Direct and converse approximation theorems in neutrosophic $L_{\delta,p}(\mathcal{U})$ space

Alaa Adnan Auad[1, *], **Mohammed A. Hilal**[2]

[1]Department of Mathematics, College of Education for pure science, University of Anbar, Iraq

[2]Middle Technical University, Technical Institute of Baquba, Diyala, Iraq

Emails: alaa.adnan.auad@uoanbar.edu.iq; mohammed_azeez_hilal@mtu.edu.iq

Abstract

A neutrosophic is a strong framework to characterize novel mathematical structures. This framework is more suitable and flexible set side by side to fuzzy sets and intuitionistic fuzzy sets. In this work, we focus on some famous mathematical spaces like $L_{\delta,p}(\mathcal{U})$ when we work on displaying a feature the immediate and contrary theorems of unrestrained functions in the space $L_{\delta,p}(\mathcal{U})$ are considered. Also, some characteristics of modification symmetric and modulus of neutrosophic smoothness have been discussed. Moreover, the identical among approximate tools such as the neutrosophic K-functional and neutrosophic modulus of softness.

Keywords: Neutrosophic K-functional; modulus of softness; unrestrained functions; neutrosophic $L_{\delta,p}(\mathcal{U})$ space and modification symmetric

1. Introduction

In 1965, the famous scientist Zadeh [1] discussed the fuzzy theory when he had matured to realize the inability of fragile groups emerging from a simple mathematical formula to deal with algebraic mathematical structures in a detailed and clear manner. At that time, Zadeh presented in his work everything surrounding this theory for the purpose of forming a coherent theory, and he wanted to develop the algebraic mathematical structures in that area. In the end, Zadeh was able to complete this task, but he indicated that this study could be developed in the future by filling in its obvious weaknesses. After that, in 1998, the American scientist Smarandj [2] discovered some of these weaknesses that Zadeh referred to and he developed and discovered this theory by presenting an interesting new theory known as the Neutrosophic theory or the theory of non-neutrality because it focuses on the principle of neutrality in daily life situations. As an extension of the blurring theory referred to above. Through this study, Samarandaj indicated that this theory takes into account all the failures and weaknesses that occurred with Zadeh, and that he worked to highlight the important strengths of this theory. Recently, this theory has received the attention of a large number of workers in the field of scientific research around the world, which has prompted them to work on linking it to other branches of mathematics, such as algebra [3,4], topology [5,6], complex analysis [7,8], and statistics [9, 10].], and other mathematics branches [11, 12]. In these papers works, the idea of Ω-BCK, BCK algebra with neutrosophic environment introduced by Zelle et al. [13]. Al-Quran et al [14] discovered the design of the complex of neutrosophic. Al-Rumaini et al [15,16] advanced the notion of the ring neutrosophic theory. Al-Sharqi et al. [17,18] introduced the notion of soft neutrosophic to relate different structures. In [19,20] Abid et al introduced many algebraic concepts with neutrosophic ideas. Alaa et al. [21,22] initiated the notion of many approximation structures that can combine with neutrosophic in the beast way. Alaa et al. [23,24] again combine the of approximation structures with many algebraic concepts. As previously stated, we can infer from the foregoing that this concept is applicable to a wide range of algebraic ideas as well as the resolution of a wide range of everyday situations. [25,26].

DOI: https://doi.org/10.54216/IJNS.250206
Received: February 02, 2024 Revised: April 28, 2024 Accepted: July 25, 2024

The concept of the degree best approximation of functions has been studied by many mathematical researchers in approximation theory [27,28]. Most of these papers relate to the functions approximated by algebraic polynomials in several spaces more details in the following sources [29]. The concept of the direct theorem was first studied by the Russian scientist Weierstrass in 1885. presented a new concept related to the approximation of functions, indicating that any continuous function can be approximated by a polynomial under the conditions of a small positive constant. In a second perspective the opposite formula of Bern-stein (1912) with the practical side is interesting. Such closely related mathematical relationships have been of attractive to the many researchers and they are certain virtuous copies in the new approximate. theory. The several authors in field of approximation theory studied direct and converse theorems of bounded functions in measurable space see [30].

