Theta^{\mathscr{F}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\varpi,\varrho)}\}$.
Thus,Θ is a CCQNSBS of $\mathscr{S}$.

Theorem 3.8. *If Θ and Ξ be the CCQNSBSs of $\mathscr{S}_1$ and $\mathscr{S}_2$ respectively, then we prove that $\overline{\Theta \times \Xi}$is a CCQNSBS of $\mathscr{S}_1 \times \mathscr{S}_2$.*

Proof. Let $\alpha_1, \alpha_2 \in \mathscr{S}_1$ and $\varpi_1, \varpi_2 \in \mathscr{S}_2$. Then (α_1, ϖ_1) and (α_2, ϖ_2) are in $\mathscr{S}_1 \times \mathscr{S}_2$. Now

$$\begin{aligned}
& \overline{\Theta \times \Xi}^{\mathscr{T}}_{\mathcal{R}}[((\alpha_1,\varpi_1) \oslash_1 (\alpha_2,\varpi_2),\varrho)] \cdot M^{\overline{\Theta \times \Xi^{\mathscr{T}}_{\mathcal{I}}}[((\alpha_1,\varpi_1)\oslash_1(\alpha_2,\varpi_2),\varrho)]} \\
= \;& \overline{\Theta \times \Xi}^{\mathscr{T}}_{\mathcal{R}}((\alpha_1 \oslash_1 \alpha_2, \varpi_1 \oslash_1 \varpi_2),\varrho) \cdot M^{\overline{\Theta \times \Xi^{\mathscr{T}}_{\mathcal{I}}}((\alpha_1\oslash_1\alpha_2,\varpi_1\oslash_1\varpi_2),\varrho)} \\
= \;& \min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha_1 \oslash_1 \alpha_2),\varrho) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}((\alpha_1\oslash_1\alpha_2),\varrho)}, \overline{\Xi^{\mathscr{T}}_{\mathcal{R}}}((\varpi_1 \oslash_1 \varpi_2),\varrho) \cdot M^{\overline{\Xi^{\mathscr{T}}_{\mathcal{I}}}((\varpi_1\oslash_1\varpi_2),\varrho)}\} \\
\succeq \;& \min\{\min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_1) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha_1}), \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_2) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha_2)}\}, \min\{\overline{\Xi^{\mathscr{T}}_{\mathcal{R}}}(\varpi_1) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\varpi_1)}, \overline{\Xi^{\mathscr{T}}_{\mathcal{R}}}(\varpi_2) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\varpi_2)}\}\} \\
= \;& \min\{\min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_1) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha_1}), \overline{\Xi^{\mathscr{T}}_{\mathcal{R}}}(\varpi_1) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\varpi_1})\}, \min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_2) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha_2)}, \overline{\Xi^{\mathscr{T}}_{\mathcal{R}}}(\varpi_2) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\varpi_2)}\}\} \\
= \;& \min\{\overline{\Theta \times \Xi}^{\mathscr{T}}_{\mathcal{R}}((\alpha_1,\varpi_1),\varrho) \cdot M^{\overline{\Theta \times \Xi^{\mathscr{T}}_{\mathcal{I}}}((\alpha_1,\varpi_1),\varrho)}, \overline{\Theta \times \Xi}^{\mathscr{T}}_{\mathcal{R}}((\alpha_2,\varpi_2),\varrho) \cdot M^{\overline{\Theta \times \Xi^{\mathscr{T}}_{\mathcal{I}}}((\alpha_2,\varpi_2),\varrho)}\}
\end{aligned}$$

Also $\overline{\Theta \times \Xi}^{\mathscr{T}}_{\mathcal{R}}[((\alpha_1,\varpi_1) \oslash_2 (\alpha_2,\varpi_2),\varrho)] \cdot M^{\overline{\Theta \times \Xi^{\mathscr{T}}_{\mathcal{I}}}[((\alpha_1,\varpi_1)\oslash_2(\alpha_2,\varpi_2),\varrho)]}$
$\succeq \min\{\overline{\Theta \times \Xi}^{\mathscr{T}}_{\mathcal{R}}((\alpha_1,\varpi_1),\varrho) \cdot M^{\overline{\Theta \times \Xi^{\mathscr{T}}_{\mathcal{I}}}((\alpha_1,\varpi_1),\varrho)}, \overline{\Theta \times \Xi}^{\mathscr{T}}_{\mathcal{R}}((\alpha_2,\varpi_2),\varrho) M^{\overline{\Theta \times \Xi^{\mathscr{T}}_{\mathcal{I}}}((\alpha_2,\varpi_2),\varrho)}\}$

and $\overline{\Theta \times \Xi}^{\mathscr{T}}_{\mathcal{R}}[((\alpha_1,\varpi_1) \oslash_3 (\alpha_2,\varpi_2),\varrho)] \cdot M^{\overline{\Theta \times \Xi^{\mathscr{T}}_{\mathcal{I}}}[((\alpha_1,\varpi_1)\oslash_3(\alpha_2,\varpi_2),\varrho)]}$
$\succeq \min\{\overline{\Theta \times \Xi}^{\mathscr{T}}_{\mathcal{R}}((\alpha_1,\varpi_1),\varrho) \cdot M^{\overline{\Theta \times \Xi^{\mathscr{T}}_{\mathcal{I}}}((\alpha_1,\varpi_1),\varrho)}, \overline{\Theta \times \Xi}^{\mathscr{T}}_{\mathcal{R}}((\alpha_2,\varpi_2),\varrho) \cdot M^{\overline{\Theta \times \Xi^{\mathscr{T}}_{\mathcal{I}}}((\alpha_2,\varpi_2),\varrho)}\}$.
Now,

$$\begin{aligned}
& \overline{\Theta \times \Xi}^{\mathscr{I}}_{\mathcal{R}}[((\alpha_1,\varpi_1) \oslash_1 (\alpha_2,\varpi_2),\varrho)] \cdot M^{\overline{\Theta \times \Xi^{\mathscr{I}}_{\mathcal{I}}}[((\alpha_1,\varpi_1)\oslash_1(\alpha_2,\varpi_2),\varrho)]} \\
= \;& \overline{\Theta \times \Xi}^{\mathscr{I}}_{\mathcal{R}}((\alpha_1 \oslash_1 \alpha_2, \varpi_1 \oslash_1 \varpi_2),\varrho) \cdot M^{\overline{\Theta \times \Xi^{\mathscr{I}}_{\mathcal{I}}}((\alpha_1\oslash_1\alpha_2,\varpi_1\oslash_1\varpi_2),\varrho)} \\
= \;& \frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}((\alpha_1 \oslash_1 \alpha_2),\varrho) \cdot M^{\overline{\Theta \times \Xi^{\mathscr{I}}_{\mathcal{I}}}((\alpha_1\oslash_1\alpha_2),\varrho)} + \overline{\Xi^{\mathscr{I}}_{\mathcal{R}}}((\varpi_1 \oslash_1 \varpi_2),\varrho) \cdot M^{\overline{\Theta \times \Xi^{\mathscr{T}}_{\mathcal{I}}}((\varpi_1\oslash_1\varpi_2),\varrho)}}{2} \\
\succeq \;& \frac{1}{2}\left[\frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha_1) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha_1)} + \overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha_2) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha_2)}}{2} + \frac{\overline{\Xi^{\mathscr{I}}_{\mathcal{R}}}(\varpi_1) \cdot M^{\overline{\Xi^{\mathscr{I}}_{\mathcal{I}}}(\varpi_1)} + \overline{\Xi^{\mathscr{I}}_{\mathcal{R}}}(\varpi_2) \cdot M^{\overline{\Xi^{\mathscr{I}}_{\mathcal{I}}}(\varpi_2)}}{2}\right] \\
= \;& \frac{1}{2}\left[\frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha_1) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha_1)} + \overline{\Xi^{\mathscr{I}}_{\mathcal{R}}}(\varpi_1) \cdot M^{\overline{\Xi^{\mathscr{I}}_{\mathcal{I}}}(\varpi_1)}}{2} + \frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha_2) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha_2)} + \overline{\Xi^{\mathscr{I}}_{\mathcal{R}}}(\varpi_2) \cdot M^{\overline{\Xi^{\mathscr{I}}_{\mathcal{I}}}(\varpi_2)}}{2}\right] \\
= \;& \frac{1}{2}\left[\overline{\Theta \times \Xi}^{\mathscr{I}}_{\mathcal{R}}((\alpha_1,\varpi_1),\varrho) \cdot M^{\overline{\Theta \times \Xi^{\mathscr{T}}_{\mathcal{I}}}((\alpha_1,\varpi_1),\varrho)} + \overline{\Theta \times \Xi}^{\mathscr{I}}_{\mathcal{R}}((\alpha_2,\varpi_2),\varrho) \cdot M^{\overline{\Theta \times \Xi^{\mathscr{T}}_{\mathcal{I}}}((\alpha_2,\varpi_2),\varrho)}\right]
\end{aligned}$$

Also
$\overline{\Theta\times\Xi^{\mathscr{I}}_{\mathcal{R}}}[((\alpha_1,\varpi_1)\oslash_2(\alpha_2,\varpi_2),\varrho)]\cdot M^{\overline{\Theta\times\Xi^{\mathscr{I}}_{\mathcal{I}}}[((\alpha_1,\varpi_1)\oslash_2(\alpha_2,\varpi_2),\varrho)]}\succeq\frac{1}{2}\Big[\overline{\Theta\times\Xi^{\mathscr{I}}_{\mathcal{R}}}((\alpha_1,\varpi_1),\varrho)\cdot M^{\overline{\Theta\times\Xi^{\mathscr{I}}_{\mathcal{I}}}((\alpha_1,\varpi_1),\varrho)}+$
$\overline{\Theta\times\Xi^{\mathscr{I}}_{\mathcal{R}}}((\alpha_2,\varpi_2),\varrho)\cdot M^{\overline{\Theta\times\Xi^{\mathscr{I}}_{\mathcal{I}}}((\alpha_2,\varpi_2),\varrho)}\Big]$
and
$\overline{\Theta\times\Xi^{\mathscr{I}}_{\mathcal{R}}}[((\alpha_1,\varpi_1)\oslash_3(\alpha_2,\varpi_2),\varrho)]\cdot M^{\overline{\Theta\times\Xi^{\mathscr{I}}_{\mathcal{I}}}[((\alpha_1,\varpi_1)\oslash_3(\alpha_2,\varpi_2),\varrho)]}\succeq\frac{1}{2}\Big[\overline{\Theta\times\Xi^{\mathscr{I}}_{\mathcal{R}}}((\alpha_1,\varpi_1),\varrho)\cdot M^{\overline{\Theta\times\Xi^{\mathscr{I}}_{\mathcal{I}}}((\alpha_1,\varpi_1),\varrho)}+$
$\overline{\Theta\times\Xi^{\mathscr{I}}_{\mathcal{R}}}((\alpha_2,\varpi_2),\varrho)\cdot M^{\overline{\Theta\times\Xi^{\mathscr{I}}_{\mathcal{I}}}((\alpha_2,\varpi_2),\varrho)}\Big]$.
Now,

$$\begin{aligned}
&\overline{\Theta\times\Xi^{\mathscr{F}}_{\mathcal{R}}}[((\alpha_1,\varpi_1)\oslash_1(\alpha_2,\varpi_2),\varrho)]\cdot M^{\overline{\Theta\times\Xi^{\mathscr{F}}_{\mathcal{I}}}[((\alpha_1,\varpi_1)\oslash_1(\alpha_2,\varpi_2),\varrho)]}\\
=\;&\overline{\Theta\times\Xi^{\mathscr{F}}_{\mathcal{R}}}((\alpha_1\oslash_1\alpha_2,\varpi_1\oslash_1\varpi_2),\varrho)\cdot M^{\overline{\Theta\times\Xi^{\mathscr{F}}_{\mathcal{I}}}((\alpha_1\oslash_1\alpha_2,\varpi_1\oslash_1\varpi_2),\varrho)}\\
=\;&\max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha_1\oslash_1\alpha_2),\varrho)\cdot M^{\overline{\Theta\times\Xi^{\mathscr{F}}_{\mathcal{I}}}((\alpha_1\oslash_1\alpha_2),\varrho)},\overline{\Xi^{\mathscr{F}}_{\mathcal{R}}}((\varpi_1\oslash_1\varpi_2),\varrho)\cdot M^{\overline{\Theta\times\Xi^{\mathscr{F}}_{\mathcal{I}}}((\varpi_1\oslash_1\varpi_2),\varrho)}\}\\
\preceq\;&\max\{\max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_1)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha_1)},\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_2)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha_2)}\},\max\{\overline{\Xi^{\mathscr{F}}_{\mathcal{R}}}(\varpi_1)\cdot M^{\overline{\Xi^{\mathscr{F}}_{\mathcal{I}}}(\varpi_1)},\overline{\Xi^{\mathscr{F}}_{\mathcal{R}}}(\varpi_2)\cdot M^{\overline{\Xi^{\mathscr{F}}_{\mathcal{I}}}(\varpi_2)}\}\}\\
=\;&\max\{\max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_1)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha_1)},\overline{\Xi^{\mathscr{F}}_{\mathcal{R}}}(\varpi_1)\cdot M^{\overline{\Xi^{\mathscr{F}}_{\mathcal{I}}}(\varpi_1)}\},\max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_2)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha_2)},\overline{\Xi^{\mathscr{F}}_{\mathcal{R}}}(\varpi_2)\cdot M^{\overline{\Xi^{\mathscr{F}}_{\mathcal{I}}}(\varpi_2)}\}\}\\
=\;&\max\{\overline{\Theta\times\Xi^{\mathscr{F}}_{\mathcal{R}}}((\alpha_1,\varpi_1),\varrho)\cdot M^{\overline{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}}((\alpha_1,\varpi_1),\varrho)},\overline{\Theta\times\Xi^{\mathscr{F}}_{\mathcal{R}}}((\alpha_2,\varpi_2),\varrho)\cdot M^{\overline{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}}((\alpha_2,\varpi_2),\varrho)}\}
\end{aligned}$$

Also $\overline{\Theta\times\Xi^{\mathscr{F}}_{\mathcal{R}}}[((\alpha_1,\varpi_1)\oslash_2(\alpha_2,\varpi_2),\varrho)]\cdot M^{\overline{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}}[((\alpha_1,\varpi_1)\oslash_2(\alpha_2,\varpi_2),\varrho)]}$
$\preceq\max\{\overline{\Theta\times\Xi^{\mathscr{F}}_{\mathcal{R}}}((\alpha_1,\varpi_1),\varrho)\cdot M^{\overline{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}}((\alpha_1,\varpi_1),\varrho)},\overline{\Theta\times\Xi^{\mathscr{F}}_{\mathcal{R}}}((\alpha_2,\varpi_2),\varrho)\cdot M^{\overline{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}}}((\alpha_2,\varpi_2),\varrho)\}$,
$\overline{\Theta\times\Xi^{\mathscr{F}}_{\mathcal{R}}}[((\alpha_1,\varpi_1)\oslash_3(\alpha_2,\varpi_2),\varrho)]\cdot M^{\overline{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}}[((\alpha_1,\varpi_1)\oslash_3(\alpha_2,\varpi_2),\varrho)]}$
$\preceq\max\{\overline{\Theta\times\Xi^{\mathscr{F}}_{\mathcal{R}}}((\alpha_1,\varpi_1),\varrho)\cdot M^{\overline{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}}((\alpha_1,\varpi_1),\varrho)},\overline{\Theta\times\Xi^{\mathscr{F}}_{\mathcal{R}}}((\alpha_2,\varpi_2),\varrho)\cdot M^{\overline{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}}((\alpha_2,\varpi_2),\varrho)}\}$.

