New approach for subbisemiring of bisemiring is applied to complex cubic anti neutrosophic set and its extension

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Abstract

We construct and analyze the concept of complex cubic anti neutrosophic subbisemiring (ComCANSBS). We analyze the important properties and homomorphic aspects of ComCANSBS. For bisemirings, we propose the ComCANSBS level sets. A complex neutrosophic subset of bisemiring Ⓢ is represented by the symbol Γ if and only if each non-empty level set $R^{(\wp,\varkappa)}$, where $R = (\overbrace{\Re_\Gamma^\top} \cdot e^{i\theta \overbrace{\Im_\Gamma^\top}}, \overbrace{\Re_\Gamma^\gimel} \cdot e^{i\theta \overbrace{\Im_\Gamma^\gimel}}, \overbrace{\Re_\Gamma^F} \cdot e^{i\theta \overbrace{\Im_\Gamma^F}}, \Re_\Gamma^\top \cdot e^{i\theta \Im_\Gamma^\top}, \Re_\Gamma^\gimel \cdot e^{i\theta \Im_\Gamma^\gimel}, \Re_\Gamma^F \cdot e^{i\theta \Im_\Gamma^F})$ is a ComCANSBS of Ⓢ. Let Υ be a ComCANSBS of bisemiring Ⓢ. If and only if Υ is a ComCANSBS of Ⓢ $\times$ Ⓢ, then Γ is a ComCANSBS of bisemiring Ⓢ. Let Γ be the strongest complex anti neutrosophic relation of bisemiring Ⓢ. We show that homomorphic images of all ComCANSBSs are ComCANSBSs, and homomorphic pre-images of all ComCANSBSs are ComCANSBSs. There are examples given to illustrate our results.

Keywords: ComCANSBS; ComCNANSBS; SBS; homomorphism

1 Introduction

Zadeh[1] developed fuzzy set (FS) theory, which works best at handling ambiguity and uncertainty. An element in an FS is considered a member if it contains a single value inside the interval. However, since resistance can exist in real-world situations, the degree of non-membership does not necessarily equal one minus the degree of membership. As FS theory advances quickly, more and more hybrid fuzzy models are being developed. Numerous uncertain theories, including FS,[1] intuitionistic FS (IFS),[2] Pythagorean FS (PFS),[3] and spherical FS (SFS),[4] have been developed as a result of the uncertainties. Sets with grades ranging from 0 to 1, referred to as MG, comprise an FS. IFS is classified as MG, in spite of Atanassov[2] stating that non-membership grades (NMG) can only have a value of 1. There is a chance that, in a decision-making process, the sum of MGs and NMGs will sometimes exceed 1. The generalized MG and NMG logic, which has a value not exceeding 1 and is determined by the square of the MGs and NMGs, was constructed by Yager[3] using PFS logic. These ideas are unable to explain the neutral state, which is neither positive nor negative. Cuong[5] discussed the image with associates. Three grading points were utilized by FS: positive, neutral, and negative. These grades added together could not total more than 1. For some purposes, it is also superior to PFS or IFS. It is an independent generalization of three models that deals with the truth, indeterminacy, and falsity of FS and IFS.

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The neutrosophic set (NS) was developed by Smarandache[6] to deal with contradicting and ambiguous data. This reasoning establishes the degree to which a statement is true, ambiguous, or untrue. The notion of the complex fuzzy set (CFS) is presented by Ramot et al.[7] There is a wide range of possible values for the membership functions of CFSs deals. The unit circle of the complex plane is extended to $[0, 1]$, but the unit circle of a fuzzy membership function is fixed. The CFS X is characterized by a membership function $\mu_X(x)$ that extends to the unit circle in the complex plane instead of only $[0, 1]$. Thus, $\mu_X(x)$ is a complex-valued function that assigns a grade of membership of the form $\eta_X(x) \cdot e^{i\tau_X(x)}$, where $i = \sqrt{-1}$, to any element x in the discourse universe. The value of $\mu_X(x)$ is defined by the two real-valued variables, $\eta_X(x)$ and $\tau_X(x)$, where $\eta_X(x) \in [0, 1]$. Semiring logic and its uses were introduced by Golan.[8] Hussian and associates[9] discussed the idea and application of bisemirings. The topic of bipolar-valued FSs and related procedures is covered by Lee.[10] Ahsan et al. investigated fuzzy semirings.[11] The notion of bisemirings was first presented by Sen et al.[12] An intuitionistic fuzzy normal subbisemiring of bisemiring was recently presented by Palanikumar et al.[13] The notion of bisemiring was established by Palanikumar et al.[14] utilizing bipolar-valued neutrosophic normal sets. Recently, many authors discussed the new concepts such as neutrosopic set, fuzzy extension set and speircal fuzzy set[15]-.[26] The following contributions are made to this study:

1. The intersection of a every ComCANSBSs is again a ComCANSBS of bisemiring $\circledS$.
2. Let Γ be a ComCANSBS of $\circledS$ and Υ be a strongest complex cubic anti neutrosophic relation of $\circledS$. Then Γ is a ComCANSBS of bisemiring $\circledS$ if and only if Υ is a ComCANSBS of $\circledS \times \circledS$.
3. $R = (\overbrace{\Re^{\top}_{\Gamma}} \cdot e^{i\theta\, \overbrace{\Im^{\top}_{\Gamma}}}, \overbrace{\Re^{\daleth}_{\Gamma}} \cdot e^{i\theta\, \overbrace{\Im^{\daleth}_{\Gamma}}}, \overbrace{\Re^{\digamma}_{\Gamma}} \cdot e^{i\theta\, \overbrace{\Im^{\digamma}_{\Gamma}}}, \Re^{\top}_{\Gamma} \cdot e^{i\theta\Im^{\top}_{\Gamma}}, \Re^{\daleth}_{\Gamma} \cdot e^{i\theta\Im^{\daleth}_{\Gamma}}, \Re^{\digamma}_{\Gamma} \cdot e^{i\theta\Im^{\digamma}_{\Gamma}})$ is a subbisemiring of $\circledS$ for all $\wp, \varkappa \in \mathbb{D}[0, 1]$.
4. The homomorphic image of every ComCANSBS is a ComCANSBS and homomorphic preimage of every ComCANSBS is a ComCANSBS.

We will look at certain aspects of the SBS and ComCANSBS concepts and make some conclusions. The article is divided into the following five sections. In Section 1, semirings and SBS are introduced. Section 2 contains information on semiring and SBS preparation. Listing of ComCANSBS properties is done in Section 3. Utilizing numerical examples are recommended for ComCANSBS evaluation. Section 4 indicates the outcome and future direction.

2 Preliminaries

Definition 2.1. [12] An algebraic structure $(\circledS, \uplus, \ominus, \odot)$ is a bisemiring,if $(\circledS, \uplus, \ominus)$ and $(\circledS, \ominus, \odot)$ are semirings, ie.,$(S, \uplus)$, $(S, \ominus)$ and $(\circledS, \odot)$ are semigroups and

1. $z_v \ominus (z_\zeta \uplus z_\eta) = (z_v \ominus z_\zeta) \uplus (z_v \ominus z_\eta)$,
2. $(z_\zeta \uplus z_\eta) \ominus z_v = (z_\zeta \ominus z_v) \uplus (z_\eta \ominus z_v)$,
3. $z_v \odot (z_\zeta \ominus z_\eta) = (z_v \odot z_\zeta) \ominus (z_v \odot z_\eta)$,
4. $(z_\zeta \ominus z_\eta) \odot z_v = (z_\zeta \odot z_v) \ominus (z_\eta \odot z_v)$, $\forall\ z_v, z_\zeta, z_\eta \in \circledS$.

Definition 2.2. [6] A NS v in the universe $\mathcal{U}$ is $v = \{x, u^{\top}_v(x), u^{\daleth}_v(x), u^{\digamma}_v(x) | x \in \mathcal{U}\}$,where $u^{\top}_v(x)$,$u^{\daleth}_v(x)$ $u^{\digamma}_v(x)$ represents the TD,ID and FD of v respectively. Consider the mapping $u^{\top}_v : \mathcal{U} \to [0, 1], u^{\daleth}_v : \mathcal{U} \to [0, 1], u^{\digamma}_v : \mathcal{U} \to [0, 1]$ and $0 \leqslant u^{\top}_v(x) + u^{\daleth}_v(x) + u^{\digamma}_v(x) \leqslant 3$.

Definition 2.3. [6] Let $\psi_1 = \langle u^{\top}_{\psi_1}, u^{\daleth}_{\psi_1}, u^{\digamma}_{\psi_1} \rangle$,$\psi_2 = \langle u^{\top}_{\psi_2}, u^{\daleth}_{\psi_2}, u^{\digamma}_{\psi_2} \rangle$ and $\psi_3 = \langle u^{\top}_{\psi_3}, u^{\daleth}_{\psi_3}, u^{\digamma}_{\psi_3} \rangle$ be the three neutrosophic numbers over $\mathcal{U}$. Then

1. $\psi_2 \ominus \psi_3 = \Big\langle \max(\chi^{\top}_{\psi_2}, u^{\top}_{\psi_3}), \min(\chi^{\daleth}_{\psi_2}, u^{\daleth}_{\psi_3}), \min(\chi^{\digamma}_{\psi_2}, u^{\digamma}_{\psi_3}) \Big\rangle$,

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2. $\psi_2 \circledast \psi_3 = \Big\langle \min(\chi^{\top}_{\psi_2}, u^{\top}_{\psi_3}), \max(\chi^{\daleth}_{\psi_2}, u^{\daleth}_{\psi_3}), \max(\chi^{F}_{\psi_2}, u^{F}_{\psi_3}) \Big\rangle$,
3. $\psi_2 \geqslant \psi_3$ iff $u^{\top}_{\psi_2} \geqslant u^{\top}_{\psi_3}$ and $u^{\daleth}_{\psi_2} \leqslant u^{\daleth}_{\psi_3}$ and $u^{F}_{\psi_2} \leqslant u^{F}_{\psi_3}$,
4. $\psi_2 = \psi_3$ iff $u^{\top}_{\psi_2} = u^{\top}_{\psi_3}$ and $u^{\daleth}_{\psi_2} = u^{\daleth}_{\psi_3}$ and $u^{F}_{\psi_2} = u^{F}_{\psi_3}$.

Definition 2.4. [6] For any NS $\psi = \{x, \chi^{\top}_{v}(x), \chi^{\daleth}_{v}(x), \chi^{F}_{v}(x)\}$ of $\mathcal{U}$. Then (ζ, τ)-cut is defined as $\{x \in U | \chi^{\top}_{v}(x) \geqslant \zeta, \chi^{\daleth}_{v}(x) \geqslant \zeta, \chi^{F}_{v}(x) \leqslant \tau\}$.

Definition 2.5. [6] Let V and Y be two NSs of S. Then Cartesian product of V and Y is defined as $V \times Y = \{\chi^{\top}_{V\times Y}(\chi, \psi), \chi^{\daleth}_{V\times Y}(\chi, \psi), \chi^{F}_{V\times Y}(\chi, \psi) | \; for \; all \; \chi, \psi \in S\}$, where
$\chi^{\top}_{V\times Y}(\chi, \psi) = \min\{\chi^{\top}_{V}(x), \chi^{\top}_{Y}(\psi)\}, \chi^{\daleth}_{V\times Y}(\chi, \psi) = \frac{\chi^{\daleth}_{V}(x) + \chi^{\daleth}_{Y}(\psi)}{2}, \chi^{F}_{V\times Y}(\chi, \psi) = \max\{\chi^{F}_{V}(x), \chi^{F}_{Y}(\psi)\}$.

Definition 2.6. A fuzzy subset v of a bisemiring $(Ⓢ, \bigcirc_1, \bigcirc_2, \bigcirc_3)$ is represents a fuzzy subbisemiring of Ⓢ if $\chi_v(\chi \bigcirc_1 \varepsilon) \geqslant \min\{\chi_v(x), \chi_v(\varepsilon)\}, \chi_v(\chi \bigcirc_2 \varepsilon) \geqslant \min\{\chi_v(x), \chi_v(\varepsilon)\}, \chi_v(\chi \bigcirc_3 \varepsilon) \geqslant \min\{\chi_v(x), \chi_v(\varepsilon)\}$,for all $\chi, \varepsilon \in Ⓢ$.

3 Complex cubic anti neutrosophic subbisemiring

Here Ⓢ denotes bisemiring unless other stated, Ψ stands for real part and Ξ stands for imaginary part and $\Bbbk = 2\pi$.

Definition 3.1. The complex cubic anti neutrosophic set(ComCANS) $\aleph$ in universal set $\mho$,
$\aleph = \{\zeta, \widehat{\Psi^{\top}_{\aleph}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{\top}_{\aleph}(\zeta, \partial)}}, \widehat{\Psi^{\daleth}_{\aleph}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{\daleth(\zeta,\partial)}_{\aleph}}}, \widehat{\Psi^{F}_{\aleph}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{F(\zeta,\partial)}_{\aleph}}}, \Psi^{\top}_{\aleph}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{\top(\zeta,\partial)}_{\aleph}}, \Psi^{\daleth}_{\aleph}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{\daleth(\zeta,\partial)}_{\aleph}}, \Psi^{F}_{\aleph}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{F(\zeta,\partial)}_{\aleph}} : \zeta \in \mho\}$, where $\widehat{\Psi^{\top}_{\aleph}}(\zeta, \partial) = [\Psi^{\top L}_{\aleph}, \Psi^{\top U}_{\aleph}]$, $\widehat{\Psi^{\daleth}_{\aleph}}(\zeta, \partial) = [\Psi^{\daleth L}_{\aleph}, \Psi^{\daleth U}_{\aleph}]$, $\widehat{\Psi^{F}_{\aleph}}(\zeta, \partial) = [\Psi^{F L}_{\aleph}, \Psi^{F U}_{\aleph}]$ and $\widehat{\Psi^{\top}_{\aleph}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{\top(\zeta,\partial)}_{\aleph}}}, \widehat{\Psi^{\daleth}_{\aleph}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{\daleth(\zeta,\partial)}_{\aleph}}}, \widehat{\Psi^{F}_{\aleph}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{F(\zeta,\partial)}_{\aleph}}} : \mho \to D[0, 1]$ and $\Psi^{\top}_{\aleph}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{\top(\zeta,\partial)}_{\aleph}}, \Psi^{\daleth}_{\aleph}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{\daleth(\zeta,\partial)}_{\aleph}}, \Psi^{F}_{\aleph}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{F(\zeta,\partial)}_{\aleph}} : \mho \to [0, 1]$ represents the truth degree,indeterminacy degree and false degree respectively.

For simplicity the symbol $\widehat{\Psi^{\top}_{\aleph}}, \widehat{\Psi^{\daleth}_{\aleph}}, \widehat{\Psi^{F}_{\aleph}}$ is ComCANS $\aleph = \{\zeta, \widehat{\Psi^{\top}_{\aleph}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{\top(\zeta,\partial)}_{\aleph}}}, \widehat{\Psi^{\daleth}_{\aleph}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{\daleth(\zeta,\partial)}_{\aleph}}}, \widehat{\Psi^{F}_{\aleph}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{F(\zeta,\partial)}_{\aleph}}}, \Psi^{\top}_{\aleph}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{\top(\zeta,\partial)}_{\aleph}}, \Psi^{\daleth}_{\aleph}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{\daleth(\zeta,\partial)}_{\aleph}}, \Psi^{F}_{\aleph}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{F(\zeta,\partial)}_{\aleph}} : \zeta \in \mho\}$.

Definition 3.2. Let $\aleph = \Big\{\zeta, \widehat{\Psi^{\top}_{\aleph}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{\top(\zeta,\partial)}_{\aleph}}}, \widehat{\Psi^{\daleth}_{\aleph}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{\daleth(\zeta,\partial)}_{\aleph}}}, \widehat{\Psi^{F}_{\aleph}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{F(\zeta,\partial)}_{\aleph}}}, \Psi^{\top}_{\aleph}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{\top(\zeta,\partial)}_{\aleph}}, \Psi^{\daleth}_{\aleph}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{\daleth(\zeta,\partial)}_{\aleph}}, \Psi^{F}_{\aleph}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{F(\zeta,\partial)}_{\aleph}}\Big\}$ and
$\Sigma = \Big\{\zeta, \widehat{\Psi^{\top}_{\Sigma}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{\top(\zeta,\partial)}_{\Sigma}}}, \widehat{\Psi^{\daleth}_{\Sigma}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{\daleth(\zeta,\partial)}_{\Sigma}}}, \widehat{\Psi^{F}_{\Sigma}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{F(\zeta,\partial)}_{\Sigma}}}, \Psi^{\top}_{\Sigma}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{\top(\zeta,\partial)}_{\Sigma}}, \Psi^{\daleth}_{\Sigma}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{\daleth(\zeta,\partial)}_{\Sigma}}, \Psi^{F}_{\Sigma}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{F(\zeta,\partial)}_{\Sigma}}\Big\}$ be two ComCANSs of $\mho$.
Then we define the intersection and union operation is defined as

(i) $\aleph \cap \Sigma = \Big\{\Big(\zeta, \max\{\widehat{\Psi^{\top}_{\aleph}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{\top(\zeta,\partial)}_{\aleph}}}, \widehat{\Psi^{\top}_{\Sigma}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{\top(\zeta,\partial)}_{\Sigma}}}\}, \max\{\widehat{\Psi^{\daleth}_{\aleph}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{\daleth(\zeta,\partial)}_{\aleph}}}, \widehat{\Psi^{\daleth}_{\Sigma}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{\daleth(\zeta,\partial)}_{\Sigma}}}\}, \min\{\widehat{\Psi^{F}_{\aleph}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{F(\zeta,\partial)}_{\aleph}}}, \widehat{\Psi^{F}_{\Sigma}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{F(\zeta,\partial)}_{\Sigma}}}\},$
$\max\{\Psi^{\top}_{\aleph}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{\top(\zeta,\partial)}_{\aleph}}, \Psi^{\top}_{\Sigma}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{\top(\zeta,\partial)}_{\Sigma}}\}, \max\{\Psi^{\daleth}_{\aleph}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{\daleth(\zeta,\partial)}_{\aleph}}, \Psi^{\daleth}_{\Sigma}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{\daleth(\zeta,\partial)}_{\Sigma}}\}, \min\{\Psi^{F}_{\aleph}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{F(\zeta,\partial)}_{\aleph}}, \Psi^{F}_{\Sigma}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{F(\zeta,\partial)}_{\Sigma}}\}\Big) \Big| \zeta \in \mho\Big\}$.

(ii) $\aleph \cup \Sigma = \Big\{\Big(\zeta, \min\{\widehat{\Psi^{\top}_{\aleph}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{\top(\zeta,\partial)}_{\aleph}}}, \widehat{\Psi^{\top}_{\Sigma}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{\top(\zeta,\partial)}_{\Sigma}}}\}, \min\{\widehat{\Psi^{\daleth}_{\aleph}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{\daleth(\zeta,\partial)}_{\aleph}}}, \widehat{\Psi^{\daleth}_{\Sigma}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{\daleth(\zeta,\partial)}_{\Sigma}}}\}, \max\{\widehat{\Psi^{F}_{\aleph}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{F(\zeta,\partial)}_{\aleph}}}, \widehat{\Psi^{F}_{\Sigma}}(\zeta, \partial) \cdot e^{i\Bbbk\, \widehat{\Xi^{F(\zeta,\partial)}_{\Sigma}}}\},$
$\min\{\Psi^{\top}_{\aleph}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{\top(\zeta,\partial)}_{\aleph}}, \Psi^{\top}_{\Sigma}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{\top(\zeta,\partial)}_{\Sigma}}\}, \min\{\Psi^{\daleth}_{\aleph}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{\daleth(\zeta,\partial)}_{\aleph}}, \Psi^{\daleth}_{\Sigma}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{\daleth(\zeta,\partial)}_{\Sigma}}\}, \max\{\Psi^{F}_{\aleph}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{F(\zeta,\partial)}_{\aleph}}, \Psi^{F}_{\Sigma}(\zeta, \partial) \cdot e^{i\Bbbk \Xi^{F(\zeta,\partial)}_{\Sigma}}\}\Big) \Big| \zeta \in \mho\Big\}$.

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Definition 3.3. For any ComCANS $\aleph = \Big\{\zeta, \widehat{\Psi_{\aleph}^{\top}}(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{\top(\zeta,\partial)}}}, \widehat{\Psi_{\aleph}^{\gimel}}(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{\gimel(\zeta,\partial)}}}, \widehat{\Psi_{\aleph}^{F}}(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{F(\zeta,\partial)}}}, \Psi_{\aleph}^{\top}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top(\zeta,\partial)}}, \Psi_{\aleph}^{\gimel}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\gimel(\zeta,\partial)}}, \Psi_{\aleph}^{F}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F(\zeta,\partial)}}\Big\}$ of a universal set $\mho$. Then (β,δ)-cut is defined as $\Big\{\zeta\in\mho \mid \widehat{\Psi_{\aleph}^{\top}}(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{\top(\zeta,\partial)}}} \leqslant \beta, \widehat{\Psi_{\aleph}^{\gimel}}(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{\gimel(\zeta,\partial)}}} \leqslant \beta, \widehat{\Psi_{\aleph}^{F}}(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{F(\zeta,\partial)}}} \geqslant \delta, \Psi_{\aleph}^{\top}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top(\zeta,\partial)}} \leqslant \beta, \Psi_{\aleph}^{\gimel}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\gimel(\zeta,\partial)}} \leqslant \beta, \Psi_{\aleph}^{F}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F(\zeta,\partial)}} \geqslant \delta\Big\}$.

Definition 3.4. The Cartesian product of $\aleph$ and Σ is defined as

$$\aleph\times\Sigma = \Big\{\widehat{\Psi_{\aleph\times\Sigma}^{\top}}((\zeta,\rho),\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph\times\Sigma}^{\top((\zeta,\rho),\partial)}}}, \widehat{\Psi_{\aleph\times\Sigma}^{\gimel}}((\zeta,\rho),\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph\times\Sigma}^{\gimel((\zeta,\rho),\partial)}}}, \widehat{\Psi_{\aleph\times\Sigma}^{F}}(\zeta,\rho)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph\times\Sigma}^{F((\zeta,\rho),\partial)}}}, \Psi_{\aleph\times\Sigma}^{\top}((\zeta,\rho),\partial)\cdot e^{i\Bbbk\Xi_{\aleph\times\Sigma}^{\top((\zeta,\rho),\partial)}}, \Psi_{\aleph\times\Sigma}^{\gimel}((\zeta,\rho),\partial)\cdot e^{i\Bbbk\Xi_{\aleph\times\Sigma}^{\gimel((\zeta,\rho),\partial)}}, \Psi_{\aleph\times\Sigma}^{F}(\zeta,\rho)\cdot e^{i\Bbbk\Xi_{\aleph\times\Sigma}^{F((\zeta,\rho),\partial)}} \mid for\ all\ \zeta,\rho\in S\Big\},$$

where $\aleph$ and Σ be the ComCANS of $\mho$

$$\left\{\begin{array}{l}
\widehat{\Psi_{\aleph\times\Sigma}^{\top}}((\zeta,\rho),\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph\times\Sigma}^{\top((\zeta,\rho),\partial)}}} = \max\Big\{\widehat{\Psi_{\aleph}^{\top}}(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{\top(\zeta,\partial)}}}, \widehat{\Psi_{\Sigma}^{\top}}(\rho,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\Sigma}^{\top(\rho,\partial)}}}\Big\}\\
\widehat{\Psi_{\aleph\times\Sigma}^{\gimel}}((\zeta,\rho),\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph\times\Sigma}^{\gimel((\zeta,\rho),\partial)}}} = \frac{\widehat{\Psi_{\aleph}^{\gimel}}(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{\gimel(\zeta,\partial)}}} + \widehat{\Psi_{\Sigma}^{\gimel}}(\rho,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\Sigma}^{\gimel(\rho,\partial)}}}}{2}\\
\widehat{\Psi_{\aleph\times\Sigma}^{F}}((\zeta,\rho),\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph\times\Sigma}^{F((\zeta,\rho),\partial)}}} = \min\Big\{\widehat{\Psi_{\aleph}^{F}}(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{F(\zeta,\partial)}}}, \widehat{\Psi_{\Sigma}^{F}}(\rho,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\Sigma}^{F(\rho,\partial)}}}\Big\}
\end{array}\right\}$$

$$\left\{\begin{array}{l}
\Psi_{\aleph\times\Sigma}^{\top}((\zeta,\rho),\partial)\cdot e^{i\Bbbk\Xi_{\aleph\times\Sigma}^{\top((\zeta,\rho),\partial)}} = \max\Big\{\Psi_{\aleph}^{\top}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top(\zeta,\partial)}}, \Psi_{\Sigma}^{\top}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\Sigma}^{\top(\rho,\partial)}}\Big\}\\
\Psi_{\aleph\times\Sigma}^{\gimel}((\zeta,\rho),\partial)\cdot e^{i\Bbbk\Xi_{\aleph\times\Sigma}^{\gimel((\zeta,\rho),\partial)}} = \frac{\Psi_{\aleph}^{\gimel}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\gimel(\zeta,\partial)}} + \Psi_{\Sigma}^{\gimel}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\Sigma}^{\gimel(\rho,\partial)}}}{2}\\
\Psi_{\aleph\times\Sigma}^{F}((\zeta,\rho),\partial)\cdot e^{i\Bbbk\Xi_{\aleph\times\Sigma}^{F((\zeta,\rho),\partial)}} = \min\Big\{\Psi_{\aleph}^{F}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F(\zeta,\partial)}}, \Psi_{\Sigma}^{F}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\Sigma}^{F(\rho,\partial)}}\Big\}
\end{array}\right\}$$