In this work, we introduced the immediately and contrary forms of unrestrained mappings via algebraic polynomials in neutrosophic weighted space, also given the proof that K-functional and modulus of smoothness are identical.

Now we present some closely related concepts with the best approximation of unconstrained functions in the weighted space

Let $\mathcal{U} = [-1,1]$, $L_p(\mathcal{U})$, p greater than or equal 1 and less than infinity, signified the space of all constrained mappings equipped limited norm as :

$$\|\phi\|_p = (\int_{\mathcal{U}} |\phi(t)|^p dt)^{\frac{1}{p}} < \infty, \text{ where } t \in \mathcal{U} . \qquad (1)$$

We denoted the space of all weighted functions by Γ, with $\delta: \mathcal{U} \longrightarrow \mathbb{R}^+$ such that $\varphi \in \Gamma$.

The space $L_{\delta,p}(\mathcal{U})$ symbolized the space of all unrestrained functions, the of this space equipped by

$$\|\phi\|_{\varepsilon,p} = (\int_{\mathcal{U}} |\phi(t)\delta(t)|^p dt)^{\frac{1}{p}} < \infty, \text{ where } t \in \mathcal{U}. \qquad (2)$$

Let Σ_i be the collection of algebraic polynomial. with order not widely i. The mark of better approximat. of unrestrained mapping in measured collection $L_{\delta,p}(\mathcal{U})$ by algebraic polynomi. $p_i \in \Sigma_i$ certain by

$$E_i(\phi)_{\varepsilon,p} = \underbrace{inf}_{p_i \in \Sigma_i} \|\phi - p_i\|_{\varepsilon,p}. \qquad (3)$$

We express the variance of $\phi \in L_{\varepsilon,p}(\mathcal{U})$ of grade $k \in \mathbb{N}$ with estimate $\xi \in \mathbb{R}$

$$\Delta_\xi^n \phi(t) = \sum_{i=0}^n (-1)^i \binom{n}{i} \phi(t + i\xi), \quad (4)$$

the modulus of softness of function $\phi \in L_{\varepsilon,p}(\mathcal{U})$with index $k \in \mathbb{N}$ is define by

$$\Omega_k(\phi,\eta)_{\varepsilon,p} = \underbrace{sup}_{|h|\leq\xi} \left\|\Delta_\eta^k \phi\right\|_{p,\eta}. \quad (5)$$

If for given function $\phi \in L_{\varepsilon,p}(\mathcal{U})$ and $n \in \mathbb{N}$, then n-derivative of the mapping f signified by $\phi^{(n)}$.

In a several of approximat. formulas the mistake is guessed by correct approximate instrument K-function. , we express it is by

$$\mathcal{F}(\phi,\eta)_{\varepsilon,p} = \underbrace{inf}_{\upsilon \in S} \left\{\|\phi - \upsilon\|_{\varepsilon,p} + \eta^n \left\|\upsilon^{(k)}\right\|_{\varepsilon,p}\right\},$$

Where $n \in \mathbb{N}$,S is subspace of$L_{\varepsilon,p}(\mathcal{U})$

K-functional was examining by P. eeter and L. ions in 1962 and defined their usual form by P. eeter in 1964.

2. Support lemmas

In this section of our current work, we will present some postulates that support our highest results:

Lemma 1.1 [20]

Let $\mathcal{P}_n \in \mathcal{J}_n$ and $1 \leq p < \infty$. Then

$$\|\mathcal{P}_n\|_p \leq c_p \, (\frac{1}{n} \sum_{k=0}^{n-1} \left|\mathcal{P}^{(k)}(t)\right|^p)^{\frac{1}{p}}.$$

Lemma 1.2

DOI: https://doi.org/10.54216/IJNS.250206
Received: February 02, 2024 Revised: April 28, 2024 Accepted: July 25, 2024

If $\in L_{\varepsilon,p}(\mathcal{U})$, there is a derivative $\phi^{(n)} \in L_{\varepsilon,p}(\mathcal{U})$ where $n \in \mathbb{N}$, then

$E_n(\phi,\eta)_{\varepsilon,p} \leq c_n E_n(\phi^{(n)},\eta)_{\varepsilon,p}$, such that $c_n$is Associated fixed on n.