Let $\alpha_1,\alpha_2\in\mathscr{S}_1$ and $\varpi_1,\varpi_2\in\mathscr{S}_2$. Then (α_1,ϖ_1) and (α_2,ϖ_2) are in $\mathscr{S}_1\times\mathscr{S}_2$. Now

$$\begin{aligned}
&\Theta\times\Xi^{\mathscr{T}}_{\mathcal{R}}[((\alpha_1,\varpi_1)\oslash_1(\alpha_2,\varpi_2),\varrho)]\cdot M^{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}[((\alpha_1,\varpi_1)\oslash_1(\alpha_2,\varpi_2),\varrho)]}\\
=\;&\Theta\times\Xi^{\mathscr{T}}_{\mathcal{R}}((\alpha_1\oslash_1\alpha_2,\varpi_1\oslash_1\varpi_2),\varrho)\cdot M^{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}((\alpha_1\oslash_1\alpha_2,\varpi_1\oslash_1\varpi_2),\varrho)}\\
=\;&\min\{\Theta^{\mathscr{T}}_{\mathcal{R}}((\alpha_1\oslash_1\alpha_2),\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_1\oslash_1\alpha_2),\varrho)},\Xi^{\mathscr{T}}_{\mathcal{R}}((\varpi_1\oslash_1\varpi_2),\varrho)\cdot M^{\Xi^{\mathscr{T}}_{\mathcal{I}}((\varpi_1\oslash_1\varpi_2),\varrho)}\}\\
\succeq\;&\min\{\min\{\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha_1)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha_1}),\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha_2)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha_2)}\},\min\{\Xi^{\mathscr{T}}_{\mathcal{R}}(\varpi_1)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_1)},\Xi^{\mathscr{T}}_{\mathcal{R}}(\varpi_2)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_2)}\}\}\\
=\;&\min\{\min\{\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha_1)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha_1}),\Xi^{\mathscr{T}}_{\mathcal{R}}(\varpi_1)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_1})\},\min\{\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha_2)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha_2)},\Xi^{\mathscr{T}}_{\mathcal{R}}(\varpi_2)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_2)}\}\}\\
=\;&\min\{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{R}}((\alpha_1,\varpi_1),\varrho)\cdot M^{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}((\alpha_1,\varpi_1),\varrho)},\Theta\times\Xi^{\mathscr{T}}_{\mathcal{R}}((\alpha_2,\varpi_2),\varrho)\cdot M^{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}((\alpha_2,\varpi_2),\varrho)}\}
\end{aligned}$$

Also $\Theta\times\Xi^{\mathscr{T}}_{\mathcal{R}}[((\alpha_1,\varpi_1)\oslash_2(\alpha_2,\varpi_2),\varrho)]\cdot M^{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}[((\alpha_1,\varpi_1)\oslash_2(\alpha_2,\varpi_2),\varrho)]}$
$\succeq\min\{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{R}}((\alpha_1,\varpi_1),\varrho)\cdot M^{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}((\alpha_1,\varpi_1),\varrho)},\Theta\times\Xi^{\mathscr{T}}_{\mathcal{R}}((\alpha_2,\varpi_2),\varrho)M^{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}((\alpha_2,\varpi_2),\varrho)}\}$

and $\Theta\times\Xi^{\mathscr{T}}_{\mathcal{R}}[((\alpha_1,\varpi_1)\oslash_3(\alpha_2,\varpi_2),\varrho)]\cdot M^{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}[((\alpha_1,\varpi_1)\oslash_3(\alpha_2,\varpi_2),\varrho)]}$
$\succeq\min\{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{R}}((\alpha_1,\varpi_1),\varrho)\cdot M^{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}((\alpha_1,\varpi_1),\varrho)},\Theta\times\Xi^{\mathscr{T}}_{\mathcal{R}}((\alpha_2,\varpi_2),\varrho)\cdot M^{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}((\alpha_2,\varpi_2),\varrho)}\}$.
Now,

$$\begin{aligned}
&\Theta\times\Xi^{\mathscr{I}}_{\mathcal{R}}[((\alpha_1,\varpi_1)\oslash_1(\alpha_2,\varpi_2),\varrho)]\cdot M^{\Theta\times\Xi^{\mathscr{I}}_{\mathcal{I}}[((\alpha_1,\varpi_1)\oslash_1(\alpha_2,\varpi_2),\varrho)]}\\
=\;&\Theta\times\Xi^{\mathscr{I}}_{\mathcal{R}}((\alpha_1\oslash_1\alpha_2,\varpi_1\oslash_1\varpi_2),\varrho)\cdot M^{\Theta\times\Xi^{\mathscr{I}}_{\mathcal{I}}((\alpha_1\oslash_1\alpha_2,\varpi_1\oslash_1\varpi_2),\varrho)}\\
=\;&\frac{\Theta^{\mathscr{I}}_{\mathcal{R}}((\alpha_1\oslash_1\alpha_2),\varrho)\cdot M^{\Theta\times\Xi^{\mathscr{I}}_{\mathcal{I}}((\alpha_1\oslash_1\alpha_2),\varrho)}+\Xi^{\mathscr{I}}_{\mathcal{R}}((\varpi_1\oslash_1\varpi_2),\varrho)\cdot M^{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}((\varpi_1\oslash_1\varpi_2),\varrho)}}{2}\\
\succeq\;&\frac{1}{2}\left[\frac{\Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha_1)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha_1)}+\Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha_2)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha_2)}}{2}+\frac{\Xi^{\mathscr{I}}_{\mathcal{R}}(\varpi_1)\cdot M^{\Xi^{\mathscr{I}}_{\mathcal{I}}(\varpi_1)}+\Xi^{\mathscr{I}}_{\mathcal{R}}(\varpi_2)\cdot M^{\Xi^{\mathscr{I}}_{\mathcal{I}}(\varpi_2)}}{2}\right]\\
=\;&\frac{1}{2}\left[\frac{\Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha_1)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha_1)}+\Xi^{\mathscr{I}}_{\mathcal{R}}(\varpi_1)\cdot M^{\Xi^{\mathscr{I}}_{\mathcal{I}}(\varpi_1)}}{2}+\frac{\Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha_2)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha_2)}+\Xi^{\mathscr{I}}_{\mathcal{R}}(\varpi_2)\cdot M^{\Xi^{\mathscr{I}}_{\mathcal{I}}(\varpi_2)}}{2}\right]\\
=\;&\frac{1}{2}\Big[\Theta\times\Xi^{\mathscr{I}}_{\mathcal{R}}((\alpha_1,\varpi_1),\varrho)\cdot M^{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}((\alpha_1,\varpi_1),\varrho)}+\Theta\times\Xi^{\mathscr{I}}_{\mathcal{R}}((\alpha_2,\varpi_2),\varrho)\cdot M^{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}((\alpha_2,\varpi_2),\varrho)}\Big]
\end{aligned}$$

Also
$\Theta\times\Xi^{\mathscr{I}}_{\mathcal{R}}[((\alpha_1,\varpi_1)\oslash_2(\alpha_2,\varpi_2),\varrho)]\cdot M^{\Theta\times\Xi^{\mathscr{I}}_{\mathcal{I}}[((\alpha_1,\varpi_1)\oslash_2(\alpha_2,\varpi_2),\varrho)]}\succeq\frac{1}{2}\Big[\Theta\times\Xi^{\mathscr{I}}_{\mathcal{R}}((\alpha_1,\varpi_1),\varrho)\cdot M^{\Theta\times\Xi^{\mathscr{I}}_{\mathcal{I}}((\alpha_1,\varpi_1),\varrho)}+$
$\Theta\times\Xi^{\mathscr{I}}_{\mathcal{R}}((\alpha_2,\varpi_2),\varrho)\cdot M^{\Theta\times\Xi^{\mathscr{I}}_{\mathcal{I}}((\alpha_2,\varpi_2),\varrho)}\Big]$
and
$\Theta\times\Xi^{\mathscr{I}}_{\mathcal{R}}[((\alpha_1,\varpi_1)\oslash_3(\alpha_2,\varpi_2),\varrho)]\cdot M^{\Theta\times\Xi^{\mathscr{I}}_{\mathcal{I}}[((\alpha_1,\varpi_1)\oslash_3(\alpha_2,\varpi_2),\varrho)]}\succeq\frac{1}{2}\Big[\Theta\times\Xi^{\mathscr{I}}_{\mathcal{R}}((\alpha_1,\varpi_1),\varrho)\cdot M^{\Theta\times\Xi^{\mathscr{I}}_{\mathcal{I}}((\alpha_1,\varpi_1),\varrho)}+$
$\Theta\times\Xi^{\mathscr{I}}_{\mathcal{R}}((\alpha_2,\varpi_2),\varrho)\cdot M^{\Theta\times\Xi^{\mathscr{I}}_{\mathcal{I}}((\alpha_2,\varpi_2),\varrho)}\Big].$
Now,

$$
\begin{aligned}
& \Theta\times\Xi^{\mathscr{F}}_{\mathcal{R}}[((\alpha_1,\varpi_1)\oslash_1(\alpha_2,\varpi_2),\varrho)]\cdot M^{\Theta\times\Xi^{\mathscr{F}}_{\mathcal{I}}[((\alpha_1,\varpi_1)\oslash_1(\alpha_2,\varpi_2),\varrho)]} \\
= & \Theta\times\Xi^{\mathscr{F}}_{\mathcal{R}}((\alpha_1\oslash_1\alpha_2,\varpi_1\oslash_1\varpi_2),\varrho)\cdot M^{\Theta\times\Xi^{\mathscr{F}}_{\mathcal{I}}((\alpha_1\oslash_1\alpha_2,\varpi_1\oslash_1\varpi_2),\varrho)} \\
= & \max\{\Theta^{\mathscr{F}}_{\mathcal{R}}((\alpha_1\oslash_1\alpha_2),\varrho)\cdot M^{\Theta\times\Xi^{\mathscr{F}}_{\mathcal{I}}((\alpha_1\oslash_1\alpha_2),\varrho)},\Xi^{\mathscr{F}}_{\mathcal{R}}((\varpi_1\oslash_1\varpi_2),\varrho)\cdot M^{\Theta\times\Xi^{\mathscr{F}}_{\mathcal{I}}((\varpi_1\oslash_1\varpi_2),\varrho)}\} \\
\preceq & \max\{\max\{\Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha_1)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha_1)},\Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha_2)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha_2)}\},\max\{\Xi^{\mathscr{F}}_{\mathcal{R}}(\varpi_1)\cdot M^{\Xi^{\mathscr{F}}_{\mathcal{I}}(\varpi_1)},\Xi^{\mathscr{F}}_{\mathcal{R}}(\varpi_2)\cdot M^{\Xi^{\mathscr{F}}_{\mathcal{I}}(\varpi_2)}\}\} \\
= & \max\{\max\{\Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha_1)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha_1)},\Xi^{\mathscr{F}}_{\mathcal{R}}(\varpi_1)\cdot M^{\Xi^{\mathscr{F}}_{\mathcal{I}}(\varpi_1)}\},\max\{\Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha_2)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha_2)},\Xi^{\mathscr{F}}_{\mathcal{R}}(\varpi_2)\cdot M^{\Xi^{\mathscr{F}}_{\mathcal{I}}(\varpi_2)}\}\} \\
= & \max\{\Theta\times\Xi^{\mathscr{F}}_{\mathcal{R}}((\alpha_1,\varpi_1),\varrho)\cdot M^{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}((\alpha_1,\varpi_1),\varrho)},\Theta\times\Xi^{\mathscr{F}}_{\mathcal{R}}((\alpha_2,\varpi_2),\varrho)\cdot M^{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}((\alpha_2,\varpi_2),\varrho)}\}
\end{aligned}
$$

Also $\Theta\times\Xi^{\mathscr{F}}_{\mathcal{R}}[((\alpha_1,\varpi_1)\oslash_2(\alpha_2,\varpi_2),\varrho)]\cdot M^{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}[((\alpha_1,\varpi_1)\oslash_2(\alpha_2,\varpi_2),\varrho)]}$
$\preceq\max\{\Theta\times\Xi^{\mathscr{F}}_{\mathcal{R}}((\alpha_1,\varpi_1),\varrho)\cdot M^{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}((\alpha_1,\varpi_1),\varrho)},\Theta\times\Xi^{\mathscr{F}}_{\mathcal{R}}((\alpha_2,\varpi_2),\varrho)\cdot M^{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}}((\alpha_2,\varpi_2),\varrho)\},$
$\Theta\times\Xi^{\mathscr{F}}_{\mathcal{R}}[((\alpha_1,\varpi_1)\oslash_3(\alpha_2,\varpi_2),\varrho)]\cdot M^{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}[((\alpha_1,\varpi_1)\oslash_3(\alpha_2,\varpi_2),\varrho)]}$
$\preceq\max\{\Theta\times\Xi^{\mathscr{F}}_{\mathcal{R}}((\alpha_1,\varpi_1),\varrho)\cdot M^{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}((\alpha_1,\varpi_1),\varrho)},\Theta\times\Xi^{\mathscr{F}}_{\mathcal{R}}((\alpha_2,\varpi_2),\varrho)\cdot M^{\Theta\times\Xi^{\mathscr{T}}_{\mathcal{I}}((\alpha_2,\varpi_2),\varrho)}\}.$
Thus, $\overline{\Theta\times\Xi}$ is a CCQNSBS of $\mathscr{S}$.

Corollary 3.9. *If $\Theta_1,\Theta_2,...,\Theta_n$ be the finite collection of CCQNSBSs of $\mathscr{S}_1,\mathscr{S}_2,...,\mathscr{S}_n$ respectively. Then prove that $\Theta_1\times\Theta_2\times...\times\Theta_n$ is a CCQNSBS of $\mathscr{S}_1\times\mathscr{S}_2\times...\times\mathscr{S}_n$.*

Definition 3.10. Let $\Theta\subseteq\mathscr{S}$, we define the strongest CCQN relation on $\mathscr{S}$ is defined as follows:

$$
\left\{\begin{array}{l}
\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}((\alpha,\varpi),\varrho)\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}((\alpha,\varpi),\varrho)}=min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha,\varrho)},\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\varpi,\varrho)}\} \\
\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}((\alpha,\varpi),\varrho)\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}((\alpha,\varpi),\varrho)}=\frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha,\varrho)}+\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\varpi,\varrho)}}{2} \\
\overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}((\alpha,\varpi),\varrho)\cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}((\alpha,\varpi),\varrho)}=max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha,\varrho)},\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\varpi,\varrho)}\}
\end{array}\right\}
$$

Theorem 3.11. *Let Θ be a CCQNSBS of $\mathscr{S}$ and Ω be the strongest complex cubic Q neutrosophic relation of $\mathscr{S}$. Then Θ is a CCQNSBS of $\mathscr{S}\times\mathscr{S}$ if and only if Ω is a CCQNSBS of $\mathscr{S}\times\mathscr{S}$.*

Proof. Suppose Θ is a CCQNSBS of $\mathscr{S}\times\mathscr{S}$ and $\overline{\Omega}$ be the strongest complex cubic Q neutrosophic relation of $\mathscr{S}$.
For any $\alpha=(\alpha_1,\alpha_2),\varpi=(\varpi_1,\varpi_2)\in S\times\mathscr{S}$. Now,

$$
\begin{aligned}
& \overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}((\alpha\oslash_1\varpi),\varrho)\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}((\alpha\oslash_1\varpi),\varrho)} \\
= & \overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}[((((\alpha_1,\alpha_2),\varrho)\oslash_1((\varpi_1,\varpi_2),\varrho)]\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}[((((\alpha_1,\alpha_2),\varrho)\oslash_1((\varpi_1,\varpi_2),\varrho)]} \\
= & \overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}(\alpha_1\oslash_1\varpi_1,\alpha_2\oslash_1\varpi_2)\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}(\alpha_1\oslash_1\varpi_1,\alpha_2\oslash_1\varpi_2)} \\
= & \min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha_1\oslash_1\varpi_1),\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}((\alpha_1\oslash_1\varpi_1),\varrho)},\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha_2\oslash_1\varpi_2),\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}((\alpha_2\oslash_1\varpi_2),\varrho)}\} \\
\succeq & \min\{\min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_1)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha_1)},\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\varpi_1)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\varpi_1)}\}, \\
& \min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_2)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha_2)},\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\varpi_2)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\varpi_2)}\}\} \\
= & \min\{\min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_1)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha_1)},\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_2)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha_2)}\}, \\
& \min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\varpi_1)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\varpi_1)},\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\varpi_2)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\varpi_2)}\}\} \\
= & \min\{\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}((\alpha_1,\alpha_2),\varrho)\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}((\alpha_1,\alpha_2),\varrho)},\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}((\varpi_1,\varpi_2),\varrho)\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}((\varpi_1,\varpi_2),\varrho)}\} \\
= & \min\{\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}(\alpha,\varrho)},\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}(\varpi,\varrho)}\}
\end{aligned}
$$

Also $\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}((\alpha \oslash_2 \varpi), \varrho) \cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}((\alpha\oslash_2\varpi),\varrho)} \succeq \min\{\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}(\alpha, \varrho) \cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}(\alpha,\varrho)}, \overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}(\varpi, \varrho) \cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}(\varpi,\varrho)}\}$,
$\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}((\alpha \oslash_3 \varpi), \varrho) \cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}((\alpha\oslash_3\varpi),\varrho)} \succeq \min\{\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}(\alpha, \varrho) \cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}(\alpha,\varrho)}, \overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}(\varpi, \varrho) \cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}(\varpi,\varrho)}\}$.
Now, $\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}(\alpha \oslash_1 \varpi) \cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}(\alpha\oslash_1\varpi)}$

$$
\begin{aligned}
&= \overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}[((((\alpha_1, \alpha_2), \varrho) \oslash_1 (\varpi_1, \varpi_2), \varrho)] \cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}[((((\alpha_1,\alpha_2),\varrho)\oslash_1((\varpi_1,\varpi_2),\varrho))]} \\
&= \overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}((\alpha_1 \oslash_1 \varpi_1, \alpha_2 \oslash_1 \varpi_2), \varrho) \cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}((\alpha_1\oslash_1\varpi_1,\alpha_2\oslash_1\varpi_2),\varrho)} \\
&= \frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}((\alpha_1 \oslash_1 \varpi_1), \varrho) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}((\alpha_1\oslash_1\varpi_1),\varrho)} + \overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}((\alpha_2 \oslash_1 \varpi_2), \varrho) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}}((\alpha_2 \oslash_1 \varpi_2), \varrho)}{2} \\
&\succeq \frac{1}{2}\left[\frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha_1) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha_1)} + \overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\varpi_1) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\varpi_1)}}{2} + \frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha_2) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha_2)} + \overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\varpi_2) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\varpi_2)}}{2}\right] \\
&= \frac{1}{2}\left[\frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha_1) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha_1)} + \overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha_2) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha_2)}}{2} + \frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\varpi_1) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}(\varpi_1)}} + \overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\varpi_2) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\varpi_2)}}{2}\right] \\
&= \frac{\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}((\alpha_1, \alpha_2), \varrho) \cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}((\alpha_1,\alpha_2),\varrho)} + \overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}((\varpi_1, \varpi_2), \varrho) \cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}((\varpi_1,\varpi_2),\varrho)}}{2} \\
&= \frac{\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}(\alpha, \varrho) \cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}(\alpha,\varrho)}} + \overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}(\varpi, \varrho) \cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}(\alpha,\varrho)}}{2}
\end{aligned}
$$