Definition 3.5. For any ComCANS $\aleph$ of Ⓢ is said to be a complex cubic anti neutrosophic SBS (ComCANSBS) of Ⓢ if

$$\left\{\begin{array}{l}
\widehat{\Psi_{\aleph}^{\top}}((\zeta\ddagger_1\rho),\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{\top}}((\zeta\ddagger_1\rho),\partial)} \leqslant \max\{\widehat{\Psi_{\aleph}^{\top}}(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{\top}}((\zeta,\partial)}, \widehat{\Psi_{\aleph}^{\top}}(\rho,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{\top}}((\rho,\partial)}\}\\
\widehat{\Psi_{\aleph}^{\top}}((\zeta\ddagger_2\rho),\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{\top}}((\zeta\ddagger_2\rho),\partial)} \leqslant \max\{\widehat{\Psi_{\aleph}^{\top}}(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{\top}}((\zeta,\partial)}, \widehat{\Psi_{\aleph}^{\top}}(\rho,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{\top}}((\rho,\partial)}\}\\
\widehat{\Psi_{\aleph}^{\top}}((\zeta\ddagger_3\rho),\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{\top}}((\zeta\ddagger_3\rho),\partial)} \leqslant \max\{\widehat{\Psi_{\aleph}^{\top}}(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{\top}}((\zeta,\partial)}, \widehat{\Psi_{\aleph}^{\top}}(\rho,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{\top}}((\rho,\partial)}\}
\end{array}\right\}$$

$$\left\{\begin{array}{l}
\widehat{\Psi_{\aleph}^{\gimel}}((\zeta\ddagger_1\rho),\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{\gimel}}((\zeta\ddagger_1\rho),\partial)} \leqslant \frac{\widehat{\Psi_{\aleph}^{\gimel}}(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{\gimel}}((\zeta,\partial)} + \widehat{\Psi_{\aleph}^{\gimel}}(\rho,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{\gimel}}((\rho,\partial)}}{2}\\
\text{OR}\\
\widehat{\Psi_{\aleph}^{\gimel}}((\zeta\ddagger_2\rho),\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{\top}}((\zeta\ddagger_2\rho),\partial)} \leqslant \frac{\widehat{\Psi_{\aleph}^{\gimel}}(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{\gimel}}((\zeta,\partial)} + \widehat{\Psi_{\aleph}^{\gimel}}(\rho,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{\gimel}}((\rho,\partial)}}{2}\\
\text{OR}\\
\widehat{\Psi_{\aleph}^{\gimel}}((\zeta\ddagger_3\rho),\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{\gimel}}((\zeta\ddagger_3\rho),\partial)} \leqslant \frac{\widehat{\Psi_{\aleph}^{\gimel}}(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{\gimel}}((\zeta,\partial)} + \widehat{\Psi_{\aleph}^{\gimel}}(\rho,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{\top}}((\rho,\partial)}}{2}
\end{array}\right\}$$

$$\left\{\begin{array}{l}
\widehat{\Psi_{\aleph}^{F}}((\zeta\ddagger_1\rho),\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{F}}((\zeta\ddagger_1\rho),\partial)} \geqslant \min\{\widehat{\Psi_{\aleph}^{F}}(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{F}}((\zeta,\partial)}, \widehat{\Psi_{\aleph}^{F}}(\rho,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{F}}((\rho,\partial)}\}\\
\widehat{\Psi_{\aleph}^{F}}((\zeta\ddagger_2\rho),\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{F}}((\zeta\ddagger_2\rho),\partial)} \geqslant \min\{\widehat{\Psi_{\aleph}^{F}}(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{F}}((\zeta,\partial)}, \widehat{\Psi_{\aleph}^{F}}(\rho,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{F}}((\rho,\partial)}\}\\
\widehat{\Psi_{\aleph}^{F}}((\zeta\ddagger_3\rho),\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{F}}((\zeta\ddagger_3\rho),\partial)} \geqslant \min\{\widehat{\Psi_{\aleph}^{F}}(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{F}}((\zeta,\partial)}, \widehat{\Psi_{\aleph}^{F}}(\rho,\partial)\cdot e^{i\Bbbk\widehat{\Xi_{\aleph}^{F}}((\rho,\partial)}\}
\end{array}\right\}$$

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$$\left\{\begin{array}{l} \Psi_{\aleph}^{\top}((\zeta \ddagger_1 \rho), \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta\ddagger_1\rho),\partial)} \leqslant \max\{\Psi_{\aleph}^{\top}(\zeta,\partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta,\partial)}, \Psi_{\aleph}^{\top}(\rho,\partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\rho,\partial)}\} \\ \Psi_{\aleph}^{\top}((\zeta \ddagger_2 \rho), \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta\ddagger_2\rho),\partial)} \leqslant \max\{\Psi_{\aleph}^{\top}(\zeta,\partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta,\partial)}, \Psi_{\aleph}^{\top}(\rho,\partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\rho,\partial)}\} \\ \Psi_{\aleph}^{\top}((\zeta \ddagger_3 \rho), \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta\ddagger_3\rho),\partial)} \leqslant \max\{\Psi_{\aleph}^{\top}(\zeta,\partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta,\partial)}, \Psi_{\aleph}^{\top}(\rho,\partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\rho,\partial)}\} \end{array}\right\}$$

$$\left\{\begin{array}{l} \Psi_{\aleph}^{\daleth}((\zeta \ddagger_1 \rho), \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}((\zeta\ddagger_1\rho),\partial)} \leqslant \frac{\Psi_{\aleph}^{\daleth}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}((\zeta,\partial)}+\Psi_{\aleph}^{\daleth}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}((\rho,\partial)}}{2} \\ \qquad\qquad \text{OR} \\ \Psi_{\aleph}^{\daleth}((\zeta \ddagger_2 \rho), \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta\ddagger_2\rho),\partial)} \leqslant \frac{\Psi_{\aleph}^{\daleth}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}((\zeta,\partial)}+\Psi_{\aleph}^{\daleth}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}((\rho,\partial)}}{2} \\ \qquad\qquad \text{OR} \\ \Psi_{\aleph}^{\daleth}((\zeta \ddagger_3 \rho), \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}((\zeta\ddagger_3\rho),\partial)} \leqslant \frac{\Psi_{\aleph}^{\daleth}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}((\zeta,\partial)}+\Psi_{\aleph}^{\daleth}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\rho,\partial)}}{2} \end{array}\right\}$$

$$\left\{\begin{array}{l} \Psi_{\aleph}^{F}((\zeta \ddagger_1 \rho), \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\zeta\ddagger_1\rho),\partial)} \geqslant \min\{\Psi_{\aleph}^{F}(\zeta,\partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\zeta,\partial)}, \Psi_{\aleph}^{F}(\rho,\partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\rho,\partial)}\} \\ \Psi_{\aleph}^{F}((\zeta \ddagger_2 \rho), \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\zeta\ddagger_2\rho),\partial)} \geqslant \min\{\Psi_{\aleph}^{F}(\zeta,\partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\zeta,\partial)}, \Psi_{\aleph}^{F}(\rho,\partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\rho,\partial)}\} \\ \Psi_{\aleph}^{F}((\zeta \ddagger_3 \rho), \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\zeta\ddagger_3\rho),\partial)} \geqslant \min\{\Psi_{\aleph}^{F}(\zeta,\partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\zeta,\partial)}, \Psi_{\aleph}^{F}(\rho,\partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\rho,\partial)}\} \end{array}\right\}$$

for all $\zeta, \rho \in \circledS$.

Example 3.6. Consider the bisemiring $\circledS = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$ with the Cayley table:

$\ddagger_1$	ς_1	ς_2	ς_3	ς_4
ς_1	ς_1	ς_1	ς_1	ς_1
ς_2	ς_1	ς_2	ς_1	ς_2
ς_3	ς_1	ς_1	ς_3	ς_3
ς_4	ς_1	ς_2	ς_3	ς_4

$\ddagger_2$	ς_1	ς_2	ς_3	ς_4
ς_1	ς_1	ς_2	ς_3	ς_4
ς_2	ς_2	ς_2	ς_4	ς_4
ς_3	ς_3	ς_4	ς_3	ς_4
ς_4	ς_4	ς_4	ς_4	ς_4

$\ddagger_3$	ς_1	ς_2	ς_3	ς_4
ς_1	ς_1	ς_1	ς_1	ς_1
ς_2	ς_1	ς_2	ς_3	ς_4
ς_3	ς_4	ς_4	ς_4	ς_4
ς_4	ς_4	ς_4	ς_4	ς_4

	$(\xi) = \varsigma_1$	$(\xi) = \varsigma_2$
$(\overparen{\Psi_{\aleph}^{\top}}, \overparen{\Xi_{\aleph}^{\top}})(\xi)$	$[0.75e^{i2\pi(0.65)}, 0.8e^{i2\pi(0.7)}]$	$[0.8e^{i2\pi(0.75)}, 0.85e^{i2\pi(0.8)}]$
$(\overparen{\Psi_{\aleph}^{\daleth}}, \overparen{\Xi_{\aleph}^{\daleth}})(\xi)$	$[0.65e^{i2\pi(0.75)}, 0.7e^{i2\pi(0.8)}]$	$[0.7e^{i2\pi(0.8)}, 0.75e^{i2\pi(0.85)}]$
$(\overparen{\Psi_{\aleph}^{F}}, \overparen{\Xi_{\aleph}^{F}})(\xi)$	$[0.75e^{i2\pi(0.6)}, 0.8e^{i2\pi(0.65)}]$	$[0.65e^{i2\pi(0.7)}, 0.7e^{i2\pi(0.75)}]$

	$(\xi) = \varsigma_3$	$(\xi) = \varsigma_4$
$(\overparen{\Psi_{\aleph}^{\top}}, \overparen{\Xi_{\aleph}^{\top}})(\xi)$	$[0.9e^{i2\pi(0.85)}, 0.95e^{i2\pi(0.9)}]$	$[0.85e^{i2\pi(0.8)}, 0.9e^{i2\pi(0.85)}]$
$(\overparen{\Psi_{\aleph}^{\daleth}}, \overparen{\Xi_{\aleph}^{\daleth}})(\xi)$	$[0.85e^{i2\pi(0.95)}, 0.9e^{i2\pi(1)}]$	$[0.75e^{i2\pi(0.85)}, 0.8e^{i2\pi(0.9)}]$
$(\overparen{\Psi_{\aleph}^{F}}, \overparen{\Xi_{\aleph}^{F}})(\xi)$	$[0.45e^{i2\pi(0.9)}, 0.55e^{i2\pi(0.95)}]$	$[0.55e^{i2\pi(0.8)}, 0.6e^{i2\pi(0.85)}]$

	$(\xi) = \varsigma_1$	$(\xi) = \varsigma_2$
$(\Psi_{\aleph}^{\top}, \Xi_{\aleph}^{\top})(\xi)$	$0.7e^{i2\pi(0.85)}$	$0.75e^{i2\pi(0.9)}$
$(\Psi_{\aleph}^{\daleth}, \Xi_{\aleph}^{\daleth})(\xi)$	$0.55e^{i2\pi(0.8)}$	$0.65e^{i2\pi(0.85)}$
$(\Psi_{\aleph}^{F}, \Xi_{\aleph}^{F})(\xi)$	$0.9e^{i2\pi(0.75)}$	$0.8e^{i2\pi(0.65)}$

	$(\xi) = \varsigma_3$	$(\xi) = \varsigma_4$
$(\Psi_{\aleph}^{\top}, \Xi_{\aleph}^{\top})(\xi)$	$0.85e^{i2\pi(1)}$	$0.8e^{i2\pi(0.95)}$
$(\Psi_{\aleph}^{\daleth}, \Xi_{\aleph}^{\daleth})(\xi)$	$0.75e^{i2\pi(0.95)}$	$0.7e^{i2\pi(0.9)}$
$(\Psi_{\aleph}^{F}, \Xi_{\aleph}^{F})(\xi)$	$0.6e^{i2\pi(0.45)}$	$0.7e^{i2\pi(0.55)}$

Hence, $\aleph$ is a ComCANSBS of $\circledS$.

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Theorem 3.7. *The intersection of a every ComCANSBSs is again a ComCANSBS of* Ⓢ.

Proof. Let $\{\overbrace{\hbar_i} : i \in I\}$ be the family of ComCANSBSs of Ⓢ and $\aleph = \bigcap_{i\in I} \overbrace{\hbar_i}$. Let $\zeta, \rho \in$ Ⓢ.
Now,

$$\begin{aligned}
\overbrace{\Psi^{\top}_{\aleph}}((\zeta \ddagger_1 \rho),\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\top}_{\aleph}}((\zeta\ddagger_1\rho),\partial)} &= \sup_{i\in I}\ \overbrace{\Psi^{\top}_{\hbar_i}}((\zeta \ddagger_1 \rho),\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\top}_{\aleph}}((\zeta\ddagger_1\rho),\partial)}\\
&\leqslant \sup_{i\in I}\ \max\{\overbrace{\Psi^{\top}_{\hbar_i}}(\zeta,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\top}_{\hbar_i}}(\zeta,\partial)}, \overbrace{\Psi^{\top}_{\hbar_i}}(\rho,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\top}_{\hbar_i}}(\rho,\partial)}\}\\
&= \max\left\{\sup_{i\in I}\ \overbrace{\Psi^{\top}_{\hbar_i}}(\zeta,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\top}_{\hbar_i}}(\zeta,\partial)}, \sup_{i\in I}\ \overbrace{\Psi^{\top}_{\hbar_i}}(\rho,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\top}_{\hbar_i}}(\rho,\partial)}\right\}\\
&= \max\{\overbrace{\Psi^{\top}_{\aleph}}(\zeta,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\top}_{\aleph}}(\zeta,\partial)}, \overbrace{\Psi^{\top}_{\aleph}}(\rho,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\top}_{\aleph}}(\rho,\partial)}\}
\end{aligned}$$

Similarly,
$\overbrace{\Psi^{\top}_{\aleph}}((\zeta \ddagger_2 \rho),\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\top}_{\aleph}}((\zeta\ddagger_2\rho),\partial)} \leqslant \max\{\overbrace{\Psi^{\top}_{\aleph}}(\zeta,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\top}_{\aleph}}(\zeta,\partial)}, \overbrace{\Psi^{\top}_{\aleph}}(\rho,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\top}_{\aleph}}(\rho,\partial)}\}$,
$\overbrace{\Psi^{\top}_{\aleph}}((\zeta \ddagger_3 \rho),\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\top}_{\aleph}}((\zeta\ddagger_3\rho),\partial)} \leqslant \max\{\overbrace{\Psi^{\top}_{\aleph}}(\zeta,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\top}_{\aleph}}(\zeta,\partial)}, \overbrace{\Psi^{\top}_{\aleph}}(\rho,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\top}_{\aleph}}(\rho,\partial)}\}$.
Now,

$$\begin{aligned}
\overbrace{\Psi^{\gimel}_{\aleph}}((\zeta \ddagger_1 \rho),\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\gimel}_{\aleph}}((\zeta\ddagger_1\rho),\partial)} &= \sup_{i\in I}\ \overbrace{\Psi^{\gimel}_{\hbar_i}}((\zeta \ddagger_1 \rho),\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\gimel}_{\aleph}}((\zeta\ddagger_1\rho),\partial)}\\
&\leqslant \sup_{i\in I}\ \frac{\overbrace{\Psi^{\gimel}_{\hbar_i}}(\zeta,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\gimel}_{\hbar_i}}(\zeta,\partial)} + \overbrace{\Psi^{\gimel}_{\hbar_i}}(\rho,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\gimel}_{\hbar_i}}(\rho,\partial)}}{2}\\
&= \frac{\sup_{i\in I}\ \overbrace{\Psi^{\gimel}_{\hbar_i}}(\zeta,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\gimel}_{\hbar_i}}(\zeta,\partial)} + \sup_{i\in I}\ \overbrace{\Psi^{\gimel}_{\hbar_i}}(\rho,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\gimel}_{\hbar_i}}(\rho,\partial)}}{2}\\
&= \frac{\overbrace{\Psi^{\gimel}_{\aleph}}(\zeta,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\gimel}_{\aleph}}(\zeta,\partial)} + \overbrace{\Psi^{\gimel}_{\aleph}}(\rho,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\gimel}_{\aleph}}(\rho,\partial)}}{2}
\end{aligned}$$

Similarly,
$\overbrace{\Psi^{\gimel}_{\aleph}}((\zeta \ddagger_2 \rho),\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\gimel}_{\aleph}}((\zeta\ddagger_2\rho),\partial)} \leqslant \frac{\overbrace{\Psi^{\gimel}_{\aleph}}(\zeta,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\gimel}_{\aleph}}(\zeta,\partial)} + \overbrace{\Psi^{\gimel}_{\aleph}}(\rho,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\gimel}_{\aleph}}(\rho,\partial)}}{2}$ and
$\overbrace{\Psi^{\gimel}_{\aleph}}((\zeta \ddagger_3 \rho),\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\gimel}_{\aleph}}((\zeta\ddagger_3\rho),\partial)} \leqslant \frac{\overbrace{\Psi^{\gimel}_{\aleph}}(\zeta,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\gimel}_{\aleph}}(\zeta,\partial)} + \overbrace{\Psi^{\gimel}_{\aleph}}(\rho,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{\gimel}_{\aleph}}(\rho,\partial)}}{2}$.
Now,

$$\begin{aligned}
\overbrace{\Psi^{F}_{\aleph}}((\zeta \ddagger_1 \rho),\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{F}_{\aleph}}((\zeta\ddagger_1\rho),\partial)} &= \inf_{i\in I}\ \overbrace{\Psi^{F}_{\hbar_i}}((\zeta \ddagger_1 \rho),\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{F}_{\aleph}}((\zeta\ddagger_1\rho),\partial)}\\
&\geqslant \inf_{i\in I}\ \min\{\overbrace{\Psi^{F}_{\hbar_i}}(\zeta,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{F}_{\hbar_i}}(\zeta,\partial)}, \overbrace{\Psi^{F}_{\hbar_i}}(\rho,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{F}_{\hbar_i}}(\rho,\partial)}\}\\
&= \min\left\{\inf_{i\in I}\ \overbrace{\Psi^{F}_{\hbar_i}}(\zeta,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{F}_{\hbar_i}}(\zeta,\partial)}, \inf_{i\in I}\ \overbrace{\Psi^{F}_{\hbar_i}}(\rho,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{F}_{\hbar_i}}(\rho,\partial)}\right\}\\
&= \min\{\overbrace{\Psi^{F}_{\aleph}}(\zeta,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{F}_{\aleph}}(\zeta,\partial)}, \overbrace{\Psi^{F}_{\aleph}}(\rho,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{F}_{\aleph}}(\rho,\partial)}\}
\end{aligned}$$

Similarly,
$\overbrace{\Psi^{F}_{\aleph}}((\zeta \ddagger_2 \rho),\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{F}_{\aleph}}((\zeta\ddagger_2\rho),\partial)} \geqslant \min\{\overbrace{\Psi^{F}_{\aleph}}(\zeta,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{F}_{\aleph}}(\zeta,\partial)}, \overbrace{\Psi^{F}_{\aleph}}(\rho,\partial)\cdot e^{i\mathbb{k}\,\overbrace{\Xi^{F}_{\aleph}}(\rho,\partial)}\}$ and

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$\widehat{\Psi_{\aleph}^{F}}((\zeta \ddagger_3 \rho), \partial) \cdot e^{i\Bbbk \widehat{\Xi_{\aleph}^{F}}((\zeta\ddagger_3\rho),\partial)} \geqslant \min\{\widehat{\Psi_{\aleph}^{F}}(\zeta, \partial) \cdot e^{i\Bbbk \widehat{\Xi_{\aleph}^{F}}(\zeta,\partial)}, \widehat{\Psi_{\aleph}^{F}}(\rho, \partial) \cdot e^{i\Bbbk \widehat{\Xi_{\aleph}^{F}}(\rho,\partial)}\}$.
Let $\{\hbar_i : i \in I\}$ be the family of ComCANSBSs of Ⓢ and $\aleph = \bigcap_{i\in I} \hbar_i$. Let $\zeta, \rho \in$ Ⓢ.

Now,

$$\begin{aligned}
\Psi_{\aleph}^{\top}((\zeta \ddagger_1 \rho), \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta\ddagger_1\rho),\partial)} &= \sup_{i\in I} \ \Psi_{\hbar_i}^{\top}((\zeta \ddagger_1 \rho), \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta\ddagger_1\rho),\partial)} \\
&\leqslant \sup_{i\in I} \ \max\{\Psi_{\hbar_i}^{\top}(\zeta, \partial) \cdot e^{i\Bbbk\Xi_{\hbar_i}^{\top}(\zeta,\partial)}, \Psi_{\hbar_i}^{\top}(\rho, \partial) \cdot e^{i\Bbbk\Xi_{\hbar_i}^{\top}(\rho,\partial)}\} \\
&= \max\left\{\sup_{i\in I} \ \Psi_{\hbar_i}^{\top}(\zeta, \partial) \cdot e^{i\Bbbk\Xi_{\hbar_i}^{\top}(\zeta,\partial)}, \sup_{i\in I} \ \Psi_{\hbar_i}^{\top}(\rho, \partial) \cdot e^{i\Bbbk\Xi_{\hbar_i}^{\top}(\rho,\partial)}\right\} \\
&= \max\{\Psi_{\aleph}^{\top}(\zeta, \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\zeta,\partial)}, \Psi_{\aleph}^{\top}(\rho, \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\rho,\partial)}\}
\end{aligned}$$

Similarly,
$\Psi_{\aleph}^{\top}((\zeta \ddagger_2 \rho), \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta\ddagger_2\rho),\partial)} \leqslant \max\{\Psi_{\aleph}^{\top}(\zeta, \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\zeta,\partial)}, \Psi_{\aleph}^{\top}(\rho, \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\rho,\partial)}\}$,
$\Psi_{\aleph}^{\top}((\zeta \ddagger_3 \rho), \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta\ddagger_3\rho),\partial)} \leqslant \max\{\Psi_{\aleph}^{\top}(\zeta, \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\zeta,\partial)}, \Psi_{\aleph}^{\top}(\rho, \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\rho,\partial)}\}$.
Now,

$$\begin{aligned}
\Psi_{\aleph}^{\daleth}((\zeta \ddagger_1 \rho), \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}((\zeta\ddagger_1\rho),\partial)} &= \sup_{i\in I} \ \Psi_{\hbar_i}^{\daleth}((\zeta \ddagger_1 \rho), \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}((\zeta\ddagger_1\rho),\partial)} \\
&\leqslant \sup_{i\in I} \ \frac{\Psi_{\hbar_i}^{\daleth}(\zeta, \partial) \cdot e^{i\Bbbk\Xi_{\hbar_i}^{\daleth}(\zeta,\partial)} + \Psi_{\hbar_i}^{\daleth}(\rho, \partial) \cdot e^{i\Bbbk\Xi_{\hbar_i}^{\daleth}(\rho,\partial)}}{2} \\
&= \frac{\sup_{i\in I} \ \Psi_{\hbar_i}^{\daleth}(\zeta, \partial) \cdot e^{i\Bbbk\Xi_{\hbar_i}^{\daleth}(\zeta,\partial)} + \sup_{i\in I} \ \Psi_{\hbar_i}^{\daleth}(\rho, \partial) \cdot e^{i\Bbbk\Xi_{\hbar_i}^{\daleth}(\rho,\partial)}}{2} \\
&= \frac{\Psi_{\aleph}^{\daleth}(\zeta, \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\zeta,\partial)} + \Psi_{\aleph}^{\daleth}(\rho, \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\rho,\partial)}}{2}
\end{aligned}$$

Similarly,
$\Psi_{\aleph}^{\daleth}((\zeta \ddagger_2 \rho), \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}((\zeta\ddagger_2\rho),\partial)} \leqslant \frac{\Psi_{\aleph}^{\daleth}(\zeta, \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\zeta,\partial)} + \Psi_{\aleph}^{\daleth}(\rho, \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\rho,\partial)}}{2}$ and
$\Psi_{\aleph}^{\daleth}((\zeta \ddagger_3 \rho), \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}((\zeta\ddagger_3\rho),\partial)} \leqslant \frac{\Psi_{\aleph}^{\daleth}(\zeta, \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\zeta,\partial)} + \Psi_{\aleph}^{\daleth}(\rho, \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\rho,\partial)}}{2}$.
Now,

$$\begin{aligned}
\Psi_{\aleph}^{F}((\zeta \ddagger_1 \rho), \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\zeta\ddagger_1\rho),\partial)} &= \inf_{i\in I} \ \Psi_{\hbar_i}^{F}((\zeta \ddagger_1 \rho), \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\zeta\ddagger_1\rho),\partial)} \\
&\geqslant \inf_{i\in I} \ \min\{\Psi_{\hbar_i}^{F}(\zeta, \partial) \cdot e^{i\Bbbk\Xi_{\hbar_i}^{F}(\zeta,\partial)}, \Psi_{\hbar_i}^{F}(\rho, \partial) \cdot e^{i\Bbbk\Xi_{\hbar_i}^{F}(\rho,\partial)}\} \\
&= \min\left\{\inf_{i\in I} \ \Psi_{\hbar_i}^{F}(\zeta, \partial) \cdot e^{i\Bbbk\Xi_{\hbar_i}^{F}(\zeta,\partial)}, \inf_{i\in I} \ \Psi_{\hbar_i}^{F}(\rho, \partial) \cdot e^{i\Bbbk\Xi_{\hbar_i}^{F}(\rho,\partial)}\right\} \\
&= \min\{\Psi_{\aleph}^{F}(\zeta, \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\zeta,\partial)}, \Psi_{\aleph}^{F}(\rho, \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\rho,\partial)}\}
\end{aligned}$$

Similarly,
$\Psi_{\aleph}^{F}((\zeta \ddagger_2 \rho), \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\zeta\ddagger_2\rho),\partial)} \geqslant \min\{\Psi_{\aleph}^{F}(\zeta, \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\zeta,\partial)}, \Psi_{\aleph}^{F}(\rho, \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\rho,\partial)}\}$ and
$\Psi_{\aleph}^{F}((\zeta \ddagger_3 \rho), \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\zeta\ddagger_3\rho),\partial)} \geqslant \min\{\Psi_{\aleph}^{F}(\zeta, \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\zeta,\partial)}, \Psi_{\aleph}^{F}(\rho, \partial) \cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\rho,\partial)}\}$.
Thus, $\aleph$ is a ComCANSBS of Ⓢ.