Proof: For $\in L_{\varepsilon,p}(\mathcal{U})$, there is $\rho_n \in \Sigma_n$ and from lemma (1.1)

$$E_n(\phi,\eta)_{\varepsilon,p} \leq \|\phi - \rho_n\|_{\varepsilon,p}$$

$$\leq c\left(\int_{\mathcal{U}} |(\phi(t) - \rho_n(t))\delta(t)|^p\right)^{\frac{1}{p}}$$

$$\leq c\left(\int_X \left|(\phi^{(k)}(t) - {\rho^{(n)}}_n(t))\delta(t)\right|^p\right)^{\frac{1}{p}}$$

$$\leq \max\{\, c_n \,\}\ E_n(\phi^{(k)},\eta)_{\varepsilon,p}.$$

So, $E_n(\phi,\eta)_{\varepsilon,p} \leq c_n E_n(\phi^{(n)},\eta)_{\varepsilon,p}$.

Lemma 1.3

If $\phi \in L_{\varepsilon,p}(\mathcal{U})$, $1 \leq p < \infty$, $n, i > 0$ and η real number, then

i. $\left\|\Delta_\eta^n\, \phi\right\|_{\varepsilon,p} \leq c_n\|\phi\|_{\varepsilon,p}$, where c_n positive unbroken conditional on n.
ii. $\left\|\Delta_\eta^{n+1}\, \phi\right\|_{\varepsilon,p} \leq c_n\left\|\Delta_\eta^n\, \phi\right\|_{\varepsilon,p}$.
iii. $\left\|\Delta_\eta^n\, \phi\right\|_{\varepsilon,p} = 0$, as $\delta \longrightarrow 0$.

Proof : (i) we have $\Delta_\eta^n\, \phi(t) = \sum_{i=0}^n (-1)^{n-i}\binom{n}{i}\phi(t + i\eta)$, $\Longrightarrow$

$$\left\|\Delta_\eta^n\, \phi\right\|_{\varepsilon,p} = \left\|\sum_{i=0}^n (-1)^{n-i}\binom{n}{i}\phi(t + i\eta)\right\|_{\varepsilon,p}$$

$$\leq \left|\sum_{i=0}^n (-1)^{n-i}\binom{n}{i}\phi(t + i\eta)\right| \|\phi\|_{\varepsilon,p}$$

$$\leq c_n\|\phi\|_{\varepsilon,p} \text{ ,where } c_n = max\left|\sum_{i=0}^n (-1)^{n-i}\binom{n}{i}\phi(t + i\eta)\right|.$$

Thus, $\left\|\Delta_\eta^n\, \phi\right\|_{\varepsilon,p} \leq c_n\|\phi\|_{\varepsilon,p}$

(ii) $\left\|\Delta_\eta^{n+1}\, \phi\right\|_{\varepsilon,p} = \left\|\Delta_\eta^n(\,\Delta_\eta^l\, \phi(.))\right\|_{p,\eta}$, from (i) , we have

$$\left\|\Delta_\eta^{n+1}\, \phi\right\|_{\varepsilon,p} \leq c_n\left\|\Delta_\eta^n\, \phi\right\|_{\varepsilon,p} .$$

(iii) its immediately from definition of modification symmetric.

Lemma 1.4

If $\phi \in L_{\varepsilon,p}(\mathcal{U})$, , $1 \leq p < \infty$, $0 < i < n$, then $\Omega_n(\phi,\eta)_{\varepsilon,p} \leq c\ \Omega_i(\phi,\eta)_{\varepsilon,p}$, where $\eta > 0$.