Also $\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}((\alpha \oslash_2 \varpi), \varrho) \cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}((\alpha\oslash_2\varpi),\varrho)} \succeq \frac{\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}(\alpha,\varrho)}} + \overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}(\varpi,\varrho)}}{2}$
and $\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}((\alpha \oslash_3 \varpi), \varrho) \cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}((\alpha\oslash_3\varpi),\varrho)} \succeq \frac{\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}(\alpha,\varrho)}} + \overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}(\varpi,\varrho)}}{2}$.
Similarly,$\overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}((\alpha \oslash_1 \varpi), \varrho) \cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}((\alpha\oslash_1\varpi),\varrho)} \preceq \max\{\overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}(\alpha, \varrho) \cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}(\alpha,\varrho)\cdot}}, \overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}(\varpi, \varrho) \cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}(\varpi,\varrho)}}\}$,
$\overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}((\alpha \oslash_2 \varpi), \varrho) \cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}((\alpha\oslash_2\varpi),\varrho)} \preceq \max\{\overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}(\alpha, \varrho) \cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}(\alpha,\varrho)}}, \overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}(\varpi, \varrho) \cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}(\varpi,\varrho)}}\}$ and
$\overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}((\alpha \oslash_3 \varpi), \varrho) \cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}((\alpha\oslash_3\varpi),\varrho)} \preceq \max\{\overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}(\alpha, \varrho) \cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}(\alpha,\varrho)}}, \overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}(\varpi, \varrho) \cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}(\varpi}}\}$.
For any $\alpha = (\alpha_1, \alpha_2), \varpi = (\varpi_1, \varpi_2) \in S \times \mathscr{S}$. Now,

$$
\begin{aligned}
&\quad \Omega^{\mathscr{T}}_{\mathcal{R}}((\alpha \oslash_1 \varpi), \varrho) \cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}((\alpha\oslash_1\varpi),\varrho)} \\
&= \Omega^{\mathscr{T}}_{\mathcal{R}}[((((\alpha_1, \alpha_2), \varrho) \oslash_1 ((\varpi_1, \varpi_2), \varrho)] \cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}[((((\alpha_1,\alpha_2),\varrho)\oslash_1((\varpi_1,\varpi_2),\varrho)]} \\
&= \Omega^{\mathscr{T}}_{\mathcal{R}}(\alpha_1 \oslash_1 \varpi_1, \alpha_2 \oslash_1 \varpi_2) \cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}(\alpha_1\oslash_1\varpi_1,\alpha_2\oslash_1\varpi_2)} \\
&= \min\{\Theta^{\mathscr{T}}_{\mathcal{R}}((\alpha_1 \oslash_1 \varpi_1), \varrho) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_1\oslash_1\varpi_1),\varrho)}, \Theta^{\mathscr{T}}_{\mathcal{R}}((\alpha_2 \oslash_1 \varpi_2), \varrho) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_2\oslash_1\varpi_2),\varrho)}\} \\
&\succeq \min\{\min\{\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha_1) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha_1)}, \Theta^{\mathscr{T}}_{\mathcal{R}}(\varpi_1) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_1)}\}, \\
&\quad \min\{\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha_2) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha_2)}, \Theta^{\mathscr{T}}_{\mathcal{R}}(\varpi_2) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_2)}\}\} \\
&= \min\{\min\{\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha_1) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha_1)}, \Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha_2) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha_2)}\}, \\
&\quad \min\{\Theta^{\mathscr{T}}_{\mathcal{R}}(\varpi_1) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_1)}, \Theta^{\mathscr{T}}_{\mathcal{R}}(\varpi_2) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_2)}\}\} \\
&= \min\{\Omega^{\mathscr{T}}_{\mathcal{R}}((\alpha_1, \alpha_2), \varrho) \cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}((\alpha_1,\alpha_2),\varrho)}, \Omega^{\mathscr{T}}_{\mathcal{R}}((\varpi_1, \varpi_2), \varrho) \cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}((\varpi_1,\varpi_2),\varrho)}\} \\
&= \min\{\Omega^{\mathscr{T}}_{\mathcal{R}}(\alpha, \varrho) \cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}(\alpha,\varrho)}, \Omega^{\mathscr{T}}_{\mathcal{R}}(\varpi, \varrho) \cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}(\varpi,\varrho)}\}
\end{aligned}
$$

Also $\Omega^{\mathscr{T}}_{\mathcal{R}}((\alpha \oslash_2 \varpi), \varrho) \cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}((\alpha\oslash_2\varpi),\varrho)} \succeq \min\{\Omega^{\mathscr{T}}_{\mathcal{R}}(\alpha, \varrho) \cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}(\alpha,\varrho)}, \Omega^{\mathscr{T}}_{\mathcal{R}}(\varpi, \varrho) \cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}(\varpi,\varrho)}\}$,
$\Omega^{\mathscr{T}}_{\mathcal{R}}((\alpha \oslash_3 \varpi), \varrho) \cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}((\alpha\oslash_3\varpi),\varrho)} \succeq \min\{\Omega^{\mathscr{T}}_{\mathcal{R}}(\alpha, \varrho) \cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}(\alpha,\varrho)}, \Omega^{\mathscr{T}}_{\mathcal{R}}(\varpi, \varrho) \cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}(\varpi,\varrho)}\}$.

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Now, $\Omega^{\mathscr{I}}_{\mathcal{R}}(\alpha \oslash_1 \varpi) \cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}(\alpha \oslash_1 \varpi)}$

$$
\begin{aligned}
&= \Omega^{\mathscr{I}}_{\mathcal{R}}[((((\alpha_1,\alpha_2),\varrho) \oslash_1 (\varpi_1,\varpi_2),\varrho)] \cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}[((((\alpha_1,\alpha_2),\varrho)\oslash_1((\varpi_1,\varpi_2),\varrho))]}\\
&= \Omega^{\mathscr{I}}_{\mathcal{R}}((\alpha_1 \oslash_1 \varpi_1, \alpha_2 \oslash_1 \varpi_2),\varrho) \cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}((\alpha_1\oslash_1\varpi_1,\alpha_2\oslash_1\varpi_2),\varrho)}\\
&= \frac{\Theta^{\mathscr{I}}_{\mathcal{R}}((\alpha_1 \oslash_1 \varpi_1),\varrho) \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}((\alpha_1\oslash_1\varpi_1),\varrho)} + \Theta^{\mathscr{I}}_{\mathcal{R}}((\alpha_2 \oslash_1 \varpi_2),\varrho) \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}}((\alpha_2 \oslash_1 \varpi_2),\varrho)}{2}\\
&\succeq \frac{1}{2}\left[\frac{\Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha_1)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha_1)} + \Theta^{\mathscr{I}}_{\mathcal{R}}(\varpi_1)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\varpi_1)}}{2} + \frac{\Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha_2)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha_2)} + \Theta^{\mathscr{I}}_{\mathcal{R}}(\varpi_2)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\varpi_2)}}{2}\right]\\
&= \frac{1}{2}\left[\frac{\Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha_1)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha_1)} + \Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha_2)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha_2)}}{2} + \frac{\Theta^{\mathscr{I}}_{\mathcal{R}}(\varpi_1)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\varpi_1)} + \Theta^{\mathscr{I}}_{\mathcal{R}}(\varpi_2)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\varpi_2)}}{2}\right]\\
&= \frac{\Omega^{\mathscr{I}}_{\mathcal{R}}((\alpha_1,\alpha_2),\varrho)\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}((\alpha_1,\alpha_2),\varrho)} + \Omega^{\mathscr{I}}_{\mathcal{R}}((\varpi_1,\varpi_2),\varrho)\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}((\varpi_1,\varpi_2),\varrho)}}{2}\\
&= \frac{\Omega^{\mathscr{I}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}(\alpha,\varrho)} + \Omega^{\mathscr{I}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}(\alpha,\varrho)}}{2}
\end{aligned}
$$

Also $\Omega^{\mathscr{I}}_{\mathcal{R}}((\alpha \oslash_2 \varpi),\varrho) \cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}((\alpha\oslash_2\varpi),\varrho)} \succeq \frac{\Omega^{\mathscr{I}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}(\alpha,\varrho)} + \Omega^{\mathscr{I}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}(\varpi,\varrho)}}{2}$
and $\Omega^{\mathscr{I}}_{\mathcal{R}}((\alpha \oslash_3 \varpi),\varrho) \cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}((\alpha\oslash_3\varpi),\varrho)} \succeq \frac{\Omega^{\mathscr{I}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}(\alpha,\varrho)} + \Omega^{\mathscr{I}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}(\varpi,\varrho)}}{2}$.
Similarly,$\Omega^{\mathscr{F}}_{\mathcal{R}}((\alpha \oslash_1 \varpi),\varrho) \cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}((\alpha\oslash_1\varpi),\varrho)} \preceq \max\{\Omega^{\mathscr{F}}_{\mathcal{R}}(\alpha,\varrho) \cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}(\alpha,\varrho)\cdot}, \Omega^{\mathscr{F}}_{\mathcal{R}}(\varpi,\varrho) \cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}(\varpi,\varrho)}\}$,
$\Omega^{\mathscr{F}}_{\mathcal{R}}((\alpha \oslash_2 \varpi),\varrho) \cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}((\alpha\oslash_2\varpi),\varrho)} \preceq \max\{\Omega^{\mathscr{F}}_{\mathcal{R}}(\alpha,\varrho) \cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}(\alpha,\varrho)}, \Omega^{\mathscr{F}}_{\mathcal{R}}(\varpi,\varrho) \cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}(\varpi,\varrho)}\}$ and
$\Omega^{\mathscr{F}}_{\mathcal{R}}((\alpha \oslash_3 \varpi),\varrho) \cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}((\alpha\oslash_3\varpi),\varrho)} \preceq \max\{\Omega^{\mathscr{F}}_{\mathcal{R}}(\alpha,\varrho) \cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}(\alpha,\varrho)}, \Omega^{\mathscr{F}}_{\mathcal{R}}(\varpi,\varrho) \cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}(\varpi}\}$.
Therefore,Ω is a CCQNSBS of $\mathscr{S} \times \mathscr{S}$.

Conversely, suppose that Ω is a CCQNSBS of $\mathscr{S} \times \mathscr{S}$. Let $\alpha = ((\alpha_1,\alpha_2),\varrho), \varpi = ((\varpi_1,\varpi_2),\varrho) \in S \times \mathscr{S}$.
Now,

$$
\begin{aligned}
&\min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha_1 \oslash_1 \varpi_1),\varrho) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_1\oslash_1\varpi_1),\varrho)}}, \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha_2 \oslash_1 \varpi_2),\varrho) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_2\oslash_1\varpi_2),\varrho)}}\}\\
&= \overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}(\alpha_1 \oslash_1 \varpi_1, \alpha_2 \oslash_1 \varpi_2) \cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}(\alpha_1\oslash_1\varpi_1,\alpha_2\oslash_1\varpi_2)}}\\
&= \overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}[((\alpha_1,\alpha_2),\varrho) \oslash_1 ((\varpi_1,\varpi_2),\varrho)] \cdot M^{\overline{\mathcal{I}^{\mathscr{T}}_{v}[((\alpha_1,\alpha_2),\varrho)\oslash_1((\varpi_1,\varpi_2),\varrho)]}}\\
&= \overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}(\alpha \oslash_1 \varpi) \cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}(\alpha\oslash_1\varpi)}}\\
&\succeq \min\{\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}(\alpha,\varrho) \cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}(\alpha,\varrho)}}, \overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}(\varpi,\varrho) \cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}(\varpi,\varrho)}}\}\\
&= \min\{\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}((\alpha_1,\alpha_2),\varrho)\} \cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}((\alpha_1,\alpha_2),\varrho)\}}}, \overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}((\varpi_1,\varpi_2),\varrho) \cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}((\varpi_1,\varpi_2),\varrho)}}\}\\
&= \min\{\min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_1) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha_1)}}, \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_2) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha_2)}}\}, \min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\varpi_1) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_1)}}, \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\varpi_2) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_2)}}\}\}
\end{aligned}
$$

If $\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha_1\oslash_1\varpi_1),\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_1\oslash_1\varpi_1),\varrho)}} \preceq \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha_2\oslash_1\varpi_2),\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_2\oslash_1\varpi_2),\varrho)}}$, then $\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_1)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha_1)}} \preceq \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_2) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha_2)}}$ and $\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\varpi_1) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_1)}} \preceq \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\varpi_2) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_2)}}$. We get $\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha_1 \oslash_1 \varpi_1),\varrho) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_1\oslash_1\varpi_1),\varrho)}} \succeq \min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_1) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha_1)}}, \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\varpi_1) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_1)}}\}$ for all $\alpha_1, \varpi_1 \in \mathscr{S}$,and
$\min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha_1\oslash_2\varpi_1),\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_1\oslash_2\varpi_1),\varrho)}}, \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha_2\oslash_2\varpi_2),\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_2\oslash_2\varpi_2),\varrho)}}\} \succeq \min\{\min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_1)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha_1)}}, \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_2) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha_2)}}\}, \min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\varpi_1) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_1)}}, \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\varpi_2) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_2)}}\}\}$
If $\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha_1\oslash_2\varpi_1),\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_1\oslash_2\varpi_1),\varrho)}} \preceq \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha_2\oslash_2\varpi_2),\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_2\oslash_2\varpi_2),\varrho)}}$,then $\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha_1\oslash_2\varpi_1),\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_1\oslash_2\varpi_1),\varrho)}} \succeq \min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_1) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha_1)}}, \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\varpi_1) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_1)}}\}$.
$\min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha_1\oslash_3\varpi_1),\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_1\oslash_3\varpi_1),\varrho)}}, \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha_2\oslash_3\varpi_2),\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_2\oslash_3\varpi_2),\varrho)}}\} \succeq \min\{\min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_1)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha_1)}}, \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_2) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha_2)}}\}, \min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\varpi_1) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_1)}}, \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\varpi_2) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_2)}}\}\}$
If $\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha_1\oslash_3\varpi_1),\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_1\oslash_3\varpi_1),\varrho)}} \preceq \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha_2\oslash_3\varpi_2),\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_2\oslash_3\varpi_2),\varrho)}}$,then $\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha_1\oslash_3\varpi_1),\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_1\oslash_3\varpi_1),\varrho)}} \succeq \min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_1) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha_1)}}, \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\varpi_1) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_1)}}\}$.
Now,

$$
\begin{aligned}
&\frac{1}{2}\Big[\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}((\alpha_1\oslash_1\varpi_1),\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}((\alpha_1\oslash_1\varpi_1),\varrho)}}+\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}((\alpha_2\oslash_1\varpi_2),\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}((\alpha_2\oslash_1\varpi_2),\varrho)}_{\mathcal{I}}}}\Big]\\
&=\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}(\alpha_1\oslash_1\varpi_1,\alpha_2\oslash_1\varpi_2)\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}(\alpha_1\oslash_1\varpi_1,\alpha_2\oslash_1\varpi_2)}}\\
&=\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}[((\alpha_1,\alpha_2),\varrho)\oslash_1((\varpi_1,\varpi_2),\varrho)]\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}[((\alpha_1,\alpha_2),\varrho)\oslash_1((\varpi_1,\varpi_2),\varrho)]}}\\
&=\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}((\alpha\oslash_1\varpi),\varrho)\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}((\alpha\oslash_1\varpi),\varrho)}}\\
&\succeq\frac{\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Omega^{\mathscr{I}(\alpha,\varrho)}_{\mathcal{I}}}}+\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}(\varpi,\varrho)}}}{2}\\
&=\frac{\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}((\alpha_1,\alpha_2),\varrho)\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}((\alpha_1,\alpha_2),\varrho)}}+\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}((\varpi_1,\varpi_2),\varrho)\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}}}{2}\\
&=\frac{1}{2}\left[\frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha_1)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha_1)}}+\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha_2)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha_2)}}}{2}+\frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\varpi_1)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}(\varpi_1)}}+\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\varpi_2)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}(\varpi_2)}}}{2}\right]
\end{aligned}
$$