Theorem 3.8. *If $\aleph$ and Σ be the ComCANSBSs of $Ⓢ_1$ and $Ⓢ_2$ respectively, then $\widehat{\aleph \times \Sigma}$ is a ComCANSBS of $Ⓢ_1 \times Ⓢ_2$.*

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Proof. Let $\zeta_1, \zeta_2 \in \circledS_1$ and $\rho_1, \rho_2 \in \circledS_2$. Then (ζ_1, ρ_1) and (ζ_2, ρ_2) are in $\circledS_1 \times \circledS_2$. Now

$$\begin{aligned}
&\overbrace{\Psi^{\top}_{\aleph\times\Sigma}}[((\zeta_1,\rho_1)\ddagger_1(\zeta_2,\rho_2),\partial)]\cdot e^{i\Bbbk\overbrace{\Xi^{\top}_{\aleph\times\Sigma}}[((\zeta_1,\rho_1)\ddagger_1(\zeta_2,\rho_2),\partial)]}\\
=\;&\overbrace{\Psi^{\top}_{\aleph\times\Sigma}}((\zeta_1\ddagger_1\zeta_2,\rho_1\ddagger_1\rho_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi^{\top}_{\aleph\times\Sigma}}((\zeta_1\ddagger_1\zeta_2,\rho_1\ddagger_1\rho_2),\partial)}\\
=\;&\max\{\overbrace{\Psi^{\top}_{\aleph}}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi^{\top}_{\aleph}}((\zeta_1\ddagger_1\zeta_2),\partial)},\overbrace{\Psi^{\top}_{\Sigma}}((\rho_1\ddagger_1\rho_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi^{\top}_{\Sigma}}((\rho_1\ddagger_1\rho_2),\partial)}\}\\
\leqslant\;&\max\{\max\{\overbrace{\Psi^{\top}_{\aleph}}(\zeta_1)\cdot e^{i\Bbbk\overbrace{\Xi^{\top}_{\aleph}}(\zeta_1)},\overbrace{\Psi^{\top}_{\aleph}}(\zeta_2)\cdot e^{i\Bbbk\overbrace{\Xi^{\top}_{\aleph}}(\zeta_2)}\},\\
&\max\{\overbrace{\Psi^{\top}_{\Sigma}}(\rho_1)\cdot e^{i\Bbbk\overbrace{\Xi^{\top}_{\aleph}}(\rho_1)},\overbrace{\Psi^{\top}_{\Sigma}}(\rho_2)\cdot e^{i\Bbbk\overbrace{\Xi^{\top}_{\aleph}}(\rho_2)}\}\}\\
=\;&\max\{\max\{\overbrace{\Psi^{\top}_{\aleph}}(\zeta_1)\cdot e^{i\Bbbk\overbrace{\Xi^{\top}_{\aleph}}(\zeta_1)},\overbrace{\Psi^{\top}_{\Sigma}}(\rho_1)\cdot e^{i\Bbbk\overbrace{\Xi^{\top}_{\aleph}}(\rho_1)}\},\\
&\max\{\overbrace{\Psi^{\top}_{\aleph}}(\zeta_2)\cdot e^{i\Bbbk\overbrace{\Xi^{\top}_{\aleph}}(\zeta_2)},\overbrace{\Psi^{\top}_{\Sigma}}(\rho_2)\cdot e^{i\Bbbk\overbrace{\Xi^{\top}_{\aleph}}(\rho_2)}\}\}\\
=\;&\max\{\overbrace{\Psi^{\top}_{\aleph\times\Sigma}}((\zeta_1,\rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi^{\top}_{\aleph\times\Sigma}}((\zeta_1,\rho_1),\partial)},\overbrace{\Psi^{\top}_{\aleph\times\Sigma}}((\zeta_2,\rho_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi^{\top}_{\aleph\times\Sigma}}((\zeta_2,\rho_2),\partial)}\}
\end{aligned}$$

Also $\overbrace{\Psi^{\top}_{\aleph\times\Sigma}}[((\zeta_1,\rho_1)\ddagger_2(\zeta_2,\rho_2),\partial)]\cdot e^{i\Bbbk\overbrace{\Xi^{\top}_{\aleph\times\Sigma}}[((\zeta_1,\rho_1)\ddagger_2(\zeta_2,\rho_2),\partial)]}$
$\leqslant\max\{\overbrace{\Psi^{\top}_{\aleph\times\Sigma}}((\zeta_1,\rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi^{\top}_{\aleph\times\Sigma}}((\zeta_1,\rho_1),\partial)},\overbrace{\Psi^{\top}_{\aleph\times\Sigma}}((\zeta_2,\rho_2),\partial)e^{i\Bbbk\overbrace{\Xi^{\top}_{\aleph\times\Sigma}}((\zeta_2,\rho_2),\partial)}\}$

and $\overbrace{\Psi^{\top}_{\aleph\times\Sigma}}[((\zeta_1,\rho_1)\ddagger_3(\zeta_2,\rho_2),\partial)]\cdot e^{i\Bbbk\overbrace{\Xi^{\top}_{\aleph\times\Sigma}}[((\zeta_1,\rho_1)\ddagger_3(\zeta_2,\rho_2),\partial)]}$
$\leqslant\max\{\overbrace{\Psi^{\top}_{\aleph\times\Sigma}}((\zeta_1,\rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi^{\top}_{\aleph\times\Sigma}}((\zeta_1,\rho_1),\partial)},\overbrace{\Psi^{\top}_{\aleph\times\Sigma}}((\zeta_2,\rho_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi^{\top}_{\aleph\times\Sigma}}((\zeta_2,\rho_2),\partial)}\}.$
Now,

$$\begin{aligned}
&\overbrace{\Psi^{\gimel}_{\aleph\times\Sigma}}[((\zeta_1,\rho_1)\ddagger_1(\zeta_2,\rho_2),\partial)]\cdot e^{i\Bbbk\overbrace{\Xi^{\gimel}_{\aleph\times\Sigma}}[((\zeta_1,\rho_1)\ddagger_1(\zeta_2,\rho_2),\partial)]}\\
=\;&\overbrace{\Psi^{\gimel}_{\aleph\times\Sigma}}((\zeta_1\ddagger_1\zeta_2,\rho_1\ddagger_1\rho_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi^{\gimel}_{\aleph\times\Sigma}}((\zeta_1\ddagger_1\zeta_2,\rho_1\ddagger_1\rho_2),\partial)}\\
=\;&\frac{\overbrace{\Psi^{\gimel}_{\aleph}}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi^{\gimel}_{\aleph\times\Sigma}}((\zeta_1\ddagger_1\zeta_2),\partial)}+\overbrace{\Psi^{\gimel}_{\Sigma}}((\rho_1\ddagger_1\rho_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi^{\top}_{\aleph\times\Sigma}}((\rho_1\ddagger_1\rho_2),\partial)}}{2}\\
\leqslant\;&\frac{1}{2}\left[\frac{\overbrace{\Psi^{\gimel}_{\aleph}}(\zeta_1)\cdot e^{i\Bbbk\overbrace{\Xi^{\gimel}_{\aleph}}(\zeta_1)}+\overbrace{\Psi^{\gimel}_{\aleph}}(\zeta_2)\cdot e^{i\Bbbk\overbrace{\Xi^{\gimel}_{\aleph}}(\zeta_2)}}{2}+\frac{\overbrace{\Psi^{\gimel}_{\Sigma}}(\rho_1)\cdot e^{i\Bbbk\overbrace{\Xi^{\gimel}_{\Sigma}}(\rho_1)}+\overbrace{\Psi^{\gimel}_{\Sigma}}(\rho_2)\cdot e^{i\Bbbk\overbrace{\Xi^{\gimel}_{\Sigma}}(\rho_2)}}{2}\right]\\
=\;&\frac{1}{2}\left[\frac{\overbrace{\Psi^{\gimel}_{\aleph}}(\zeta_1)\cdot e^{i\Bbbk\overbrace{\Xi^{\gimel}_{\aleph}}(\zeta_1)}+\overbrace{\Psi^{\gimel}_{\Sigma}}(\rho_1)\cdot e^{i\Bbbk\overbrace{\Xi^{\gimel}_{\Sigma}}(\rho_1)}}{2}+\frac{\overbrace{\Psi^{\gimel}_{\aleph}}(\zeta_2)\cdot e^{i\Bbbk\overbrace{\Xi^{\gimel}_{\aleph}}(\zeta_2)}+\overbrace{\Psi^{\gimel}_{\Sigma}}(\rho_2)\cdot e^{i\Bbbk\overbrace{\Xi^{\gimel}_{\Sigma}}(\rho_2)}}{2}\right]\\
=\;&\frac{1}{2}\left[\overbrace{\Psi^{\gimel}_{\aleph\times\Sigma}}((\zeta_1,\rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi^{\top}_{\aleph\times\Sigma}}((\zeta_1,\rho_1),\partial)}+\overbrace{\Psi^{\gimel}_{\aleph\times\Sigma}}((\zeta_2,\rho_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi^{\top}_{\aleph\times\Sigma}}((\zeta_2,\rho_2),\partial)}\right]
\end{aligned}$$

Also
$\overbrace{\Psi^{\gimel}_{\aleph\times\Sigma}}[((\zeta_1,\rho_1)\ddagger_2(\zeta_2,\rho_2),\partial)]\cdot e^{i\Bbbk\overbrace{\Xi^{\gimel}_{\aleph\times\Sigma}}[((\zeta_1,\rho_1)\ddagger_2(\zeta_2,\rho_2),\partial)]}\leqslant\frac{1}{2}\Big[\overbrace{\Psi^{\gimel}_{\aleph\times\Sigma}}((\zeta_1,\rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi^{\gimel}_{\aleph\times\Sigma}}((\zeta_1,\rho_1),\partial)}+$
$\overbrace{\Psi^{\gimel}_{\aleph\times\Sigma}}((\zeta_2,\rho_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi^{\gimel}_{\aleph\times\Sigma}}((\zeta_2,\rho_2),\partial)}\Big]$
and
$\overbrace{\Psi^{\gimel}_{\aleph\times\Sigma}}[((\zeta_1,\rho_1)\ddagger_3(\zeta_2,\rho_2),\partial)]\cdot e^{i\Bbbk\overbrace{\Xi^{\gimel}_{\aleph\times\Sigma}}[((\zeta_1,\rho_1)\ddagger_3(\zeta_2,\rho_2),\partial)]}\leqslant\frac{1}{2}\Big[\overbrace{\Psi^{\gimel}_{\aleph\times\Sigma}}((\zeta_1,\rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi^{\gimel}_{\aleph\times\Sigma}}((\zeta_1,\rho_1),\partial)}+$

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$\overbrace{\Psi^{\daleth}_{\aleph\times\Sigma}}((\zeta_2,\rho_2),\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{\daleth}_{\aleph\times\Sigma}}((\zeta_2,\rho_2),\partial)}\Big]$.
Now,

$$\begin{aligned}
&\overbrace{\Psi^{F}_{\aleph\times\Sigma}}[((\zeta_1,\rho_1)\ddagger_1(\zeta_2,\rho_2),\partial)]\cdot e^{i\Bbbk\,\overbrace{\Xi^{F}_{\aleph\times\Sigma}}[((\zeta_1,\rho_1)\ddagger_1(\zeta_2,\rho_2),\partial)]}\\
=\;&\overbrace{\Psi^{F}_{\aleph\times\Sigma}}((\zeta_1\ddagger_1\zeta_2,\rho_1\ddagger_1\rho_2),\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{F}_{\aleph\times\Sigma}}((\zeta_1\ddagger_1\zeta_2,\rho_1\ddagger_1\rho_2),\partial)}\\
=\;&\min\{\overbrace{\Psi^{F}_{\aleph}}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{F}_{\aleph\times\Sigma}}((\zeta_1\ddagger_1\zeta_2),\partial)},\overbrace{\Psi^{F}_{\Sigma}}((\rho_1\ddagger_1\rho_2),\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{F}_{\aleph\times\Sigma}}((\rho_1\ddagger_1\rho_2),\partial)}\}\\
\geqslant\;&\min\{\min\{\overbrace{\Psi^{F}_{\aleph}}(\zeta_1)\cdot e^{i\Bbbk\,\overbrace{\Xi^{F}_{\aleph}}(\zeta_1)},\overbrace{\Psi^{F}_{\aleph}}(\zeta_2)\cdot e^{i\Bbbk\,\overbrace{\Xi^{F}_{\aleph}}(\zeta_2)}\},\\
&\min\{\overbrace{\Psi^{F}_{\Sigma}}(\rho_1)\cdot e^{i\Bbbk\,\overbrace{\Xi^{F}_{\Sigma}}(\rho_1)},\overbrace{\Psi^{F}_{\Sigma}}(\rho_2)\cdot e^{i\Bbbk\,\overbrace{\Xi^{F}_{\Sigma}}(\rho_2)}\}\}\\
=\;&\min\{\min\{\overbrace{\Psi^{F}_{\aleph}}(\zeta_1)\cdot e^{i\Bbbk\,\overbrace{\Xi^{F}_{\aleph}}(\zeta_1)},\overbrace{\Psi^{F}_{\Sigma}}(\rho_1)\cdot e^{i\Bbbk\,\overbrace{\Xi^{F}_{\Sigma}}(\rho_1)}\},\\
&\min\{\overbrace{\Psi^{F}_{\aleph}}(\zeta_2)\cdot e^{i\Bbbk\,\overbrace{\Xi^{F}_{\aleph}}(\zeta_2)},\overbrace{\Psi^{F}_{\Sigma}}(\rho_2)\cdot e^{i\Bbbk\,\overbrace{\Xi^{F}_{\Sigma}}(\rho_2)}\}\}\\
=\;&\min\{\overbrace{\Psi^{F}_{\aleph\times\Sigma}}((\zeta_1,\rho_1),\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{\top}_{\aleph\times\Sigma}}((\zeta_1,\rho_1),\partial)},\overbrace{\Psi^{F}_{\aleph\times\Sigma}}((\zeta_2,\rho_2),\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{\top}_{\aleph\times\Sigma}}((\zeta_2,\rho_2),\partial)}\}
\end{aligned}$$

Also $\overbrace{\Psi^{F}_{\aleph\times\Sigma}}[((\zeta_1,\rho_1)\ddagger_2(\zeta_2,\rho_2),\partial)]\cdot e^{i\Bbbk\,\overbrace{\Xi^{\top}_{\aleph\times\Sigma}}[((\zeta_1,\rho_1)\ddagger_2(\zeta_2,\rho_2),\partial)]}$
$\geqslant\min\{\overbrace{\Psi^{F}_{\aleph\times\Sigma}}((\zeta_1,\rho_1),\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{\top}_{\aleph\times\Sigma}}((\zeta_1,\rho_1),\partial)},\overbrace{\Psi^{F}_{\aleph\times\Sigma}}((\zeta_2,\rho_2),\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{\top}_{\aleph\times\Sigma}}}((\zeta_2,\rho_2),\partial)\}$,
$\overbrace{\Psi^{F}_{\aleph\times\Sigma}}[((\zeta_1,\rho_1)\ddagger_3(\zeta_2,\rho_2),\partial)]\cdot e^{i\Bbbk\,\overbrace{\Xi^{\top}_{\aleph\times\Sigma}}[((\zeta_1,\rho_1)\ddagger_3(\zeta_2,\rho_2),\partial)]}$
$\geqslant\min\{\overbrace{\Psi^{F}_{\aleph\times\Sigma}}((\zeta_1,\rho_1),\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{\top}_{\aleph\times\Sigma}}((\zeta_1,\rho_1),\partial)},\overbrace{\Psi^{F}_{\aleph\times\Sigma}}((\zeta_2,\rho_2),\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{\top}_{\aleph\times\Sigma}}((\zeta_2,\rho_2),\partial)}\}$.

Let $\zeta_1,\zeta_2\in\circledS_1$ and $\rho_1,\rho_2\in\circledS_2$. Then (ζ_1,ρ_1) and (ζ_2,ρ_2) are in $\circledS_1\times\circledS_2$. Now

$$\begin{aligned}
&\Psi^{\top}_{\aleph\times\Sigma}[((\zeta_1,\rho_1)\ddagger_1(\zeta_2,\rho_2),\partial)]\cdot e^{i\Bbbk\Xi^{\top}_{\aleph\times\Sigma}[((\zeta_1,\rho_1)\ddagger_1(\zeta_2,\rho_2),\partial)]}\\
=\;&\Psi^{\top}_{\aleph\times\Sigma}((\zeta_1\ddagger_1\zeta_2,\rho_1\ddagger_1\rho_2),\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph\times\Sigma}((\zeta_1\ddagger_1\zeta_2,\rho_1\ddagger_1\rho_2),\partial)}\\
=\;&\max\{\Psi^{\top}_{\aleph}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}((\zeta_1\ddagger_1\zeta_2),\partial)},\Psi^{\top}_{\Sigma}((\rho_1\ddagger_1\rho_2),\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\Sigma}((\rho_1\ddagger_1\rho_2),\partial)}\}\\
\leqslant\;&\max\{\max\{\Psi^{\top}_{\aleph}(\zeta_1)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}(\zeta_1)},\Psi^{\top}_{\aleph}(\zeta_2)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}(\zeta_2)}\},\max\{\Psi^{\top}_{\Sigma}(\rho_1)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}(\rho_1)},\Psi^{\top}_{\Sigma}(\rho_2)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}(\rho_2)}\}\}\\
=\;&\max\{\max\{\Psi^{\top}_{\aleph}(\zeta_1)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}(\zeta_1)},\Psi^{\top}_{\Sigma}(\rho_1)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}(\rho_1)}\},\max\{\Psi^{\top}_{\aleph}(\zeta_2)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}(\zeta_2)},\Psi^{\top}_{\Sigma}(\rho_2)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}(\rho_2)}\}\}\\
=\;&\max\{\Psi^{\top}_{\aleph\times\Sigma}((\zeta_1,\rho_1),\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph\times\Sigma}((\zeta_1,\rho_1),\partial)},\Psi^{\top}_{\aleph\times\Sigma}((\zeta_2,\rho_2),\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph\times\Sigma}((\zeta_2,\rho_2),\partial)}\}
\end{aligned}$$

Also $\Psi^{\top}_{\aleph\times\Sigma}[((\zeta_1,\rho_1)\ddagger_2(\zeta_2,\rho_2),\partial)]\cdot e^{i\Bbbk\Xi^{\top}_{\aleph\times\Sigma}[((\zeta_1,\rho_1)\ddagger_2(\zeta_2,\rho_2),\partial)]}$
$\leqslant\max\{\Psi^{\top}_{\aleph\times\Sigma}((\zeta_1,\rho_1),\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph\times\Sigma}((\zeta_1,\rho_1),\partial)},\Psi^{\top}_{\aleph\times\Sigma}((\zeta_2,\rho_2),\partial)e^{i\Bbbk\Xi^{\top}_{\aleph\times\Sigma}((\zeta_2,\rho_2),\partial)}\}$

and $\Psi^{\top}_{\aleph\times\Sigma}[((\zeta_1,\rho_1)\ddagger_3(\zeta_2,\rho_2),\partial)]\cdot e^{i\Bbbk\Xi^{\top}_{\aleph\times\Sigma}[((\zeta_1,\rho_1)\ddagger_3(\zeta_2,\rho_2),\partial)]}$
$\leqslant\max\{\Psi^{\top}_{\aleph\times\Sigma}((\zeta_1,\rho_1),\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph\times\Sigma}((\zeta_1,\rho_1),\partial)},\Psi^{\top}_{\aleph\times\Sigma}((\zeta_2,\rho_2),\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph\times\Sigma}((\zeta_2,\rho_2),\partial)}\}$.

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Now,

$$\begin{aligned}
&\Psi^{ℷ}_{\aleph\times\Sigma}[((\zeta_1,\rho_1)\ddagger_1(\zeta_2,\rho_2),\partial)]\cdot e^{i\Bbbk\Xi^{ℷ}_{\aleph\times\Sigma}[((\zeta_1,\rho_1)\ddagger_1(\zeta_2,\rho_2),\partial)]}\\
=\;&\Psi^{ℷ}_{\aleph\times\Sigma}((\zeta_1\ddagger_1\zeta_2,\rho_1\ddagger_1\rho_2),\partial)\cdot e^{i\Bbbk\Xi^{ℷ}_{\aleph\times\Sigma}((\zeta_1\ddagger_1\zeta_2,\rho_1\ddagger_1\rho_2),\partial)}\\
=\;&\frac{\Psi^{ℷ}_{\aleph}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk\Xi^{ℷ}_{\aleph\times\Sigma}((\zeta_1\ddagger_1\zeta_2),\partial)}+\Psi^{ℷ}_{\Sigma}((\rho_1\ddagger_1\rho_2),\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph\times\Sigma}((\rho_1\ddagger_1\rho_2),\partial)}}{2}\\
\leqslant\;&\frac{1}{2}\left[\frac{\Psi^{ℷ}_{\aleph}(\zeta_1)\cdot e^{i\Bbbk\Xi^{ℷ}_{\aleph}(\zeta_1)}+\Psi^{ℷ}_{\aleph}(\zeta_2)\cdot e^{i\Bbbk\Xi^{ℷ}_{\aleph}(\zeta_2)}}{2}+\frac{\Psi^{ℷ}_{\Sigma}(\rho_1)\cdot e^{i\Bbbk\Xi^{ℷ}_{\Sigma}(\rho_1)}+\Psi^{ℷ}_{\Sigma}(\rho_2)\cdot e^{i\Bbbk\Xi^{ℷ}_{\Sigma}(\rho_2)}}{2}\right]\\
=\;&\frac{1}{2}\left[\frac{\Psi^{ℷ}_{\aleph}(\zeta_1)\cdot e^{i\Bbbk\Xi^{ℷ}_{\aleph}(\zeta_1)}+\Psi^{ℷ}_{\Sigma}(\rho_1)\cdot e^{i\Bbbk\Xi^{ℷ}_{\Sigma}(\rho_1)}}{2}+\frac{\Psi^{ℷ}_{\aleph}(\zeta_2)\cdot e^{i\Bbbk\Xi^{ℷ}_{\aleph}(\zeta_2)}+\Psi^{ℷ}_{\Sigma}(\rho_2)\cdot e^{i\Bbbk\Xi^{ℷ}_{\Sigma}(\rho_2)}}{2}\right]\\
=\;&\frac{1}{2}\left[\Psi^{ℷ}_{\aleph\times\Sigma}((\zeta_1,\rho_1),\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph\times\Sigma}((\zeta_1,\rho_1),\partial)}+\Psi^{ℷ}_{\aleph\times\Sigma}((\zeta_2,\rho_2),\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph\times\Sigma}((\zeta_2,\rho_2),\partial)}\right]
\end{aligned}$$

Also
$\Psi^{ℷ}_{\aleph\times\Sigma}[((\zeta_1,\rho_1)\ddagger_2(\zeta_2,\rho_2),\partial)]\cdot e^{i\Bbbk\Xi^{ℷ}_{\aleph\times\Sigma}[((\zeta_1,\rho_1)\ddagger_2(\zeta_2,\rho_2),\partial)]}\leqslant\frac{1}{2}\Big[\Psi^{ℷ}_{\aleph\times\Sigma}((\zeta_1,\rho_1),\partial)\cdot e^{i\Bbbk\Xi^{ℷ}_{\aleph\times\Sigma}((\zeta_1,\rho_1),\partial)}+\Psi^{ℷ}_{\aleph\times\Sigma}((\zeta_2,\rho_2),\partial)\cdot e^{i\Bbbk\Xi^{ℷ}_{\aleph\times\Sigma}((\zeta_2,\rho_2),\partial)}\Big]$
and
$\Psi^{ℷ}_{\aleph\times\Sigma}[((\zeta_1,\rho_1)\ddagger_3(\zeta_2,\rho_2),\partial)]\cdot e^{i\Bbbk\Xi^{ℷ}_{\aleph\times\Sigma}[((\zeta_1,\rho_1)\ddagger_3(\zeta_2,\rho_2),\partial)]}\leqslant\frac{1}{2}\Big[\Psi^{ℷ}_{\aleph\times\Sigma}((\zeta_1,\rho_1),\partial)\cdot e^{i\Bbbk\Xi^{ℷ}_{\aleph\times\Sigma}((\zeta_1,\rho_1),\partial)}+\Psi^{ℷ}_{\aleph\times\Sigma}((\zeta_2,\rho_2),\partial)\cdot e^{i\Bbbk\Xi^{ℷ}_{\aleph\times\Sigma}((\zeta_2,\rho_2),\partial)}\Big]$.
Now,

$$\begin{aligned}
&\Psi^{F}_{\aleph\times\Sigma}[((\zeta_1,\rho_1)\ddagger_1(\zeta_2,\rho_2),\partial)]\cdot e^{i\Bbbk\Xi^{F}_{\aleph\times\Sigma}[((\zeta_1,\rho_1)\ddagger_1(\zeta_2,\rho_2),\partial)]}\\
=\;&\Psi^{F}_{\aleph\times\Sigma}((\zeta_1\ddagger_1\zeta_2,\rho_1\ddagger_1\rho_2),\partial)\cdot e^{i\Bbbk\Xi^{F}_{\aleph\times\Sigma}((\zeta_1\ddagger_1\zeta_2,\rho_1\ddagger_1\rho_2),\partial)}\\
=\;&\min\{\Psi^{F}_{\aleph}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk\Xi^{F}_{\aleph\times\Sigma}((\zeta_1\ddagger_1\zeta_2),\partial)},\Psi^{F}_{\Sigma}((\rho_1\ddagger_1\rho_2),\partial)\cdot e^{i\Bbbk\Xi^{F}_{\aleph\times\Sigma}((\rho_1\ddagger_1\rho_2),\partial)}\}\\
\geqslant\;&\min\{\min\{\Psi^{F}_{\aleph}(\zeta_1)\cdot e^{i\Bbbk\Xi^{F}_{\aleph}(\zeta_1)},\Psi^{F}_{\aleph}(\zeta_2)\cdot e^{i\Bbbk\Xi^{F}_{\aleph}(\zeta_2)}\},\min\{\Psi^{F}_{\Sigma}(\rho_1)\cdot e^{i\Bbbk\Xi^{F}_{\Sigma}(\rho_1)},\Psi^{F}_{\Sigma}(\rho_2)\cdot e^{i\Bbbk\Xi^{F}_{\Sigma}(\rho_2)}\}\}\\
=\;&\min\{\min\{\Psi^{F}_{\aleph}(\zeta_1)\cdot e^{i\Bbbk\Xi^{F}_{\aleph}(\zeta_1)},\Psi^{F}_{\Sigma}(\rho_1)\cdot e^{i\Bbbk\Xi^{F}_{\Sigma}(\rho_1)}\},\min\{\Psi^{F}_{\aleph}(\zeta_2)\cdot e^{i\Bbbk\Xi^{F}_{\aleph}(\zeta_2)},\Psi^{F}_{\Sigma}(\rho_2)\cdot e^{i\Bbbk\Xi^{F}_{\Sigma}(\rho_2)}\}\}\\
=\;&\min\{\Psi^{F}_{\aleph\times\Sigma}((\zeta_1,\rho_1),\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph\times\Sigma}((\zeta_1,\rho_1),\partial)},\Psi^{F}_{\aleph\times\Sigma}((\zeta_2,\rho_2),\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph\times\Sigma}((\zeta_2,\rho_2),\partial)}\}
\end{aligned}$$

Also $\Psi^{F}_{\aleph\times\Sigma}[((\zeta_1,\rho_1)\ddagger_2(\zeta_2,\rho_2),\partial)]\cdot e^{i\Bbbk\Xi^{\top}_{\aleph\times\Sigma}[((\zeta_1,\rho_1)\ddagger_2(\zeta_2,\rho_2),\partial)]}$
$\geqslant\min\{\Psi^{F}_{\aleph\times\Sigma}((\zeta_1,\rho_1),\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph\times\Sigma}((\zeta_1,\rho_1),\partial)},\Psi^{F}_{\aleph\times\Sigma}((\zeta_2,\rho_2),\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph\times\Sigma}}((\zeta_2,\rho_2),\partial)\}$,
$\Psi^{F}_{\aleph\times\Sigma}[((\zeta_1,\rho_1)\ddagger_3(\zeta_2,\rho_2),\partial)]\cdot e^{i\Bbbk\Xi^{\top}_{\aleph\times\Sigma}[((\zeta_1,\rho_1)\ddagger_3(\zeta_2,\rho_2),\partial)]}$
$\geqslant\min\{\Psi^{F}_{\aleph\times\Sigma}((\zeta_1,\rho_1),\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph\times\Sigma}((\zeta_1,\rho_1),\partial)},\Psi^{F}_{\aleph\times\Sigma}((\zeta_2,\rho_2),\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph\times\Sigma}((\zeta_2,\rho_2),\partial)}\}$.
Thus, $\overbrace{\aleph\times\Sigma}$ is a ComCANSBS of Ⓢ.