Proof : We have

$$\left\|\Delta_\eta^n\, \phi\right\|_{\varepsilon,p} = \left\|\Delta_\eta^{n-1}(\,\Delta_\eta^l\, \phi(.))\right\|_{\varepsilon,p} \leq c\left\|\Delta_\eta^l\, \phi(.)\right\|_{\varepsilon,p} .$$

$$\Longrightarrow sup\left\|\Delta_\eta^n\, \phi\right\|_{\varepsilon,p} \leq csup\left\|\Delta_\eta^n\, \phi\right\|_{\varepsilon,p}$$

$$\Longrightarrow \Omega_n(\phi,\eta)_{\varepsilon,p} \leq c\ \Omega_i(\phi,\eta)_{\varepsilon,p}.$$

Lemma 1.5 [19]

Let $0 < n < k$& $k \in \mathbb{N}$. Then $\forall\ \rho_n \in \Sigma_n$

$$\frac{C(k)}{n}\left\|\rho_n^{(k)}\right\|_{\varepsilon,p} \leq \left\|\Delta_{\frac{1}{n}}^k \rho_n\right\|_{\varepsilon,p} \leq C(k)\left\|\rho_n^{(k)}\right\|_{\varepsilon,p}.$$

3. The main effects

DOI: https://doi.org/10.54216/IJNS.250206
Received: February 02, 2024 Revised: April 28, 2024 Accepted: July 25, 2024

Theorem 2.1

Let $\phi \in L_{\varepsilon,p}(\mathcal{U})$. Then $n > 0$ & $\forall\ i \in \mathbb{N}$, the next form is hold:

$E_i(\phi,\eta)_{\varepsilon,p} \le c_n\, \Omega_l(\phi,\eta)_{\varepsilon,p}$, $\eta = \frac{1}{i}$, where c_n is absolute constant conditional on n.

Proof:

Let $\rho_i(t)$ be algebraic polynomial of order not more than i and sufficient the following situations:

$\int_{-1}^{1} \rho_i(t)\ \ dt = 1$, for $i = 1,2,3, \dots$

$\int_{-1}^{1} |t|^n |\rho_i(t)|\ dxt\ \le c_n \frac{1}{(i+1)^n}$, $i = 0,1,2,3, \dots$.

We will study the situation of $j \in \mathbb{N}$:

Set, $\mathcal{M}_{i-1}(y) = (-1)^{j+1} \int_{-1}^{1} \rho_i(t) \sum_{n=1}^{j} (-1)^1 \binom{j}{n} f(t-ny)dt$ of order less than i.

So, , $\mathcal{M}_{i-1}(y) - \rho_{i-1}(t) = (-1)^j \int_{-1}^{1} \rho_i(t) \sum_{n=1}^{j} (-1)^1 \binom{j}{n} f(t-ny)dt$

$$= (-1) \int_{-1}^{1} \rho_i(t) \Delta_x^r\ \phi(t)dt.$$

$$\begin{aligned}
E_i(\phi,\eta)_{\varepsilon,p} &\le \|\phi - \rho_{i-1}(.)\|_{\varepsilon,p} \le c_n \|\phi - \rho_{i-1}(.)\|_{\varepsilon,p} \\
&= c_n \left\| (-1) \int_{-1}^{1} \rho_{i-1}(t) \Delta_t^j\ \phi(t)dt \right\|_{\varepsilon,p} \\
&\le c_n (-1)^j \int_{-1}^{1} |\rho_{i-1}(t)| \left\| \Delta_t^j \phi(t) \right\|_{\varepsilon,p}\ dt \\
&\le c_n (-1)^j \int_{-1}^{1} |\rho_{i-1}(t)| sup \left\| \Delta_t^j \phi(t) \right\|_{\varepsilon,p}\ dt \\
&\le c_n \int |\rho_{i-1}(t)| \Omega_i(\phi,\eta)_{\varepsilon,p}\ dt \\
&\le c_n \Omega_i(\phi,\eta)_{\varepsilon,p,}\ .
\end{aligned}$$

Theorem 2.2

Let $\phi \in L_{\varepsilon,p}(\mathcal{U})$ & , $i \in \mathbb{N}$. Then

$\Omega_i(\phi,\eta)_{\varepsilon,p,} \le i^{-n} \sum_{i=1}^{n} (i^n - (i-1)^n)\ \ E_i(\phi,\eta)_{\varepsilon,p,}.$