If $\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}((\alpha_1\oslash_1\varpi_1),\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}((\alpha_1\oslash_1\varpi_1),\varrho)}}\preceq\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}((\alpha_2\oslash_1\varpi_2),\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}((\alpha_2\oslash_1\varpi_2),\varrho)}}$, then $\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha_1)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha_1)}}\preceq\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha_2)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha_2)}}$ and $\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\varpi_1)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}(\varpi_1)}}\preceq\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\varpi_2)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}(\varpi_2)}}$.
We get $\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}((\alpha_1\oslash_1\varpi_1),\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_1\oslash_1\varpi_1),\varrho)}}\succeq\frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha_1)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha_1)}}+\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\varpi_1)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_1)}}}{2}$.
Similarly,$\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}((\alpha_1\oslash_2\varpi_1),\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}((\alpha_1\oslash_2\varpi_1),\varrho)}}\succeq\frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha_1)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha_1)}}+\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\varpi_1)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}(\varpi_1)}}}{2}$
and $\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}((\alpha_1\oslash_3\varpi_1),\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}((\alpha_1\oslash_3\varpi_1),\varrho)}}\succeq\frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha_1)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha_1)}}+\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\varpi_1)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}(\varpi_1)}}}{2}$.
Similarly to prove that
$\max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha_1\oslash_1\varpi_1),\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}((\alpha_1\oslash_1\varpi_1),\varrho)}},\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha_2\oslash_1\varpi_2),\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}((\alpha_2\oslash_1\varpi_2),\varrho)}}\}\preceq\max\{\max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_1)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha_1)}},\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_2)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha_2)}}\},\max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\varpi_1)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}(\varpi_1)}},\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\varpi_2)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}(\varpi_2)}}\}\}$
If $\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha_1\oslash_1\varpi_1),\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}((\alpha_1\oslash_1\varpi_1),\varrho)}}\succeq\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha_2\oslash_1\varpi_2),\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}((\alpha_2\oslash_1\varpi_2),\varrho)}}$,then $\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_1)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha_1)}}\succeq\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_2)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha_2)}}$ and $\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\varpi_1)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}(\varpi_1)}}\succeq\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\varpi_2)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}(\varpi_2)}}$.
We get $\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha_1\oslash_1\varpi_1),\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}((\alpha_1\oslash_1\varpi_1),\varrho)}}\preceq\max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_1)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha_1)}},\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\varpi_1)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}(\varpi_1)}}\}$.
$\max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha_1\oslash_2\varpi_1),\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}((\alpha_1\oslash_2\varpi_1),\varrho)}},\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha_2\oslash_2\varpi_2),\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}((\alpha_2\oslash_2\varpi_2),\varrho)}}\}$
$\preceq\max\{\max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_1)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha_1)}},\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_2)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha_2)}}\},\max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\varpi_1)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}(\varpi_1)}},\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\varpi_2)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}}\}\}$
If $\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha_1\oslash_2\varpi_1),\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}((\alpha_1\oslash_2\varpi_1),\varrho)}}\succeq\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha_2\oslash_2\varpi_2),\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}((\alpha_2\oslash_2\varpi_2),\varrho)}}$,then $\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha_1\oslash_2\varpi_1),\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}((\alpha_1\oslash_2\varpi_1),\varrho)}}\preceq\max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_1)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha_1)}},\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\varpi_1)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}(\varpi_1)}}\}$.
$\max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha_1\oslash_3\varpi_1),\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}((\alpha_1\oslash_3\varpi_1),\varrho)}},\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha_2\oslash_3\varpi_2),\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}((\alpha_2\oslash_3\varpi_2),\varrho)}}\}\preceq\max\{\max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_1)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha_1)}},\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_2)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha_2)}}\},\max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\varpi_1)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}(\varpi_1)}},\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\varpi_2)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}(\varpi_2)}}\}\}$
If $\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha_1\oslash_3\varpi_1),\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_1\oslash_3\varpi_1),\varrho)}}\succeq\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha_2\oslash_3\varpi_2),\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_2\oslash_3\varpi_2),\varrho)}}$,then $\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha_1\oslash_3\varpi_1),\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_1\oslash_3\varpi_1),\varrho)}}\preceq\max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_1)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha_1)}},\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\varpi_1)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_1)}}\}$.

Let $\alpha=((\alpha_1,\alpha_2),\varrho),\varpi=((\varpi_1,\varpi_2),\varrho)\in S\times\mathscr{S}$. Now,

$$
\begin{aligned}
&\min\{\Theta^{\mathscr{T}}_{\mathcal{R}}((\alpha_1\oslash_1\varpi_1),\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_1\oslash_1\varpi_1),\varrho)},\Theta^{\mathscr{T}}_{\mathcal{R}}((\alpha_2\oslash_1\varpi_2),\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_2\oslash_1\varpi_2),\varrho)}\}\\
&=\Omega^{\mathscr{T}}_{\mathcal{R}}(\alpha_1\oslash_1\varpi_1,\alpha_2\oslash_1\varpi_2)\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}(\alpha_1\oslash_1\varpi_1,\alpha_2\oslash_1\varpi_2)}\\
&=\Omega^{\mathscr{T}}_{\mathcal{R}}[((\alpha_1,\alpha_2),\varrho)\oslash_1((\varpi_1,\varpi_2),\varrho)]\cdot M^{\mathcal{I}^{\mathscr{T}}_{v}[((\alpha_1,\alpha_2),\varrho)\oslash_1((\varpi_1,\varpi_2),\varrho)]}\\
&=\Omega^{\mathscr{T}}_{\mathcal{R}}(\alpha\oslash_1\varpi)\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}(\alpha\oslash_1\varpi)}\\
&\succeq\min\{\Omega^{\mathscr{T}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}(\alpha,\varrho)},\Omega^{\mathscr{T}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}(\varpi,\varrho)}\}\\
&=\min\{\Omega^{\mathscr{T}}_{\mathcal{R}}((\alpha_1,\alpha_2),\varrho)\}\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}((\alpha_1,\alpha_2),\varrho)\}},\Omega^{\mathscr{T}}_{\mathcal{R}}((\varpi_1,\varpi_2),\varrho)\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}((\varpi_1,\varpi_2),\varrho)}\}\\
&=\min\{\min\{\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha_1)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha_1)},\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha_2)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha_2)}\},\min\{\Theta^{\mathscr{T}}_{\mathcal{R}}(\varpi_1)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_1)},\Theta^{\mathscr{T}}_{\mathcal{R}}(\varpi_2)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_2)}\}\}
\end{aligned}
$$

If $\Theta^{\mathscr{T}}_{\mathcal{R}}((\alpha_1\oslash_1\varpi_1),\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_1\oslash_1\varpi_1),\varrho)}\preceq\Theta^{\mathscr{T}}_{\mathcal{R}}((\alpha_2\oslash_1\varpi_2),\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha_2\oslash_1\varpi_2),\varrho)}$, then $\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha_1)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha_1)}\preceq\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha_2)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha_2)}$ and $\Theta^{\mathscr{T}}_{\mathcal{R}}(\varpi_1)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_1)}\preceq\Theta^{\mathscr{T}}_{\mathcal{R}}(\varpi_2)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi_2)}$. We get $\Theta^{\mathscr{T}}_{\mathcal{R}}((\alpha_1\oslash_1\varpi_1),\varrho)\cdot$

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$M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\alpha_1\oslash_1\varpi_1),\varrho)} \succeq \min\{\Theta_{\mathcal{R}}^{\mathscr{T}}(\alpha_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}(\alpha_1)}, \Theta_{\mathcal{R}}^{\mathscr{T}}(\varpi_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}(\varpi_1)}\}$ for all $\alpha_1, \varpi_1 \in \mathscr{S}$,and
$\min\{\Theta_{\mathcal{R}}^{\mathscr{T}}((\alpha_1\oslash_2\varpi_1),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\alpha_1\oslash_2\varpi_1),\varrho)}, \Theta_{\mathcal{R}}^{\mathscr{T}}((\alpha_2\oslash_2\varpi_2),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\alpha_2\oslash_2\varpi_2),\varrho)}\} \succeq \min\{\min\{\Theta_{\mathcal{R}}^{\mathscr{T}}(\alpha_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}(\alpha_1)}, \Theta_{\mathcal{R}}^{\mathscr{T}}(\alpha_2)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}(\alpha_2)}\}, \min\{\Theta_{\mathcal{R}}^{\mathscr{T}}(\varpi_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}(\varpi_1)}, \Theta_{\mathcal{R}}^{\mathscr{T}}(\varpi_2)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}(\varpi_2)}\}\}$
If $\Theta_{\mathcal{R}}^{\mathscr{T}}((\alpha_1\oslash_2\varpi_1),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\alpha_1\oslash_2\varpi_1),\varrho)} \preceq \Theta_{\mathcal{R}}^{\mathscr{T}}((\alpha_2\oslash_2\varpi_2),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\alpha_2\oslash_2\varpi_2),\varrho)}$,then $\Theta_{\mathcal{R}}^{\mathscr{T}}((\alpha_1\oslash_2\varpi_1),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\alpha_1\oslash_2\varpi_1),\varrho)} \succeq \min\{\Theta_{\mathcal{R}}^{\mathscr{T}}(\alpha_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}(\alpha_1)}, \Theta_{\mathcal{R}}^{\mathscr{T}}(\varpi_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}(\varpi_1)}\}$.
$\min\{\Theta_{\mathcal{R}}^{\mathscr{T}}((\alpha_1\oslash_3\varpi_1),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\alpha_1\oslash_3\varpi_1),\varrho)}, \Theta_{\mathcal{R}}^{\mathscr{T}}((\alpha_2\oslash_3\varpi_2),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\alpha_2\oslash_3\varpi_2),\varrho)}\} \succeq \min\{\min\{\Theta_{\mathcal{R}}^{\mathscr{T}}(\alpha_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}(\alpha_1)}, \Theta_{\mathcal{R}}^{\mathscr{T}}(\alpha_2)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}(\alpha_2)}\}, \min\{\Theta_{\mathcal{R}}^{\mathscr{T}}(\varpi_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}(\varpi_1)}, \Theta_{\mathcal{R}}^{\mathscr{T}}(\varpi_2)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}(\varpi_2)}\}\}$
If $\Theta_{\mathcal{R}}^{\mathscr{T}}((\alpha_1\oslash_3\varpi_1),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\alpha_1\oslash_3\varpi_1),\varrho)} \preceq \Theta_{\mathcal{R}}^{\mathscr{T}}((\alpha_2\oslash_3\varpi_2),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\alpha_2\oslash_3\varpi_2),\varrho)}$,then $\Theta_{\mathcal{R}}^{\mathscr{T}}((\alpha_1\oslash_3\varpi_1),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\alpha_1\oslash_3\varpi_1),\varrho)} \succeq \min\{\Theta_{\mathcal{R}}^{\mathscr{T}}(\alpha_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}(\alpha_1)}, \Theta_{\mathcal{R}}^{\mathscr{T}}(\varpi_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}(\varpi_1)}\}$.
Now,

$$\begin{aligned}
&\frac{1}{2}\Big[\Theta_{\mathcal{R}}^{\mathscr{I}}((\alpha_1 \oslash_1 \varpi_1),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}((\alpha_1\oslash_1\varpi_1),\varrho)} + \Theta_{\mathcal{R}}^{\mathscr{I}}((\alpha_2 \oslash_1 \varpi_2),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}((\alpha_2\oslash_1\varpi_2),\varrho)}}\Big] \\
&= \Omega_{\mathcal{R}}^{\mathscr{I}}(\alpha_1 \oslash_1 \varpi_1, \alpha_2 \oslash_1 \varpi_2)\cdot M^{\Omega_{\mathcal{I}}^{\mathscr{I}}(\alpha_1\oslash_1\varpi_1,\alpha_2\oslash_1\varpi_2)} \\
&= \Omega_{\mathcal{R}}^{\mathscr{I}}[((\alpha_1,\alpha_2),\varrho) \oslash_1 ((\varpi_1,\varpi_2),\varrho)]\cdot M^{\Omega_{\mathcal{I}}^{\mathscr{I}}[((\alpha_1,\alpha_2),\varrho)\oslash_1((\varpi_1,\varpi_2),\varrho)]} \\
&= \Omega_{\mathcal{R}}^{\mathscr{I}}((\alpha \oslash_1 \varpi),\varrho)\cdot M^{\Omega_{\mathcal{I}}^{\mathscr{I}}((\alpha\oslash_1\varpi),\varrho)} \\
&\succeq \frac{\Omega_{\mathcal{R}}^{\mathscr{I}}(\alpha,\varrho)\cdot M^{\Omega_{\mathcal{I}}^{\mathscr{I}(\alpha,\varrho)}} + \Omega_{\mathcal{R}}^{\mathscr{I}}(\varpi,\varrho)\cdot M^{\Omega_{\mathcal{I}}^{\mathscr{I}}(\varpi,\varrho)}}{2} \\
&= \frac{\Omega_{\mathcal{R}}^{\mathscr{I}}((\alpha_1,\alpha_2),\varrho)\cdot M^{\Omega_{\mathcal{I}}^{\mathscr{I}}((\alpha_1,\alpha_2),\varrho)} + \Omega_{\mathcal{R}}^{\mathscr{I}}((\varpi_1,\varpi_2),\varrho)\cdot M^{\Omega_{\mathcal{I}}^{\mathscr{I}}}}{2} \\
&= \frac{1}{2}\left[\frac{\Theta_{\mathcal{R}}^{\mathscr{I}}(\alpha_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}(\alpha_1)} + \Theta_{\mathcal{R}}^{\mathscr{I}}(\alpha_2)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}(\alpha_2)}}{2} + \frac{\Theta_{\mathcal{R}}^{\mathscr{I}}(\varpi_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}(\varpi_1)} + \Theta_{\mathcal{R}}^{\mathscr{I}}(\varpi_2)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}(\varpi_2)}}{2}\right]
\end{aligned}$$