Corollary 3.9. *If* $\aleph_1,\aleph_2,...,\aleph_n$ *be the finite collection of ComCANSBSs of* Ⓢ$_1$, Ⓢ$_2$, ..., Ⓢ$_n$ *respectively. Then* $\aleph_1\times\aleph_2\times...\times\aleph_n$ *is a ComCANSBS of* Ⓢ$_1\times$ Ⓢ$_2\times...\times$ Ⓢ$_n$.

Definition 3.10. Let $\aleph\subseteq$ Ⓢ,the strongest ComCAN relation on Ⓢ is

$$\left\{\begin{array}{l}
\overbrace{\Psi^{\top}_{\hbar}}((\zeta,\rho),\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{\top}_{\hbar}}((\zeta,\rho),\partial)}=min\{\overbrace{\Psi^{\top}_{\aleph}}(\zeta,\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{\top}_{\aleph}}(\zeta,\partial)},\overbrace{\Psi^{\top}_{\aleph}}(\rho,\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{\top}_{\aleph}}(\rho,\partial)}\}\\
\overbrace{\Psi^{ℷ}_{\hbar}}((\zeta,\rho),\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{ℷ}_{\hbar}}((\zeta,\rho),\partial)}=\frac{\overbrace{\Psi^{ℷ}_{\aleph}}(\zeta,\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{ℷ}_{\aleph}}(\zeta,\partial)}+\overbrace{\Psi^{ℷ}_{\aleph}}(\rho,\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{ℷ}_{\aleph}}(\rho,\partial)}}{2}\\
\overbrace{\Psi^{F}_{\hbar}}((\zeta,\rho),\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{F}_{\hbar}}((\zeta,\rho),\partial)}=max\{\overbrace{\Psi^{F}_{\aleph}}(\zeta,\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{F}_{\aleph}}(\zeta,\partial)},\overbrace{\Psi^{F}_{\aleph}}(\rho,\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{F}_{\aleph}}(\rho,\partial)}\}
\end{array}\right\}$$

Theorem 3.11. *Let* $\aleph$ *be a ComCANSBS of* Ⓢ *and* $\hbar$ *be a strongest complex cubic anti neutrosophic relation of* Ⓢ. *Then* $\aleph$ *is a ComCANSBS of* Ⓢ × Ⓢ *if and only if* $\hbar$ *is a ComCANSBS of* Ⓢ × Ⓢ.

Proof. Suppose $\aleph$ is a ComCANSBS of Ⓢ × Ⓢ and $\overbrace{\hbar}$ be the strongest complex cubic anti neutrosophic relation of Ⓢ.

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For any $\zeta = (\zeta_1, \zeta_2), \rho = (\rho_1, \rho_2) \in S \times Ⓢ$. Now,

$$\begin{aligned}
& \overbrace{\Psi_\hbar^\top}((\zeta \ddagger_1 \rho), \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\top}((\zeta\ddagger_1\rho),\partial)} \\
= \;& \overbrace{\Psi_\hbar^\top}[(((((\zeta_1, \zeta_2), \partial) \ddagger_1 ((\rho_1, \rho_2), \partial)] \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\top}[(((((\zeta_1,\zeta_2),\partial)\ddagger_1((\rho_1,\rho_2),\partial)]} \\
= \;& \overbrace{\Psi_\hbar^\top}(\zeta_1 \ddagger_1 \rho_1, \zeta_2 \ddagger_1 \rho_2) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\top}(\zeta_1\ddagger_1\rho_1,\zeta_2\ddagger_1\rho_2)} \\
= \;& \max\{\overbrace{\Psi_\aleph^\top}((\zeta_1 \ddagger_1 \rho_1), \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\aleph^\top}((\zeta_1\ddagger_1\rho_1),\partial)}, \overbrace{\Psi_\aleph^\top}((\zeta_2 \ddagger_1 \rho_2), \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\aleph^\top}((\zeta_2\ddagger_1\rho_2),\partial)}\} \\
\leqslant \;& \max\{\max\{\overbrace{\Psi_\aleph^\top}(\zeta_1) \cdot e^{i\Bbbk \overbrace{\Xi_\aleph^\top}(\zeta_1)}, \overbrace{\Psi_\aleph^\top}(\rho_1) \cdot e^{i\Bbbk \overbrace{\Xi_\aleph^\top}(\rho_1)}\}, \\
& \max\{\overbrace{\Psi_\aleph^\top}(\zeta_2) \cdot e^{i\Bbbk \overbrace{\Xi_\aleph^\top}(\zeta_2)}, \overbrace{\Psi_\aleph^\top}(\rho_2) \cdot e^{i\Bbbk \overbrace{\Xi_\aleph^\top}(\rho_2)}\}\} \\
= \;& \max\{\max\{\overbrace{\Psi_\aleph^\top}(\zeta_1) \cdot e^{i\Bbbk \overbrace{\Xi_\aleph^\top}(\zeta_1)}, \overbrace{\Psi_\aleph^\top}(\zeta_2) \cdot e^{i\Bbbk \overbrace{\Xi_\aleph^\top}(\zeta_2)}\}, \\
& \max\{\overbrace{\Psi_\aleph^\top}(\rho_1) \cdot e^{i\Bbbk \overbrace{\Xi_\aleph^\top}(\rho_1)}, \overbrace{\Psi_\aleph^\top}(\rho_2) \cdot e^{i\Bbbk \overbrace{\Xi_\aleph^\top}(\rho_2)}\}\} \\
= \;& \max\{\overbrace{\Psi_\hbar^\top}((\zeta_1, \zeta_2), \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\top}((\zeta_1,\zeta_2),\partial)}, \overbrace{\Psi_\hbar^\top}((\rho_1, \rho_2), \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\top}((\rho_1,\rho_2),\partial)}\} \\
= \;& \max\{\overbrace{\Psi_\hbar^\top}(\zeta, \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\top}(\zeta,\partial)}, \overbrace{\Psi_\hbar^\top}(\rho, \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\top}(\rho,\partial)}\}
\end{aligned}$$

Also $\overbrace{\Psi_\hbar^\top}((\zeta \ddagger_2 \rho), \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\top}((\zeta\ddagger_2\rho),\partial)} \leqslant \max\{\overbrace{\Psi_\hbar^\top}(\zeta, \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\top}(\zeta,\partial)}, \overbrace{\Psi_\hbar^\top}(\rho, \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\top}(\rho,\partial)}\}$, $\overbrace{\Psi_\hbar^\top}((\zeta \ddagger_3 \rho), \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\top}((\zeta\ddagger_3\rho),\partial)} \leqslant \max\{\overbrace{\Psi_\hbar^\top}(\zeta, \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\top}(\zeta,\partial)}, \overbrace{\Psi_\hbar^\top}(\rho, \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\top}(\rho,\partial)}\}$.

Now, $\overbrace{\Psi_\hbar^\daleth}(\zeta \ddagger_1 \rho) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\daleth}(\zeta\ddagger_1\rho)}$

$$\begin{aligned}
&= \overbrace{\Psi_\hbar^\daleth}[(((((\zeta_1, \zeta_2), \partial) \ddagger_1 (\rho_1, \rho_2), \partial)] \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\daleth}[(((((\zeta_1,\zeta_2),\partial)\ddagger_1((\rho_1,\rho_2),\partial))]} \\
&= \overbrace{\Psi_\hbar^\daleth}((\zeta_1 \ddagger_1 \rho_1, \zeta_2 \ddagger_1 \rho_2), \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\daleth}((\zeta_1\ddagger_1\rho_1,\zeta_2\ddagger_1\rho_2),\partial)} \\
&= \frac{\overbrace{\Psi_\aleph^\daleth}((\zeta_1 \ddagger_1 \rho_1), \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\aleph^\daleth}((\zeta_1\ddagger_1\rho_1),\partial)} + \overbrace{\Psi_\aleph^\daleth}((\zeta_2 \ddagger_1 \rho_2), \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\aleph^\daleth}}((\zeta_2 \ddagger_1 \rho_2), \partial)}{2} \\
&\leqslant \frac{1}{2}\left[\frac{\overbrace{\Psi_\aleph^\daleth}(\zeta_1) \cdot e^{i\Bbbk \overbrace{\Xi_\aleph^\daleth}(\zeta_1)} + \overbrace{\Psi_\aleph^\daleth}(\rho_1) \cdot e^{i\Bbbk \overbrace{\Xi_\aleph^\daleth}(\rho_1)}}{2} + \frac{\overbrace{\Psi_\aleph^\daleth}(\zeta_2) \cdot e^{i\Bbbk \overbrace{\Xi_\aleph^\daleth}(\zeta_2)} + \overbrace{\Psi_\aleph^\daleth}(\rho_2) \cdot e^{i\Bbbk \overbrace{\Xi_\aleph^\daleth}(\rho_2)}}{2}\right] \\
&= \frac{1}{2}\left[\frac{\overbrace{\Psi_\aleph^\daleth}(\zeta_1) \cdot e^{i\Bbbk \overbrace{\Xi_\aleph^\daleth}(\zeta_1)} + \overbrace{\Psi_\aleph^\daleth}(\zeta_2) \cdot e^{i\Bbbk \overbrace{\Xi_\aleph^\daleth}(\zeta_2)}}{2} + \frac{\overbrace{\Psi_\aleph^\daleth}(\rho_1) \cdot e^{i\Bbbk \overbrace{\Xi_\aleph^\daleth(\rho_1)}} + \overbrace{\Psi_\aleph^\daleth}(\rho_2) \cdot e^{i\Bbbk \overbrace{\Xi_\aleph^\daleth}(\rho_2)}}{2}\right] \\
&= \frac{\overbrace{\Psi_\hbar^\daleth}((\zeta_1, \zeta_2), \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\daleth}((\zeta_1,\zeta_2),\partial)} + \overbrace{\Psi_\hbar^\daleth}((\rho_1, \rho_2), \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\daleth}((\rho_1,\rho_2),\partial)}}{2} \\
&= \frac{\overbrace{\Psi_\hbar^\daleth}(\zeta, \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\daleth(\zeta, \partial)}} + \overbrace{\Psi_\hbar^\daleth}(\rho, \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\daleth}(\zeta,\partial)}}{2}
\end{aligned}$$

Also $\overbrace{\Psi_\hbar^\daleth}((\zeta \ddagger_2 \rho), \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\daleth}((\zeta\ddagger_2\rho),\partial)} \leqslant \frac{\overbrace{\Psi_\hbar^\daleth}(\zeta,\partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\daleth(\zeta, \partial)}} + \overbrace{\Psi_\hbar^\daleth}(\rho,\partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\daleth}(\rho,\partial)}}{2}$

and $\overbrace{\Psi_\hbar^\daleth}((\zeta \ddagger_3 \rho), \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\daleth}((\zeta\ddagger_3\rho),\partial)} \leqslant \frac{\overbrace{\Psi_\hbar^\daleth}(\zeta,\partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\daleth(\zeta, \partial)}} + \overbrace{\Psi_\hbar^\daleth}(\rho,\partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^\daleth}(\rho,\partial)}}{2}$.

Similarly, $\overbrace{\Psi_\hbar^F}((\zeta \ddagger_1 \rho), \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^F}((\zeta\ddagger_1\rho),\partial)} \geqslant \min\{\overbrace{\Psi_\hbar^F}(\zeta, \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^F(\zeta, \partial)\cdot}}, \overbrace{\Psi_\hbar^F}(\rho, \partial) \cdot e^{i\Bbbk \overbrace{\Xi_\hbar^F(\rho, \partial)}}\}$,

$\widehat{\Psi_{\hbar}^{F}}((\zeta \ddagger_2 \rho),\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{F}}((\zeta\ddagger_2\rho),\partial)} \geqslant \min\{\widehat{\Psi_{\hbar}^{F}}(\zeta,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{F}(\zeta,\partial)}}, \widehat{\Psi_{\hbar}^{F}}(\rho,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{F}(\rho,\partial)}}\}$ and
$\widehat{\Psi_{\hbar}^{F}}((\zeta \ddagger_3 \rho),\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{F}}((\zeta\ddagger_3\rho),\partial)} \geqslant \min\{\widehat{\Psi_{\hbar}^{F}}(\zeta,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{F}(\zeta,\partial)}}, \widehat{\Psi_{\hbar}^{F}}(\rho,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{\daleth}(\rho}}\}$.
For any $\zeta=(\zeta_1,\zeta_2), \rho=(\rho_1,\rho_2)\in Ⓢ\times Ⓢ$. Now,

$$\begin{aligned}
& \Psi_{\hbar}^{\top}((\zeta \ddagger_1 \rho),\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\top}((\zeta\ddagger_1\rho),\partial)} \\
= \;& \Psi_{\hbar}^{\top}[(((((\zeta_1,\zeta_2),\partial)\ddagger_1((\rho_1,\rho_2),\partial)]\cdot e^{i\Bbbk\Xi_{\hbar}^{\top}[(((((\zeta_1,\zeta_2),\partial)\ddagger_1((\rho_1,\rho_2),\partial)]} \\
= \;& \Psi_{\hbar}^{\top}(\zeta_1\ddagger_1\rho_1,\zeta_2\ddagger_1\rho_2)\cdot e^{i\Bbbk\Xi_{\hbar}^{\top}(\zeta_1\ddagger_1\rho_1,\zeta_2\ddagger_1\rho_2)} \\
= \;& \max\{\Psi_{\aleph}^{\top}((\zeta_1\ddagger_1\rho_1),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta_1\ddagger_1\rho_1),\partial)}, \Psi_{\aleph}^{\top}((\zeta_2\ddagger_1\rho_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta_2\ddagger_1\rho_2),\partial)}\} \\
\leqslant \;& \max\{\max\{\Psi_{\aleph}^{\top}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\zeta_1)}, \Psi_{\aleph}^{\top}(\rho_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\rho_1)}\}, \\
& \max\{\Psi_{\aleph}^{\top}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\zeta_2)}, \Psi_{\aleph}^{\top}(\rho_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\rho_2)}\}\} \\
= \;& \max\{\max\{\Psi_{\aleph}^{\top}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\zeta_1)}, \Psi_{\aleph}^{\top}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\zeta_2)}\}, \\
& \max\{\Psi_{\aleph}^{\top}(\rho_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\rho_1)}, \Psi_{\aleph}^{\top}(\rho_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\rho_2)}\}\} \\
= \;& \max\{\Psi_{\hbar}^{\top}((\zeta_1,\zeta_2),\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\top}((\zeta_1,\zeta_2),\partial)}, \Psi_{\hbar}^{\top}((\rho_1,\rho_2),\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\top}((\rho_1,\rho_2),\partial)}\} \\
= \;& \max\{\Psi_{\hbar}^{\top}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\top}(\zeta,\partial)}, \Psi_{\hbar}^{\top}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\top}(\rho,\partial)}\}
\end{aligned}$$

Also $\Psi_{\hbar}^{\top}((\zeta\ddagger_2\rho),\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\top}((\zeta\ddagger_2\rho),\partial)} \leqslant \max\{\Psi_{\hbar}^{\top}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\top}(\zeta,\partial)}, \Psi_{\hbar}^{\top}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\top}(\rho,\partial)}\}$,
$\Psi_{\hbar}^{\top}((\zeta\ddagger_3\rho),\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\top}((\zeta\ddagger_3\rho),\partial)} \leqslant \max\{\Psi_{\hbar}^{\top}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\top}(\zeta,\partial)}, \Psi_{\hbar}^{\top}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\top}(\rho,\partial)}\}$.
Now, $\Psi_{\hbar}^{\daleth}(\zeta\ddagger_1\rho)\cdot e^{i\Bbbk\Xi_{\hbar}^{\daleth}(\zeta\ddagger_1\rho)}$

$$\begin{aligned}
&= \Psi_{\hbar}^{\daleth}[(((((\zeta_1,\zeta_2),\partial)\ddagger_1(\rho_1,\rho_2),\partial)]\cdot e^{i\Bbbk\Xi_{\hbar}^{\daleth}[(((((\zeta_1,\zeta_2),\partial)\ddagger_1((\rho_1,\rho_2),\partial))]} \\
&= \Psi_{\hbar}^{\daleth}((\zeta_1\ddagger_1\rho_1,\zeta_2\ddagger_1\rho_2),\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\daleth}((\zeta_1\ddagger_1\rho_1,\zeta_2\ddagger_1\rho_2),\partial)} \\
&= \frac{\Psi_{\aleph}^{\daleth}((\zeta_1\ddagger_1\rho_1),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}((\zeta_1\ddagger_1\rho_1),\partial)} + \Psi_{\aleph}^{\daleth}((\zeta_2\ddagger_1\rho_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}}((\zeta_2\ddagger_1\rho_2),\partial)}{2} \\
&\leqslant \frac{1}{2}\left[\frac{\Psi_{\aleph}^{\daleth}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\zeta_1)} + \Psi_{\aleph}^{\daleth}(\rho_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\rho_1)}}{2} + \frac{\Psi_{\aleph}^{\daleth}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\zeta_2)} + \Psi_{\aleph}^{\daleth}(\rho_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\rho_2)}}{2}\right] \\
&= \frac{1}{2}\left[\frac{\Psi_{\aleph}^{\daleth}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\zeta_1)} + \Psi_{\aleph}^{\daleth}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\zeta_2)}}{2} + \frac{\Psi_{\aleph}^{\daleth}(\rho_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\rho_1)} + \Psi_{\aleph}^{\daleth}(\rho_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\rho_2)}}{2}\right] \\
&= \frac{\Psi_{\hbar}^{\daleth}((\zeta_1,\zeta_2),\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\daleth}((\zeta_1,\zeta_2),\partial)} + \Psi_{\hbar}^{\daleth}((\rho_1,\rho_2),\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\daleth}((\rho_1,\rho_2),\partial)}}{2} \\
&= \frac{\Psi_{\hbar}^{\daleth}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\daleth}(\zeta,\partial)} + \Psi_{\hbar}^{\daleth}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\daleth}(\zeta,\partial)}}{2}
\end{aligned}$$

Also $\Psi_{\hbar}^{\daleth}((\zeta\ddagger_2\rho),\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\daleth}((\zeta\ddagger_2\rho),\partial)} \leqslant \frac{\Psi_{\hbar}^{\daleth}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\daleth}(\zeta,\partial)} + \Psi_{\hbar}^{\daleth}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\daleth}(\rho,\partial)}}{2}$
and $\Psi_{\hbar}^{\daleth}((\zeta\ddagger_3\rho),\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\daleth}((\zeta\ddagger_3\rho),\partial)} \leqslant \frac{\Psi_{\hbar}^{\daleth}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\daleth}(\zeta,\partial)} + \Psi_{\hbar}^{\daleth}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\daleth}(\rho,\partial)}}{2}$.
Similarly, $\Psi_{\hbar}^{F}((\zeta\ddagger_1\rho),\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{F}((\zeta\ddagger_1\rho),\partial)} \geqslant \min\{\Psi_{\hbar}^{F}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{F}(\zeta,\partial)\cdot}, \Psi_{\hbar}^{F}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{F}(\rho,\partial)}\}$,
$\Psi_{\hbar}^{F}((\zeta\ddagger_2\rho),\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{F}((\zeta\ddagger_2\rho),\partial)} \geqslant \min\{\Psi_{\hbar}^{F}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{F}(\zeta,\partial)}, \Psi_{\hbar}^{F}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{F}(\rho,\partial)}\}$ and
$\Psi_{\hbar}^{F}((\zeta\ddagger_3\rho),\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{F}((\zeta\ddagger_3\rho),\partial)} \geqslant \min\{\Psi_{\hbar}^{F}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{F}(\zeta,\partial)}, \Psi_{\hbar}^{F}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\daleth}(\rho}\}$.
Therefore, $\hbar$ is a ComCANSBS of $Ⓢ\times Ⓢ$.
Conversely, suppose that $\hbar$ is a ComCANSBS of $Ⓢ\times Ⓢ$. Let $\zeta=((\zeta_1,\zeta_2),\partial), \rho=((\rho_1,\rho_2),\partial)\in Ⓢ\times Ⓢ$.

