



On Some Results about Complex Order Bessel's Equation

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Abstract

In this paper, we derive the Bessel's equation of complex order $(n + i)$ from the classical well-known Bessel's equation. In addition, we generalize that recurrence relation from Bessel's equation of order (n) to Bessel equation of complex order $(n + i)$. On the other hand, we present an algorithm to solve the novel complex order equation with many illustrated examples that explain the validity of our approach.

Keywords: Bessel's equation; complex order; differential equation; partial derivation

1. Introduction

The linear differential equation of the second order and the following variable coefficients:

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0, \quad n \geq 0 \quad \dots(1.1)$$

(1.1) is known as the Bessel's equation of order n and this equation appears a lot in Applied Mathematics, Physics and engineering problems. Any solution that achieves this equation is called the Bessel functions of rank n . When solving this equation, if the value of n is an integer, the general solution is as follows:

$$y(x) = c_1 J_n(x) + c_2 y_n(x) \quad \dots(1.2)$$

Since $y_n(x), J_n(x)$ are Bessel functions of the first and second types, respectively, and c_2, c_1 are optional constants, and:

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} \quad \dots(1.3)$$

$$y_n(x) = \frac{1}{\sin n\pi} [J_n(x) \cos n\pi - J_{-n}(x)] \quad \dots(1.4)$$

If n is not an integer, the general solution is as follows:

$$y(x) = c_1 J_n(x) + c_2 J_{-n}(x)$$

Since $J_{-n}(x), J_n(x)$ are Bessel functions of the first kind of rank $-n, n$, respectively, and c_2, c_1 are optional constants, and if:

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} \quad \dots(1.5)$$

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{-n+2r} \quad \dots(1.6)$$

Bernoulli was the first to give the concept of the Bessel function in 1732, using the zero-order function as a solution to the problem of oscillation of a one-sided suspended chain, and later the French scientist Willem Bessel (1784-1846) introduced a generalization of this equation used by other mathematicians such as Euler and Lord Rayleigh..... Etc.

Bessel's equation was discussed in many cases, for example, of zero rank, $\frac{1}{2}$ rank, 1 rank, integral rank, and solutions were obtained for these cases ([3], [4], [7], [8], [9], [11])

Our study will be summarized in this research on the case where n is a non integer, as the study is carried out on the Bessel equation of nodal rank $(n + i)$ and the study of some of its properties.

Definition (1.1): Regular singular point [2].

If we have the following differential equation:

$$\frac{d^2y}{dx^2} + p_1(x) \frac{dy}{dx} + p_2(x)y = 0$$

If both $p_1(x)$ and $p_2(x)$ are functions that are not analytic at point x_0 , then this point is called the Singular point of the differential equation. If the functions $(x - x_0)p_1(x)$ and $(x - x_0)^2p_2(x)$ are analytic functions at point x_0 , then this point is called a regular singular point.

For a differential equation, if one or both of them is not an analytical function at the point x_0 , this point is called an irregular singular point.

Definition (1.2): Gamma Function [1]:

defined as:

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$$

Which approaches when $n > 0$. There is also another Formula:

$$\Gamma(n + 1) = n\Gamma(n)$$

It is also known as the factorial function, thus:

$$\Gamma(n + 1) = n!$$

Definition (1.3): Wronskian determinant. [8]

If $y_2(x), y_1(x)$ are real-valued or nodal functions, the wronskian determinant is equal to:

$$w(y_1(x), y_2(x)) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

If $w(y_1(x), y_2(x)) = 0$, then the two solutions are linearly dependent, and if $w(y_1(x), y_2(x)) \neq 0$, then the two solutions are linearly independent.

2. Solve the Bessel equation of nodal rank by the method of power sequences

Let be a linear differential equation of the second order and the following variable coefficients:

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - (n + i)^2)y = 0, \quad n \geq 0 \quad \dots(2.1)$$

Bessel equation of rank $(n + i)$. Point $x = 0$ regular singular point we begin solving equation (2.1) by the Frobenius method by assuming that:

$$y = x^m(a_0 + a_1x + a_2x^2 + a_3x^3 + \dots), \quad m > 0 \quad \dots(2.2)$$

$$y = \sum_{r=0}^{\infty} a_r x^{m+r}, \quad a_r \neq 0$$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r(m+r)x^{m+r-1} \quad \dots(2.3)$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r(m+r)(m+r-1)x^{m+r-2} \quad \dots(2.4)$$