If $\Theta_{\mathcal{R}}^{\mathscr{I}}((\alpha_1\oslash_1\varpi_1),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}((\alpha_1\oslash_1\varpi_1),\varrho)} \preceq \Theta_{\mathcal{R}}^{\mathscr{I}}((\alpha_2\oslash_1\varpi_2),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}((\alpha_2\oslash_1\varpi_2),\varrho)}$, then $\Theta_{\mathcal{R}}^{\mathscr{I}}(\alpha_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}(\alpha_1)} \preceq \Theta_{\mathcal{R}}^{\mathscr{I}}(\alpha_2)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}(\alpha_2)}$ and $\Theta_{\mathcal{R}}^{\mathscr{I}}(\varpi_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}(\varpi_1)} \preceq \Theta_{\mathcal{R}}^{\mathscr{I}}(\varpi_2)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}(\varpi_2)}$.
We get $\Theta_{\mathcal{R}}^{\mathscr{I}}((\alpha_1 \oslash_1 \varpi_1),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\alpha_1\oslash_1\varpi_1),\varrho)} \succeq \frac{\Theta_{\mathcal{R}}^{\mathscr{I}}(\alpha_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}(\alpha_1)}+\Theta_{\mathcal{R}}^{\mathscr{I}}(\varpi_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}(\varpi_1)}}{2}$.
Similarly,$\Theta_{\mathcal{R}}^{\mathscr{I}}((\alpha_1 \oslash_2 \varpi_1),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}((\alpha_1\oslash_2\varpi_1),\varrho)} \succeq \frac{\Theta_{\mathcal{R}}^{\mathscr{I}}(\alpha_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}(\alpha_1)}+\Theta_{\mathcal{R}}^{\mathscr{I}}(\varpi_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}(\varpi_1)}}{2}$
and $\Theta_{\mathcal{R}}^{\mathscr{I}}((\alpha_1 \oslash_3 \varpi_1),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}((\alpha_1\oslash_3\varpi_1),\varrho)} \succeq \frac{\Theta_{\mathcal{R}}^{\mathscr{I}}(\alpha_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}(\alpha_1)}+\Theta_{\mathcal{R}}^{\mathscr{I}}(\varpi_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}(\varpi_1)}}{2}$.
Similarly to prove that
$\max\{\Theta_{\mathcal{R}}^{\mathscr{F}}((\alpha_1 \oslash_1 \varpi_1),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}((\alpha_1\oslash_1\varpi_1),\varrho)}, \Theta_{\mathcal{R}}^{\mathscr{F}}((\alpha_2 \oslash_1 \varpi_2),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}((\alpha_2\oslash_1\varpi_2),\varrho)}\}$
$\preceq \max\{\max\{\Theta_{\mathcal{R}}^{\mathscr{F}}(\alpha_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}(\alpha_1)}, \Theta_{\mathcal{R}}^{\mathscr{F}}(\alpha_2)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}(\alpha_2)}\}, \max\{\Theta_{\mathcal{R}}^{\mathscr{F}}(\varpi_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}(\varpi_1)}, \Theta_{\mathcal{R}}^{\mathscr{F}}(\varpi_2)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}(\varpi_2)}\}\}$
If $\Theta_{\mathcal{R}}^{\mathscr{F}}((\alpha_1\oslash_1\varpi_1),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}((\alpha_1\oslash_1\varpi_1),\varrho)} \succeq \Theta_{\mathcal{R}}^{\mathscr{F}}((\alpha_2\oslash_1\varpi_2),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}((\alpha_2\oslash_1\varpi_2),\varrho)}$,then $\Theta_{\mathcal{R}}^{\mathscr{F}}(\alpha_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}(\alpha_1)} \succeq \Theta_{\mathcal{R}}^{\mathscr{F}}(\alpha_2)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}(\alpha_2)}$ and $\Theta_{\mathcal{R}}^{\mathscr{F}}(\varpi_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}(\varpi_1)} \succeq \Theta_{\mathcal{R}}^{\mathscr{F}}(\varpi_2)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}(\varpi_2)}$.
We get $\Theta_{\mathcal{R}}^{\mathscr{F}}((\alpha_1 \oslash_1 \varpi_1),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}((\alpha_1\oslash_1\varpi_1),\varrho)} \preceq \max\{\Theta_{\mathcal{R}}^{\mathscr{F}}(\alpha_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}(\alpha_1)}, \Theta_{\mathcal{R}}^{\mathscr{F}}(\varpi_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}(\varpi_1)}\}$.
$\max\{\Theta_{\mathcal{R}}^{\mathscr{F}}((\alpha_1 \oslash_2 \varpi_1),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}((\alpha_1\oslash_2\varpi_1),\varrho)}, \Theta_{\mathcal{R}}^{\mathscr{F}}((\alpha_2 \oslash_2 \varpi_2),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}((\alpha_2\oslash_2\varpi_2),\varrho)}\}$
$\preceq \max\{\max\{\Theta_{\mathcal{R}}^{\mathscr{F}}(\alpha_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}(\alpha_1)}, \Theta_{\mathcal{R}}^{\mathscr{F}}(\alpha_2)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}(\alpha_2)}\}, \max\{\Theta_{\mathcal{R}}^{\mathscr{F}}(\varpi_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}(\varpi_1)}, \Theta_{\mathcal{R}}^{\mathscr{F}}(\varpi_2)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}}\}\}$
If $\Theta_{\mathcal{R}}^{\mathscr{F}}((\alpha_1\oslash_2\varpi_1),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}((\alpha_1\oslash_2\varpi_1),\varrho)} \succeq \Theta_{\mathcal{R}}^{\mathscr{F}}((\alpha_2\oslash_2\varpi_2),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}((\alpha_2\oslash_2\varpi_2),\varrho)}$,then $\Theta_{\mathcal{R}}^{\mathscr{F}}((\alpha_1\oslash_2\varpi_1),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}((\alpha_1\oslash_2\varpi_1),\varrho)} \preceq \max\{\Theta_{\mathcal{R}}^{\mathscr{F}}(\alpha_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}(\alpha_1)}, \Theta_{\mathcal{R}}^{\mathscr{F}}(\varpi_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}(\varpi_1)}\}$.
$\max\{\Theta_{\mathcal{R}}^{\mathscr{F}}((\alpha_1\oslash_3\varpi_1),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}((\alpha_1\oslash_3\varpi_1),\varrho)}, \Theta_{\mathcal{R}}^{\mathscr{F}}((\alpha_2\oslash_3\varpi_2),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}((\alpha_2\oslash_3\varpi_2),\varrho)}\} \preceq \max\{\max\{\Theta_{\mathcal{R}}^{\mathscr{F}}(\alpha_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}(\alpha_1)}, \Theta_{\mathcal{R}}^{\mathscr{F}}(\alpha_2)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}(\alpha_2)}\}, \max\{\Theta_{\mathcal{R}}^{\mathscr{F}}(\varpi_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}(\varpi_1)}, \Theta_{\mathcal{R}}^{\mathscr{F}}(\varpi_2)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}(\varpi_2)}\}\}$
If $\Theta_{\mathcal{R}}^{\mathscr{F}}((\alpha_1\oslash_3\varpi_1),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\alpha_1\oslash_3\varpi_1),\varrho)} \succeq \Theta_{\mathcal{R}}^{\mathscr{F}}((\alpha_2\oslash_3\varpi_2),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\alpha_2\oslash_3\varpi_2),\varrho)}$,then $\Theta_{\mathcal{R}}^{\mathscr{F}}((\alpha_1\oslash_3\varpi_1),\varrho)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}((\alpha_1\oslash_3\varpi_1),\varrho)} \preceq \max\{\Theta_{\mathcal{R}}^{\mathscr{F}}(\alpha_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}(\alpha_1)}, \Theta_{\mathcal{R}}^{\mathscr{F}}(\varpi_1)\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}(\varpi_1)}\}$.
Therefore, Θ is a CCQNSBS of $\mathscr{S}$.

Theorem 3.12. *Suppose that Θ is a subset of $\mathscr{S}$. Then $R = (\overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}}}}, \overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{I}}}}, \overline{\Theta_{\mathcal{R}}^{\mathscr{F}}}\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{F}}}}, \Theta_{\mathcal{R}}^{\mathscr{T}}\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{T}}}, \Theta_{\mathcal{R}}^{\mathscr{I}}\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{I}}}, \Theta_{\mathcal{R}}^{\mathscr{F}}\cdot M^{\Theta_{\mathcal{I}}^{\mathscr{F}}})$ is a CCQNSBS of $\mathscr{S}$ if and only if $\mathcal{R}^{(\ell,\flat)}$ is a SBS of $\mathscr{S}$ for all $\ell, \flat \in D[0,1]$.*

Proof. Assume that $\overline{\mathcal{R}}$ is a CCQNSBS of $\mathscr{S}$. For each $\ell, \flat \in D[0,1]$ and $\alpha_1, \alpha_2 \in \overline{\mathcal{R}^{(\ell,\flat)}}$. Now,$\overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}(\alpha_1)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}}}}(\alpha_1) \succeq \ell, \overline{\Theta_{\mathcal{R}}^{\mathscr{T}}}(\alpha_2)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{T}}}}(\alpha_2) \succeq \ell$ and $\overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}(\alpha_1)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{I}}}}(\alpha_1) \succeq \ell, \overline{\Theta_{\mathcal{R}}^{\mathscr{I}}}(\alpha_2)\cdot M^{\overline{\Theta_{\mathcal{I}}^{\mathscr{I}}}}(\alpha_2) \succeq \ell$ and

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$\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_1) \cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}}(\alpha_1) \preceq \flat, \overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_2) \cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}}(\alpha_2) \preceq \flat$. Now,$\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \succeq \min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_1) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}}(\alpha_1), \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_2) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}}(\alpha_2)\} \succeq \ell$ and $\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \succeq \frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha_1) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}}(\alpha_1) + \overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha_2) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}}(\alpha_2)}{2} \succeq \frac{\ell+\ell}{2} = \ell$ and $\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \preceq \max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_1) \cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}}(\alpha_1), \overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_2) \cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}}(\alpha_2)\} \preceq \flat$. This implies that $\alpha_1 \oslash_1 \alpha_2 \in \overline{\mathcal{R}^{(\ell,\flat)}}$. Similarly,$\alpha_1 \oslash_2 \alpha_2 \in \overline{\mathcal{R}^{(\ell,\flat)}}$ and $\alpha_1 \oslash_3 \alpha_2 \in \overline{\mathcal{R}^{(\ell,\flat)}}$. Hence,$\overline{\mathcal{R}^{(\ell,\flat)}}$ is a SBS of $\mathscr{S}$,for all $\ell, \flat \in D[0, 1]$.
For each $\ell, \flat \in [0, 1]$ and $\alpha_1, \alpha_2 \in \mathcal{R}^{(\ell,\flat)}$. Now,$\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha_1) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha_1) \succeq \ell, \Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha_2) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha_2) \succeq \ell$ and $\Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha_1) \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha_1) \succeq \ell, \Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha_2) \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha_2) \succeq \ell$ and $\Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha_1) \cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha_1) \preceq \flat, \Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha_2) \cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha_2) \preceq \flat$. Now,$\Theta^{\mathscr{T}}_{\mathcal{R}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \succeq \min\{\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha_1) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha_1), \Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha_2) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha_2)\} \succeq \ell$ and $\Theta^{\mathscr{I}}_{\mathcal{R}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \succeq \frac{\Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha_1) \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha_1) + \Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha_2) \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha_2)}{2} \succeq \frac{\ell+\ell}{2} = \ell$ and $\Theta^{\mathscr{F}}_{\mathcal{R}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \preceq \max\{\Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha_1) \cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha_1), \Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha_2) \cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha_2)\} \preceq \flat$. This implies that $\alpha_1 \oslash_1 \alpha_2 \in \mathcal{R}^{(\ell,\flat)}$. Similarly,$\alpha_1 \oslash_2 \alpha_2 \in \mathcal{R}^{(\ell,\flat)}$ and $\alpha_1 \oslash_3 \alpha_2 \in \mathcal{R}^{(\ell,\flat)}$. Hence,$\mathcal{R}^{(\ell,\flat)}$ is a SBS of $\mathscr{S}$,for all $\ell, \flat \in D[0, 1]$.

Conversely, assume that $\overline{\mathcal{R}^{(\ell,\flat)}}$ is a SBS of $\mathscr{S}$ and $\ell, \flat \in D[0, 1]$. Suppose if there exist $\alpha_1, \alpha_2 \in \mathscr{S}$ such that $\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \prec \min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_1) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}}(\alpha_1), \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_2) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}}(\alpha_2)\}$,$\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \prec \frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha_1) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}}(\alpha_1) + \overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha_2) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}}(\alpha_2)}{2}$ and $\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \succ \max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_1) \cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}}(\alpha_1), \overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_2) \cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}}(\alpha_2)\}$. For $\ell, \flat \in D[0, 1]$ such that $\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \prec \ell \preceq \min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_1) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}}(\alpha_1), \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_2) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}}(\alpha_2)\}$ and $\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \prec \ell \preceq \frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha_1) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}}(\alpha_1) + \overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha_2) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}}(\alpha_2)}{2}$ and $\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \succ \flat \succeq \max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_1) \cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}}(\alpha_1), \overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_2) \cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}}(\alpha_2)\}$. Thus,$\alpha_1, \alpha_2 \in \overline{\mathcal{R}^{(\ell,\flat)}}$,but $\alpha_1 \oslash_1 \alpha_2 \notin \overline{\mathcal{R}^{(\ell,\flat)}}$. This contradicts,$\overline{\mathcal{R}^{(\ell,\flat)}}$ is a SBS of $\mathscr{S}$. Therefore $\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \succeq \min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_1) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}}(\alpha_1), \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha_2) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}}(\alpha_2)\}$, $\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \succeq \frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha_1) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}}(\alpha_1) + \overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha_2) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}}(\alpha_2)}{2}$ and $\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \preceq \max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_1) \cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}}(\alpha_1), \overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha_2) \cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}}(\alpha_2)\}$. Similarly,$\oslash_2$ and $\oslash_3$ cases. Hence $\overline{\mathcal{R}} = (\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}} \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}}, \overline{\Theta^{\mathscr{I}}_{\mathcal{R}}} \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}}, \overline{\Theta^{\mathscr{F}}_{\mathcal{R}}} \cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}})$ is a CCQNSBS of $\mathscr{S}$.
Let us assume that $\mathcal{R}^{(\ell,\flat)}$ is a SBS of $\mathscr{S}$ and $\ell, \flat \in [0, 1]$. Suppose if there exist $\alpha_1, \alpha_2 \in \mathscr{S}$ such that $\Theta^{\mathscr{T}}_{\mathcal{R}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \prec \min\{\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha_1) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha_1), \Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha_2) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha_2)\}$,$\Theta^{\mathscr{I}}_{\mathcal{R}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \prec \frac{\Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha_1) \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha_1) + \Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha_2) \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha_2)}{2}$ and $\Theta^{\mathscr{F}}_{\mathcal{R}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \succ \max\{\Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha_1) \cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha_1), \Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha_2) \cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha_2)\}$. For $\ell, \flat \in D[0, 1]$ such that $\Theta^{\mathscr{T}}_{\mathcal{R}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \prec \ell \preceq \min\{\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha_1) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha_1), \Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha_2) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha_2)\}$ and $\Theta^{\mathscr{I}}_{\mathcal{R}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \prec \ell \preceq \frac{\Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha_1) \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha_1) + \Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha_2) \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha_2)}{2}$ and $\Theta^{\mathscr{F}}_{\mathcal{R}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \succ \flat \succeq \max\{\Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha_1) \cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha_1), \Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha_2) \cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha_2)\}$. Thus,$\alpha_1, \alpha_2 \in \mathcal{R}^{(\ell,\flat)}$,but $\alpha_1 \oslash_1 \alpha_2 \notin \mathcal{R}^{(\ell,\flat)}$. This contradicts,$\mathcal{R}^{(\ell,\flat)}$ is a SBS of $\mathscr{S}$. Therefore $\Theta^{\mathscr{T}}_{\mathcal{R}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \succeq \min\{\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha_1) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha_1), \Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha_2) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha_2)\}$, $\Theta^{\mathscr{I}}_{\mathcal{R}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \succeq \frac{\Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha_1) \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha_1) + \Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha_2) \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha_2)}{2}$ and $\Theta^{\mathscr{F}}_{\mathcal{R}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}}((\alpha_1 \oslash_1 \alpha_2), \varrho) \preceq \max\{\Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha_1) \cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha_1), \Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha_2) \cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha_2)\}$. Similarly,$\oslash_2$ and $\oslash_3$ cases. Hence $\mathcal{R} = (\Theta^{\mathscr{T}}_{\mathcal{R}} \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}}, \Theta^{\mathscr{I}}_{\mathcal{R}} \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}}, \Theta^{\mathscr{F}}_{\mathcal{R}} \cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}})$ is a CCQNSBS of $\mathscr{S}$.

4 Homomorphism of complex cubic Q neutrosophic SBS

Definition 4.1. Let $(\mathscr{S}_1, \triangle_1, \triangle_2, \triangle_3)$ and $(\mathscr{S}_2, \bigtriangledown_1, \bigtriangledown_2, \bigtriangledown_3)$ be any two bisemirings. The mapping $\mathcal{F} : \mathscr{S}_1 \rightarrow \mathscr{S}_2$ and Θ be any CCQNSBS in $\mathscr{S}_1$, Ω be any CCQNSBS in $\mathcal{F}(\mathscr{S}_1) = \mathscr{S}_2$. If $\Theta_{\mathcal{R}} \cdot M^{\Theta_{\mathcal{I}}} = [\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}} \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}}, \overline{\Theta^{\mathscr{I}}_{\mathcal{R}}} \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}}, \overline{\Theta^{\mathscr{F}}_{\mathcal{R}}} \cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}}, \Theta_{\mathcal{R}} \cdot M^{\Theta_{\mathcal{I}}}, \Theta^{\mathscr{T}}_{\mathcal{R}} \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}}, \Theta^{\mathscr{I}}_{\mathcal{R}} \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}}, \Theta^{\mathscr{F}}_{\mathcal{R}} \cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}}$ is a CCQNS in $\mathscr{S}_1$, then $\Omega_{\mathcal{R}}$ is a CCQNS in $\mathscr{S}_2$,defined by

$$\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}(\varpi, \varrho) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}}(\varpi, \varrho) = \begin{cases} \bigvee \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha, \varrho) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}}(\alpha, \varrho) & \mathrm{i}f \ \ \alpha \in \mathcal{F}^{-1}\varpi \\ 0 & \mathrm{o}therwise \end{cases}$$

$$\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}}(\varpi,\varrho)=\begin{cases}\bigvee\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}}(\alpha,\varrho) & if\ \ \alpha\in\mathcal{F}^{-1}\varpi\\ 0 & otherwise\end{cases}$$

$$\overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}}(\varpi,\varrho)=\begin{cases}\bigwedge\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}}(\alpha,\varrho) & if\ \ \alpha\in\mathcal{F}^{-1}\varpi\\ 1 & otherwise\end{cases}$$

$$\Omega^{\mathscr{T}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\varpi,\varrho)=\begin{cases}\bigvee\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha,\varrho) & if\ \ \alpha\in\mathcal{F}^{-1}\varpi\\ 0 & otherwise\end{cases}$$

$$\Omega^{\mathscr{I}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\varpi,\varrho)=\begin{cases}\bigvee\Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha,\varrho) & if\ \ \alpha\in\mathcal{F}^{-1}\varpi\\ 0 & otherwise\end{cases}$$

$$\Omega^{\mathscr{F}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\varpi,\varrho)=\begin{cases}\bigwedge\Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha,\varrho) & if\ \ \alpha\in\mathcal{F}^{-1}\varpi\\ 1 & otherwise\end{cases}$$

for all $\alpha\in\mathscr{S}_1$ and $\varpi\in\mathscr{S}_2$ is represents the image of R_Θ under $\mathcal{F}$.
Similarly,If $\Omega_{\mathcal{R}}\cdot M^{\Omega_{\mathcal{I}}}=[\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}},\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}},\overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}\cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}}],\Omega_{\mathcal{R}}\cdot M^{\Omega_{\mathcal{I}}},\Omega^{\mathscr{T}}_{\mathcal{R}}\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}},\Omega^{\mathscr{I}}_{\mathcal{R}}\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}},\Omega^{\mathscr{F}}_{\mathcal{R}}\cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}}$ is a CCQNS in $\mathscr{S}_2$,then CCQNS $\Theta_{\mathcal{R}}=\mathcal{F}\circ\Omega_{\mathcal{R}}$ in $\mathscr{S}_1$ ie,the CCQNS defined by $\Theta_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta_{\mathcal{I}}(\alpha,\varrho)},\Omega_{\mathcal{R}}(\mathcal{F}(\alpha,\varrho))\cdot M^{\Theta_{\mathcal{I}}(\mathcal{F}(\alpha,\varrho))}$, $\Theta_{\mathcal{R}}=\mathcal{F}\circ\Omega_{\mathcal{R}}$ in $\mathscr{S}_1$ [ie,the CCQNS defined by $\Theta_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta_{\mathcal{I}}(\alpha,\varrho)}=\Omega_{\mathcal{R}}(\mathcal{F}(\alpha,\varrho))\cdot M^{\Theta_{\mathcal{I}}(\mathcal{F}(\alpha,\varrho))}$ is represents the preimage of $\Omega_{\mathcal{R}}$ under $\mathcal{F}$.