Now,

$$\max\{\overbrace{\Psi_{\aleph}^{\top}}((\zeta_1 \ddagger_1 \rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}((\zeta_1 \ddagger_1 \rho_1),\partial)}},\overbrace{\Psi_{\aleph}^{\top}}((\zeta_2 \ddagger_1 \rho_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}((\zeta_2 \ddagger_1 \rho_2),\partial)}}\}$$
$$=\overbrace{\Psi_{\hbar}^{\top}}(\zeta_1 \ddagger_1 \rho_1,\zeta_2 \ddagger_1 \rho_2)\cdot e^{i\Bbbk\overbrace{\Xi_{\hbar}^{\top}(\zeta_1 \ddagger_1 \rho_1,\zeta_2 \ddagger_1 \rho_2)}}$$
$$=\overbrace{\Psi_{\hbar}^{\top}}[((\zeta_1,\zeta_2),\partial)\ddagger_1((\rho_1,\rho_2),\partial)]\cdot e^{i\Bbbk\overbrace{\Xi_{v}^{\top}[((\zeta_1,\zeta_2),\partial)\ddagger_1((\rho_1,\rho_2),\partial)]}}$$
$$=\overbrace{\Psi_{\hbar}^{\top}}(\zeta \ddagger_1 \rho)\cdot e^{i\Bbbk\overbrace{\Xi_{\hbar}^{\top}(\zeta \ddagger_1 \rho)}}$$
$$\leqslant\max\{\overbrace{\Psi_{\hbar}^{\top}}(\zeta,\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\hbar}^{\top}(\zeta,\partial)}},\overbrace{\Psi_{\hbar}^{\top}}(\rho,\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\hbar}^{\top}(\rho,\partial)}}\}$$
$$=\max\{\overbrace{\Psi_{\hbar}^{\top}}((\zeta_1,\zeta_2),\partial)\}\cdot e^{i\Bbbk\overbrace{\Xi_{\hbar}^{\top}((\zeta_1,\zeta_2),\partial)\}}},\overbrace{\Psi_{\hbar}^{\top}}((\rho_1,\rho_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\hbar}^{\top}}((\rho_1,\rho_2),\partial)}\}$$
$$=\max\{\max\{\overbrace{\Psi_{\aleph}^{\top}}(\zeta_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}(\zeta_1)},\overbrace{\Psi_{\aleph}^{\top}}(\zeta_2)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}(\zeta_2)}\},\max\{\overbrace{\Psi_{\aleph}^{\top}}(\rho_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}(\rho_1)}},\overbrace{\Psi_{\aleph}^{\top}}(\rho_2)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}(\rho_2)}}\}\}$$

If $\overbrace{\Psi_{\aleph}^{\top}}((\zeta_1 \ddagger_1 \rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}((\zeta_1 \ddagger_1 \rho_1),\partial)}}\geqslant\overbrace{\Psi_{\aleph}^{\top}}((\zeta_2 \ddagger_1 \rho_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}((\zeta_2 \ddagger_1 \rho_2),\partial)}}$, then $\overbrace{\Psi_{\aleph}^{\top}}(\zeta_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}(\zeta_1)}\geqslant\overbrace{\Psi_{\aleph}^{\top}}(\zeta_2)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}(\zeta_2)}$ and $\overbrace{\Psi_{\aleph}^{\top}}(\rho_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}(\rho_1)}}\geqslant\overbrace{\Psi_{\aleph}^{\top}}(\rho_2)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}(\rho_2)}}$. We get $\overbrace{\Psi_{\aleph}^{\top}}((\zeta_1 \ddagger_1 \rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}((\zeta_1 \ddagger_1 \rho_1),\partial)}}\leqslant\max\{\overbrace{\Psi_{\aleph}^{\top}}(\zeta_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}(\zeta_1)},\overbrace{\Psi_{\aleph}^{\top}}(\rho_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}(\rho_1)}}\}$ for all $\zeta_1,\rho_1\in Ⓢ$,and $\max\{\overbrace{\Psi_{\aleph}^{\top}}((\zeta_1\ddagger_2\rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}((\zeta_1 \ddagger_2 \rho_1),\partial)}},\overbrace{\Psi_{\aleph}^{\top}}((\zeta_2\ddagger_2\rho_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}((\zeta_2 \ddagger_2 \rho_2),\partial)}}\}\leqslant\max\{\max\{\overbrace{\Psi_{\aleph}^{\top}}(\zeta_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}(\zeta_1)},\overbrace{\Psi_{\aleph}^{\top}}(\zeta_2)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}(\zeta_2)}\},\max\{\overbrace{\Psi_{\aleph}^{\top}}(\rho_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}(\rho_1)}},\overbrace{\Psi_{\aleph}^{\top}}(\rho_2)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}(\rho_2)}}\}\}$

If $\overbrace{\Psi_{\aleph}^{\top}}((\zeta_1 \ddagger_2 \rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}((\zeta_1 \ddagger_2 \rho_1),\partial)}}\geqslant\overbrace{\Psi_{\aleph}^{\top}}((\zeta_2 \ddagger_2 \rho_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}((\zeta_2 \ddagger_2 \rho_2),\partial)}}$,then $\overbrace{\Psi_{\aleph}^{\top}}((\zeta_1 \ddagger_2 \rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}((\zeta_1 \ddagger_2 \rho_1),\partial)}}\leqslant\max\{\overbrace{\Psi_{\aleph}^{\top}}(\zeta_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}(\zeta_1)},\overbrace{\Psi_{\aleph}^{\top}}(\rho_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}(\rho_1)}}\}$.

$\max\{\overbrace{\Psi_{\aleph}^{\top}}((\zeta_1 \ddagger_3 \rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}((\zeta_1\ddagger_3\rho_1),\partial)},\overbrace{\Psi_{\aleph}^{\top}}((\zeta_2 \ddagger_3 \rho_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}((\zeta_2\ddagger_3\rho_2),\partial)}\}\leqslant\max\{\max\{\overbrace{\Psi_{\aleph}^{\top}}(\zeta_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}(\zeta_1)},\overbrace{\Psi_{\aleph}^{\top}}(\zeta_2)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}(\zeta_2)}\},\max\{\overbrace{\Psi_{\aleph}^{\top}}(\rho_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}(\rho_1)},\overbrace{\Psi_{\aleph}^{\top}}(\rho_2)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}(\rho_2)}\}\}$

If $\overbrace{\Psi_{\aleph}^{\top}}((\zeta_1 \ddagger_3 \rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}((\zeta_1\ddagger_3\rho_1),\partial)}\geqslant\overbrace{\Psi_{\aleph}^{\top}}((\zeta_2 \ddagger_3 \rho_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}((\zeta_2\ddagger_3\rho_2),\partial)}$,then $\overbrace{\Psi_{\aleph}^{\top}}((\zeta_1 \ddagger_3 \rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}((\zeta_1\ddagger_3\rho_1),\partial)}\leqslant\max\{\overbrace{\Psi_{\aleph}^{\top}}(\zeta_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}(\zeta_1)},\overbrace{\Psi_{\aleph}^{\top}}(\rho_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}(\rho_1)}}\}$.

Now,

$$\frac{1}{2}\Big[\overbrace{\Psi_{\aleph}^{\beth}}((\zeta_1 \ddagger_1 \rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\beth}((\zeta_1 \ddagger_1 \rho_1),\partial)}}+\overbrace{\Psi_{\aleph}^{\beth}}((\zeta_2 \ddagger_1 \rho_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\beth((\zeta_2\ddagger_1\rho_2),\partial)}}}\Big]$$
$$=\overbrace{\Psi_{\hbar}^{\beth}}(\zeta_1 \ddagger_1 \rho_1,\zeta_2 \ddagger_1 \rho_2)\cdot e^{i\Bbbk\overbrace{\Xi_{\hbar}^{\beth}(\zeta_1 \ddagger_1 \rho_1,\zeta_2 \ddagger_1 \rho_2)}}$$
$$=\overbrace{\Psi_{\hbar}^{\beth}}[((\zeta_1,\zeta_2),\partial)\ddagger_1((\rho_1,\rho_2),\partial)]\cdot e^{i\Bbbk\overbrace{\Xi_{\hbar}^{\beth}[((\zeta_1,\zeta_2),\partial)\ddagger_1((\rho_1,\rho_2),\partial)]}}$$
$$=\overbrace{\Psi_{\hbar}^{\beth}}((\zeta \ddagger_1 \rho),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\hbar}^{\beth}((\zeta \ddagger_1 \rho),\partial)}}$$
$$\leqslant\frac{\overbrace{\Psi_{\hbar}^{\beth}}(\zeta,\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\hbar}^{\beth(\zeta,\partial)}}}+\overbrace{\Psi_{\hbar}^{\beth}}(\rho,\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\hbar}^{\beth}(\rho,\partial)}}}{2}$$
$$=\frac{\overbrace{\Psi_{\hbar}^{\beth}}((\zeta_1,\zeta_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\hbar}^{\beth}((\zeta_1,\zeta_2),\partial)}}+\overbrace{\Psi_{\hbar}^{\beth}}((\rho_1,\rho_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\hbar}^{\beth}}}}{2}$$
$$=\frac{1}{2}\left[\frac{\overbrace{\Psi_{\aleph}^{\beth}}(\zeta_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\beth}}(\zeta_1)}+\overbrace{\Psi_{\aleph}^{\beth}}(\zeta_2)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\beth}}(\zeta_2)}}{2}+\frac{\overbrace{\Psi_{\aleph}^{\beth}}(\rho_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\beth}(\rho_1)}}+\overbrace{\Psi_{\aleph}^{\beth}}(\rho_2)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\beth}(\rho_2)}}}{2}\right]$$

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If $\overbrace{\Psi_{\aleph}^{\daleth}}((\zeta_1\ddagger_1\rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\daleth}}((\zeta_1\ddagger_1\rho_1),\partial)} \geqslant \overbrace{\Psi_{\aleph}^{\daleth}}((\zeta_2\ddagger_1\rho_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\daleth}((\zeta_2\ddagger_1\rho_2),\partial)}}$, then $\overbrace{\Psi_{\aleph}^{\daleth}}(\zeta_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\daleth}}(\zeta_1)} \geqslant \overbrace{\Psi_{\aleph}^{\daleth}}(\zeta_2)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\daleth}}(\zeta_2)}$ and $\overbrace{\Psi_{\aleph}^{\daleth}}(\rho_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\daleth}(\rho_1)}} \geqslant \overbrace{\Psi_{\aleph}^{\daleth}}(\rho_2)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\daleth}(\rho_2)}}$.

We get $\overbrace{\Psi_{\aleph}^{\daleth}}((\zeta_1\ddagger_1\rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}((\zeta_1\ddagger_1\rho_1),\partial)} \leqslant \frac{\overbrace{\Psi_{\aleph}^{\daleth}}(\zeta_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\daleth}}(\zeta_1)}+\overbrace{\Psi_{\aleph}^{\daleth}}(\rho_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}(\rho_1)}}}{2}$.

Similarly, $\overbrace{\Psi_{\aleph}^{\daleth}}((\zeta_1\ddagger_2\rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\daleth}((\zeta_1\ddagger_2\rho_1),\partial)}} \leqslant \frac{\overbrace{\Psi_{\aleph}^{\daleth}}(\zeta_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\daleth}}(\zeta_1)}+\overbrace{\Psi_{\aleph}^{\daleth}}(\rho_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\daleth}}(\rho_1)}}{2}$

and $\overbrace{\Psi_{\aleph}^{\daleth}}((\zeta_1\ddagger_3\rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\daleth}((\zeta_1\ddagger_3\rho_1),\partial)}} \leqslant \frac{\overbrace{\Psi_{\aleph}^{\daleth}}(\zeta_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\daleth}}(\zeta_1)}+\overbrace{\Psi_{\aleph}^{\daleth}}(\rho_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\daleth}(\rho_1)}}}{2}$.
Similarly to prove that

$\min\{\overbrace{\Psi_{\aleph}^{F}}((\zeta_1\ddagger_1\rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}((\zeta_1\ddagger_1\rho_1),\partial)}}, \overbrace{\Psi_{\aleph}^{F}}((\zeta_2\ddagger_1\rho_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}((\zeta_2\ddagger_1\rho_2),\partial)}}\} \geqslant \min\{\min\{\overbrace{\Psi_{\aleph}^{F}}(\zeta_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}}(\zeta_1)}, \overbrace{\Psi_{\aleph}^{F}}(\zeta_2)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}}(\zeta_2)}\}, \min\{\overbrace{\Psi_{\aleph}^{F}}(\rho_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}(\rho_1)}}, \overbrace{\Psi_{\aleph}^{F}}(\rho_2)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}(\rho_2)}}\}\}$

If $\overbrace{\Psi_{\aleph}^{F}}((\zeta_1\ddagger_1\rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}((\zeta_1\ddagger_1\rho_1),\partial)}} \leqslant \overbrace{\Psi_{\aleph}^{F}}((\zeta_2\ddagger_1\rho_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}((\zeta_2\ddagger_1\rho_2),\partial)}}$, then $\overbrace{\Psi_{\aleph}^{F}}(\zeta_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}}(\zeta_1)} \leqslant \overbrace{\Psi_{\aleph}^{F}}(\zeta_2)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}}(\zeta_2)}$ and $\overbrace{\Psi_{\aleph}^{F}}(\rho_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}(\rho_1)}} \leqslant \overbrace{\Psi_{\aleph}^{F}}(\rho_2)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}(\rho_2)}}$.

We get $\overbrace{\Psi_{\aleph}^{F}}((\zeta_1\ddagger_1\rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}((\zeta_1\ddagger_1\rho_1),\partial)}} \geqslant \min\{\overbrace{\Psi_{\aleph}^{F}}(\zeta_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}}(\zeta_1)}, \overbrace{\Psi_{\aleph}^{F}}(\rho_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}(\rho_1)}}\}$.

$\min\{\overbrace{\Psi_{\aleph}^{F}}((\zeta_1\ddagger_2\rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}((\zeta_1\ddagger_2\rho_1),\partial)}}, \overbrace{\Psi_{\aleph}^{F}}((\zeta_2\ddagger_2\rho_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}((\zeta_2\ddagger_2\rho_2),\partial)}}\}$
$\geqslant \min\{\min\{\overbrace{\Psi_{\aleph}^{F}}(\zeta_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}}(\zeta_1)}, \overbrace{\Psi_{\aleph}^{F}}(\zeta_2)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}}(\zeta_2)}\}, \min\{\overbrace{\Psi_{\aleph}^{F}}(\rho_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}(\rho_1)}}, \overbrace{\Psi_{\aleph}^{F}}(\rho_2)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}}}\}\}$

If $\overbrace{\Psi_{\aleph}^{F}}((\zeta_1\ddagger_2\rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}((\zeta_1\ddagger_2\rho_1),\partial)}} \leqslant \overbrace{\Psi_{\aleph}^{F}}((\zeta_2\ddagger_2\rho_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}((\zeta_2\ddagger_2\rho_2),\partial)}}$, then $\overbrace{\Psi_{\aleph}^{F}}((\zeta_1\ddagger_2\rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}((\zeta_1\ddagger_2\rho_1),\partial)}} \geqslant \min\{\overbrace{\Psi_{\aleph}^{F}}(\zeta_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}}(\zeta_1)}, \overbrace{\Psi_{\aleph}^{F}}(\rho_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}(\rho_1)}}\}$.

$\min\{\overbrace{\Psi_{\aleph}^{F}}((\zeta_1\ddagger_3\rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}((\zeta_1\ddagger_3\rho_1),\partial)}}, \overbrace{\Psi_{\aleph}^{F}}((\zeta_2\ddagger_3\rho_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}((\zeta_2\ddagger_3\rho_2),\partial)}}\} \geqslant \min\{\min\{\overbrace{\Psi_{\aleph}^{F}}(\zeta_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}}(\zeta_1)}, \overbrace{\Psi_{\aleph}^{F}}(\zeta_2)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}}(\zeta_2)}\}, \min\{\overbrace{\Psi_{\aleph}^{F}}(\rho_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}(\rho_1)}}, \overbrace{\Psi_{\aleph}^{F}}(\rho_2)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}(\rho_2)}}\}\}$

If $\overbrace{\Psi_{\aleph}^{F}}((\zeta_1\ddagger_3\rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}((\zeta_1\ddagger_3\rho_1),\partial)}} \leqslant \overbrace{\Psi_{\aleph}^{F}}((\zeta_2\ddagger_3\rho_2),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}((\zeta_2\ddagger_3\rho_2),\partial)}}$, then $\overbrace{\Psi_{\aleph}^{F}}((\zeta_1\ddagger_3\rho_1),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}((\zeta_1\ddagger_3\rho_1),\partial)}} \geqslant \min\{\overbrace{\Psi_{\aleph}^{F}}(\zeta_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}}(\zeta_1)}, \overbrace{\Psi_{\aleph}^{F}}(\rho_1)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}(\rho_1)}}\}$.

Let $\zeta=((\zeta_1,\zeta_2),\partial), \rho=((\rho_1,\rho_2),\partial)\in ⓢ\times ⓢ$. Now,

$$\begin{aligned}
&\max\{\Psi_{\aleph}^{\top}((\zeta_1\ddagger_1\rho_1),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta_1\ddagger_1\rho_1),\partial)}, \Psi_{\aleph}^{\top}((\zeta_2\ddagger_1\rho_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta_2\ddagger_1\rho_2),\partial)}\}\\
&=\Psi_{\hbar}^{\top}(\zeta_1\ddagger_1\rho_1,\zeta_2\ddagger_1\rho_2)\cdot e^{i\Bbbk\Xi_{\hbar}^{\top}(\zeta_1\ddagger_1\rho_1,\zeta_2\ddagger_1\rho_2)}\\
&=\Psi_{\hbar}^{\top}[((\zeta_1,\zeta_2),\partial)\ddagger_1((\rho_1,\rho_2),\partial)]\cdot e^{i\Bbbk\Xi_{v}^{\top}[((\zeta_1,\zeta_2),\partial)\ddagger_1((\rho_1,\rho_2),\partial)]}\\
&=\Psi_{\hbar}^{\top}(\zeta\ddagger_1\rho)\cdot e^{i\Bbbk\Xi_{\hbar}^{\top}(\zeta\ddagger_1\rho)}\\
&\leqslant\max\{\Psi_{\hbar}^{\top}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\top}(\zeta,\partial)}, \Psi_{\hbar}^{\top}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\top}(\rho,\partial)}\}\\
&=\max\{\Psi_{\hbar}^{\top}((\zeta_1,\zeta_2),\partial)\}\cdot e^{i\Bbbk\Xi_{\hbar}^{\top}((\zeta_1,\zeta_2),\partial)\}}, \Psi_{\hbar}^{\top}((\rho_1,\rho_2),\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\top}((\rho_1,\rho_2),\partial)}\}\\
&=\max\{\max\{\Psi_{\aleph}^{\top}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\zeta_1)}, \Psi_{\aleph}^{\top}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\zeta_2)}\}, \max\{\Psi_{\aleph}^{\top}(\rho_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\rho_1)}, \Psi_{\aleph}^{\top}(\rho_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\rho_2)}\}\}
\end{aligned}$$

If $\Psi_{\aleph}^{\top}((\zeta_1\ddagger_1\rho_1),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta_1\ddagger_1\rho_1),\partial)} \geqslant \Psi_{\aleph}^{\top}((\zeta_2\ddagger_1\rho_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta_2\ddagger_1\rho_2),\partial)}$, then $\Psi_{\aleph}^{\top}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\zeta_1)} \geqslant \Psi_{\aleph}^{\top}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\zeta_2)}$ and $\Psi_{\aleph}^{\top}(\rho_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\rho_1)} \geqslant \Psi_{\aleph}^{\top}(\rho_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\rho_2)}$. We get $\Psi_{\aleph}^{\top}((\zeta_1\ddagger_1\rho_1),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta_1\ddagger_1\rho_1),\partial)} \leqslant \max\{\Psi_{\aleph}^{\top}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\zeta_1)}, \Psi_{\aleph}^{\top}(\rho_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\rho_1)}\}$ for all $\zeta_1,\rho_1\in ⓢ$,and
$\max\{\Psi_{\aleph}^{\top}((\zeta_1\ddagger_2\rho_1),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta_1\ddagger_2\rho_1),\partial)}, \Psi_{\aleph}^{\top}((\zeta_2\ddagger_2\rho_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta_2\ddagger_2\rho_2),\partial)}\} \leqslant \max\{\max\{\Psi_{\aleph}^{\top}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\zeta_1)}, \Psi_{\aleph}^{\top}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\zeta_2)}\}, \max\{\Psi_{\aleph}^{\top}(\rho_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\rho_1)}, \Psi_{\aleph}^{\top}(\rho_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\rho_2)}\}\}$
If $\Psi_{\aleph}^{\top}((\zeta_1\ddagger_2\rho_1),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta_1\ddagger_2\rho_1),\partial)} \geqslant \Psi_{\aleph}^{\top}((\zeta_2\ddagger_2\rho_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta_2\ddagger_2\rho_2),\partial)}$,then $\Psi_{\aleph}^{\top}((\zeta_1\ddagger_2\rho_1),\partial)\cdot$

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$e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta_1\ddagger_2\rho_1),\partial)} \leqslant \max\{\Psi_{\aleph}^{\top}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\zeta_1)},\Psi_{\aleph}^{\top}(\rho_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\rho_1)}\}$.
$\max\{\Psi_{\aleph}^{\top}((\zeta_1 \ddagger_3 \rho_1),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta_1\ddagger_3\rho_1),\partial)},\Psi_{\aleph}^{\top}((\zeta_2 \ddagger_3 \rho_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta_2\ddagger_3\rho_2),\partial)}\} \leqslant \max\{\max\{\Psi_{\aleph}^{\top}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\zeta_1)},\Psi_{\aleph}^{\top}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\zeta_2)}\},\max\{\Psi_{\aleph}^{\top}(\rho_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\rho_1)},\Psi_{\aleph}^{\top}(\rho_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\rho_2)}\}\}$
If $\Psi_{\aleph}^{\top}((\zeta_1 \ddagger_3 \rho_1),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta_1\ddagger_3\rho_1),\partial)} \geqslant \Psi_{\aleph}^{\top}((\zeta_2 \ddagger_3 \rho_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta_2\ddagger_3\rho_2),\partial)}$,then $\Psi_{\aleph}^{\top}((\zeta_1 \ddagger_3 \rho_1),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta_1\ddagger_3\rho_1),\partial)} \leqslant \max\{\Psi_{\aleph}^{\top}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\zeta_1)},\Psi_{\aleph}^{\top}(\rho_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\rho_1)}\}$.
Now,

$$\begin{aligned}
&\frac{1}{2}\Big[\Psi_{\aleph}^{\daleth}((\zeta_1 \ddagger_1 \rho_1),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}((\zeta_1\ddagger_1\rho_1),\partial)} + \Psi_{\aleph}^{\daleth}((\zeta_2 \ddagger_1 \rho_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth((\zeta_2\ddagger_1\rho_2),\partial)}}\Big]\\
&= \Psi_{\hbar}^{\daleth}(\zeta_1 \ddagger_1 \rho_1, \zeta_2 \ddagger_1 \rho_2)\cdot e^{i\Bbbk\Xi_{\hbar}^{\daleth}(\zeta_1\ddagger_1\rho_1,\zeta_2\ddagger_1\rho_2)}\\
&= \Psi_{\hbar}^{\daleth}[((\zeta_1,\zeta_2),\partial) \ddagger_1 ((\rho_1,\rho_2),\partial)]\cdot e^{i\Bbbk\Xi_{\hbar}^{\daleth}[((\zeta_1,\zeta_2),\partial)\ddagger_1((\rho_1,\rho_2),\partial)]}\\
&= \Psi_{\hbar}^{\daleth}((\zeta \ddagger_1 \rho),\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\daleth}((\zeta\ddagger_1\rho),\partial)}\\
&\leqslant \frac{\Psi_{\hbar}^{\daleth}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\daleth(\zeta,\partial)}} + \Psi_{\hbar}^{\daleth}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\daleth}(\rho,\partial)}}{2}\\
&= \frac{\Psi_{\hbar}^{\daleth}((\zeta_1,\zeta_2),\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\daleth}((\zeta_1,\zeta_2),\partial)} + \Psi_{\hbar}^{\daleth}((\rho_1,\rho_2),\partial)\cdot e^{i\Bbbk\Xi_{\hbar}^{\daleth}}}{2}\\
&= \frac{1}{2}\left[\frac{\Psi_{\aleph}^{\daleth}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\zeta_1)} + \Psi_{\aleph}^{\daleth}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\zeta_2)}}{2} + \frac{\Psi_{\aleph}^{\daleth}(\rho_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\rho_1)} + \Psi_{\aleph}^{\daleth}(\rho_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\rho_2)}}{2}\right]
\end{aligned}$$

If $\Psi_{\aleph}^{\daleth}((\zeta_1 \ddagger_1 \rho_1),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}((\zeta_1\ddagger_1\rho_1),\partial)} \geqslant \Psi_{\aleph}^{\daleth}((\zeta_2 \ddagger_1 \rho_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}((\zeta_2\ddagger_1\rho_2),\partial)}$, then $\Psi_{\aleph}^{\daleth}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\zeta_1)} \geqslant \Psi_{\aleph}^{\daleth}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\zeta_2)}$ and $\Psi_{\aleph}^{\daleth}(\rho_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\rho_1)} \geqslant \Psi_{\aleph}^{\daleth}(\rho_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\rho_2)}$.
We get $\Psi_{\aleph}^{\daleth}((\zeta_1 \ddagger_1 \rho_1),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta_1\ddagger_1\rho_1),\partial)} \leqslant \frac{\Psi_{\aleph}^{\daleth}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\zeta_1)}+\Psi_{\aleph}^{\daleth}(\rho_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\rho_1)}}{2}$.
Similarly,$\Psi_{\aleph}^{\daleth}((\zeta_1 \ddagger_2 \rho_1),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}((\zeta_1\ddagger_2\rho_1),\partial)} \leqslant \frac{\Psi_{\aleph}^{\daleth}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\zeta_1)}+\Psi_{\aleph}^{\daleth}(\rho_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\rho_1)}}{2}$
and $\Psi_{\aleph}^{\daleth}((\zeta_1 \ddagger_3 \rho_1),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}((\zeta_1\ddagger_3\rho_1),\partial)} \leqslant \frac{\Psi_{\aleph}^{\daleth}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\zeta_1)}+\Psi_{\aleph}^{\daleth}(\rho_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\rho_1)}}{2}$.
Similarly to prove that
$\min\{\Psi_{\aleph}^{F}((\zeta_1 \ddagger_1 \rho_1),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\zeta_1\ddagger_1\rho_1),\partial)},\Psi_{\aleph}^{F}((\zeta_2 \ddagger_1 \rho_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\zeta_2\ddagger_1\rho_2),\partial)}\}$
$\geqslant \min\{\min\{\Psi_{\aleph}^{F}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\zeta_1)},\Psi_{\aleph}^{F}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\zeta_2)}\},\min\{\Psi_{\aleph}^{F}(\rho_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\rho_1)},\Psi_{\aleph}^{F}(\rho_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\rho_2)}\}\}$
If $\Psi_{\aleph}^{F}((\zeta_1 \ddagger_1 \rho_1),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\zeta_1\ddagger_1\rho_1),\partial)} \leqslant \Psi_{\aleph}^{F}((\zeta_2 \ddagger_1 \rho_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\zeta_2\ddagger_1\rho_2),\partial)}$,then $\Psi_{\aleph}^{F}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\zeta_1)} \leqslant \Psi_{\aleph}^{F}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\zeta_2)}$ and $\Psi_{\aleph}^{F}(\rho_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\rho_1)} \leqslant \Psi_{\aleph}^{F}(\rho_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\rho_2)}$.
We get $\Psi_{\aleph}^{F}((\zeta_1 \ddagger_1 \rho_1),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\zeta_1\ddagger_1\rho_1),\partial)} \geqslant \min\{\Psi_{\aleph}^{F}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\zeta_1)},\Psi_{\aleph}^{F}(\rho_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\rho_1)}\}$.
$\min\{\Psi_{\aleph}^{F}((\zeta_1 \ddagger_2 \rho_1),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\zeta_1\ddagger_2\rho_1),\partial)},\Psi_{\aleph}^{F}((\zeta_2 \ddagger_2 \rho_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\zeta_2\ddagger_2\rho_2),\partial)}\}$
$\geqslant \min\{\min\{\Psi_{\aleph}^{F}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\zeta_1)},\Psi_{\aleph}^{F}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\zeta_2)}\},\min\{\Psi_{\aleph}^{F}(\rho_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\rho_1)},\Psi_{\aleph}^{F}(\rho_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}}\}\}$
If $\Psi_{\aleph}^{F}((\zeta_1 \ddagger_2 \rho_1),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\zeta_1\ddagger_2\rho_1),\partial)} \leqslant \Psi_{\aleph}^{F}((\zeta_2 \ddagger_2 \rho_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\zeta_2\ddagger_2\rho_2),\partial)}$,then $\Psi_{\aleph}^{F}((\zeta_1 \ddagger_2 \rho_1),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\zeta_1\ddagger_2\rho_1),\partial)} \geqslant \min\{\Psi_{\aleph}^{F}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\zeta_1)},\Psi_{\aleph}^{F}(\rho_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\rho_1)}\}$.
$\min\{\Psi_{\aleph}^{F}((\zeta_1 \ddagger_3 \rho_1),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\zeta_1\ddagger_3\rho_1),\partial)},\Psi_{\aleph}^{F}((\zeta_2 \ddagger_3 \rho_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\zeta_2\ddagger_3\rho_2),\partial)}\}$
$\geqslant \min\{\min\{\Psi_{\aleph}^{F}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\zeta_1)},\Psi_{\aleph}^{F}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\zeta_2)}\},\min\{\Psi_{\aleph}^{F}(\rho_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\rho_1)},\Psi_{\aleph}^{F}(\rho_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\rho_2)}\}\}$
If $\Psi_{\aleph}^{F}((\zeta_1 \ddagger_3 \rho_1),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta_1\ddagger_3\rho_1),\partial)} \leqslant \Psi_{\aleph}^{F}((\zeta_2 \ddagger_3 \rho_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta_2\ddagger_3\rho_2),\partial)}$,then $\Psi_{\aleph}^{F}((\zeta_1 \ddagger_3 \rho_1),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta_1\ddagger_3\rho_1),\partial)} \geqslant \min\{\Psi_{\aleph}^{F}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\zeta_1)},\Psi_{\aleph}^{F}(\rho_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\rho_1)}\}$.
Therefore, $\aleph$ is a ComCANSBS of Ⓢ.