We substitute equation (2.2) and equation (2.4) into equation (2.1) and get:

$$\begin{aligned}
 & x^2 \sum_{r=0}^{\infty} a_r(m+r)(m+r-1)x^{m+r-2} + x \sum_{r=0}^{\infty} a_r(m+r)x^{m+r-1} + (x^2 - (n+i)^2) \sum_{r=0}^{\infty} a_r x^{m+r} = 0 \\
 & \sum_{r=0}^{\infty} a_r(m+r)(m+r-1)x^{m+r} + \sum_{r=0}^{\infty} a_r(m+r)x^{m+r} + \sum_{r=0}^{\infty} a_r x^{m+r-2} - (n+i)^2 \sum_{r=0}^{\infty} a_r x^{m+r} = 0 \\
 & \sum_{r=0}^{\infty} a_r(m+r)(m+r-1)x^{m+r} + \sum_{r=0}^{\infty} a_r(m+r)x^{m+r} + \sum_{r=2}^{\infty} a_{r-2}x^{m+r} - (n+i)^2 \sum_{r=0}^{\infty} a_r x^{m+r} = 0 \\
 & m(m-1)a_0x^m + m(m+1)a_1x^{m+1} + ma_0x^m + (m+1)a_1x^{m+1} - (n+i)^2a_0x^m - (n+i)^2a_1x^{m+1} \\
 & + \sum_{r=2}^{\infty} [(m+r)(m+r-1) + (m+r) - (n+i)^2]a_r + a_{r-2}x^{m+r} = 0
 \end{aligned}$$

Equating the coefficient x^m , we get:

$$\begin{aligned}
 [m(m-1) + m - (n+i)^2]a_0 &= 0 \quad , \quad a_0 \neq 0 \\
 m^2 - (n+i)^2 &= 0 \Rightarrow m^2 = (n+i)^2
 \end{aligned}$$

Either $m = n+i$ or $m = -(n+i)$.

Equating the coefficient x^{m+1} , we get:

$$[m(m+1) + m + 1 - (n+i)^2]a_1 = 0$$

The first case: when $m = n+i$ we get:

$$[2m+1]a_1 = 0$$

Since $2m+1 \neq 0$ because $m > 0$.

So, $a_1 = 0$

Equating the coefficient x^{m+r} , we get:

$$\begin{aligned}
 [(m+r)(m+r-1) + (m+r) - (n+i)^2]a_r + a_{r-2} &= 0 \quad , \quad a_0 \neq 0 \\
 [m^2 + 2mr + r^2 - (n+i)^2]a_r &= -a_{r-2}
 \end{aligned}$$

Since $m = n+i$, we get:

$$\begin{aligned}
 [(n+i)^2 + 2(n+i)r + r^2 - (n+i)^2]a_r &= -a_{r-2} \\
 r[2(n+i) + r]a_r &= -a_{r-2}
 \end{aligned}$$

$$a_r = \frac{-a_{r-2}}{r[2(n+i)+r]} \quad , \quad r \geq 2 \quad \dots(2.5)$$

$$a_2 = \frac{-a_{r-2}}{2[2(n+i)+2]} \Rightarrow a_2 = \frac{-a_0}{2^2[(n+i)+1]}$$

$$a_3 = \frac{-a_1}{3[2(n+i)+3]} \because a_1 = 0 \Rightarrow a_3 = 0$$

$$a_4 = \frac{-a_2}{4[2(n+i)+4]} \Rightarrow a_4 = \frac{-1}{4[2(n+i)+4]} \cdot \frac{-a_0}{2^2[(n+i)+1]}$$

$$a_4 = \frac{a_0}{4[2(n+i)+4] 2^2[(n+i)+1]}$$

$$a_5 = \frac{-a_3}{5[2(n+i)+5]} \because a_3 = 0 \Rightarrow a_5 = 0$$

Then:

$$a_1 = a_3 = a_5 = a_7 = \dots = 0$$

And therefore:

$$a_{2n-1} = 0, \quad n = 1, 2, 3, \dots$$

This means that the coefficients of equation (2.1) are only even coefficients. So, we substitute r in the iterative formula (2.5) by $2k$, we get:

$$\begin{aligned} a_{2k} &= \frac{-a_{2k-2}}{2k[2(n+i) + 2k]}, \quad k = 1, 2, 3, 4, \dots \\ a_{2k} &= \frac{-a_{2k-2}}{4k[(n+i) + k]} \\ k = 1 &\Rightarrow a_2 = \frac{-a_0}{4[(n+i) + 1]} \\ k = 2 &\Rightarrow a_4 = \frac{-a_2}{4.2[(n+i) + 2]} \Rightarrow a_4 = \frac{-1}{4.2[(n+i) + 2]} \cdot \frac{-a_0}{4[(n+i) + 1]} \\ &\Rightarrow a_4 = \frac{a_0}{4^2.2.1[(n+i) + 2][(n+i) + 1]} \\ k = 3 &\Rightarrow a_6 = \frac{-a_4}{4.3[(n+i) + 3]} \\ &\Rightarrow a_6 = \frac{-1}{4.3[(n+i) + 3]} \cdot \frac{a_0}{4^2.2.1[(n+i) + 2][(n+i) + 1]} \\ &\Rightarrow a_6 = \frac{-a_0}{4^2.3.2.1[(n+i) + 3][(n+i) + 2][(n+i) + 1]} \end{aligned}$$