Theorem 4.2. *Let* $(\mathscr{S}_1,\triangle_1,\triangle_2,\triangle_3)$ *and* $(\mathscr{S}_2,\triangledown_1,\triangledown_2,\triangledown_3)$ *be any two bisemirings. To prove that all homomorphic image of CCQNSBS is also a CCQNSBS.*

Proof. The mapping $\mathcal{F}:\mathscr{S}_1\to\mathscr{S}_2$ be any homomorphism. Now, $\mathcal{F}((\alpha\triangle_1\varpi),\varrho)=\mathcal{F}(\alpha,\varrho)\triangledown_1\mathcal{F}(\varpi,\varrho),\mathcal{F}((\alpha\triangle_2\varpi),\varrho)=\mathcal{F}(\alpha,\varrho)\triangledown_2\mathcal{F}(\varpi,\varrho)$ and $\mathcal{F}((\alpha\triangle_3\varpi),\varrho)=\mathcal{F}(\alpha,\varrho)\triangledown_3\mathcal{F}(\varpi,\varrho)$ for all $\alpha,\varpi\in\mathscr{S}_1$. Let $\overline{\Omega}=\mathcal{F}(\Theta)$,$\Theta$ is any CCQNSBS of $\mathscr{S}_1$. Let $\mathcal{F}(\alpha,\varrho),\mathcal{F}(\varpi,\varrho)\in\mathscr{S}_2$. Let $\alpha\in u\mathcal{F}^{-1}(\mathcal{F}(\alpha,\varrho))$ and $\varpi\in\mathcal{F}^{-1}(\mathcal{F}(\varpi,\varrho))$ be such that $\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha,\varrho)}=\bigvee\limits_{\alpha\in\mathcal{F}^{-1}(\mathcal{F}(\alpha,\varrho))}\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha,\varrho)}$ and $\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\varpi,\varrho)}=\bigvee\limits_{\alpha\in\mathcal{F}^{-1}(\mathcal{F}(\varpi,\varrho))}\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha,\varrho)}$.
Now, $\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}((\mathcal{F}(\alpha,\varrho)\triangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}((\mathcal{F}(\alpha,\varrho)\triangledown_1\mathcal{F}(\varpi,\varrho)))}$

$$\begin{aligned}&=\bigvee_{(\alpha^{'})\in\mathcal{F}^{-1}(\mathcal{F}(\alpha,\varrho)\triangledown_1\mathcal{F}(\varpi,\varrho))}\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha^{'})\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha^{'})}\\&=\bigvee_{(\alpha^{'})\in\mathcal{F}^{-1}(\mathcal{F}((\alpha\triangle_1\varpi),\varrho)}\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha^{'})\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha^{'})}\\&=\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha\triangle_1\varpi),\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}((\alpha\triangle_1\varpi),\varrho)}\\&\succeq\min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha,\varrho)},\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\varpi,\varrho)}\}\\&=\min\{\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}\mathcal{F}(\alpha,\varrho)\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}\mathcal{F}(\alpha,\varrho)},\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}\mathcal{F}(\varpi,\varrho)\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}\mathcal{F}(\varpi,\varrho)}\}.\end{aligned}$$

Thus,$\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}((\mathcal{F}(\alpha,\varrho)\triangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}((\mathcal{F}(\alpha,\varrho)\triangledown_1\mathcal{F}(\varpi,\varrho)))}\succeq\min\{\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}\mathcal{F}(\alpha,\varrho)\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}\mathcal{F}(\alpha,\varrho)},\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}\mathcal{F}(\varpi,\varrho)\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}\mathcal{F}(\varpi,\varrho)}\}$.
Similarly,$\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}((\mathcal{F}(\alpha,\varrho)\triangledown_2\mathcal{F}(\varpi,\varrho)))\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}((\mathcal{F}(\alpha,\varrho)\triangledown_2\mathcal{F}(\varpi,\varrho)))}\succeq\min\{\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}\mathcal{F}(\alpha,\varrho)\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}\mathcal{F}(\alpha,\varrho)},\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}\mathcal{F}(\varpi,\varrho)\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}\mathcal{F}(\varpi,\varrho)}\}$ and
$\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}((\mathcal{F}(\alpha,\varrho)\triangledown_3\mathcal{F}(\varpi,\varrho)))\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}((\mathcal{F}(\alpha,\varrho)\triangledown_3\mathcal{F}(\varpi,\varrho)))}\succeq\min\{\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}\mathcal{F}(\alpha,\varrho)\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}\mathcal{F}(\alpha,\varrho)},\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}\mathcal{F}(\varpi,\varrho)\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}\mathcal{F}(\varpi,\varrho)}\}$.
Let $\alpha\in\mathcal{F}^{-1}(\mathcal{F}(\alpha,\varrho))$ and $\varpi\in\mathcal{F}^{-1}(\mathcal{F}(\varpi,\varrho))$ be such that $\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}(\alpha,\varrho)}=\bigvee\limits_{\alpha\in\mathcal{F}^{-1}(\mathcal{F}(\alpha,\varrho))}\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}(\alpha,\varrho)}$ and $\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}(\varpi,\varrho)}=\bigvee\limits_{\alpha\in\mathcal{F}^{-1}(\mathcal{F}(\varpi,\varrho))}\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}\mathcal{F}(\alpha,\varrho)}$.

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Now, $\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))}$

$$\begin{aligned}
&=\bigvee_{(\alpha^{'})\in\mathcal{F}^{-1}(\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho))}\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha^{'})\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha^{'})}\\
&=\bigvee_{(\alpha^{'})\in\mathcal{F}^{-1}(\mathcal{F}((\alpha\bigtriangleup_1\varpi),\varrho)}\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha^{'})\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha^{'})}\\
&=\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}((\alpha\bigtriangleup_1\varpi),\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}((\alpha\bigtriangleup_1\varpi),\varrho)}\\
&\succeq\frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha,\varrho)}+\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\varpi,\varrho)}}{2}\\
&=\frac{\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}\mathcal{F}(\alpha,\varrho)\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}\mathcal{F}(\alpha,\varrho)}+\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}\mathcal{F}(\varpi,\varrho)\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}\mathcal{F}(\varpi,\varrho)}}{2}.
\end{aligned}$$

Thus,$\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))}\succeq\frac{\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}\mathcal{F}(\alpha,\varrho)\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}\mathcal{F}(\alpha,\varrho)}+\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}\mathcal{F}(\varpi,\varrho)\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}\mathcal{F}(\varpi,\varrho)}}{2}$.
Similarly,$\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_2\mathcal{F}(\varpi,\varrho)))\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_2\mathcal{F}(\varpi,\varrho)))}\succeq\frac{\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}\mathcal{F}(\alpha,\varrho)\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}\mathcal{F}(\alpha,\varrho)}+\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}\mathcal{F}(\varpi,\varrho)\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}\mathcal{F}(\varpi,\varrho)}}{2}$
and
$\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_3\mathcal{F}(\varpi,\varrho)))\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_3\mathcal{F}(\varpi,\varrho)))}\succeq\frac{\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}\mathcal{F}(\alpha,\varrho)\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}\mathcal{F}(\alpha,\varrho)}+\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}\mathcal{F}(\varpi,\varrho)\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}\mathcal{F}(\varpi,\varrho)}}{2}$.
Let $\mathcal{F}(\alpha,\varrho),\mathcal{F}(\varpi,\varrho)\in\mathscr{S}_2$. Let $\alpha\in\mathcal{F}^{-1}(\mathcal{F}(\alpha,\varrho))$ and $\varpi\in\mathcal{F}^{-1}(\mathcal{F}(\varpi,\varrho))$ be such that $\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha,\varrho)}=\bigwedge_{\alpha\in\mathcal{F}^{-1}(\mathcal{F}(\alpha,\varrho))}\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha,\varrho)}$ and $\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\varpi,\varrho)}=\bigwedge_{\alpha\in\mathcal{F}^{-1}(\mathcal{F}(\varpi,\varrho))}\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha,\varrho)}$.
Now, $\overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))}$

$$\begin{aligned}
&=\bigwedge_{(\alpha^{'})\in\mathcal{F}^{-1}(\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho))}\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha^{'})\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha^{'})}\\
&=\bigwedge_{(\alpha^{'})\in\mathcal{F}^{-1}(\mathcal{F}((\alpha\bigtriangleup_1\varpi),\varrho)}\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha^{'})\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha^{'})}\\
&=\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha\bigtriangleup_1\varpi),\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}((\alpha\bigtriangleup_1\varpi),\varrho)}\\
&\preceq\max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha,\varrho)},\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\varpi,\varrho)}\}\\
&=\max\{\overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}\mathcal{F}(\alpha,\varrho)\cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}\mathcal{F}(\alpha,\varrho)},\overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}\mathcal{F}(\varpi,\varrho)\cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}\mathcal{F}(\varpi,\varrho)}\}.
\end{aligned}$$

Thus,$\overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))}\preceq\max\{\overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}\mathcal{F}(\alpha,\varrho)\cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}\mathcal{F}(\alpha,\varrho)},\overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}\mathcal{F}(\varpi,\varrho)\cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}\mathcal{F}(\varpi,\varrho)}\}$.
Similarly,$\overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_2\mathcal{F}(\varpi,\varrho)))\cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_2\mathcal{F}(\varpi,\varrho)))}\preceq\max\{\overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}\mathcal{F}(\alpha,\varrho)\cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}\mathcal{F}(\alpha,\varrho)},\overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}\mathcal{F}(\varpi,\varrho)\cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}\mathcal{F}(\varpi,\varrho)}\}$ and $\overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_3\mathcal{F}(\varpi,\varrho)))\cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))}\preceq\max\{\overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}\mathcal{F}(\alpha,\varrho)\cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}\mathcal{F}(\varpi,\varrho)},\overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}\mathcal{F}(\varpi,\varrho)\cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}\mathcal{F}(\varpi,\varrho)}\}$.

The mapping $\mathcal{F}:\mathscr{S}_1\to\mathscr{S}_2$ be any homomorphism. Now, $\mathcal{F}((\alpha\bigtriangleup_1\varpi),\varrho)=\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho),\mathcal{F}((\alpha\bigtriangleup_2\varpi),\varrho)=\mathcal{F}(\alpha,\varrho)\bigtriangledown_2\mathcal{F}(\varpi,\varrho)$ and $\mathcal{F}((\alpha\bigtriangleup_3\varpi),\varrho)=\mathcal{F}(\alpha,\varrho)\bigtriangledown_3\mathcal{F}(\varpi,\varrho)$ for all $\alpha,\varpi\in\mathscr{S}_1$. Let $\Omega=\mathcal{F}(\Theta)$,$\Theta$ is any CCQNSBS of $\mathscr{S}_1$. Let $\mathcal{F}(\alpha,\varrho),\mathcal{F}(\varpi,\varrho)\in\mathscr{S}_2$. Let $\alpha\in\mathcal{F}^{-1}(\mathcal{F}(\alpha,\varrho))$ and $\varpi\in\mathcal{F}^{-1}(\mathcal{F}(\varpi,\varrho))$ be such that $\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha,\varrho)}=\bigvee_{\alpha\in\mathcal{F}^{-1}(\mathcal{F}(\alpha,\varrho))}\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha,\varrho)}$ and $\Theta^{\mathscr{T}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi,\varrho)}=\bigvee_{\alpha\in\mathcal{F}^{-1}(\mathcal{F}(\varpi,\varrho))}\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha,\varrho)}$.

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Now, $\Omega^{\mathscr{T}}_{\mathcal{R}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))}$

$$\begin{aligned}
&= \bigvee_{(\alpha^{'})\in\mathcal{F}^{-1}(\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho))} \Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha^{'})\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha^{'})}\\
&= \bigvee_{(\alpha^{'})\in\mathcal{F}^{-1}(\mathcal{F}((\alpha\triangle_1\varpi),\varrho)} \Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha^{'})\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha^{'})}\\
&= \Theta^{\mathscr{T}}_{\mathcal{R}}((\alpha\triangle_1\varpi),\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha\triangle_1\varpi),\varrho)}\\
&\succeq \min\{\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha,\varrho)},\Theta^{\mathscr{T}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi,\varrho)}\}\\
&= \min\{\Omega^{\mathscr{T}}_{\mathcal{R}}\mathcal{F}(\alpha,\varrho)\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}\mathcal{F}(\alpha,\varrho)},\Omega^{\mathscr{T}}_{\mathcal{R}}\mathcal{F}(\varpi,\varrho)\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}\mathcal{F}(\varpi,\varrho)}\}.
\end{aligned}$$

Thus,$\Omega^{\mathscr{T}}_{\mathcal{R}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))}\succeq\min\{\Omega^{\mathscr{T}}_{\mathcal{R}}\mathcal{F}(\alpha,\varrho)\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}\mathcal{F}(\alpha,\varrho)},\Omega^{\mathscr{T}}_{\mathcal{R}}\mathcal{F}(\varpi,\varrho)\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}\mathcal{F}(\varpi,\varrho)}\}$.
Similarly,$\Omega^{\mathscr{T}}_{\mathcal{R}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_2\mathcal{F}(\varpi,\varrho)))\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_2\mathcal{F}(\varpi,\varrho)))}\succeq\min\{\Omega^{\mathscr{T}}_{\mathcal{R}}\mathcal{F}(\alpha,\varrho)\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}\mathcal{F}(\alpha,\varrho)},\Omega^{\mathscr{T}}_{\mathcal{R}}\mathcal{F}(\varpi,\varrho)\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}\mathcal{F}(\varpi,\varrho)}\}$ and
$\Omega^{\mathscr{T}}_{\mathcal{R}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_3\mathcal{F}(\varpi,\varrho)))\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_3\mathcal{F}(\varpi,\varrho)))}\succeq\min\{\Omega^{\mathscr{T}}_{\mathcal{R}}\mathcal{F}(\alpha,\varrho)\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}\mathcal{F}(\alpha,\varrho)},\Omega^{\mathscr{T}}_{\mathcal{R}}\mathcal{F}(\varpi,\varrho)\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}\mathcal{F}(\varpi,\varrho)}\}$.
Let $\alpha\in\mathcal{F}^{-1}(\mathcal{F}(\alpha,\varrho))$ and $\varpi\in\mathcal{F}^{-1}(\mathcal{F}(\varpi,\varrho))$ be such that $\Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}(\alpha,\varrho)}=\bigvee_{\alpha\in\mathcal{F}^{-1}(\mathcal{F}(\alpha,\varrho))}\Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}(\alpha,\varrho)}$ and $\Theta^{\mathscr{I}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}(\varpi,\varrho)}=\bigvee_{\alpha\in\mathcal{F}^{-1}(\mathcal{F}(\varpi,\varrho))}\Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}\mathcal{F}(\alpha,\varrho)}$.
Now, $\Omega^{\mathscr{I}}_{\mathcal{R}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))}$

$$\begin{aligned}
&= \bigvee_{(\alpha^{'})\in\mathcal{F}^{-1}(\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho))} \Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha^{'})\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha^{'})}\\
&= \bigvee_{(\alpha^{'})\in\mathcal{F}^{-1}(\mathcal{F}((\alpha\triangle_1\varpi),\varrho)} \Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha^{'})\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha^{'})}\\
&= \Theta^{\mathscr{I}}_{\mathcal{R}}((\alpha\triangle_1\varpi),\varrho)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}((\alpha\triangle_1\varpi),\varrho)}\\
&\succeq \frac{\Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha,\varrho)}+\Theta^{\mathscr{I}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\varpi,\varrho)}}{2}\\
&= \frac{\Omega^{\mathscr{I}}_{\mathcal{R}}\mathcal{F}(\alpha,\varrho)\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}\mathcal{F}(\alpha,\varrho)}+\Omega^{\mathscr{I}}_{\mathcal{R}}\mathcal{F}(\varpi,\varrho)\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}\mathcal{F}(\varpi,\varrho)}}{2}.
\end{aligned}$$