Theorem 3.12. *Suppose that $\aleph$ is a subset of Ⓢ. Then $R = (\overbrace{\Psi_{\aleph}^{\top}}\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}}, \overbrace{\Psi_{\aleph}^{\daleth}}\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\daleth}}}, \overbrace{\Psi_{\aleph}^{F}}\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}}}, \Psi_{\aleph}^{\top}\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}}, \Psi_{\aleph}^{\daleth}\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}}, \Psi_{\aleph}^{F}\cdot e^{i\Bbbk\Xi_{\aleph}^{F}})$ is a ComCANSBS of Ⓢ if and only if $\Psi^{(\beta,\delta)}$ is a SBS of Ⓢ for all $\beta,\delta \in D[0,1]$.*

Proof. Assume that $\overbrace{\Psi}$ is a ComCANSBS of Ⓢ. For each $\beta,\delta \in D[0,1]$ and $\zeta_1,\zeta_2 \in \overbrace{\Psi^{(\beta,\delta)}}$. Now, $\overbrace{\Psi_{\aleph}^{\top}}(\zeta_1)\cdot e^{i\Bbbk}\overbrace{\Xi_{\aleph}^{\top}}(\zeta_1) \leqslant \beta$, $\overbrace{\Psi_{\aleph}^{\top}}(\zeta_2)\cdot e^{i\Bbbk}\overbrace{\Xi_{\aleph}^{\top}}(\zeta_2) \leqslant \beta$ and $\overbrace{\Psi_{\aleph}^{\daleth}}(\zeta_1)\cdot e^{i\Bbbk}\overbrace{\Xi_{\aleph}^{\daleth}}(\zeta_1) \leqslant \beta$, $\overbrace{\Psi_{\aleph}^{\daleth}}(\zeta_2)\cdot e^{i\Bbbk}\overbrace{\Xi_{\aleph}^{\daleth}}(\zeta_2) \leqslant \beta$ and $\overbrace{\Psi_{\aleph}^{F}}(\zeta_1)\cdot e^{i\Bbbk}\overbrace{\Xi_{\aleph}^{F}}(\zeta_1) \geqslant \delta$, $\overbrace{\Psi_{\aleph}^{F}}(\zeta_2)\cdot e^{i\Bbbk}\overbrace{\Xi_{\aleph}^{F}}(\zeta_2) \geqslant \delta$. Now, $\overbrace{\Psi_{\aleph}^{\top}}((\zeta_1 \ddagger_1 \zeta_2),\partial)\cdot e^{i\Bbbk}\overbrace{\Xi_{\aleph}^{\top}}((\zeta_1 \ddagger_1 \zeta_2),\partial) \leqslant$

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$\max\{\widehat{\Psi_{\aleph}^{\top}}(\zeta_1)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\top}}(\zeta_1),\widehat{\Psi_{\aleph}^{\top}}(\zeta_2)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\top}}(\zeta_2)\}\leqslant\beta$ and $\widehat{\Psi_{\aleph}^{\daleth}}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\daleth}}((\zeta_1\ddagger_1\zeta_2),\partial)\leqslant \frac{\widehat{\Psi_{\aleph}^{\daleth}}(\zeta_1)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\daleth}}(\zeta_1)+\widehat{\Psi_{\aleph}^{\daleth}}(\zeta_2)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\daleth}}(\zeta_2)}{2}\leqslant\frac{\beta+\beta}{2}=\beta$ and $\widehat{\Psi_{\aleph}^{F}}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{F}}((\zeta_1\ddagger_1\zeta_2),\partial)\geqslant \min\{\widehat{\Psi_{\aleph}^{F}}(\zeta_1)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{F}}(\zeta_1),\widehat{\Psi_{\aleph}^{F}}(\zeta_2)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{F}}(\zeta_2)\}\geqslant\delta$. This implies that $\zeta_1\ddagger_1\zeta_2\in\widehat{\Psi^{(\beta,\delta)}}$. Similarly,$\zeta_1\ddagger_2\zeta_2\in\widehat{\Psi^{(\beta,\delta)}}$ and $\zeta_1\ddagger_3\zeta_2\in\widehat{\Psi^{(\beta,\delta)}}$. Hence,$\widehat{\Psi^{(\beta,\delta)}}$ is a SBS of Ⓢ,for all $\beta,\delta\in D[0,1]$.
For each $\beta,\delta\in[0,1]$ and $\zeta_1,\zeta_2\in\Psi^{(\beta,\delta)}$. Now,$\Psi_{\aleph}^{\top}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}}(\zeta_1)\leqslant\beta,\Psi_{\aleph}^{\top}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}}(\zeta_2)\leqslant\beta$ and $\Psi_{\aleph}^{\daleth}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}}(\zeta_1)\leqslant\beta,\Psi_{\aleph}^{\daleth}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}}(\zeta_2)\leqslant\beta$ and $\Psi_{\aleph}^{F}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}}(\zeta_1)\geqslant\delta,\Psi_{\aleph}^{F}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}}(\zeta_2)\geqslant\delta$. Now,$\Psi_{\aleph}^{\top}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}}((\zeta_1\ddagger_1\zeta_2),\partial)\leqslant\max\{\Psi_{\aleph}^{\top}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}}(\zeta_1),\Psi_{\aleph}^{\top}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}}(\zeta_2)\}\leqslant\beta$ and $\Psi_{\aleph}^{\daleth}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}}((\zeta_1\ddagger_1\zeta_2),\partial)\leqslant\frac{\Psi_{\aleph}^{\daleth}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}}(\zeta_1)+\Psi_{\aleph}^{\daleth}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}}(\zeta_2)}{2}\leqslant\frac{\beta+\beta}{2}=\beta$ and $\Psi_{\aleph}^{F}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}}((\zeta_1\ddagger_1\zeta_2),\partial)\geqslant\min\{\Psi_{\aleph}^{F}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}}(\zeta_1),\Psi_{\aleph}^{F}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}}(\zeta_2)\}\geqslant\delta$. This implies that $\zeta_1\ddagger_1\zeta_2\in\Psi^{(\beta,\delta)}$. Similarly,$\zeta_1\ddagger_2\zeta_2\in\Psi^{(\beta,\delta)}$ and $\zeta_1\ddagger_3\zeta_2\in\Psi^{(\beta,\delta)}$. Hence,$\Psi^{(\beta,\delta)}$ is a SBS of Ⓢ,for all $\beta,\delta\in D[0,1]$.

Conversely, assume that $\widehat{\Psi^{(\beta,\delta)}}$ is a SBS of Ⓢ and $\beta,\delta\in D[0,1]$. Suppose if there exist $\zeta_1,\zeta_2\in$ Ⓢ such that $\widehat{\Psi_{\aleph}^{\top}}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\top}}((\zeta_1\ddagger_1\zeta_2),\partial)>\max\{\widehat{\Psi_{\aleph}^{\top}}(\zeta_1)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\top}}(\zeta_1),\widehat{\Psi_{\aleph}^{\top}}(\zeta_2)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\top}}(\zeta_2)\}$, $\widehat{\Psi_{\aleph}^{\daleth}}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\daleth}}((\zeta_1\ddagger_1\zeta_2),\partial)>\frac{\widehat{\Psi_{\aleph}^{\daleth}}(\zeta_1)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\daleth}}(\zeta_1)+\widehat{\Psi_{\aleph}^{\daleth}}(\zeta_2)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\daleth}}(\zeta_2)}{2}$ and $\widehat{\Psi_{\aleph}^{F}}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{F}}((\zeta_1\ddagger_1\zeta_2),\partial)<\min\{\widehat{\Psi_{\aleph}^{F}}(\zeta_1)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{F}}(\zeta_1),\widehat{\Psi_{\aleph}^{F}}(\zeta_2)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{F}}(\zeta_2)\}$. For $\beta,\delta\in D[0,1]$ such that $\widehat{\Psi_{\aleph}^{\top}}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\top}}((\zeta_1\ddagger_1\zeta_2),\partial)>\beta\geqslant\max\{\widehat{\Psi_{\aleph}^{\top}}(\zeta_1)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\top}}(\zeta_1),\widehat{\Psi_{\aleph}^{\top}}(\zeta_2)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\top}}(\zeta_2)\}$ and $\widehat{\Psi_{\aleph}^{\daleth}}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\daleth}}((\zeta_1\ddagger_1\zeta_2),\partial)>\beta\geqslant\frac{\widehat{\Psi_{\aleph}^{\daleth}}(\zeta_1)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\daleth}}(\zeta_1)+\widehat{\Psi_{\aleph}^{\daleth}}(\zeta_2)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\daleth}}(\zeta_2)}{2}$ and $\widehat{\Psi_{\aleph}^{F}}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{F}}((\zeta_1\ddagger_1\zeta_2),\partial)<\delta\leqslant\min\{\widehat{\Psi_{\aleph}^{F}}(\zeta_1)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{F}}(\zeta_1),\widehat{\Psi_{\aleph}^{F}}(\zeta_2)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{F}}(\zeta_2)\}$. Thus,$\zeta_1,\zeta_2\in\widehat{\Psi^{(\beta,\delta)}}$,but $\zeta_1\ddagger_1\zeta_2\notin\widehat{\Psi^{(\beta,\delta)}}$. This contradicts,$\widehat{\Psi^{(\beta,\delta)}}$ is a SBS of Ⓢ. Therefore $\widehat{\Psi_{\aleph}^{\top}}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\top}}((\zeta_1\ddagger_1\zeta_2),\partial)\leqslant\max\{\widehat{\Psi_{\aleph}^{\top}}(\zeta_1)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\top}}(\zeta_1),\widehat{\Psi_{\aleph}^{\top}}(\zeta_2)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\top}}(\zeta_2)\}$, $\widehat{\Psi_{\aleph}^{\daleth}}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\daleth}}((\zeta_1\ddagger_1\zeta_2),\partial)\leqslant\frac{\widehat{\Psi_{\aleph}^{\daleth}}(\zeta_1)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\daleth}}(\zeta_1)+\widehat{\Psi_{\aleph}^{\daleth}}(\zeta_2)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\daleth}}(\zeta_2)}{2}$ and $\widehat{\Psi_{\aleph}^{F}}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{F}}((\zeta_1\ddagger_1\zeta_2),\partial)\geqslant\min\{\widehat{\Psi_{\aleph}^{F}}(\zeta_1)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{F}}(\zeta_1),\widehat{\Psi_{\aleph}^{F}}(\zeta_2)\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{F}}(\zeta_2)\}$. Similarly,$\ddagger_2$ and $\ddagger_3$ cases.
Hence $\widehat{\Psi}=(\widehat{\Psi_{\aleph}^{\top}}\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\top}},\widehat{\Psi_{\aleph}^{\daleth}}\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\daleth}},\widehat{\Psi_{\aleph}^{F}}\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{F}})$ is a ComCANSBS of Ⓢ.
Let us assume that $\Psi^{(\beta,\delta)}$ is a SBS of Ⓢ and $\beta,\delta\in[0,1]$. Suppose if there exist $\zeta_1,\zeta_2\in$ Ⓢ such that $\Psi_{\aleph}^{\top}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}}((\zeta_1\ddagger_1\zeta_2),\partial)>\max\{\Psi_{\aleph}^{\top}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}}(\zeta_1),\Psi_{\aleph}^{\top}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}}(\zeta_2)\}$,$\Psi_{\aleph}^{\daleth}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}}((\zeta_1\ddagger_1\zeta_2),\partial)>\frac{\Psi_{\aleph}^{\daleth}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}}(\zeta_1)+\Psi_{\aleph}^{\daleth}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}}(\zeta_2)}{2}$ and $\Psi_{\aleph}^{F}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}}((\zeta_1\ddagger_1\zeta_2),\partial)<\min\{\Psi_{\aleph}^{F}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}}(\zeta_1),\Psi_{\aleph}^{F}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}}(\zeta_2)\}$. For $\beta,\delta\in D[0,1]$ such that $\Psi_{\aleph}^{\top}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}}((\zeta_1\ddagger_1\zeta_2),\partial)>\beta\geqslant\max\{\Psi_{\aleph}^{\top}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}}(\zeta_1),\Psi_{\aleph}^{\top}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}}(\zeta_2)\}$ and $\Psi_{\aleph}^{\daleth}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}}((\zeta_1\ddagger_1\zeta_2),\partial)>\beta\geqslant\frac{\Psi_{\aleph}^{\daleth}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}}(\zeta_1)+\Psi_{\aleph}^{\daleth}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}}(\zeta_2)}{2}$ and $\Psi_{\aleph}^{F}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}}((\zeta_1\ddagger_1\zeta_2),\partial)<\delta\leqslant\min\{\Psi_{\aleph}^{F}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}}(\zeta_1),\Psi_{\aleph}^{F}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}}(\zeta_2)\}$. Thus,$\zeta_1,\zeta_2\in\Psi^{(\beta,\delta)}$,but $\zeta_1\ddagger_1\zeta_2\notin\Psi^{(\beta,\delta)}$. This contradicts,$\Psi^{(\beta,\delta)}$ is a SBS of Ⓢ. Therefore $\Psi_{\aleph}^{\top}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}}((\zeta_1\ddagger_1\zeta_2),\partial)\leqslant\max\{\Psi_{\aleph}^{\top}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}}(\zeta_1),\Psi_{\aleph}^{\top}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}}(\zeta_2)\}$, $\Psi_{\aleph}^{\daleth}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}}((\zeta_1\ddagger_1\zeta_2),\partial)\leqslant\frac{\Psi_{\aleph}^{\daleth}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}}(\zeta_1)+\Psi_{\aleph}^{\daleth}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}}(\zeta_2)}{2}$ and $\Psi_{\aleph}^{F}((\zeta_1\ddagger_1\zeta_2),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}}((\zeta_1\ddagger_1\zeta_2),\partial)\geqslant\min\{\Psi_{\aleph}^{F}(\zeta_1)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}}(\zeta_1),\Psi_{\aleph}^{F}(\zeta_2)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}}(\zeta_2)\}$. Similarly,$\ddagger_2$ and $\ddagger_3$ cases.
Hence $\Psi=(\Psi_{\aleph}^{\top}\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}},\Psi_{\aleph}^{\daleth}\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}},\Psi_{\aleph}^{F}\cdot e^{i\Bbbk\Xi_{\aleph}^{F}})$ is a ComCANSBS of Ⓢ.

Definition 3.13. Let $(Ⓢ_1,\ominus_1,\ominus_2,\ominus_3)$ and $(Ⓢ_2,\circledast_1,\circledast_2,\circledast_3)$ be any two bisemirings. The mapping $\nabla: Ⓢ_1\to Ⓢ_2$ and $\aleph$ be any ComCANSBS in $Ⓢ_1$, $\hbar$ be any ComCANSBS in $\nabla(Ⓢ_1)=Ⓢ_2$. If $\Psi_{\aleph}\cdot e^{i\Bbbk\Xi_{\aleph}}=[\widehat{\Psi_{\aleph}^{\top}}\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\top}},\widehat{\Psi_{\aleph}^{\daleth}}\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{\daleth}},\widehat{\Psi_{\aleph}^{F}}\cdot e^{i\Bbbk}\widehat{\Xi_{\aleph}^{F}},\Psi_{\aleph}\cdot e^{i\Bbbk\Xi_{\aleph}},\Psi_{\aleph}^{\top}\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}},\Psi_{\aleph}^{\daleth}\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}},\Psi_{\aleph}^{F}\cdot e^{i\Bbbk\Xi_{\aleph}^{F}}$ is a ComCANS in $Ⓢ_1$, then $\Psi_{\hbar}$ is a ComCANS in $Ⓢ_2$,defined by

$$\overbrace{\Psi_{\hbar}^{\top}}(\rho,\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}}(\rho,\partial)=\begin{cases}\inf\overbrace{\Psi_{\aleph}^{\top}}(\zeta,\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}}(\zeta,\partial) & if\ \ \zeta\in\nabla^{-1}\rho\\ 0 & otherwise\end{cases}$$

$$\overbrace{\Psi_{\hbar}^{\daleth}}(\rho,\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\daleth}}}(\rho,\partial)=\begin{cases}\inf\overbrace{\Psi_{\aleph}^{\daleth}}(\zeta,\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\daleth}}}(\zeta,\partial) & if\ \ \zeta\in\nabla^{-1}\rho\\ 0 & otherwise\end{cases}$$

$$\overbrace{\Psi_{\hbar}^{F}}(\rho,\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}}}(\rho,\partial)=\begin{cases}\sup\overbrace{\Psi_{\aleph}^{F}}(\zeta,\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{F}}}(\zeta,\partial) & if\ \ \zeta\in\nabla^{-1}\rho\\ 1 & otherwise\end{cases}$$

$$\Psi_{\hbar}^{\top}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}}(\rho,\partial)=\begin{cases}\inf\Psi_{\aleph}^{\top}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}}(\zeta,\partial) & if\ \ \zeta\in\nabla^{-1}\rho\\ 0 & otherwise\end{cases}$$

$$\Psi_{\hbar}^{\daleth}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}}(\rho,\partial)=\begin{cases}\inf\Psi_{\aleph}^{\daleth}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}}(\zeta,\partial) & if\ \ \zeta\in\nabla^{-1}\rho\\ 0 & otherwise\end{cases}$$

$$\Psi_{\hbar}^{F}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}}(\rho,\partial)=\begin{cases}\sup\Psi_{\aleph}^{F}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}}(\zeta,\partial) & if\ \ \zeta\in\nabla^{-1}\rho\\ 1 & otherwise\end{cases}$$

for all $\zeta\in Ⓢ_1$ and $\rho\in Ⓢ_2$ is represents the image of $R_{\aleph}$ under ∇.

Similarly,If $\Psi_{\hbar}\cdot e^{i\Bbbk\Xi_{\hbar}}=[\overbrace{\Psi_{\hbar}^{\top}}\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}},\overbrace{\Psi_{\hbar}^{\daleth}}\cdot e^{i\Bbbk\overbrace{\Xi_{\hbar}^{\daleth}}},\overbrace{\Psi_{\hbar}^{F}}\cdot e^{i\Bbbk\overbrace{\Xi_{\hbar}^{F}}}]$, $\Psi_{\hbar}\cdot e^{i\Bbbk\Xi_{\hbar}},\Psi_{\hbar}^{\top}\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}},\Psi_{\hbar}^{\daleth}\cdot e^{i\Bbbk\Xi_{\hbar}^{\daleth}},\Psi_{\hbar}^{F}\cdot e^{i\Bbbk\Xi_{\hbar}^{F}}$ is a ComCANS in $Ⓢ_2$,then ComCANS $\Psi_{\aleph}=\nabla\circ\Psi_{\hbar}$ in $Ⓢ_1$ ie,the ComCANS defined by $\Psi_{\aleph}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}(\zeta,\partial)}$, $\Psi_{\hbar}(\nabla(\zeta,\partial))\cdot e^{i\Bbbk\Xi_{\aleph}(\nabla(\zeta,\partial))}$, $\Psi_{\aleph}=\nabla\circ\Psi_{\hbar}$ in $Ⓢ_1$ [ie,the ComCANS defined by $\Psi_{\aleph}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}(\zeta,\partial)}=\Psi_{\hbar}(\nabla(\zeta,\partial))\cdot e^{i\Bbbk\Xi_{\aleph}(\nabla(\zeta,\partial))}$ is represents the preimage of $\Psi_{\hbar}$ under ∇.

Theorem 3.14. *The homomorphic image of every ComCANSBS is a ComCANSBS.*

Proof. The mapping $\nabla:Ⓢ_1\rightarrow Ⓢ_2$ be any homomorphism. Now, $\nabla((\zeta\ominus_1\rho),\partial)=\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial),\nabla((\zeta\ominus_2\rho),\partial)=\nabla(\zeta,\partial)\circledast_2\nabla(\rho,\partial)$ and $\nabla((\zeta\ominus_3\rho),\partial)=\nabla(\zeta,\partial)\circledast_3\nabla(\rho,\partial)$ for all $\zeta,\rho\in Ⓢ_1$. Let $\overbrace{\hbar}=\nabla(\aleph)$,$\aleph$ is any ComCANSBS of $Ⓢ_1$. Let $\nabla(\zeta,\partial),\nabla(\rho,\partial)\in Ⓢ_2$. Let $\zeta\in u\nabla^{-1}(\nabla(\zeta,\partial))$ and $\rho\in\nabla^{-1}(\nabla(\rho,\partial))$ be such that $\overbrace{\Psi_{\aleph}^{\top}}(\zeta,\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}(\zeta,\partial)}=\inf_{\zeta\in\nabla^{-1}(\nabla(\zeta,\partial))}\overbrace{\Psi_{\aleph}^{\top}}(\zeta,\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}(\zeta,\partial)}$ and $\overbrace{\Psi_{\aleph}^{\top}}(\rho,\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}(\rho,\partial)}=\inf_{\zeta\in\nabla^{-1}(\nabla(\rho,\partial))}\overbrace{\Psi_{\aleph}^{\top}}(\zeta,\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}(\zeta,\partial)}$.

Now, $\overbrace{\Psi_{\hbar}^{\top}}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))}$

$$\begin{aligned}&=\inf_{(\zeta')\in\nabla^{-1}(\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial))}\overbrace{\Psi_{\aleph}^{\top}}(\zeta')\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}(\zeta')}\\&=\inf_{(\zeta')\in\nabla^{-1}(\nabla((\zeta\ominus_1\rho),\partial)}\overbrace{\Psi_{\aleph}^{\top}}(\zeta')\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}(\zeta')}\\&=\overbrace{\Psi_{\aleph}^{\top}}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}((\zeta\ominus_1\rho),\partial)}\\&\leqslant\max\{\overbrace{\Psi_{\aleph}^{\top}}(\zeta,\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}(\zeta,\partial)},\overbrace{\Psi_{\aleph}^{\top}}(\rho,\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\aleph}^{\top}}(\rho,\partial)}\}\\&=\max\{\overbrace{\Psi_{\hbar}^{\top}}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\hbar}^{\top}}\nabla(\zeta,\partial)},\overbrace{\Psi_{\hbar}^{\top}}\nabla(\rho,\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\hbar}^{\top}}\nabla(\rho,\partial)}\}.\end{aligned}$$

Thus, $\overbrace{\Psi_{\hbar}^{\top}}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\overbrace{\Xi_{\hbar}^{\top}}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))}\leqslant\max\{\overbrace{\Psi_{\hbar}^{\top}}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\hbar}^{\top}}\nabla(\zeta,\partial)},\overbrace{\Psi_{\hbar}^{\top}}\nabla(\rho,\partial)\cdot e^{i\Bbbk\overbrace{\Xi_{\hbar}^{\top}}\nabla(\rho,\partial)}\}$.

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Similarly, $\widehat{\Psi_{\hbar}^{\top}}((\nabla(\zeta,\partial)\circledast_2\nabla(\rho,\partial)))\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{\top}}((\nabla(\zeta,\partial)\circledast_2\nabla(\rho,\partial)))} \leqslant \max\{\widehat{\Psi_{\hbar}^{\top}}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{\top}}\nabla(\zeta,\partial)}, \widehat{\Psi_{\hbar}^{\top}}\nabla(\rho,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{\top}}\nabla(\rho,\partial)}\}$ and

$\widehat{\Psi_{\hbar}^{\top}}((\nabla(\zeta,\partial)\circledast_3\nabla(\rho,\partial)))\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{\top}}((\nabla(\zeta,\partial)\circledast_3\nabla(\rho,\partial)))} \leqslant \max\{\widehat{\Psi_{\hbar}^{\top}}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{\top}}\nabla(\zeta,\partial)}, \widehat{\Psi_{\hbar}^{\top}}\nabla(\rho,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{\top}}\nabla(\rho,\partial)}\}$.