Therefore:

$$a_{2k} = \frac{(-1)^k}{4^k k! [(n+i) + k] \dots [(n+i) + 1]}$$

We substitute the coefficients in equation (2.2) with $m = n + i$, and get:

$$\begin{aligned} y &= x^{n+i} \left(a_0 - \frac{a_0}{4[(n+i) + 1]} x^2 + \frac{a_0}{4^2.2.1[(n+i) + 2][(n+i) + 1]} x^4 \right. \\ &\quad \left. - \frac{-a_0}{4^3.3.2.1[(n+i) + 3][(n+i) + 2][(n+i) + 1]} x^6 + \dots \right) \\ y &= a_0 x^{n+i} \left(1 - \frac{1}{4[(n+i) + 1]} x^2 + \frac{1}{4^2.2.1[(n+i) + 2][(n+i) + 1]} x^4 \right. \\ &\quad \left. - \frac{1}{4^3.3.2.1[(n+i) + 3][(n+i) + 2][(n+i) + 1]} x^6 + \dots \right) \end{aligned} \quad \dots (2.6)$$

If we choose the value of the constant a_0 in the form [3]:

$$a_0 = \frac{1}{2^{n+i}(n+i)!}$$

We substitute equation (2.7) into equation (2.6), and get:

$$\begin{aligned} y &= \frac{1}{2^{n+i}(n+i)!} x^{n+i} \left(1 - \frac{1}{4[(n+i) + 1]} x^2 + \frac{1}{4^2.2.1[(n+i) + 2][(n+i) + 1]} x^4 \right. \\ &\quad \left. - \frac{1}{4^3.3.2.1[(n+i) + 3][(n+i) + 2][(n+i) + 1]} x^6 + \dots \right) \\ y &= \left(\frac{x}{2} \right)^{n+i} \left[\frac{1}{(n+i)!} - \frac{1}{4[(n+i) + 1]} x^2 - \frac{1}{4^2.2.1[(n+i) + 2]} x^4 - \frac{1}{4^3.3.2.1[(n+i) + 3]} x^6 + \dots \right] \end{aligned} \quad \dots (2.8)$$

Since $[(n+i) + 1] = (n+i)!$ [12], thus:

$$y = \left(\frac{x}{2}\right)^{n+i} \left[\frac{1}{[(n+i+1)]} - \frac{1}{4[(n+i+2)]} x^2 + \frac{1}{4^2 \cdot 2.1[(n+i+3)]} x^4 - \frac{1}{4^3 \cdot 3.2.1[n+i+4]} x^6 + \dots \right]$$

$$y = \left(\frac{x}{2}\right)^{n+i} \left[\frac{1}{[(n+i+1)]} - \frac{1}{4[(n+i+2)]} \left(\frac{x}{2}\right)^2 + \frac{1}{2.1[(n+i+3)]} \left(\frac{x}{2}\right)^4 - \frac{1}{3.2.1[n+i+4]} \left(\frac{x}{2}\right)^6 + \dots \right]$$

$$y_{n+i} = J_{n+i}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r![(n+i+r+1)]} \left(\frac{x}{2}\right)^{n+i+2r} \quad \dots(2.9)$$

$J_{n+i}(x)$ is called a Bessel function of the first type and rank $n+i$. to find the other solution of Bessel's equation we take $m = -(n+i)$ the iterative formula (2.5) becomes:

$$a_r = \frac{-a_{r-2}}{r[r-2(n+i)]} \quad , \quad r \geq 2 \quad , \quad r \neq 2(n+i)$$

Using the same steps that we performed to obtain $J_{n+i}(x)$, we get the other solution to Bessel's equation, which takes the following form:

$$J_{-(n+i)}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r![-(n+i)+r+1]} \left(\frac{x}{2}\right)^{-(n+i)+2r} \quad \dots(2.10)$$

This function is called a Bessel function of the first type and rank $-(n+i)$ and therefore the general solution of the Bessel equation of nodal rank $n+i$ has the following form:

$$y(x) = c_1 J_{n+i}(x) + c_2 J_{-(n+i)}(x)$$

Example (1): Find the general solution of the following Bessel equation:

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - (3+i)^2)y = 0$$

Solution:

Bessel equation of Rank $3+i$ and the general solution of this equation is:

$$y(x) = c_1 J_{3+i}(x) + c_2 J_{-3-i}(x)$$

Since c_1 and c_2 are optional constants, thus:

$$J_{3+i}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r![(3+i+r+1)]} \left(\frac{x}{2}\right)^{3+i+2r}$$

$$J_{3+i}(x) = \left(\frac{x}{2}\right)^{3+i} \left[\frac{1}{0![(3+i+1)]} - \frac{1}{1![(3+i+2)]} \left(\frac{x}{2}\right)^2 + \frac{1}{2![(3+i+3)]} \left(\frac{x}{2}\right)^4 - \frac{1}{3![(3+i+4)]} \left(\frac{x}{2}\right)^6 + \dots \right]$$

$$J_{3+i}(x) = \left(\frac{x}{2}\right)^{3+i} \left[\frac{1}{[(4+i)]} - \frac{1}{4[(5+i)]} x^2 + \frac{1}{4^2 \cdot 2.1[(6+i)]} x^4 - \frac{1}{4^3 \cdot 3.2.1[(7+i)]} x^6 + \dots \right]$$

$$J_{3+i}(x) = \left(\frac{x}{2}\right)^{3+i} \left[\frac{1}{(3+i)!} - \frac{1}{4(4+i)!} x^2 + \frac{1}{4^2 \cdot 2.1(5+i)!} x^4 - \frac{1}{4^3 \cdot 3.2.1(6+i)!} x^6 + \dots \right]$$

$$J_{3+i}(x) = \left(\frac{x}{2}\right)^{3+i} \left[\frac{1}{1.55 + 4.98i} - \frac{1}{4(1.22 + 21.47i)} x^2 + \frac{1}{32(-15.37 + 108.57i)} x^4 - \frac{1}{384(-20079 + 636.05i)} x^6 + \dots \right]$$

$$J_{3+i}(x) = \left(\frac{x}{2}\right)^{3+i} [0.0569 - 0.18306i - (0.00065 - 0.0116i)x^2 - (0.00003 + 0.0002i)x^4 + (0.000001 + 0.000003i)x^6 + \dots]$$

$$J_{-3-i}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r![-3-i+r+1]} \left(\frac{x}{2}\right)^{-3-i+2r}$$

$$\begin{aligned}
 J_{-3-i}(x) &= \left(\frac{x}{2}\right)^{-3-i} \left[\frac{1}{0! [(-3-i+1)]} - \frac{1}{1! [(-3-i+2)]} \left(\frac{x}{2}\right)^2 + \frac{1}{2! [(-3-i+3)]} \left(\frac{x}{2}\right)^4 \right. \\
 &\quad \left. - \frac{1}{3! [(-3-i+4)]} \left(\frac{x}{2}\right)^6 + \dots \right] \\
 J_{-3-i}(x) &= \left(\frac{x}{2}\right)^{-3-i} \left[\frac{1}{[(-2-1)]} - \frac{1}{4[(-1-i)]} x^2 + \frac{1}{4^2 \cdot 2.1[(-i)]} x^4 - \frac{1}{4^3 \cdot 3.2.1[(1-i)]} x^6 + \dots \right] \\
 J_{-3-i}(x) &= \left(\frac{x}{2}\right)^{-3-i} \left[\frac{1}{(-3-i)!} - \frac{1}{4(-2-i)!} x^2 + \frac{1}{4^2 \cdot 2.1(-1-i)!} x^4 - \frac{1}{4^3 \cdot 3.2.1(-i)!} x^6 + \dots \right] \\
 J_{-3-i}(x) &= \left(\frac{x}{2}\right)^{-3-i} \left[\frac{10}{1.339 + 0.963i} - \frac{1}{2(-0.343 - 0.653i)} x^2 + \frac{1}{32(0.155 + 0.498i)} x^4 \right. \\
 &\quad \left. - \frac{1}{384(0.498 + 0.155i)} x^6 + \dots \right] \\
 J_{-3-i}(x) &= \left(\frac{x}{2}\right)^{-3-i} [4.9224 - 3.5401i + (0.3152 - 0.6001i)x^2 - (0.0178 + 0.0572i)x^4 - (0.0047 \\
 &\quad - 0.0014i)x^6 + \dots]
 \end{aligned}$$

Case (2.1):

If $J_{-(n+i)}(x), J_{n+i}(x)$ are Bessel functions of the first and second types, respectively, then these functions are linearly independent of equation (2.1) when $x \neq 0$. And that the general solution of equation (2.1) has the form:

$$y(x) = c_1 J_{n+i}(x) + c_2 J_{-(n+i)}(x)$$

Since c_1 and c_2 are optional constants.