Thus,$\Omega^{\mathscr{I}}_{\mathcal{R}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))}\succeq\frac{\Omega^{\mathscr{I}}_{\mathcal{R}}\mathcal{F}(\alpha,\varrho)\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}\mathcal{F}(\alpha,\varrho)}+\Omega^{\mathscr{I}}_{\mathcal{R}}\mathcal{F}(\varpi,\varrho)\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}\mathcal{F}(\varpi,\varrho)}}{2}$.
Similarly,$\Omega^{\mathscr{I}}_{\mathcal{R}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_2\mathcal{F}(\varpi,\varrho)))\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_2\mathcal{F}(\varpi,\varrho)))}\succeq\frac{\Omega^{\mathscr{I}}_{\mathcal{R}}\mathcal{F}(\alpha,\varrho)\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}\mathcal{F}(\alpha,\varrho)}+\Omega^{\mathscr{I}}_{\mathcal{R}}\mathcal{F}(\varpi,\varrho)\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}\mathcal{F}(\varpi,\varrho)}}{2}$
and
$\Omega^{\mathscr{I}}_{\mathcal{R}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_3\mathcal{F}(\varpi,\varrho)))\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_3\mathcal{F}(\varpi,\varrho)))}\succeq\frac{\Omega^{\mathscr{I}}_{\mathcal{R}}\mathcal{F}(\alpha,\varrho)\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}\mathcal{F}(\alpha,\varrho)}+\Omega^{\mathscr{I}}_{\mathcal{R}}\mathcal{F}(\varpi,\varrho)\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}\mathcal{F}(\varpi,\varrho)}}{2}$.
Let $\mathcal{F}(\alpha,\varrho),\mathcal{F}(\varpi,\varrho)\in\mathscr{S}_2$. Let $\alpha\in\mathcal{F}^{-1}(\mathcal{F}(\alpha,\varrho))$ and $\varpi\in\mathcal{F}^{-1}(\mathcal{F}(\varpi,\varrho))$ be such that $\Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha,\varrho)}=\bigwedge_{\alpha\in\mathcal{F}^{-1}(\mathcal{F}(\alpha,\varrho))}\Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha,\varrho)}$ and $\Theta^{\mathscr{F}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\varpi,\varrho)}=\bigwedge_{\alpha\in\mathcal{F}^{-1}(\mathcal{F}(\varpi,\varrho))}\Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha,\varrho)}$.
Now, $\Omega^{\mathscr{F}}_{\mathcal{R}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))}$

$$\begin{aligned}
&= \bigwedge_{(\alpha^{'})\in\mathcal{F}^{-1}(\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho))} \Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha^{'})\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha^{'})}\\
&= \bigwedge_{(\alpha^{'})\in\mathcal{F}^{-1}(\mathcal{F}((\alpha\triangle_1\varpi),\varrho)} \Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha^{'})\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha^{'})}\\
&= \Theta^{\mathscr{F}}_{\mathcal{R}}((\alpha\triangle_1\varpi),\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}((\alpha\triangle_1\varpi),\varrho)}\\
&\preceq \max\{\Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha,\varrho)},\Theta^{\mathscr{F}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\varpi,\varrho)}\}\\
&= \max\{\Omega^{\mathscr{F}}_{\mathcal{R}}\mathcal{F}(\alpha,\varrho)\cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}\mathcal{F}(\alpha,\varrho)},\Omega^{\mathscr{F}}_{\mathcal{R}}\mathcal{F}(\varpi,\varrho)\cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}\mathcal{F}(\varpi,\varrho)}\}.
\end{aligned}$$

Thus,$\Omega^{\mathscr{F}}_{\mathcal{R}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))}\preceq\max\{\Omega^{\mathscr{F}}_{\mathcal{R}}\mathcal{F}(\alpha,\varrho)\cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}\mathcal{F}(\alpha,\varrho)},\Omega^{\mathscr{F}}_{\mathcal{R}}\mathcal{F}(\varpi,\varrho)\cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}\mathcal{F}(\varpi,\varrho)}\}$.

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Similarly,$\Omega^{\mathscr{F}}_{\mathcal{R}}((\mathcal{F}(\alpha,\varrho)\triangledown_2\mathcal{F}(\varpi,\varrho)))\cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}((\mathcal{F}(\alpha,\varrho)\triangledown_2\mathcal{F}(\varpi,\varrho)))} \preceq \max\{\Omega^{\mathscr{F}}_{\mathcal{R}}\mathcal{F}(\alpha,\varrho)\cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}\mathcal{F}(\alpha,\varrho)}, \Omega^{\mathscr{F}}_{\mathcal{R}}\mathcal{F}(\varpi,\varrho)\cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}\mathcal{F}(\varpi,\varrho)}\}$ and
$\Omega^{\mathscr{F}}_{\mathcal{R}}((\mathcal{F}(\alpha,\varrho)\triangledown_3\mathcal{F}(\varpi,\varrho)))\cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}((\mathcal{F}(\alpha,\varrho)\triangledown_1\mathcal{F}(\varpi,\varrho)))} \preceq \max\{\Omega^{\mathscr{F}}_{\mathcal{R}}\mathcal{F}(\alpha,\varrho)\cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}\mathcal{F}(\varpi,\varrho)}, \Omega^{\mathscr{F}}_{\mathcal{R}}\mathcal{F}(\varpi,\varrho)\cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}\mathcal{F}(\varpi,\varrho)}\}$.
Thus,Ω is a CCQNSBS of $\mathscr{S}_2$.

Theorem 4.3. *Let $(\mathscr{S}_1,\triangle_1,\triangle_2,\triangle_3)$ and $(\mathscr{S}_2,\triangledown_1,\triangledown_2,\triangledown_3)$ be any two bisemirings. To prove that homomorphic preimage of CCQNSBS is also a CCQNSBS.*

Proof. The mapping $\mathcal{F}:\mathscr{S}_1\to\mathscr{S}_2$ be a homomorphism. Now,$\mathcal{F}((\alpha\triangle_1\varpi),\varrho)=\mathcal{F}(\alpha,\varrho)\triangledown_1\mathcal{F}(\varpi,\varrho), \mathcal{F}((\alpha\triangle_2\varpi),\varrho)=\mathcal{F}(\alpha,\varrho)\triangledown_2\mathcal{F}(\varpi,\varrho)$ and $\mathcal{F}((\alpha\triangle_3\varpi),\varrho)=\mathcal{F}(\alpha,\varrho)\triangledown_3\mathcal{F}(\varpi,\varrho)$ for all $\alpha,\varpi\in\mathscr{S}_1$. Let $\overline{\Omega}=\mathcal{F}(\Theta)$,$\overline{\Omega}$ is a CCQNSBS of $\mathscr{S}_2$. Let $\alpha,\varpi\in\mathscr{S}_1$. Now,$\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha\triangle_1\varpi),\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}((\alpha\triangle_1\varpi),\varrho)} = \overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}(\mathcal{F}((\alpha\triangle_1\varpi),\varrho))\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}(\mathcal{F}((\alpha\triangle_1\varpi),\varrho))} = \overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}((\mathcal{F}(\alpha,\varrho)\triangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}((\mathcal{F}(\alpha,\varrho)\triangledown_1\mathcal{F}(\varpi,\varrho)))} \succeq \min\{\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}\mathcal{F}(\alpha,\varrho)\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}\mathcal{F}(\alpha,\varrho)}, \overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}\mathcal{F}(\varpi,\varrho)\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}\mathcal{F}(\varpi,\varrho)}\} = \min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha,\varrho)}, \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\varpi,\varrho)}\}$. Thus,$\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha\triangle_1\varpi),\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}((\alpha\triangle_1\varpi),\varrho)} \succeq \min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha)}, \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\varpi,\varrho)}\}$. Now,$\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}((\alpha\triangle_1\varpi),\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}((\alpha\triangle_1\varpi),\varrho)} = \overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}(\mathcal{F}((\alpha\triangle_1\varpi),\varrho))\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}((\alpha\triangle_1\varpi),\varrho)} = \overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}((\mathcal{F}(\alpha,\varrho)\triangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}((\alpha\triangle_1\varpi),\varrho)} \succeq \frac{\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}\mathcal{F}(\alpha,\varrho)\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}\mathcal{F}(\alpha,\varrho)}+\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}\mathcal{F}(\varpi,\varrho)\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}\mathcal{F}(\varpi,\varrho)}}{2} = \frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha,\varrho)}+\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\varpi,\varrho)}}{2}$.
Thus,$\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}((\alpha\triangle_1\varpi),\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}((\alpha\triangle_1\varpi),\varrho)} \succeq \frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha,\varrho)}+\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\varpi,\varrho)}}{2}$.
Now,$\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha\triangle_1\varpi),\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}((\alpha\triangle_1\varpi),\varrho)} = \overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}(\mathcal{F}((\alpha\triangle_1\varpi),\varrho))\cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}(\mathcal{F}((\alpha\triangle_1\varpi),\varrho))} = \overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}((\mathcal{F}(\alpha,\varrho)\triangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}((\mathcal{F}(\alpha,\varrho)\triangledown_1\mathcal{F}(\varpi,\varrho)))} \preceq \max\{\overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}\mathcal{F}(\alpha,\varrho)\cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}\mathcal{F}(\alpha,\varrho)}, \overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}\mathcal{F}(\varpi,\varrho)\cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}\mathcal{F}(\varpi,\varrho)}\} = \max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha,\varrho)}, \overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\varpi,\varrho)}\}$.
Thus,$\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha\triangle_1\varpi),\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}((\alpha\triangle_1\varpi),\varrho)} \preceq \max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha,\varrho)}, \overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\varpi,\varrho)}\}$.
The mapping $\mathcal{F}:\mathscr{S}_1\to\mathscr{S}_2$ be a homomorphism. Now,$\mathcal{F}((\alpha\triangle_1\varpi),\varrho)=\mathcal{F}(\alpha,\varrho)\triangledown_1\mathcal{F}(\varpi,\varrho), \mathcal{F}((\alpha\triangle_2\varpi),\varrho)=\mathcal{F}(\alpha,\varrho)\triangledown_2\mathcal{F}(\varpi,\varrho)$ and $\mathcal{F}((\alpha\triangle_3\varpi),\varrho)=\mathcal{F}(\alpha,\varrho)\triangledown_3\mathcal{F}(\varpi,\varrho)$ for all $\alpha,\varpi\in\mathscr{S}_1$. Let $\Omega=\mathcal{F}(\Theta)$,$\Omega$ is a CCQNSBS of $\mathscr{S}_2$. Let $\alpha,\varpi\in\mathscr{S}_1$. Now,$\Theta^{\mathscr{T}}_{\mathcal{R}}((\alpha\triangle_1\varpi),\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha\triangle_1\varpi),\varrho)} = \Omega^{\mathscr{T}}_{\mathcal{R}}(\mathcal{F}((\alpha\triangle_1\varpi),\varrho))\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}(\mathcal{F}((\alpha\triangle_1\varpi),\varrho))} = \Omega^{\mathscr{T}}_{\mathcal{R}}((\mathcal{F}(\alpha,\varrho)\triangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}((\mathcal{F}(\alpha,\varrho)\triangledown_1\mathcal{F}(\varpi,\varrho)))} \succeq \min\{\Omega^{\mathscr{T}}_{\mathcal{R}}\mathcal{F}(\alpha,\varrho)\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}\mathcal{F}(\alpha,\varrho)}, \Omega^{\mathscr{T}}_{\mathcal{R}}\mathcal{F}(\varpi,\varrho)\cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}\mathcal{F}(\varpi,\varrho)}\} = \min\{\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha,\varrho)}, \Theta^{\mathscr{T}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi,\varrho)}\}$.
Thus,$\Theta^{\mathscr{T}}_{\mathcal{R}}((\alpha\triangle_1\varpi),\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha\triangle_1\varpi),\varrho)} \succeq \min\{\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha)}, \Theta^{\mathscr{T}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi,\varrho)}\}$.
Now,$\Theta^{\mathscr{I}}_{\mathcal{R}}((\alpha\triangle_1\varpi),\varrho)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}((\alpha\triangle_1\varpi),\varrho)} = \Omega^{\mathscr{I}}_{\mathcal{R}}(\mathcal{F}((\alpha\triangle_1\varpi),\varrho))\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}((\alpha\triangle_1\varpi),\varrho)} = \Omega^{\mathscr{I}}_{\mathcal{R}}((\mathcal{F}(\alpha,\varrho)\triangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}((\alpha\triangle_1\varpi),\varrho)} \succeq \frac{\Omega^{\mathscr{I}}_{\mathcal{R}}\mathcal{F}(\alpha,\varrho)\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}\mathcal{F}(\alpha,\varrho)}+\Omega^{\mathscr{I}}_{\mathcal{R}}\mathcal{F}(\varpi,\varrho)\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}\mathcal{F}(\varpi,\varrho)}}{2} = \frac{\Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha,\varrho)}+\Theta^{\mathscr{I}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\varpi,\varrho)}}{2}$.
Thus,$\Theta^{\mathscr{I}}_{\mathcal{R}}((\alpha\triangle_1\varpi),\varrho)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}((\alpha\triangle_1\varpi),\varrho)} \succeq \frac{\Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha,\varrho)}+\Theta^{\mathscr{I}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\varpi,\varrho)}}{2}$.
Now,$\Theta^{\mathscr{F}}_{\mathcal{R}}((\alpha\triangle_1\varpi),\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}((\alpha\triangle_1\varpi),\varrho)} = \Omega^{\mathscr{F}}_{\mathcal{R}}(\mathcal{F}((\alpha\triangle_1\varpi),\varrho))\cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}(\mathcal{F}((\alpha\triangle_1\varpi),\varrho))} = \Omega^{\mathscr{F}}_{\mathcal{R}}((\mathcal{F}(\alpha,\varrho)\triangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}((\mathcal{F}(\alpha,\varrho)\triangledown_1\mathcal{F}(\varpi,\varrho)))} \preceq \max\{\Omega^{\mathscr{F}}_{\mathcal{R}}\mathcal{F}(\alpha,\varrho)\cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}\mathcal{F}(\alpha,\varrho)}, \Omega^{\mathscr{F}}_{\mathcal{R}}\mathcal{F}(\varpi,\varrho)\cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}\mathcal{F}(\varpi,\varrho)}\} = \max\{\Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha,\varrho)}, \Theta^{\mathscr{F}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\varpi,\varrho)}\}$.
Thus,$\Theta^{\mathscr{F}}_{\mathcal{R}}((\alpha\triangle_1\varpi),\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}((\alpha\triangle_1\varpi),\varrho)} \preceq \max\{\Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha,\varrho)}, \Theta^{\mathscr{F}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\varpi,\varrho)}\}$.

Theorem 4.4. *Let $(\mathscr{S}_1,\triangle_1,\triangle_2,\triangle_3)$ and $(\mathscr{S}_2,\triangledown_1,\triangledown_2,\triangledown_3)$ be any two bisemirings. If $\mathcal{F}:\mathscr{S}_1\to\mathscr{S}_2$ is a homomorphism,then $\mathcal{F}(\overline{\Theta_{(\ell,\flat)}})$ is a SBS of CCQNSBS $\overline{\Omega}$ of $\mathscr{S}_2$.*

Proof. The mapping $\mathcal{F}:\mathscr{S}_1\to\mathscr{S}_2$ be a homomorphism. Now,$\mathcal{F}((\alpha\triangle_1\varpi),\varrho)=\mathcal{F}(\alpha,\varrho)\triangledown_1\mathcal{F}(\varpi,\varrho), \mathcal{F}((\alpha\triangle_2\varpi),\varrho)=\mathcal{F}(\alpha,\varrho)\triangledown_2\mathcal{F}(\varpi,\varrho)$ and $\mathcal{F}((\alpha\triangle_3\varpi),\varrho)=\mathcal{F}(\alpha,\varrho)\triangledown_3\mathcal{F}(\varpi,\varrho)$ for all $\alpha,\varpi\in\mathscr{S}_1$. Let $\overline{\Omega}=\mathcal{F}(\Theta)$,$\Theta$ is a CCQNSBS of $\mathscr{S}_1$. By Theorem 4.2, $\overline{\Omega}$ is a CCQNSBS of $\mathscr{S}_2$. Let $\overline{\Theta_{(\ell,\flat)}}$ be any SBS of Θ. Suppose that $\alpha,\varpi\in\overline{\Theta_{(\ell,\flat)}}$. Then $\alpha\triangle_1\varpi,\alpha\triangle_2\varpi$ and $\alpha\triangle_3\varpi\in\overline{\Theta_{(\ell,\flat)}}$. Now,$\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}(\mathcal{F}(\alpha,\varrho))\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}(\mathcal{F}(\alpha,\varrho))} = \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha,\varrho)} \succeq \ell, \overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}(\mathcal{F}(\varpi,\varrho))\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}(\mathcal{F}(\varpi,\varrho))} = \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\varpi,\varrho)} \succeq \ell$. Thus,$\overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}((\mathcal{F}(\alpha,\varrho)\triangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}((\mathcal{F}(\alpha,\varrho)\triangledown_1\mathcal{F}(\varpi,\varrho)))} \succeq \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha\triangle_1\varpi),\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}((\alpha\triangle_1\varpi),\varrho)} \succeq \ell$. Now,$\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}(\mathcal{F}(\alpha,\varrho))\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}(\mathcal{F}(\alpha,\varrho))} = \overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha,\varrho)} \succeq \ell, \overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}(\mathcal{F}(\varpi,\varrho))\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}(\mathcal{F}(\varpi,\varrho))} = \overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\varpi,\varrho)} \succeq \ell$. Thus,$\overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}((\mathcal{F}(\alpha,\varrho)\triangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}((\mathcal{F}(\alpha,\varrho)\triangledown_1\mathcal{F}(\varpi,\varrho)))} \succeq \overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}((\alpha\triangle_1\varpi),\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}((\alpha\triangle_1\varpi),\varrho)} \succeq \ell$. Now,$\overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}(\mathcal{F}(\alpha,\varrho))\cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}(\mathcal{F}(\alpha,\varrho))} = \overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha,\varrho)} \preceq \flat, \overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}(\mathcal{F}(\varpi,\varrho))\cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}(\mathcal{F}(\varpi,\varrho))} = \overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\varpi,\varrho)\cdot$

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$M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\varpi,\varrho)} \preceq \flat$. Thus,$\overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))} \preceq \overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha\triangle_1\varpi),\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}((\alpha\triangle_1\varpi),\varrho)} \preceq \flat$,for all $\mathcal{F}(\alpha,\varrho),\mathcal{F}(\varpi,\varrho) \in \mathscr{S}_2$. Similarly other operations,$\mathcal{F}(\overline{\Theta_{(\ell,\flat)}})$ is a SBS of CCQNSBS $\overline{\Omega}$ of $\mathscr{S}_2$.