Let $\zeta\in\nabla^{-1}(\nabla(\zeta,\partial))$ and $\rho\in\nabla^{-1}(\nabla(\rho,\partial))$ be such that $\widehat{\Psi_{\aleph}^{\gimel}}(\zeta,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{\top}}(\zeta,\partial)} = \inf_{\zeta\in\nabla^{-1}(\nabla(\zeta,\partial))}\widehat{\Psi_{\aleph}^{\gimel}}(\zeta,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{\top}}(\zeta,\partial)}$ and $\widehat{\Psi_{\aleph}^{\gimel}}(\rho,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{\top}}(\rho,\partial)} = \inf_{\zeta\in\nabla^{-1}(\nabla(\rho,\partial))}\widehat{\Psi_{\aleph}^{\gimel}}(\zeta,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{\top}}\nabla(\zeta,\partial)}$.

Now, $\widehat{\Psi_{\hbar}^{\gimel}}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{\gimel}}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))}$

$$
\begin{aligned}
&= \inf_{(\zeta')\in\nabla^{-1}(\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial))}\widehat{\Psi_{\aleph}^{\gimel}}(\zeta')\cdot e^{i\Bbbk\,\widehat{\Xi_{\aleph}^{\gimel}}(\zeta')}\\
&= \inf_{(\zeta')\in\nabla^{-1}(\nabla((\zeta\ominus_1\rho),\partial)}\widehat{\Psi_{\aleph}^{\gimel}}(\zeta')\cdot e^{i\Bbbk\,\widehat{\Xi_{\aleph}^{\gimel}}(\zeta')}\\
&= \widehat{\Psi_{\aleph}^{\gimel}}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\aleph}^{\gimel}}((\zeta\ominus_1\rho),\partial)}\\
&\leqslant \frac{\widehat{\Psi_{\aleph}^{\gimel}}(\zeta,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\aleph}^{\gimel}}(\zeta,\partial)} + \widehat{\Psi_{\aleph}^{\gimel}}(\rho,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\aleph}^{\gimel}}(\rho,\partial)}}{2}\\
&= \frac{\widehat{\Psi_{\hbar}^{\gimel}}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{\gimel}}\nabla(\zeta,\partial)} + \widehat{\Psi_{\hbar}^{\gimel}}\nabla(\rho,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{\gimel}}\nabla(\rho,\partial)}}{2}.
\end{aligned}
$$

Thus,

$\widehat{\Psi_{\hbar}^{\gimel}}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{\gimel}}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))} \leqslant \frac{\widehat{\Psi_{\hbar}^{\gimel}}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{\gimel}}\nabla(\zeta,\partial)} + \widehat{\Psi_{\hbar}^{\gimel}}\nabla(\rho,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{\gimel}}\nabla(\rho,\partial)}}{2}$.
Similarly,

$\widehat{\Psi_{\hbar}^{\gimel}}((\nabla(\zeta,\partial)\circledast_2\nabla(\rho,\partial)))\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{\gimel}}((\nabla(\zeta,\partial)\circledast_2\nabla(\rho,\partial)))} \leqslant \frac{\widehat{\Psi_{\hbar}^{\gimel}}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{\gimel}}\nabla(\zeta,\partial)} + \widehat{\Psi_{\hbar}^{\gimel}}\nabla(\rho,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{\gimel}}\nabla(\rho,\partial)}}{2}$
and

$\widehat{\Psi_{\hbar}^{\gimel}}((\nabla(\zeta,\partial)\circledast_3\nabla(\rho,\partial)))\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{\gimel}}((\nabla(\zeta,\partial)\circledast_3\nabla(\rho,\partial)))} \leqslant \frac{\widehat{\Psi_{\hbar}^{\gimel}}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{\gimel}}\nabla(\zeta,\partial)} + \widehat{\Psi_{\hbar}^{\gimel}}\nabla(\rho,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{\gimel}}\nabla(\rho,\partial)}}{2}$.

Let $\nabla(\zeta,\partial),\nabla(\rho,\partial)\in Ⓢ_2$. Let $\zeta\in\nabla^{-1}(\nabla(\zeta,\partial))$ and $\rho\in\nabla^{-1}(\nabla(\rho,\partial))$ be such that $\widehat{\Psi_{\aleph}^{F}}(\zeta,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\aleph}^{F}}(\zeta,\partial)} = \sup_{\zeta\in\nabla^{-1}(\nabla(\zeta,\partial))}\widehat{\Psi_{\aleph}^{F}}(\zeta,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\aleph}^{F}}(\zeta,\partial)}$ and $\widehat{\Psi_{\aleph}^{F}}(\rho,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\aleph}^{F}}(\rho,\partial)} = \sup_{\zeta\in\nabla^{-1}(\nabla(\rho,\partial))}\widehat{\Psi_{\aleph}^{F}}(\zeta,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\aleph}^{F}}(\zeta,\partial)}$. Now, $\widehat{\Psi_{\hbar}^{F}}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{F}}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))}$

$$
\begin{aligned}
&= \sup_{(\zeta')\in\nabla^{-1}(\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial))}\widehat{\Psi_{\aleph}^{F}}(\zeta')\cdot e^{i\Bbbk\,\widehat{\Xi_{\aleph}^{F}}(\zeta')}\\
&= \sup_{(\zeta')\in\nabla^{-1}(\nabla((\zeta\ominus_1\rho),\partial)}\widehat{\Psi_{\aleph}^{F}}(\zeta')\cdot e^{i\Bbbk\,\widehat{\Xi_{\aleph}^{F}}(\zeta')}\\
&= \widehat{\Psi_{\aleph}^{F}}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\aleph}^{F}}((\zeta\ominus_1\rho),\partial)}\\
&\geqslant \min\{\widehat{\Psi_{\aleph}^{F}}(\zeta,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\aleph}^{F}}(\zeta,\partial)}, \widehat{\Psi_{\aleph}^{F}}(\rho,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\aleph}^{F}}(\rho,\partial)}\}\\
&= \min\{\widehat{\Psi_{\hbar}^{F}}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{F}}\nabla(\zeta,\partial)}, \widehat{\Psi_{\hbar}^{F}}\nabla(\rho,\partial)\cdot e^{i\Bbbk\,\widehat{\Xi_{\hbar}^{F}}\nabla(\rho,\partial)}\}.
\end{aligned}
$$

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Thus, $\overparen{\Psi^{F}_{\hbar}}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\ \overparen{\Xi^{F}_{\hbar}}\ ((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))} \geqslant \min\{\overparen{\Psi^{F}_{\hbar}}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\ \overparen{\Xi^{F}_{\hbar}}\ \nabla(\zeta,\partial)}, \overparen{\Psi^{F}_{\hbar}}\nabla(\rho,\partial)\cdot e^{i\Bbbk\ \overparen{\Xi^{F}_{\hbar}}\ \nabla(\rho,\partial)}\}$. Similarly,

$\overparen{\Psi^{F}_{\hbar}}((\nabla(\zeta,\partial)\circledast_2\nabla(\rho,\partial)))\cdot e^{i\Bbbk\ \overparen{\Xi^{F}_{\hbar}}\ ((\nabla(\zeta,\partial)\circledast_2\nabla(\rho,\partial)))} \geqslant \min\{\overparen{\Psi^{F}_{\hbar}}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\ \overparen{\Xi^{F}_{\hbar}}\ \nabla(\zeta,\partial)}, \overparen{\Psi^{F}_{\hbar}}\nabla(\rho,\partial)\cdot e^{i\Bbbk\ \overparen{\Xi^{F}_{\hbar}}\ \nabla(\rho,\partial)}\}$ and

$\overparen{\Psi^{F}_{\hbar}}((\nabla(\zeta,\partial)\circledast_3\nabla(\rho,\partial)))\cdot e^{i\Bbbk\ \overparen{\Xi^{F}_{\hbar}}\ ((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))} \geqslant \min\{\overparen{\Psi^{F}_{\hbar}}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\ \overparen{\Xi^{F}_{\hbar}}\ \nabla(\rho,\partial)}, \overparen{\Psi^{F}_{\hbar}}\nabla(\rho,\partial)\cdot e^{i\Bbbk\ \overparen{\Xi^{F}_{\hbar}}\ \nabla(\rho,\partial)}\}$.

The mapping $\nabla : \circledS_1 \to \circledS_2$ be any homomorphism. Now, $\nabla((\zeta\ominus_1\rho),\partial) = \nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)$, $\nabla((\zeta\ominus_2\rho),\partial) = \nabla(\zeta,\partial)\circledast_2\nabla(\rho,\partial)$ and $\nabla((\zeta\ominus_3\rho),\partial) = \nabla(\zeta,\partial)\circledast_3\nabla(\rho,\partial)$ for all $\zeta,\rho\in\circledS_1$. Let $\hbar = \nabla(\aleph)$, $\aleph$ is any ComCANSBS of $\circledS_1$. Let $\nabla(\zeta,\partial),\nabla(\rho,\partial)\in\circledS_2$. Let $\zeta\in\nabla^{-1}(\nabla(\zeta,\partial))$ and $\rho\in\nabla^{-1}(\nabla(\rho,\partial))$ be such that $\Psi^{\top}_{\aleph}(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}(\zeta,\partial)} = \inf\limits_{\zeta\in\nabla^{-1}(\nabla(\zeta,\partial))}\Psi^{\top}_{\aleph}(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}(\zeta,\partial)}$ and $\Psi^{\top}_{\aleph}(\rho,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}(\rho,\partial)} = \inf\limits_{\zeta\in\nabla^{-1}(\nabla(\rho,\partial))}\Psi^{\top}_{\aleph}(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}(\zeta,\partial)}$. Now, $\Psi^{\top}_{\hbar}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))}$

$$\begin{aligned}
&= \inf_{(\zeta')\in\nabla^{-1}(\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial))}\Psi^{\top}_{\aleph}(\zeta')\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}(\zeta')}\\
&= \inf_{(\zeta')\in\nabla^{-1}(\nabla((\zeta\ominus_1\rho),\partial)}\Psi^{\top}_{\aleph}(\zeta')\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}(\zeta')}\\
&= \Psi^{\top}_{\aleph}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}((\zeta\ominus_1\rho),\partial)}\\
&\leqslant \max\{\Psi^{\top}_{\aleph}(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}(\zeta,\partial)}, \Psi^{\top}_{\aleph}(\rho,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}(\rho,\partial)}\}\\
&= \max\{\Psi^{\top}_{\hbar}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\hbar}\nabla(\zeta,\partial)}, \Psi^{\top}_{\hbar}\nabla(\rho,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\hbar}\nabla(\rho,\partial)}\}.
\end{aligned}$$

Thus, $\Psi^{\top}_{\hbar}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\Xi^{\top}_{\hbar}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))} \leqslant \max\{\Psi^{\top}_{\hbar}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\hbar}\nabla(\zeta,\partial)}, \Psi^{\top}_{\hbar}\nabla(\rho,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\hbar}\nabla(\rho,\partial)}\}$.

Similarly, $\Psi^{\top}_{\hbar}((\nabla(\zeta,\partial)\circledast_2\nabla(\rho,\partial)))\cdot e^{i\Bbbk\Xi^{\top}_{\hbar}((\nabla(\zeta,\partial)\circledast_2\nabla(\rho,\partial)))} \leqslant \max\{\Psi^{\top}_{\hbar}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\hbar}\nabla(\zeta,\partial)}, \Psi^{\top}_{\hbar}\nabla(\rho,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\hbar}\nabla(\rho,\partial)}\}$ and

$\Psi^{\top}_{\hbar}((\nabla(\zeta,\partial)\circledast_3\nabla(\rho,\partial)))\cdot e^{i\Bbbk\Xi^{\top}_{\hbar}((\nabla(\zeta,\partial)\circledast_3\nabla(\rho,\partial)))} \leqslant \max\{\Psi^{\top}_{\hbar}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\hbar}\nabla(\zeta,\partial)}, \Psi^{\top}_{\hbar}\nabla(\rho,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\hbar}\nabla(\rho,\partial)}\}$.

Let $\zeta\in\nabla^{-1}(\nabla(\zeta,\partial))$ and $\rho\in\nabla^{-1}(\nabla(\rho,\partial))$ be such that $\Psi^{\daleth}_{\aleph}(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\hbar}(\zeta,\partial)} = \inf\limits_{\zeta\in\nabla^{-1}(\nabla(\zeta,\partial))}\Psi^{\daleth}_{\aleph}(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\hbar}(\zeta,\partial)}$ and $\Psi^{\daleth}_{\aleph}(\rho,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\hbar}(\rho,\partial)} = \inf\limits_{\zeta\in\nabla^{-1}(\nabla(\rho,\partial))}\Psi^{\daleth}_{\aleph}(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\hbar}\nabla(\zeta,\partial)}$.

Now, $\Psi^{\daleth}_{\hbar}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\Xi^{\daleth}_{\hbar}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))}$

$$\begin{aligned}
&= \inf_{(\zeta')\in\nabla^{-1}(\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial))}\Psi^{\daleth}_{\aleph}(\zeta')\cdot e^{i\Bbbk\Xi^{\daleth}_{\aleph}(\zeta')}\\
&= \inf_{(\zeta')\in\nabla^{-1}(\nabla((\zeta\ominus_1\rho),\partial)}\Psi^{\daleth}_{\aleph}(\zeta')\cdot e^{i\Bbbk\Xi^{\daleth}_{\aleph}(\zeta')}\\
&= \Psi^{\daleth}_{\aleph}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\Xi^{\daleth}_{\aleph}((\zeta\ominus_1\rho),\partial)}\\
&\leqslant \frac{\Psi^{\daleth}_{\aleph}(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{\daleth}_{\aleph}(\zeta,\partial)} + \Psi^{\daleth}_{\aleph}(\rho,\partial)\cdot e^{i\Bbbk\Xi^{\daleth}_{\aleph}(\rho,\partial)}}{2}\\
&= \frac{\Psi^{\daleth}_{\hbar}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{\daleth}_{\hbar}\nabla(\zeta,\partial)} + \Psi^{\daleth}_{\hbar}\nabla(\rho,\partial)\cdot e^{i\Bbbk\Xi^{\daleth}_{\hbar}\nabla(\rho,\partial)}}{2}.
\end{aligned}$$

Thus, $\Psi^{\daleth}_{\hbar}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\Xi^{\daleth}_{\hbar}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))} \leqslant \frac{\Psi^{\daleth}_{\hbar}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{\daleth}_{\hbar}\nabla(\zeta,\partial)}+\Psi^{\daleth}_{\hbar}\nabla(\rho,\partial)\cdot e^{i\Bbbk\Xi^{\daleth}_{\hbar}\nabla(\rho,\partial)}}{2}$. Similarly,

$\Psi^{\daleth}_{\hbar}((\nabla(\zeta,\partial)\circledast_2\nabla(\rho,\partial)))\cdot e^{i\Bbbk\Xi^{\daleth}_{\hbar}((\nabla(\zeta,\partial)\circledast_2\nabla(\rho,\partial)))} \leqslant \frac{\Psi^{\daleth}_{\hbar}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{\daleth}_{\hbar}\nabla(\zeta,\partial)}+\Psi^{\daleth}_{\hbar}\nabla(\rho,\partial)\cdot e^{i\Bbbk\Xi^{\daleth}_{\hbar}\nabla(\rho,\partial)}}{2}$ and

$\Psi^{\daleth}_{\hbar}((\nabla(\zeta,\partial)\circledast_3\nabla(\rho,\partial)))\cdot e^{i\Bbbk\Xi^{\daleth}_{\hbar}((\nabla(\zeta,\partial)\circledast_3\nabla(\rho,\partial)))} \leqslant \frac{\Psi^{\daleth}_{\hbar}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{\daleth}_{\hbar}\nabla(\zeta,\partial)}+\Psi^{\daleth}_{\hbar}\nabla(\rho,\partial)\cdot e^{i\Bbbk\Xi^{\daleth}_{\hbar}\nabla(\rho,\partial)}}{2}$.

Let $\nabla(\zeta,\partial),\nabla(\rho,\partial)\in\circledS_2$. Let $\zeta\in\nabla^{-1}(\nabla(\zeta,\partial))$ and $\rho\in\nabla^{-1}(\nabla(\rho,\partial))$ be such that $\Psi^{F}_{\aleph}(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{F}_{\aleph}(\zeta,\partial)} = \sup\limits_{\zeta\in\nabla^{-1}(\nabla(\zeta,\partial))}\Psi^{F}_{\aleph}(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{F}_{\aleph}(\zeta,\partial)}$ and $\Psi^{F}_{\aleph}(\rho,\partial)\cdot e^{i\Bbbk\Xi^{F}_{\aleph}(\rho,\partial)} = \sup\limits_{\zeta\in\nabla^{-1}(\nabla(\rho,\partial))}\Psi^{F}_{\aleph}(\zeta,\partial)\cdot$

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$e^{i\Bbbk\Xi^{F}_{\aleph}(\zeta,\partial)}$.
Now, $\Psi^{F}_{\hbar}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\Xi^{F}_{\hbar}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))}$

$$\begin{aligned}
&= \sup_{(\zeta')\in\nabla^{-1}(\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial))}\Psi^{F}_{\aleph}(\zeta')\cdot e^{i\Bbbk\Xi^{F}_{\aleph}(\zeta')}\\
&= \sup_{(\zeta')\in\nabla^{-1}(\nabla((\zeta\ominus_1\rho),\partial)}\Psi^{F}_{\aleph}(\zeta')\cdot e^{i\Bbbk\Xi^{F}_{\aleph}(\zeta')}\\
&= \Psi^{F}_{\aleph}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\Xi^{F}_{\aleph}((\zeta\ominus_1\rho),\partial)}\\
&\geqslant \min\{\Psi^{F}_{\aleph}(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{F}_{\aleph}(\zeta,\partial)},\Psi^{F}_{\aleph}(\rho,\partial)\cdot e^{i\Bbbk\Xi^{F}_{\aleph}(\rho,\partial)}\}\\
&= \min\{\Psi^{F}_{\hbar}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{F}_{\hbar}\nabla(\zeta,\partial)},\Psi^{F}_{\hbar}\nabla(\rho,\partial)\cdot e^{i\Bbbk\Xi^{F}_{\hbar}\nabla(\rho,\partial)}\}.
\end{aligned}$$

Thus,$\Psi^{F}_{\hbar}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\Xi^{F}_{\hbar}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))}\geqslant\min\{\Psi^{F}_{\hbar}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{F}_{\hbar}\nabla(\zeta,\partial)},\Psi^{F}_{\hbar}\nabla(\rho,\partial)\cdot e^{i\Bbbk\Xi^{F}_{\hbar}\nabla(\rho,\partial)}\}$.
Similarly,$\Psi^{F}_{\hbar}((\nabla(\zeta,\partial)\circledast_2\nabla(\rho,\partial)))\cdot e^{i\Bbbk\Xi^{F}_{\hbar}((\nabla(\zeta,\partial)\circledast_2\nabla(\rho,\partial)))}\geqslant\min\{\Psi^{F}_{\hbar}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{F}_{\hbar}\nabla(\zeta,\partial)},\Psi^{F}_{\hbar}\nabla(\rho,\partial)\cdot e^{i\Bbbk\Xi^{F}_{\hbar}\nabla(\rho,\partial)}\}$ and
$\Psi^{F}_{\hbar}((\nabla(\zeta,\partial)\circledast_3\nabla(\rho,\partial)))\cdot e^{i\Bbbk\Xi^{F}_{\hbar}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))}\geqslant\min\{\Psi^{F}_{\hbar}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{F}_{\hbar}\nabla(\rho,\partial)},\Psi^{F}_{\hbar}\nabla(\rho,\partial)\cdot e^{i\Bbbk\Xi^{F}_{\hbar}\nabla(\rho,\partial)}\}$.
Thus,$\hbar$ is a ComCANSBS of $Ⓢ_2$.

Theorem 3.15. *The homomorphic preimage of every ComCANSBS is a ComCANSBS.*

Proof. The mapping $\nabla: Ⓢ_1\to Ⓢ_2$ be a homomorphism. Now,$\nabla((\zeta\ominus_1\rho),\partial)=\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial),\nabla((\zeta\ominus_2\rho),\partial)=\nabla(\zeta,\partial)\circledast_2\nabla(\rho,\partial)$ and $\nabla((\zeta\ominus_3\rho),\partial)=\nabla(\zeta,\partial)\circledast_3\nabla(\rho,\partial)$ for all $\zeta,\rho\in Ⓢ_1$. Let $\widehat{\hbar}=\nabla(\aleph)$, $\widehat{\hbar}$ is a ComCANSBS of $Ⓢ_2$. Let $\zeta,\rho\in Ⓢ_1$. Now, $\widehat{\Psi^{\top}_{\aleph}}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\widehat{\Xi^{\top}_{\aleph}}((\zeta\ominus_1\rho),\partial)}=\widehat{\Psi^{\top}_{\hbar}}(\nabla((\zeta\ominus_1\rho),\partial))\cdot e^{i\Bbbk\widehat{\Xi^{\top}_{\hbar}}(\nabla((\zeta\ominus_1\rho),\partial))}=\widehat{\Psi^{\top}_{\hbar}}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\widehat{\Xi^{\top}_{\hbar}}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))}\leqslant\max\{\widehat{\Psi^{\top}_{\hbar}}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi^{\top}_{\hbar}}\nabla(\zeta,\partial)},\widehat{\Psi^{\top}_{\hbar}}\nabla(\rho,\partial)\cdot e^{i\Bbbk\widehat{\Xi^{\top}_{\hbar}}\nabla(\rho,\partial)}\}=\max\{\widehat{\Psi^{\top}_{\aleph}}(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi^{\top}_{\aleph}}(\zeta,\partial)},\widehat{\Psi^{\top}_{\aleph}}(\rho,\partial)\cdot e^{i\Bbbk\widehat{\Xi^{\top}_{\aleph}}(\rho,\partial)}\}$.
Thus, $\widehat{\Psi^{\top}_{\aleph}}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\widehat{\Xi^{\top}_{\aleph}}((\zeta\ominus_1\rho),\partial)}\leqslant\max\{\widehat{\Psi^{\top}_{\aleph}}(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi^{\top}_{\aleph}}(\zeta)},\widehat{\Psi^{\top}_{\aleph}}(\rho,\partial)\cdot e^{i\Bbbk\widehat{\Xi^{\top}_{\aleph}}(\rho,\partial)}\}$.
Now, $\widehat{\Psi^{\daleth}_{\aleph}}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\widehat{\Xi^{\daleth}_{\aleph}}((\zeta\ominus_1\rho),\partial)}=\widehat{\Psi^{\daleth}_{\hbar}}(\nabla((\zeta\ominus_1\rho),\partial))\cdot e^{i\Bbbk\widehat{\Xi^{\daleth}_{\hbar}}((\zeta\ominus_1\rho),\partial)}=\widehat{\Psi^{\daleth}_{\hbar}}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\widehat{\Xi^{\daleth}_{\hbar}}((\zeta\ominus_1\rho),\partial)}\leqslant\frac{\widehat{\Psi^{\daleth}_{\hbar}}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi^{\daleth}_{\hbar}}\nabla(\zeta,\partial)}+\widehat{\Psi^{\daleth}_{\hbar}}\nabla(\rho,\partial)\cdot e^{i\Bbbk\widehat{\Xi^{\daleth}_{\hbar}}\nabla(\rho,\partial)}}{2}=\frac{\widehat{\Psi^{\daleth}_{\aleph}}(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi^{\daleth}_{\aleph}}(\zeta,\partial)}+\widehat{\Psi^{\daleth}_{\aleph}}(\rho,\partial)\cdot e^{i\Bbbk\widehat{\Xi^{\daleth}_{\aleph}}(\rho,\partial)}}{2}$.

Thus, $\widehat{\Psi^{\daleth}_{\aleph}}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\widehat{\Xi^{\daleth}_{\aleph}}((\zeta\ominus_1\rho),\partial)}\leqslant\frac{\widehat{\Psi^{\daleth}_{\aleph}}(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi^{\daleth}_{\aleph}}(\zeta,\partial)}+\widehat{\Psi^{\daleth}_{\aleph}}(\rho,\partial)\cdot e^{i\Bbbk\widehat{\Xi^{\daleth}_{\aleph}}(\rho,\partial)}}{2}$.
Now, $\widehat{\Psi^{F}_{\aleph}}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\widehat{\Xi^{F}_{\aleph}}((\zeta\ominus_1\rho),\partial)}=\widehat{\Psi^{F}_{\hbar}}(\nabla((\zeta\ominus_1\rho),\partial))\cdot e^{i\Bbbk\widehat{\Xi^{F}_{\hbar}}(\nabla((\zeta\ominus_1\rho),\partial))}=\widehat{\Psi^{F}_{\hbar}}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\widehat{\Xi^{F}_{\hbar}}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))}\geqslant\min\{\widehat{\Psi^{F}_{\hbar}}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi^{F}_{\hbar}}\nabla(\zeta,\partial)},\widehat{\Psi^{F}_{\hbar}}\nabla(\rho,\partial)\cdot e^{i\Bbbk\widehat{\Xi^{F}_{\hbar}}\nabla(\rho,\partial)}\}=\min\{\widehat{\Psi^{F}_{\aleph}}(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi^{F}_{\aleph}}(\zeta,\partial)},\widehat{\Psi^{F}_{\aleph}}(\rho,\partial)\cdot e^{i\Bbbk\widehat{\Xi^{F}_{\aleph}}(\rho,\partial)}\}$.
Thus, $\widehat{\Psi^{F}_{\aleph}}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\widehat{\Xi^{F}_{\aleph}}((\zeta\ominus_1\rho),\partial)}\geqslant\min\{\widehat{\Psi^{F}_{\aleph}}(\zeta,\partial)\cdot e^{i\Bbbk\widehat{\Xi^{F}_{\aleph}}(\zeta,\partial)},\widehat{\Psi^{F}_{\aleph}}(\rho,\partial)\cdot e^{i\Bbbk\widehat{\Xi^{F}_{\aleph}}(\rho,\partial)}\}$.