Proof:

To prove that Bessel functions $J_{-(n+i)}(x), J_{n+i}(x)$ are linearly independent functions it suffices to show that the Wronskian is not equal to zero.

$$W(J_{n+i}(x), J_{-(n+i)}(x)) = \begin{vmatrix} J_{n+i}(x) & J_{-(n+i)}(x) \\ J'_{n+i}(x) & J'_{-(n+i)}(x) \end{vmatrix}$$

Since $J_{-(n+i)}(x), J_{n+i}(x)$ satisfies equation (2.1)

$$\begin{aligned}
 x^2 J''_{n+i}(x) + x J'_{n+i}(x) + (x^2 - (n+i)^2) J_{n+i}(x) &= 0 \\
 x^2 J''_{-(n+i)}(x) + x J'_{-(n+i)}(x) + (x^2 - (n+i)^2) J_{-(n+i)}(x) &= 0
 \end{aligned}$$

We multiply these two equations by $J_{n+i}(x), J_{-(n+i)}(x)$ respectively and subtract them from each other and divide the result by x , we get:

$$x(J_{n+i}(x)J''_{-(n+i)}(x) - J_{-(n+i)}(x)J''_{n+i}(x) + J_{n+i}(x)J'_{-(n+i)}(x) - J_{-(n+i)}(x)J'_{n+i}(x)) = 0$$

This is equivalent to:

$$\frac{d}{dx}(x(J_{n+i}(x)J'_{-(n+i)}(x) - J_{-(n+i)}(x)J'_{n+i}(x))) = \frac{d}{dx}(x W(J_{n+i}(x), J_{-(n+i)}(x))) = 0$$

$$W(J_{n+i}(x), J_{-(n+i)}(x)) = \frac{c}{x}, \quad x \neq 0$$

$$W(J_{n+i}(x), J_{-(n+i)}(x)) \neq 0$$

So, the Bessel functions $J_{-(n+i)}(x), J_{n+i}(x)$ are linearly independent.

Theorem (2.2):

If

$$J_{n+i}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! [(n+i+r+1)]} \left(\frac{x}{2}\right)^{n+i+2r}, \quad n > -1$$

Then:

$$\lim_{x \rightarrow 0} \frac{J_{n+i}(x)}{x^{n+i}} = \frac{1}{2^{n+i}[(n+i+1)]}$$

Proof: from the equation (2.8):

$$\begin{aligned} J_{n+i}(x) &= \left(\frac{x}{2}\right)^{n+i} \left[\frac{1}{\Gamma(n+i+1)} - \frac{1}{4\Gamma(n+i+2)}x^2 + \frac{1}{4^2 \cdot 2.1\Gamma(n+i+3)}x^4 - \frac{1}{4^3 \cdot 3.2.1\Gamma(n+i+4)}x^6 + \dots \right] \\ \frac{J_{n+i}(x)}{x^{n+i}} &= \frac{1}{2^{n+i}[(n+i+1)]} \left[1 - \frac{1}{4(n+i+1)}x^2 + \frac{1}{4^2 \cdot 2.1(n+i+2)(n+i+1)}x^4 \right. \\ &\quad \left. - \frac{1}{4^3 \cdot 3.2.1(n+i+3)(n+i+2)(n+i+1)}x^6 + \dots \right] \\ \lim_{x \rightarrow 0} \frac{J_{n+i}(x)}{x^{n+i}} &= \lim_{x \rightarrow 0} \frac{1}{x^{n+i}[(n+i+1)]} \left[1 - \frac{1}{4(n+i+1)}x^2 + \frac{1}{4^2 \cdot 2.1(n+i+2)(n+i+1)}x^4 \right. \\ &\quad \left. - \frac{1}{4^3 \cdot 3.2.1(n+i+3)(n+i+2)(n+i+1)}x^6 + \dots \right] \\ \lim_{x \rightarrow 0} \frac{J_{n+i}(x)}{x^{n+i}} &= \frac{1}{2^{n+i}[(n+i+1)]} \end{aligned}$$

case (2.2): the following relationship connects Bessel functions of the first type.

$$J_{-(n+i)}(x) = (-1)^{n+i} J_{n+i}(x)$$

Proof:

$$\begin{aligned} J_{-(n+i)}(x) &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r![-(n+i)+r+1]} \left(\frac{x}{2}\right)^{-(n+i)+2r} = \sum_{r=0}^{(n+i)-1} \frac{(-1)^r}{r![-(n+i)+r+1]} \left(\frac{x}{2}\right)^{-(n+i)+2r} + \sum_{r=n+i}^{\infty} \frac{(-1)^r}{r![-(n+i)+r+1]} \left(\frac{x}{2}\right)^{-(n+i)+2r} = 0 + \\ &\sum_{r=n+i}^{\infty} \frac{(-1)^r}{r![-(n+i)+r+1]} \left(\frac{x}{2}\right)^{-(n+i)+2r} \end{aligned}$$