Proof :

For any$j \in \mathbb{Z}$ and $k \in \mathbb{R}$, we must

$\Delta_\eta^n\, \phi(t) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \phi(t + i\eta)$, $t, t + jk \in \mathcal{U}$,

since $\phi \in L_{\varepsilon,p}(\mathcal{U})$, & $\forall\ \varepsilon > 0$, $\exists\ \mathscr{g} \in \mathbb{Z}^+ \ni\ \mathscr{g} > n$ and $\rho_i\ \in \Sigma_i$ is better approximat. of ϕsuch that

$E_i(\phi,\eta)_{\varepsilon,p,}\ \le \|\phi - \rho_i\|_{\varepsilon,p} < \frac{1}{2j} \varepsilon.$

From lemma 1.3, we have

$$\begin{aligned}
\left\| \Delta_\eta^i\, \phi(t) \right\|_{\varepsilon,p} &\le \left\| \Delta_\eta^i (\phi - \rho_i) \right\|_{p,\eta} + \left\| \Delta_\eta^i (\rho_i) \right\|_{\varepsilon,p} \\
&\le \|\phi - \rho_i\|_{p,\eta} + \|\Delta_l^r (t_n)\|_{p,\eta} \\
&\le 2^r \|f - t_n\|_{\varepsilon,p} + \left\| \Delta_\eta^i (\rho_i)) \right\|_{\varepsilon,p} \\
&\le \epsilon + \|\Delta_l^r (t_n)\|_{p,\eta} \qquad (*)
\end{aligned}$$

So, $\left\| \Delta_\eta^i (\rho_i)) \right\|_{\varepsilon,p} \le \left\| \Delta_\eta^i (\rho_i - \mathcal{A}_{i-1}) + \Delta_\eta^i (\mathcal{A}_{i-1}) \right\|_{\varepsilon,p}$

$$\le \left\| 2(\rho_i - \mathcal{A}_{i-1}) + \Delta_\eta^i (\mathcal{A}_{i-1}) \right\|_{\varepsilon,p}$$

DOI: https://doi.org/10.54216/IJNS.250206
Received: February 02, 2024 Revised: April 28, 2024 Accepted: July 25, 2024

$$\leq \left\| 2(\rho_i - \mathcal{A}_{i-1}) + 2^i \sum_{\ell=1}^{i-1} \ell^{j} (\mathcal{A}_\ell) \right\|_{\varepsilon,p}$$

we set, $(\rho_i - \mathcal{A}_{i-1}) = \mathcal{K}_{i-1}, \Longrightarrow$ $\|\Delta_l(\rho_i)\|_{\varepsilon,p} \leq \left\| 2^\ell \sum_{\ell=i}^{\mathcal{N}} \mathcal{K}_\ell + \frac{1}{2^\ell} \sum_{\ell=1}^{i-1} \ell^i \mathcal{K}_\ell \right\|_{\varepsilon,p}$ (**)

So ,

$\left\| 2^\ell \sum_{\ell=i}^{\mathcal{N}} \mathcal{K}_\ell + \frac{1}{2^\ell} \sum_{\ell=1}^{i-1} \ell^i \mathcal{K}_\ell \right\|_{\varepsilon,p} \leq$

$\frac{\ell}{2^\ell} \left\| 2^\ell \sum_{\ell=i}^{\mathcal{N}} \mathcal{K}_\ell + \frac{1}{2^\ell} \sum_{\ell=1}^{i-1} (\ell^i - (\ell-1)^i) \sum_{j=\ell}^{\mathcal{N}} \mathcal{K}_\ell - (i-1)^r \sum_{\ell=i}^{\mathcal{N}} \mathcal{K}_\ell \right\|_{\varepsilon,p}$

$$\leq \frac{\ell}{2^\ell} \left\| \frac{1}{2^\ell} \sum_{\ell=1}^{i-1} (\ell^i - (\ell-1)^i) \sum_{j=\ell}^{\mathcal{N}} \mathcal{K}_\ell \right\|_{\varepsilon,p}$$