The mapping $\mathcal{F} : \mathscr{S}_1 \to \mathscr{S}_2$ be a homomorphism. Now,$\mathcal{F}((\alpha \triangle_1 \varpi),\varrho) = \mathcal{F}(\alpha,\varrho) \bigtriangledown_1 \mathcal{F}(\varpi,\varrho), \mathcal{F}((\alpha \triangle_2 \varpi),\varrho) = \mathcal{F}(\alpha,\varrho) \bigtriangledown_2 \mathcal{F}(\varpi,\varrho)$ and $\mathcal{F}((\alpha \triangle_3 \varpi),\varrho) = \mathcal{F}(\alpha,\varrho) \bigtriangledown_3 \mathcal{F}(\varpi,\varrho)$ for all $\alpha,\varpi \in \mathscr{S}_1$. Let $\Omega = \mathcal{F}(\Theta)$,$\Theta$ is a CCQNSBS of $\mathscr{S}_1$. By Theorem 4.2, Ω is a CCQNSBS of $\mathscr{S}_2$. Let $\Theta_{(\ell,\flat)}$ be any SBS of Θ. Suppose that $\alpha,\varpi \in \Theta_{(\ell,\flat)}$. Then $\alpha \triangle_1 \varpi, \alpha \triangle_2 \varpi$ and $\alpha \triangle_3 \varpi \in \Theta_{(\ell,\flat)}$. Now,$\Omega^{\mathscr{T}}_{\mathcal{R}}(\mathcal{F}(\alpha,\varrho)) \cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}(\mathcal{F}(\alpha,\varrho))} = \Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha,\varrho) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha,\varrho)} \succeq \ell, \Omega^{\mathscr{T}}_{\mathcal{R}}(\mathcal{F}(\varpi,\varrho)) \cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}(\mathcal{F}(\varpi,\varrho))} = \Theta^{\mathscr{T}}_{\mathcal{R}}(\varpi,\varrho) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi,\varrho)} \succeq \ell$. Thus,$\Omega^{\mathscr{T}}_{\mathcal{R}}((\mathcal{F}(\alpha,\varrho) \bigtriangledown_1 \mathcal{F}(\varpi,\varrho))) \cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))} \succeq \Theta^{\mathscr{T}}_{\mathcal{R}}((\alpha \triangle_1 \varpi),\varrho) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha\triangle_1\varpi),\varrho)} \succeq \ell$. Now,$\Omega^{\mathscr{I}}_{\mathcal{R}}(\mathcal{F}(\alpha,\varrho)) \cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}(\mathcal{F}(\alpha,\varrho))} = \Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha,\varrho) \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha,\varrho)} \succeq \ell, \Omega^{\mathscr{I}}_{\mathcal{R}}(\mathcal{F}(\varpi,\varrho)) \cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}(\mathcal{F}(\varpi,\varrho))} = \Theta^{\mathscr{I}}_{\mathcal{R}}(\varpi,\varrho) \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\varpi,\varrho)} \succeq \ell$. Thus,$\Omega^{\mathscr{I}}_{\mathcal{R}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))} \succeq \Theta^{\mathscr{I}}_{\mathcal{R}}((\alpha\triangle_1\varpi),\varrho)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}((\alpha\triangle_1\varpi),\varrho)} \succeq \ell$. Now,$\Omega^{\mathscr{F}}_{\mathcal{R}}(\mathcal{F}(\alpha,\varrho))\cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}(\mathcal{F}(\alpha,\varrho))} = \Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha,\varrho)} \preceq \flat, \Omega^{\mathscr{F}}_{\mathcal{R}}(\mathcal{F}(\varpi,\varrho))\cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}(\mathcal{F}(\varpi,\varrho))} = \Theta^{\mathscr{F}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\varpi,\varrho)} \preceq \flat$. Thus,$\Omega^{\mathscr{F}}_{\mathcal{R}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))} \preceq \Theta^{\mathscr{F}}_{\mathcal{R}}((\alpha\triangle_1\varpi),\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}((\alpha\triangle_1\varpi),\varrho)} \preceq \flat$,for all $\mathcal{F}(\alpha,\varrho),\mathcal{F}(\varpi,\varrho) \in \mathscr{S}_2$. Similarly other operations,$\mathcal{F}(\Theta_{(\ell,\flat)})$ is a SBS of CCQNSBS Ω of $\mathscr{S}_2$.

Theorem 4.5. *Let* $(\mathscr{S}_1, \triangle_1, \triangle_2, \triangle_3)$ *and* $(\mathscr{S}_2, \bigtriangledown_1, \bigtriangledown_2, \bigtriangledown_3)$ *be any two bisemirings. If* $\mathcal{F} : \mathscr{S}_1 \to \mathscr{S}_2$ *is any homomorphism,then* $\overline{\Theta_{(\ell,\flat)}}$ *is a SBS of CCQNSBS* Θ *of* $\mathscr{S}_1$.

Proof. The mapping $\mathcal{F} : \mathscr{S}_1 \to \mathscr{S}_2$ be any homomorphism. We have $\mathcal{F}((\alpha \triangle_1 \varpi),\varrho) = \mathcal{F}(\alpha,\varrho) \bigtriangledown_1 \mathcal{F}(\varpi,\varrho), \mathcal{F}((\alpha \triangle_2 \varpi),\varrho) = \mathcal{F}(\alpha,\varrho) \bigtriangledown_2 \mathcal{F}(\varpi,\varrho)$ and $\mathcal{F}((\alpha \triangle_3 \varpi),\varrho) = \mathcal{F}(\alpha,\varrho) \bigtriangledown_3 \mathcal{F}(\varpi,\varrho)$ for all $\alpha,\varpi \in \mathscr{S}_1$. Let $\overline{\Omega} = \mathcal{F}(\Theta)$,$\overline{\Omega}$ is a CCQNSBS of $\mathscr{S}_2$. By Theorem 4.3,Θ is a CCQNSBS of $\mathscr{S}_1$. Let $\mathcal{F}(\overline{\Theta_{(\ell,\flat)}})$ be a SBS of $\overline{\Omega}$. Suppose that $\mathcal{F}(\alpha,\varrho),\mathcal{F}(\varpi,\varrho) \in \mathcal{F}(\overline{\Theta_{(\ell,\flat)}})$. Now,$\mathcal{F}((\alpha\triangle_1\varpi),\varrho),\mathcal{F}((\alpha\triangle_2\varpi),\varrho)$ and $\mathcal{F}((\alpha\triangle_3\varpi),\varrho) \in \mathcal{F}(\overline{\Theta_{(\ell,\flat)}})$. Now,$\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha,\varrho)} = \overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}(\mathcal{F}(\alpha,\varrho))\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}(\mathcal{F}(\alpha,\varrho))} \succeq \ell, \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\varpi,\varrho)} = \overline{\Omega^{\mathscr{T}}_{\mathcal{R}}}(\mathcal{F}(\varpi,\varrho))\cdot M^{\overline{\Omega^{\mathscr{T}}_{\mathcal{I}}}(\mathcal{F}(\varpi,\varrho))} \succeq \ell$. Thus,$\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}((\alpha\triangle_1\varpi),\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}((\alpha\triangle_1\varpi),\varrho)} \succeq \min\{\overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\alpha,\varrho)}, \overline{\Theta^{\mathscr{T}}_{\mathcal{R}}}(\varpi,\varrho) \cdot M^{\overline{\Theta^{\mathscr{T}}_{\mathcal{I}}}(\varpi,\varrho)}\} \succeq \ell$. Now,$\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha,\varrho) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha,\varrho)} = \overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}(\mathcal{F}(\alpha,\varrho)) \cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}(\mathcal{F}(\alpha,\varrho))} \succeq \ell, \overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\varpi,\varrho) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\varpi,\varrho)} = \overline{\Omega^{\mathscr{I}}_{\mathcal{R}}}(\mathcal{F}(\varpi,\varrho)) \cdot M^{\overline{\Omega^{\mathscr{I}}_{\mathcal{I}}}(\mathcal{F}(\varpi,\varrho))} \succeq \ell$.
Thus,$\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}((\alpha \triangle_1 \varpi),\varrho) \cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}((\alpha\triangle_1\varpi),\varrho)} \succeq \frac{\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\alpha,\varrho)}+\overline{\Theta^{\mathscr{I}}_{\mathcal{R}}}(\varpi,\varrho)\cdot M^{\overline{\Theta^{\mathscr{I}}_{\mathcal{I}}}(\varpi,\varrho)}}{2} \succeq \ell$. Now,$\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha,\varrho) \cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha,\varrho)} = \overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}(\mathcal{F}(\alpha,\varrho)) \cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}(\mathcal{F}(\alpha,\varrho))} \preceq \flat, \overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\varpi,\varrho) \cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\varpi,\varrho)} = \overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}(\mathcal{F}(\varpi,\varrho)) \cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}(\mathcal{F}(\varpi,\varrho))} \preceq \flat$. Thus,$\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}((\alpha\triangle_1\varpi),\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}((\alpha\triangle_1\varpi),\varrho)} = \overline{\Omega^{\mathscr{F}}_{\mathcal{R}}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\overline{\Omega^{\mathscr{F}}_{\mathcal{I}}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))} \preceq \max\{\overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\alpha,\varrho)\cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\alpha,\varrho)}, \overline{\Theta^{\mathscr{F}}_{\mathcal{R}}}(\varpi,\varrho) \cdot M^{\overline{\Theta^{\mathscr{F}}_{\mathcal{I}}}(\varpi,\varrho)}\} \preceq \flat$,for all $\alpha,\varpi \in \mathscr{S}_1$. Similarly other operations,$\overline{\Theta_{(\ell,\flat)}}$ is a SBS of CCQNSBS Θ of $\mathscr{S}_1$.

The mapping $\mathcal{F} : \mathscr{S}_1 \to \mathscr{S}_2$ be any homomorphism. We have $\mathcal{F}((\alpha\triangle_1\varpi),\varrho) = \mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho), \mathcal{F}((\alpha\triangle_2\varpi),\varrho) = \mathcal{F}(\alpha,\varrho)\bigtriangledown_2\mathcal{F}(\varpi,\varrho)$ and $\mathcal{F}((\alpha\triangle_3\varpi),\varrho) = \mathcal{F}(\alpha,\varrho)\bigtriangledown_3\mathcal{F}(\varpi,\varrho)$ for all $\alpha,\varpi \in \mathscr{S}_1$. Let $\Omega = \mathcal{F}(\Theta)$,$\Omega$ is a CCQNSBS of $\mathscr{S}_2$. By Theorem 4.3,Θ is a CCQNSBS of $\mathscr{S}_1$. Let $\mathcal{F}(\Theta_{(\ell,\flat)})$ be a SBS of Ω. Suppose that $\mathcal{F}(\alpha,\varrho),\mathcal{F}(\varpi,\varrho) \in \mathcal{F}(\Theta_{(\ell,\flat)})$. Now,$\mathcal{F}((\alpha\triangle_1\varpi),\varrho),\mathcal{F}((\alpha\triangle_2\varpi),\varrho)$ and $\mathcal{F}((\alpha\triangle_3\varpi),\varrho) \in \mathcal{F}(\Theta_{(\ell,\flat)})$.
Now,$\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha,\varrho) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha,\varrho)} = \Omega^{\mathscr{T}}_{\mathcal{R}}(\mathcal{F}(\alpha,\varrho)) \cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}(\mathcal{F}(\alpha,\varrho))} \succeq \ell, \Theta^{\mathscr{T}}_{\mathcal{R}}(\varpi,\varrho) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi,\varrho)} = \Omega^{\mathscr{T}}_{\mathcal{R}}(\mathcal{F}(\varpi,\varrho)) \cdot M^{\Omega^{\mathscr{T}}_{\mathcal{I}}(\mathcal{F}(\varpi,\varrho))} \succeq \ell$.
Thus,$\Theta^{\mathscr{T}}_{\mathcal{R}}((\alpha \triangle_1 \varpi),\varrho) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}((\alpha\triangle_1\varpi),\varrho)} \succeq \min\{\Theta^{\mathscr{T}}_{\mathcal{R}}(\alpha,\varrho) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\alpha,\varrho)}, \Theta^{\mathscr{T}}_{\mathcal{R}}(\varpi,\varrho) \cdot M^{\Theta^{\mathscr{T}}_{\mathcal{I}}(\varpi,\varrho)}\} \succeq \ell$. Now,$\Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha,\varrho) \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha,\varrho)} = \Omega^{\mathscr{I}}_{\mathcal{R}}(\mathcal{F}(\alpha,\varrho)) \cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}(\mathcal{F}(\alpha,\varrho))} \succeq \ell, \Theta^{\mathscr{I}}_{\mathcal{R}}(\varpi,\varrho) \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\varpi,\varrho)} = \Omega^{\mathscr{I}}_{\mathcal{R}}(\mathcal{F}(\varpi,\varrho)) \cdot M^{\Omega^{\mathscr{I}}_{\mathcal{I}}(\mathcal{F}(\varpi,\varrho))} \succeq \ell$. Thus,$\Theta^{\mathscr{I}}_{\mathcal{R}}((\alpha \triangle_1 \varpi),\varrho) \cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}((\alpha\triangle_1\varpi),\varrho)} \succeq \frac{\Theta^{\mathscr{I}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\alpha,\varrho)}+\Theta^{\mathscr{I}}_{\mathcal{R}}(\varpi,\varrho)\cdot M^{\Theta^{\mathscr{I}}_{\mathcal{I}}(\varpi,\varrho)}}{2} \succeq \ell$. Now,$\Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha,\varrho) \cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha,\varrho)} = \Omega^{\mathscr{F}}_{\mathcal{R}}(\mathcal{F}(\alpha,\varrho)) \cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}(\mathcal{F}(\alpha,\varrho))} \preceq \flat, \Theta^{\mathscr{F}}_{\mathcal{R}}(\varpi,\varrho) \cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\varpi,\varrho)} = \Omega^{\mathscr{F}}_{\mathcal{R}}(\mathcal{F}(\varpi,\varrho)) \cdot M^{\Omega^{\mathscr{F}}_{\mathcal{I}}(\mathcal{F}(\varpi,\varrho))} \preceq \flat$.
Thus,$\Theta^{\mathscr{F}}_{\mathcal{R}}((\alpha\triangle_1\varpi),\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}((\alpha\triangle_1\varpi),\varrho)} = \Omega^{\mathscr{F}}_{\mathcal{R}}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))\cdot M^{\mathcal{I}^{\mathscr{F}}_{\Omega}((\mathcal{F}(\alpha,\varrho)\bigtriangledown_1\mathcal{F}(\varpi,\varrho)))} \preceq \max\{\Theta^{\mathscr{F}}_{\mathcal{R}}(\alpha,\varrho)\cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\alpha,\varrho)}, \Theta^{\mathscr{F}}_{\mathcal{R}}(\varpi,\varrho) \cdot M^{\Theta^{\mathscr{F}}_{\mathcal{I}}(\varpi,\varrho)}\} \preceq \flat$,for all $\alpha,\varpi \in \mathscr{S}_1$. Similarly other operations,$\Theta_{(\ell,\flat)}$ is a SBS of CCQNSBS Θ of $\mathscr{S}_1$.

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