Now, we put instead of $r = (n+i) + k$, we get:

$$\begin{aligned} J_{-(n+i)}(x) &= \sum_{k=0}^{\infty} \frac{(-1)^{(n+i)+k}}{((n+i)+k)![(k+1)]} \left(\frac{x}{2}\right)^{(n+i)+2k} = (-1)^{n+i} \sum_{k=0}^{\infty} \frac{(-1)^k}{k![(n+i+k+1)]} \left(\frac{x}{2}\right)^{n+i+2k} \\ J_{-(n+i)}(x) &= (-1)^{n+i} J_{n+i}(x) \end{aligned}$$

3. Resatioler neceerrcu

Resatioler neceerrcu are mentioned in [3]. In the case of n is an integer. We will consider these relations in the nodal case $(n+i)$

Case (3.1):

If:

$$J_{n+i} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r![(n+i+r+1)]} \left(\frac{x}{2}\right)^{n+i+2r}$$

Then:

1. $xJ_{n+i} = (n+i)J_{n+i} - xJ_{n+1+i}$
2. $xJ_{n+i} = -(n+i)J_{n+i} + xJ_{n-1+i}$

Proof of relation 1:

$$xJ_{n+i} = (n+i)J_{n+i} - xJ_{n+1+i}$$

It is known that:

$$J_{n+i} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r![(n+i+r+1)]} \left(\frac{x}{2}\right)^{n+i+2r}$$

$$\begin{aligned}
 J_{n+i} &= \sum_{r=0}^{\infty} \frac{(-1)^r (n+i+2r)}{r! [(n+i+r+1)]} \left(\frac{x}{2}\right)^{n+i+2r-1} \frac{1}{2} \\
 &= (n+i) \sum_{r=0}^{\infty} \frac{(-1)^r}{r! [(n+i+r+1)]} \left(\frac{x}{2}\right)^{n+i+2r-1} \frac{1}{2} \\
 &\quad + \sum_{r=0}^{\infty} \frac{(-1)^r 2r}{r! [(n+i+r+1)]} \left(\frac{x}{2}\right)^{n+i+2r-1} \frac{1}{2} \\
 xJ_{n+i} &= x(n+1) \sum_{r=0}^{\infty} \frac{(-1)^r}{r! [(n+i+r+1)]} \left(\frac{x}{2}\right)^{n+i+2r} \frac{1}{2} \\
 &\quad + x \sum_{r=1}^{\infty} \frac{(-1)^r 2}{(r-1)! [(n+i+r+1)]} \left(\frac{x}{2}\right)^{n+i+2r-1} \frac{1}{2} \\
 &= x(n+i) \sum_{r=0}^{\infty} \frac{(-1)^r}{r! [(n+i+r+1)]} \left(\frac{x}{2}\right)^{n+i+2r-1} \frac{1}{2} \\
 &\quad + x \sum_{r=1}^{\infty} \frac{(-1)^r}{(r-1)! [(n+i+r+1)]} \left(\frac{x}{2}\right)^{n+i+2r-1} \\
 &= (n+i)J_{n+i} + x \sum_{r=1}^{\infty} \frac{(-1)^r}{(r-1)! [(n+i+r+1)]} \left(\frac{x}{2}\right)^{n+i+2r-1}
 \end{aligned}$$

We assume that $s = r - 1$.

$$\begin{aligned}
 xJ_{n+i} &= (n+i)J_{n+i} + x \sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{s! [(n+i+s+2)]} \left(\frac{x}{2}\right)^{n+i+2s+1} \\
 xJ_{n+i} &= (n+i)J_{n+i} - x \sum_{s=0}^{\infty} \frac{(-1)^s}{s! [(n+i+s+2)]} \left(\frac{x}{2}\right)^{n+i+2s+1} \\
 xJ_{n+i} &= (n+i)J_{n+i} - xJ_{n+1+i}
 \end{aligned}$$