$\leq \frac{\ell}{2^\ell} \left\| \sum_{\ell=1}^{i-1} (\ell^i - (\ell-1)^i) E_\ell(\rho_i) \right\|_{\varepsilon,p}$ (***)

From (*), (**) and (***) , for $\varepsilon > 0$, we obtain

$$\|\Delta_l(\rho_i)\|_{\varepsilon,p} \leq \frac{C_\ell}{i^\ell} \left\| \sum_{u=1}^{n-1} \sum_{\ell=1}^{i-1} (\ell^i - (\ell-1)^i) E_\ell(\rho_i) \right\|_{\varepsilon,p} + \varepsilon$$

Also ,sup $\|\Delta_l(\rho_i)\|_{\varepsilon,p} \leq \frac{C_\ell}{i^\ell} \left\| \sum_{u=1}^{n-1} \sum_{\ell=1}^{i-1} (\ell^i - (\ell-1)^i) E_\ell(\rho_i) \right\|_{\varepsilon,p} + \varepsilon$

$\Longrightarrow \Omega_i(\phi, \eta)_{\varepsilon,p} \leq i^{-n} \sum_{i=1}^{n} (i^n - (i-1)^n) \quad E_i(\phi, \eta)_{\varepsilon,p}.$

Theorem 2.3

Let $\phi \in L_{\varepsilon,p}(\mathcal{U})$, $\in \mathbb{N}$ and α_i& β_i are positive constants. Then

$\alpha_i \Omega_i(\phi, \eta)_{\varepsilon,p} \leq \mathcal{F}(f, \frac{1}{n})_{p,\eta} \leq \beta_i \, \Omega_i(\phi, \eta)_{\varepsilon,p}.$

Proof:

Take ω random mapping in the collection $L_{\varepsilon,p}(\mathcal{U})$, $\ni \omega^{(i)} \in L_{\varepsilon,p}(\mathcal{U})$. By lemma 1.3 & lemma 1.4, we obtain

$$\Omega_i(\phi, \eta)_{\varepsilon,p} \leq \Omega_i(\phi - \omega, \eta)_{\varepsilon,p} + \Omega_i(\omega, \eta)_{\varepsilon,p}$$

$$\leq \beta_i \|\phi - \omega\|_{\varepsilon,p} + \Omega_i(\omega, \eta)_{\varepsilon,p}$$

$$\leq \beta_i \|\phi - \omega\|_{p,\eta} + \beta_i \Omega_i(\omega^{(i)}, \eta)_{,\varepsilon p}$$

$$\leq \beta_i \|\phi - \omega\|_{p,\eta} + \beta_{i+1} \left\| \omega^{(i)} \right\|_{\varepsilon p}$$

We chosen the infimum over all $\omega \in L_{\varepsilon,p}(\mathcal{U})$, we get

$$\Omega_i(\phi, \eta)_{\varepsilon,p} \leq \beta_i \left\{ \|\phi - \omega_i\|_{p,\eta} + \beta_{i+1} \left\| {\omega_i}^{(n)} \right\|_{\varepsilon p} \right\}$$

$\Longrightarrow \alpha_i \Omega_i(\phi, \eta)_{\varepsilon,p} \leq \mathcal{F}(\phi, \eta)_{\varepsilon,p}$ (#)

Consider $\omega_i \in \Sigma_i$, we have

$$\left\| \omega_i^{(n)} \right\|_{\varepsilon,p} \leq \frac{\beta_i}{\eta^i} \left\| \Delta_\eta^i \omega_i \right\|_{\varepsilon,p} \leq \beta_i \left\| \Delta_\eta^i (\omega_i - \phi) \right\|_{\varepsilon,p} + \beta_i \left\| \Delta_\eta^i \omega_i \phi \right\|_{\varepsilon,p}$$

$$\leq \beta_i \left(\|\phi - \omega_i\|_{\varepsilon,p} + \left\| \Delta_\delta^k \phi \right\|_{p,\eta} \right) \leq \beta_i \left(E_i(\phi, \eta)_{\varepsilon,p} + \Omega_i(\phi, \eta)_{\varepsilon,p} \right)$$

$\leq \beta_i \Omega_i(\phi, \eta)_{\varepsilon,p}$.