The mapping $\nabla: Ⓢ_1\to Ⓢ_2$ be a homomorphism. Now,$\nabla((\zeta\ominus_1\rho),\partial)=\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial),\nabla((\zeta\ominus_2\rho),\partial)=\nabla(\zeta,\partial)\circledast_2\nabla(\rho,\partial)$ and $\nabla((\zeta\ominus_3\rho),\partial)=\nabla(\zeta,\partial)\circledast_3\nabla(\rho,\partial)$ for all $\zeta,\rho\in Ⓢ_1$. Let $\hbar=\nabla(\aleph)$,$\hbar$ is a ComCANSBS of $Ⓢ_2$. Let $\zeta,\rho\in Ⓢ_1$. Now,$\Psi^{\top}_{\aleph}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}((\zeta\ominus_1\rho),\partial)}=\Psi^{\top}_{\hbar}(\nabla((\zeta\ominus_1\rho),\partial))\cdot e^{i\Bbbk\Xi^{\top}_{\hbar}(\nabla((\zeta\ominus_1\rho),\partial))}=\Psi^{\top}_{\hbar}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\Xi^{\top}_{\hbar}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))}\leqslant\max\{\Psi^{\top}_{\hbar}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\hbar}\nabla(\zeta,\partial)},\Psi^{\top}_{\hbar}\nabla(\rho,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\hbar}\nabla(\rho,\partial)}\}=\max\{\Psi^{\top}_{\aleph}(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}(\zeta,\partial)},\Psi^{\top}_{\aleph}(\rho,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}(\rho,\partial)}\}$. Thus,$\Psi^{\top}_{\aleph}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}((\zeta\ominus_1\rho),\partial)}\leqslant\max\{\Psi^{\top}_{\aleph}(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}(\zeta)},\Psi^{\top}_{\aleph}(\rho,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}(\rho,\partial)}\}$.
Now,$\Psi^{\daleth}_{\aleph}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\Xi^{\daleth}_{\aleph}((\zeta\ominus_1\rho),\partial)}=\Psi^{\daleth}_{\hbar}(\nabla((\zeta\ominus_1\rho),\partial))\cdot e^{i\Bbbk\Xi^{\daleth}_{\hbar}((\zeta\ominus_1\rho),\partial)}=\Psi^{\daleth}_{\hbar}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot$

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$e^{i\Bbbk\Xi^{\gimel}_{\hbar}((\zeta\ominus_1\rho),\partial)} \leqslant \frac{\Psi^{\gimel}_{\hbar}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{\gimel}_{\hbar}\nabla(\zeta,\partial)}+\Psi^{\gimel}_{\hbar}\nabla(\rho,\partial)\cdot e^{i\Bbbk\Xi^{\gimel}_{\hbar}\nabla(\rho,\partial)}}{2} = \frac{\Psi^{\gimel}_{\aleph}(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{\gimel}_{\aleph}(\zeta,\partial)}+\Psi^{\gimel}_{\aleph}(\rho,\partial)\cdot e^{i\Bbbk\Xi^{\gimel}_{\aleph}(\rho,\partial)}}{2}$.
Thus, $\Psi^{\gimel}_{\aleph}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\Xi^{\gimel}_{\aleph}((\zeta\ominus_1\rho),\partial)} \leqslant \frac{\Psi^{\gimel}_{\aleph}(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{\gimel}_{\aleph}(\zeta,\partial)}+\Psi^{\gimel}_{\aleph}(\rho,\partial)\cdot e^{i\Bbbk\Xi^{\gimel}_{\aleph}(\rho,\partial)}}{2}$.
Now, $\Psi^{F}_{\aleph}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\Xi^{F}_{\aleph}((\zeta\ominus_1\rho),\partial)} = \Psi^{F}_{\hbar}(\nabla((\zeta\ominus_1\rho),\partial))\cdot e^{i\Bbbk\Xi^{F}_{\hbar}(\nabla((\zeta\ominus_1\rho),\partial))} = \Psi^{F}_{\hbar}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\Xi^{F}_{\hbar}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))} \geqslant \min\{\Psi^{F}_{\hbar}\nabla(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{F}_{\hbar}\nabla(\zeta,\partial)}, \Psi^{F}_{\hbar}\nabla(\rho,\partial)\cdot e^{i\Bbbk\Xi^{F}_{\hbar}\nabla(\rho,\partial)}\} = \min\{\Psi^{F}_{\aleph}(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{F}_{\aleph}(\zeta,\partial)}, \Psi^{F}_{\aleph}(\rho,\partial)\cdot e^{i\Bbbk\Xi^{F}_{\aleph}(\rho,\partial)}\}$. Thus, $\Psi^{F}_{\aleph}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\Xi^{F}_{\aleph}((\zeta\ominus_1\rho),\partial)} \geqslant \min\{\Psi^{F}_{\aleph}(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{F}_{\aleph}(\zeta,\partial)}, \Psi^{F}_{\aleph}(\rho,\partial)\cdot e^{i\Bbbk\Xi^{F}_{\aleph}(\rho,\partial)}\}$.

Theorem 3.16. *If $\nabla:\circledS_1\to\circledS_2$ is a homomorphism,then $\nabla(\overbrace{\aleph_{(\beta,\delta)}})$ is a SBS of ComCANSBS $\overbrace{\hbar}$ of $\circledS_2$.*

Proof. The mapping $\nabla:\circledS_1\to\circledS_2$ be a homomorphism. Now, $\nabla((\zeta\ominus_1\rho),\partial) = \nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)$, $\nabla((\zeta\ominus_2\rho),\partial) = \nabla(\zeta,\partial)\circledast_2\nabla(\rho,\partial)$ and $\nabla((\zeta\ominus_3\rho),\partial) = \nabla(\zeta,\partial)\circledast_3\nabla(\rho,\partial)$ for all $\zeta,\rho\in\circledS_1$. Let $\overbrace{\hbar} = \nabla(\aleph)$, $\aleph$ is a ComCANSBS of $\circledS_1$. By Theorem 3.14, $\overbrace{\hbar}$ is a ComCANSBS of $\circledS_2$. Let $\overbrace{\aleph_{(\beta,\delta)}}$ be any SBS of $\aleph$. Suppose that $\zeta,\rho\in\overbrace{\aleph_{(\beta,\delta)}}$. Then $\zeta\ominus_1\rho, \zeta\ominus_2\rho$ and $\zeta\ominus_3\rho\in\overbrace{\aleph_{(\beta,\delta)}}$. Now, $\overbrace{\Psi^{\top}_{\hbar}}(\nabla(\zeta,\partial))\cdot e^{i\Bbbk\,\overbrace{\Xi^{\top}_{\hbar}}(\nabla(\zeta,\partial))} = \overbrace{\Psi^{\top}_{\aleph}}(\zeta,\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{\top}_{\aleph}}(\zeta,\partial)} \leqslant \beta$, $\overbrace{\Psi^{\top}_{\hbar}}(\nabla(\rho,\partial))\cdot e^{i\Bbbk\,\overbrace{\Xi^{\top}_{\hbar}}(\nabla(\rho,\partial))} = \overbrace{\Psi^{\top}_{\aleph}}(\rho,\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{\top}_{\aleph}}(\rho,\partial)} \leqslant \beta$. Thus, $\overbrace{\Psi^{\top}_{\hbar}}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\,\overbrace{\Xi^{\top}_{\hbar}}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))} \leqslant \overbrace{\Psi^{\top}_{\aleph}}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{\top}_{\aleph}}((\zeta\ominus_1\rho),\partial)} \leqslant \beta$. Now, $\overbrace{\Psi^{\gimel}_{\hbar}}(\nabla(\zeta,\partial))\cdot e^{i\Bbbk\,\overbrace{\Xi^{\gimel}_{\hbar}}(\nabla(\zeta,\partial))} = \overbrace{\Psi^{\gimel}_{\aleph}}(\zeta,\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{\gimel}_{\aleph}}(\zeta,\partial)} \leqslant \beta$, $\overbrace{\Psi^{\gimel}_{\hbar}}(\nabla(\rho,\partial))\cdot e^{i\Bbbk\,\overbrace{\Xi^{\gimel}_{\hbar}}(\nabla(\rho,\partial))} = \overbrace{\Psi^{\gimel}_{\aleph}}(\rho,\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{\gimel}_{\aleph}}(\rho,\partial)} \leqslant \beta$. Thus, $\overbrace{\Psi^{\gimel}_{\hbar}}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\,\overbrace{\Xi^{\gimel}_{\hbar}}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))} \leqslant \overbrace{\Psi^{\gimel}_{\aleph}}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{\gimel}_{\aleph}}((\zeta\ominus_1\rho),\partial)} \leqslant \beta$. Now, $\overbrace{\Psi^{F}_{\hbar}}(\nabla(\zeta,\partial))\cdot e^{i\Bbbk\,\overbrace{\Xi^{F}_{\hbar}}(\nabla(\zeta,\partial))} = \overbrace{\Psi^{F}_{\aleph}}(\zeta,\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{F}_{\aleph}}(\zeta,\partial)} \geqslant \delta$, $\overbrace{\Psi^{F}_{\hbar}}(\nabla(\rho,\partial))\cdot e^{i\Bbbk\,\overbrace{\Xi^{F}_{\hbar}}(\nabla(\rho,\partial))} = \overbrace{\Psi^{F}_{\aleph}}(\rho,\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{F}_{\aleph}}(\rho,\partial)} \geqslant \delta$. Thus, $\overbrace{\Psi^{F}_{\hbar}}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\,\overbrace{\Xi^{F}_{\hbar}}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))} \geqslant \overbrace{\Psi^{F}_{\aleph}}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{F}_{\aleph}}((\zeta\ominus_1\rho),\partial)} \geqslant \delta$,for all $\nabla(\zeta,\partial),\nabla(\rho,\partial)\in\circledS_2$. Similarly other operations, $\nabla(\overbrace{\aleph_{(\beta,\delta)}})$ is a SBS of ComCANSBS $\overbrace{\hbar}$ of $\circledS_2$.

The mapping $\nabla:\circledS_1\to\circledS_2$ be a homomorphism. Now, $\nabla((\zeta\ominus_1\rho),\partial) = \nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)$, $\nabla((\zeta\ominus_2\rho),\partial) = \nabla(\zeta,\partial)\circledast_2\nabla(\rho,\partial)$ and $\nabla((\zeta\ominus_3\rho),\partial) = \nabla(\zeta,\partial)\circledast_3\nabla(\rho,\partial)$ for all $\zeta,\rho\in\circledS_1$. Let $\hbar = \nabla(\aleph)$, $\aleph$ is a ComCANSBS of $\circledS_1$. By Theorem 3.14, $\hbar$ is a ComCANSBS of $\circledS_2$. Let $\aleph_{(\beta,\delta)}$ be any SBS of $\aleph$. Suppose that $\zeta,\rho\in\aleph_{(\beta,\delta)}$. Then $\zeta\ominus_1\rho, \zeta\ominus_2\rho$ and $\zeta\ominus_3\rho\in\aleph_{(\beta,\delta)}$. Now, $\Psi^{\top}_{\hbar}(\nabla(\zeta,\partial))\cdot e^{i\Bbbk\Xi^{\top}_{\hbar}(\nabla(\zeta,\partial))} = \Psi^{\top}_{\aleph}(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}(\zeta,\partial)} \leqslant \beta$, $\Psi^{\top}_{\hbar}(\nabla(\rho,\partial))\cdot e^{i\Bbbk\Xi^{\top}_{\hbar}(\nabla(\rho,\partial))} = \Psi^{\top}_{\aleph}(\rho,\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}(\rho,\partial)} \leqslant \beta$. Thus, $\Psi^{\top}_{\hbar}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\Xi^{\top}_{\hbar}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))} \leqslant \Psi^{\top}_{\aleph}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\Xi^{\top}_{\aleph}((\zeta\ominus_1\rho),\partial)} \leqslant \beta$. Now, $\Psi^{\gimel}_{\hbar}(\nabla(\zeta,\partial))\cdot e^{i\Bbbk\Xi^{\gimel}_{\hbar}(\nabla(\zeta,\partial))} = \Psi^{\gimel}_{\aleph}(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{\gimel}_{\aleph}(\zeta,\partial)} \leqslant \beta$, $\Psi^{\gimel}_{\hbar}(\nabla(\rho,\partial))\cdot e^{i\Bbbk\Xi^{\gimel}_{\hbar}(\nabla(\rho,\partial))} = \Psi^{\gimel}_{\aleph}(\rho,\partial)\cdot e^{i\Bbbk\Xi^{\gimel}_{\aleph}(\rho,\partial)} \leqslant \beta$. Thus, $\Psi^{\gimel}_{\hbar}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\Xi^{\gimel}_{\hbar}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))} \leqslant \Psi^{\gimel}_{\aleph}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\Xi^{\gimel}_{\aleph}((\zeta\ominus_1\rho),\partial)} \leqslant \beta$. Now, $\Psi^{F}_{\hbar}(\nabla(\zeta,\partial))\cdot e^{i\Bbbk\Xi^{F}_{\hbar}(\nabla(\zeta,\partial))} = \Psi^{F}_{\aleph}(\zeta,\partial)\cdot e^{i\Bbbk\Xi^{F}_{\aleph}(\zeta,\partial)} \geqslant \delta$, $\Psi^{F}_{\hbar}(\nabla(\rho,\partial))\cdot e^{i\Bbbk\Xi^{F}_{\hbar}(\nabla(\rho,\partial))} = \Psi^{F}_{\aleph}(\rho,\partial)\cdot e^{i\Bbbk\Xi^{F}_{\aleph}(\rho,\partial)} \geqslant \delta$. Thus, $\Psi^{F}_{\hbar}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\Xi^{F}_{\hbar}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))} \geqslant \Psi^{F}_{\aleph}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\Xi^{F}_{\aleph}((\zeta\ominus_1\rho),\partial)} \geqslant \delta$,for all $\nabla(\zeta,\partial),\nabla(\rho,\partial)\in\circledS_2$. Similarly other operations, $\nabla(\aleph_{(\beta,\delta)})$ is a SBS of ComCANSBS $\hbar$ of $\circledS_2$.

Theorem 3.17. *If $\nabla:\circledS_1\to\circledS_2$ is any homomorphism,then $\overbrace{\aleph_{(\beta,\delta)}}$ is a SBS of ComCANSBS $\aleph$ of $\circledS_1$.*

Proof. The mapping $\nabla:\circledS_1\to\circledS_2$ be any homomorphism. We have $\nabla((\zeta\ominus_1\rho),\partial) = \nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)$, $\nabla((\zeta\ominus_2\rho),\partial) = \nabla(\zeta,\partial)\circledast_2\nabla(\rho,\partial)$ and $\nabla((\zeta\ominus_3\rho),\partial) = \nabla(\zeta,\partial)\circledast_3\nabla(\rho,\partial)$ for all $\zeta,\rho\in\circledS_1$. Let $\overbrace{\hbar} = \nabla(\aleph)$, $\overbrace{\hbar}$ is a ComCANSBS of $\circledS_2$. By Theorem 3.15, $\aleph$ is a ComCANSBS of $\circledS_1$. Let $\nabla(\overbrace{\aleph_{(\beta,\delta)}})$ be a SBS of $\overbrace{\hbar}$. Suppose that $\nabla(\zeta,\partial),\nabla(\rho,\partial)\in\nabla(\overbrace{\aleph_{(\beta,\delta)}})$. Now, $\nabla((\zeta\ominus_1\rho),\partial)$, $\nabla((\zeta\ominus_2\rho),\partial)$ and $\nabla((\zeta\ominus_3\rho),\partial)\in\nabla(\overbrace{\aleph_{(\beta,\delta)}})$. Now, $\overbrace{\Psi^{\top}_{\aleph}}(\zeta,\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{\top}_{\aleph}}(\zeta,\partial)} = \overbrace{\Psi^{\top}_{\hbar}}(\nabla(\zeta,\partial))\cdot e^{i\Bbbk\,\overbrace{\Xi^{\top}_{\hbar}}(\nabla(\zeta,\partial))} \leqslant \beta$, $\overbrace{\Psi^{\top}_{\aleph}}(\rho,\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{\top}_{\aleph}}(\rho,\partial)} = \overbrace{\Psi^{\top}_{\hbar}}(\nabla(\rho,\partial))\cdot e^{i\Bbbk\,\overbrace{\Xi^{\top}_{\hbar}}(\nabla(\rho,\partial))} \leqslant \beta$. Thus, $\overbrace{\Psi^{\top}_{\aleph}}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{\top}_{\aleph}}((\zeta\ominus_1\rho),\partial)} \leqslant \max\{\overbrace{\Psi^{\top}_{\aleph}}(\zeta,\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{\top}_{\aleph}}(\zeta,\partial)}, \overbrace{\Psi^{\top}_{\aleph}}(\rho,\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{\top}_{\aleph}}(\rho,\partial)}\} \leqslant \beta$. Now, $\overbrace{\Psi^{\gimel}_{\aleph}}(\zeta,\partial)\cdot e^{i\Bbbk\,\overbrace{\Xi^{\gimel}_{\aleph}}(\zeta,\partial)} = \overbrace{\Psi^{\gimel}_{\hbar}}(\nabla(\zeta,\partial))\cdot$

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$e^{i\Bbbk\ \widehat{\Xi_{\hbar}^{\daleth}}(\nabla(\zeta,\partial))} \leqslant \beta$, $\widehat{\Psi_{\aleph}^{\daleth}}(\rho,\partial)\cdot e^{i\Bbbk\ \widehat{\Xi_{\aleph}^{\daleth}}(\rho,\partial)} = \widehat{\Psi_{\hbar}^{\daleth}}(\nabla(\rho,\partial))\cdot e^{i\Bbbk\ \widehat{\Xi_{\hbar}^{\daleth}}(\nabla(\rho,\partial))} \leqslant \beta.$

Thus, $\widehat{\Psi_{\aleph}^{\daleth}}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\ \widehat{\Xi_{\aleph}^{\daleth}}((\zeta\ominus_1\rho),\partial)} \leqslant \frac{\widehat{\Psi_{\aleph}^{\daleth}}(\zeta,\partial)\cdot e^{i\Bbbk\ \widehat{\Xi_{\aleph}^{\daleth}}(\zeta,\partial)} + \widehat{\Psi_{\aleph}^{\daleth}}(\rho,\partial)\cdot e^{i\Bbbk\ \widehat{\Xi_{\aleph}^{\daleth}}(\rho,\partial)}}{2} \leqslant \beta$. Now, $\widehat{\Psi_{\aleph}^{F}}(\zeta,\partial)\cdot e^{i\Bbbk\ \widehat{\Xi_{\aleph}^{F}}(\zeta,\partial)} = \widehat{\Psi_{\hbar}^{F}}(\nabla(\zeta,\partial))\cdot e^{i\Bbbk\ \widehat{\Xi_{\hbar}^{F}}(\nabla(\zeta,\partial))} \geqslant \delta$, $\widehat{\Psi_{\aleph}^{F}}(\rho,\partial)\cdot e^{i\Bbbk\ \widehat{\Xi_{\aleph}^{F}}(\rho,\partial)} = \widehat{\Psi_{\hbar}^{F}}(\nabla(\rho,\partial))\cdot e^{i\Bbbk\ \widehat{\Xi_{\hbar}^{F}}(\nabla(\rho,\partial))} \geqslant \delta$.

Thus, $\widehat{\Psi_{\aleph}^{F}}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\ \widehat{\Xi_{\aleph}^{F}}((\zeta\ominus_1\rho),\partial)} = \widehat{\Psi_{\hbar}^{F}}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\ \widehat{\Xi_{\hbar}^{F}}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))}$
$\geqslant \min\{\widehat{\Psi_{\aleph}^{F}}(\zeta,\partial)\cdot e^{i\Bbbk\ \widehat{\Xi_{\aleph}^{F}}(\zeta,\partial)}, \widehat{\Psi_{\aleph}^{F}}(\rho,\partial)\cdot e^{i\Bbbk\ \widehat{\Xi_{\aleph}^{F}}(\rho,\partial)}\} \geqslant \delta$, for all $\zeta,\rho\in Ⓢ_1$.
Similarly other operations, $\widehat{\aleph_{(\beta,\delta)}}$ is a SBS of ComCANSBS $\aleph$ of $Ⓢ_1$.
The mapping $\nabla: Ⓢ_1 \to Ⓢ_2$ be any homomorphism. We have $\nabla((\zeta\ominus_1\rho),\partial) = \nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)$, $\nabla((\zeta\ominus_2\rho),\partial) = \nabla(\zeta,\partial)\circledast_2\nabla(\rho,\partial)$ and $\nabla((\zeta\ominus_3\rho),\partial) = \nabla(\zeta,\partial)\circledast_3\nabla(\rho,\partial)$ for all $\zeta,\rho\in Ⓢ_1$. Let $\hbar = \nabla(\aleph)$, $\hbar$ is a ComCANSBS of $Ⓢ_2$. By Theorem 3.15, $\aleph$ is a ComCANSBS of $Ⓢ_1$. Let $\nabla(\aleph_{(\beta,\delta)})$ be a SBS of $\hbar$. Suppose that $\nabla(\zeta,\partial),\nabla(\rho,\partial)\in\nabla(\aleph_{(\beta,\delta)})$. Now, $\nabla((\zeta\ominus_1\rho),\partial),\nabla((\zeta\ominus_2\rho),\partial)$ and $\nabla((\zeta\ominus_3\rho),\partial)\in\nabla(\aleph_{(\beta,\delta)})$. Now, $\Psi_{\aleph}^{\top}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\zeta,\partial)} = \Psi_{\hbar}^{\top}(\nabla(\zeta,\partial))\cdot e^{i\Bbbk\Xi_{\hbar}^{\top}(\nabla(\zeta,\partial))} \leqslant \beta$, $\Psi_{\aleph}^{\top}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\rho,\partial)} = \Psi_{\hbar}^{\top}(\nabla(\rho,\partial))\cdot e^{i\Bbbk\Xi_{\hbar}^{\top}(\nabla(\rho,\partial))} \leqslant \beta$.
Thus, $\Psi_{\aleph}^{\top}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}((\zeta\ominus_1\rho),\partial)} \leqslant \max\{\Psi_{\aleph}^{\top}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\zeta,\partial)}, \Psi_{\aleph}^{\top}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\top}(\rho,\partial)}\} \leqslant \beta$. Now, $\Psi_{\aleph}^{\daleth}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\zeta,\partial)} = \Psi_{\hbar}^{\daleth}(\nabla(\zeta,\partial))\cdot e^{i\Bbbk\Xi_{\hbar}^{\daleth}(\nabla(\zeta,\partial))} \leqslant \beta$, $\Psi_{\aleph}^{\daleth}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\rho,\partial)} = \Psi_{\hbar}^{\daleth}(\nabla(\rho,\partial))\cdot e^{i\Bbbk\Xi_{\hbar}^{\daleth}(\nabla(\rho,\partial))} \leqslant \beta$.
Thus, $\Psi_{\aleph}^{\daleth}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}((\zeta\ominus_1\rho),\partial)} \leqslant \frac{\Psi_{\aleph}^{\daleth}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\zeta,\partial)} + \Psi_{\aleph}^{\daleth}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{\daleth}(\rho,\partial)}}{2} \leqslant \beta$. Now, $\Psi_{\aleph}^{F}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\zeta,\partial)} = \Psi_{\hbar}^{F}(\nabla(\zeta,\partial))\cdot e^{i\Bbbk\Xi_{\hbar}^{F}(\nabla(\zeta,\partial))} \geqslant \delta$, $\Psi_{\aleph}^{F}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\rho,\partial)} = \Psi_{\hbar}^{F}(\nabla(\rho,\partial))\cdot e^{i\Bbbk\Xi_{\hbar}^{F}(\nabla(\rho,\partial))} \geqslant \delta$.
Thus, $\Psi_{\aleph}^{F}((\zeta\ominus_1\rho),\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}((\zeta\ominus_1\rho),\partial)} = \Psi_{\hbar}^{F}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))\cdot e^{i\Bbbk\Xi_{\hbar}^{F}((\nabla(\zeta,\partial)\circledast_1\nabla(\rho,\partial)))} \geqslant$
$\min\{\Psi_{\aleph}^{F}(\zeta,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\zeta,\partial)}, \Psi_{\aleph}^{F}(\rho,\partial)\cdot e^{i\Bbbk\Xi_{\aleph}^{F}(\rho,\partial)}\} \geqslant \delta$, for all $\zeta,\rho\in Ⓢ_1$.
Similarly other operations, $\aleph_{(\beta,\delta)}$ is a SBS of ComCANSBS $\aleph$ of $Ⓢ_1$.

4 Conclusion and future direction

In this work a new type of ComCANSBS is presented. The complex cubic neutrosophicsubbisemiring, which expresses three grades in terms of a complex number, takes a fresh approach to the idea of three grades. Since a complex form of three grades reduces three grades to a two-dimensional parameter, it represents a paradigm shift. A complex interval value-containing neutrosophic SBS was defined. ComCANSBS level sets and ComCNANSBS level sets were defined. Our goal is to apply the cubic set to bisemiring. Additionally, a study of the characteristics of different conversions is done. In addition to data analysis, we are trying to handle images and signals, thus it makes sense to use F-transforms to expand the use of new fuzzy structures. Thus, we ought to consider applying soft set ComCANSBS, soft set ComCNANSBS in the future.

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