Proof of relation 2:

$$xJ_{n+i} = (n+i)J_{n+i} + xJ_{n-1+i}$$

It is known that:

$$\begin{aligned}
 J_{n+i} &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r! [(n+i+r+1)]} \left(\frac{x}{2}\right)^{n+i+2r} \\
 J_{n+i} &= \sum_{r=0}^{\infty} \frac{(-1)^r (n+i+2r)}{r! [(n+i+r+1)]} \left(\frac{x}{2}\right)^{n+i+2r-1} \frac{1}{2} \\
 &= \sum_{r=0}^{\infty} \frac{(-1)^r [2(n+i) + 2r - (n+i)]}{r! [(n+i+r+1)]} \left(\frac{x}{2}\right)^{n+i+2r-1} \frac{1}{2} \\
 xJ_{n+i} &= \sum_{r=0}^{\infty} \frac{(-1)^r [2(n+i) + 2r - (n+i)]}{r! [(n+i+r+1)]} \left(\frac{x}{2}\right)^{n+i+2r} \\
 &= \sum_{r=1}^{\infty} \frac{(-1)^r [2(n+i) + 2r]}{r! [(n+i+r+1)]} \left(\frac{x}{2}\right)^{n+i+2r}
 \end{aligned}$$

$$\begin{aligned}
 & -(n+i) \sum_{r=0}^{\infty} \frac{(-1)^r}{r! [(n+i+r+1)]} \left(\frac{x}{2}\right)^{n+i+2r} \\
 xJ_{n+i} &= \sum_{r=0}^{\infty} \frac{(-1)^r 2[(n+i)+r]}{r! [(n+i+r+1)]} \left(\frac{x}{2}\right)^{n+i+2r} - (n+i)J_{n+i} \\
 xJ_{n+i} &= \sum_{r=0}^{\infty} \frac{(-1)^r 2[(n+i)+r]}{r! (n+i+r)[(n+i+r)]} \left(\frac{x}{2}\right)^{n+i+2r} - (n+i)J_{n+i} \\
 xJ_{n+i} &= \sum_{r=0}^{\infty} \frac{(-1)^r 2}{r! [(n+i+r)]} \left(\frac{x}{2}\right)^{n+i+2r} - (n+i)J_{n+i} \\
 xJ_{n+i} &= x \sum_{r=0}^{\infty} \frac{(-1)^r}{r! [(n+i+r)]} \left(\frac{x}{2}\right)^{n+i+2r-1} - (n+i)J_{n+i} \\
 xJ_{n+i} &= x \sum_{r=0}^{\infty} \frac{(-1)^r}{r! [(n+i+r)]} \left(\frac{x}{2}\right)^{n-1+i+2r} - (n+i)J_{n+i} \\
 xJ_{n+i} &= xJ_{n-1+i} - (n+i)J_{n+i}
 \end{aligned}$$

Result (3.2):

If

$$J_{n+i} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! [(n+i+r+1)]} \left(\frac{x}{2}\right)^{n+i+2r}$$

Then:

3. $2J_{n+i} = J_{n-1+i} - J_{n+1+i}$
4. $2(n+i)J_{n+i} = x(J_{n-1+i} + J_{n+1+i})$

Proof of relation 3:

$$2J_{n+i} = J_{n-1+i} - J_{n+1+i}$$

From case (3.1), we get:

$$xJ_{n+i} = (n+i)J_{n+i} - xJ_{n+1+i} \quad \text{..(3.1)}$$

$$xJ_{n+i} = -(n+i)J_{n+i} + xJ_{n-1+i} \quad \text{..(3.2)}$$

Adding (3.1) and (3.2), we get:

$$2xJ_{n+i} = -xJ_{n+1+i} + xJ_{n-1+i}$$

By dividing on x , $x \neq 0$

$$2J_{n+i} = -J_{n+1+i} + J_{n-1+i}$$

$$2J_{n+i} = J_{n+1+i} - J_{n+1+i}$$

Proof of relation 4:

$$2(n+i)J_{n+i} = x(J_{n-1+i} + J_{n+1+i})$$

From case (3.1), we get:

$$xJ_{n+i} = (n+i)J_{n+i} - xJ_{n+1+i} \quad \text{..(3.3)}$$

$$xJ_{n+i} = -(n+i)J_{n+i} + xJ_{n-1+i} \quad \text{..(3.4)}$$

Substituting equation (3.4) into equation (3.3), we obtain:

$$-(n+i)J_{n+i} + xJ_{n-1+i} = (n+i)J_{n+i} - xJ_{n+1+i}$$

$$\begin{aligned} 2(n+i) J_{n+i} - x J_{n+1+i} - x J_{n-1+i} &= 0 \\ 2(n+i) J_{n+i} &= x J_{n+1+i} + x J_{n-1+i} \\ 2(n+i) J_{n+i} &= x(J_{n+1+i} + x J_{n-1+i}) \end{aligned}$$

Case (3.3):

If:

$$J_{n+i} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! [(n+i+r+1)]} \left(\frac{x}{2}\right)^{n+i+2r}$$

Then:

$$\begin{aligned} 5. \quad \frac{d}{dx} (x^{-(n+i)} J_{n+i}) &= -x^{-(n+i)} J_{n+1+i} \\ 6. \quad \frac{d}{dx} (x^{(n+i)} J_{n+i}) &= x^{(n+i)} J_{n-1+i} \end{aligned}$$