From above inequality, we get

$$\mathcal{F}(\phi, \eta)_{\varepsilon,p} \leq \|\phi - \omega_i\|_{\varepsilon,p} + \eta^n \left\| {\omega_i}^{(n)} \right\|_{\varepsilon,p} \leq E_i(\phi, \eta)_{\varepsilon,p} + \Omega_i(\phi, \eta)_{\varepsilon,p}$$

$\leq \beta_i \Omega_i(\phi, \eta)_{\varepsilon,p}$. (##)

So, from (#) and (##),the proof is completed.

4. Inferences

DOI: https://doi.org/10.54216/IJNS.250206
Received: February 02, 2024 Revised: April 28, 2024 Accepted: July 25, 2024

The immediately and reversed forms of any mapping in the neutrosophic space$L_{\varepsilon,p}(\mathcal{U})$, $1 \leq p < \infty$ by algebraic polynomi. Have been established, which is the better approximate. assessment on $[0,1]$. Also, approximate tools such as K-functional and modulus of smoothness are equivalents.

Reference

[1] Zadeh, L. A. (1965). Fuzzy sets. Information Control, 8(3): 338-353.

[2] Smarandache, F. A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Lgics, American Research Press, Reoboth, NM, USA, 1998.

[3] Jamiatun Nadwa Ismail et al. The Integrated Novel Framework: Linguistic Variables in Pythagorean Neutrosophic Set with DEMATEL for Enhanced Decision Support. Int. J. Neutrosophic Sci., vol. 21, no. 2, pp. 129-141, 2023.

[4] Abed, M. M., Al-Jumaili, A. F., Al-sharqi, F. G. Some mathematical structures in a topological group. Journal of Algebra and Applied Mathematics. 2018, 16(2), 99-117.

[5] Al-Sharqi, F. G., Abed, M. M., & Mhassin, A. A. (2018). On polish groups and their applications. Journal of Engineering and Applied Sciences, 13(18), 7533-7536.

[6] F. Al-Sharqi, A.G. Ahmad, A. Al-Quran, Interval-valued neutrosophic soft expert set from real space to complex space, Computer Modeling in Engineering and Sciences, vol. 132(1), pp. 267–293, 2022.

[7] Hammad, F. N., & Abed, M. M. (2021, March). A new results of injective module with divisible property. In Journal of Physics: Conference Series (Vol. 1818, No. 1, p. 012168). IOP Publishing.

[8] A. Auad and A. Klaeif, Best co-proximal and Co-chebyshev of unbounded functions, Journal of Physics: Conference Series, 2322 (2022) 012033.

[9] Abed, M. M., & Al-Sharqi, F. G. (2018, May). Classical Artinian module and related topics. In Journal of Physics: Conference Series (Vol. 1003, No. 1, p. 012065). IOP Publishing.

[10] Jumaili, A. M. A., Auad, A. A., & Abed, M. M. (2019). A New Type of Strongly Faint Continuous Mappings and Their Applications in Topological Spaces. Journal of Engineering and Applied Sciences, 14(3), 905-912.

[11] Sankari, H., and Abobala, M., "Solving Three Conjectures About Neutrosophic Quadruple Vector Spaces", Neutrosophic Sets and Systems, Vol., pp., 2020.

[12] Bera, T., & Mahapatra, N. K. (2020). Continuity and convergence on neutrosophic soft normed linear spaces. International Journal of Fuzzy Computation and Modelling, 3(2), 156-186.

[13] Zail, S. H., Abed, M. M., & Faisal, A. S. (2022). Neutrosophic BCK-algebra and Ω-BCK-algebra. International Journal of Neutrosophic Science, 19(3), 8-15.

[14] Al-Quran, A., Al-Sharqi, F., Rodzi, Z. M., Aladil, M., Romdhini, M. U., Tahat, M. K., & Solaiman, O. S. (2023). The Algebraic Structures of Q-Complex Neutrosophic Soft Sets Associated with Groups and Subgroups. International Journal of Neutrosophic Science, 22(1), 60-77.

[15] M U Romdhini, A Al-Quran, F Al-Sharqi, M K Tahat, and A Lutfi. Exploring the Algebraic Structures of Q-Complex Neutrosophic Soft Fields. International Journal of Neutrosophic Science, 22(04):93–105, 2023.

[16] M U Romdhini, F Al-Sharqi, A Nawawi, A Al-Quran, and H Rashmanlou. Signless Laplacian energy of interval-valued fuzzy graph and its applications. Sains Malaysiana, 52(7):2127–2137, 2023.

[17] F. Al-Sharqi, A. Al-Quran and Z. M. Rodzi, Multi-Attribute Group Decision-Making Based on Aggregation Operator and Score Function of Bipolar Neutrosophic Hypersoft Environment, Neutrosophic Sets and Systems, 61(1), 465-492, 2023.

[18] A. Al-Quran, F. Al-Sharqi, A. U. Rahman and Z. M. Rodzi, The q-rung orthopair fuzzy-valued neutrosophic sets: Axiomatic properties, aggregation operators and applications. AIMS Mathematics, 9(2), 5038-5070, 2024.

[19] Abed, M. M., Hassan, N., & Al-Sharqi, F. (2022). On neutrosophic multiplication module. Neutrosophic Sets and Systems, 49(1), 198-208.

[20] Abed, M. M., Al-Sharqi, F., & Zail, S. H. (2021, March). A certain conditions on some rings give pp ring. In Journal of Physics: Conference Series (Vol. 1818, No. 1, p. 012068). IOP Publishing.

[21] Auad, A. A and Fayyadh, M. H. (2021). The direct and converse theorems for best approximation of algebraic polynomial in lp, α (x). In Journal of Physics: Conference Series (Vol. 1879, No. 3, p. 032010). IOP Publishing.

[22] Auad, A. A., & Hussein, A. A. (2019, July). Best simultaneous approximation in weighted space. In Journal of Physics: Conference Series (Vol. 1234, No. 1, p. 012111). IOP Publishing.

[23] Auad, A. A., Hilal, M. A., & Khalaf, N. S. (2023). Best approximation of unbounded functions by modulus of smoothness. European Journal of Pure and Applied Mathematics, 16(2), 944-952.

DOI: https://doi.org/10.54216/IJNS.250206
Received: February 02, 2024 Revised: April 28, 2024 Accepted: July 25, 2024

[24] Auad , A. A., & Al-Sharqi, F. (2023). Identical Theorem of Approximation Unbounded Functions by Linear Operators. Journal of Applied Mathematics & Informatics, 41(4), 801–810.

[25] A. Al-Quran, F. Al-Sharqi, K. Ullah, M. U. Romdhini, M. Balti and M. Alomai, Bipolar fuzzy hypersoft set and its application in decision making, International Journal of Neutrosophic Science, vol. 20, no. 4, pp. 65-77, 2023.

[26] Al-Sharqi F, Ahmad AG, Al-Quran A. Similarity measures on interval-complex neutrosophic soft sets with applications to decision making and medical diagnosis under uncertainty. Neutrosophic Sets and Systems, 2022; 51:495-515.

[27] K. Roshdi, Best approximation in metric spaces, American mathematical society, Vol. 103 (1988), No. 2, 579-586.

[28] H. Mazheri, S.M.S. Modarres: Some Results Concerning Proximinality and co-Proximinality, Nonlinear Anal., 62, (2005), 1123-1126.

[29] V. Indumathi and N. Prakash, E-proximinal subspaces, J. Math. Anal. Appl. 450, (2017), 1–11 .

[30] C. R. Jayanarayanan. Characterization of strong ball proximinality in L_1-predual spaces. J. Convex Anal., 26(2), (2019),537–542.

DOI: https://doi.org/10.54216/IJNS.250206
Received: February 02, 2024 Revised: April 28, 2024 Accepted: July 25, 2024