Proof of relation 5:

$$\frac{d}{dx} (x^{-(n+i)} J_{n+i}) = -x^{-(n+i)} J_{n+1+i}$$

It is known that:

$$\begin{aligned} J_{n+i} &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r! [(n+i+r+1)]} \left(\frac{x}{2}\right)^{n+i+2r} \\ x^{-(n+i)} J_{n+i} &= \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{r! 2^{n+i+2r} [(n+i+r+1)]} \\ \frac{d}{dx} (x^{-(n+i)} J_{n+i}) &= \sum_{r=0}^{\infty} \frac{(-1)^r 2r x^{2r-1}}{r! 2^{n+i+2r} [(n+i+r+1)]} \\ &= \sum_{r=1}^{\infty} \frac{(-1)^r 2 x^{2r-1}}{(r-1)! 2^{n+i+2r} [(n+i+r+1)]} \end{aligned}$$

We assume that $r = k + 1$.

$$\begin{aligned} &= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} 2 x^{2k+1}}{k! 2^{n+i+2k+2} [(n+i+k+2)]} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} 2 x^{n+i+2k+1} x^{-(n+i)}}{k! 2^{n+i+2k+1} 2 [(n+i+k+2)]} \\ &= x^{-(n+i)} = \sum_{k=0}^{\infty} \frac{(-1)^k (-1) x^{n+i+2k+1}}{k! 2^{n+i+2k+1} [(n+i+k+2)]} \\ &= -x^{-(n+i)} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{n+i+2k+1}}{k! 2^{n+i+2k+1} [(n+i+k+2)]} \\ &= -x^{-(n+i)} J_{n+1+i} \end{aligned}$$

Proof of relation 6:

$$\frac{d}{dx} (x^{(n+i)} J_{n+i}) = x^{(n+i)} J_{n-1+i}$$

It is known that:

$$\begin{aligned}
 J_{n+i} &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r! [(n+i+r+1)]} \left(\frac{x}{2}\right)^{n+i+2r} \\
 x^{(n+i)} J_{n+i} &= \sum_{r=0}^{\infty} \frac{(-1)^r x^{2(n+i)+2r}}{r! 2^{n+i+2r} [(n+i+r+1)]} \\
 \frac{d}{dx} [x^{n+i} J_{n+i}] &= \sum_{r=0}^{\infty} \frac{(-1)^r (2(n+i)+2r) x^{2(n+i)+2r-1}}{r! 2^{n+i+2r} [(n+i+r+1)]} \\
 \frac{d}{dx} [x^{n+i} J_{n+i}] &= x^{n+i} \sum_{r=0}^{\infty} \frac{(-1)^r 2(n+i) x^{n+i+2r-1}}{r! 2^{n+i+2r} [(n+i+r+1)]} \\
 &\quad + x^{n+i} \sum_{r=0}^{\infty} \frac{(-1)^r 2r x^{n+i+2r-1}}{r! 2^{n+i+2r} [(n+i+r+1)]} \\
 \frac{d}{dx} [x^{n+i} J_{n+i}] &= x^{n+i} \sum_{r=0}^{\infty} \frac{(-1)^r (n+i) x^{n+i+2r-1}}{r! 2^{n+i+2r-1} [(n+i+r+1)]} \\
 &\quad + x^{n+i} \sum_{r=0}^{\infty} \frac{(-1)^r r x^{n+i+2r-1}}{r! 2^{n+i+2r-1} [(n+i+r+1)]} \\
 &= x^{n+i} \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+i+2r-1} (n+i+r)}{r! [(n+i+r+1)]} \\
 &= x^{n+i} \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+i+2r-1} (n+i+r)}{r! (n+i+r) [(n+i+r+1)]} \\
 &= x^{n+i} \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+i+2r-1}}{r! [(n+i+r)]} \\
 \frac{d}{dx} (x^{(n+i)} J_{n+i}) &= x^{(n+i)} J_{n-1+i}
 \end{aligned}$$

4. Illustrative examples as an application of previous relations

Example (1): Write the Bessel function J_{5+i} in the function of both Bessel functions J_{1+i}, J_{2+i}

Solution: the fourth relation of the result (3.2)

$$2(n+i)J_{n+i} = x(J_{n-1+i} + J_{n+1+i}) \quad \dots(3.5)$$

When $n = 4$

$$\begin{aligned}
 2(4+i)J_{4+i} &= x(J_{3+i} + J_{5+i}) \\
 \left(\frac{8+2i}{x}\right)J_{4+i} &= (J_{3+i} + J_{5+i}) \\
 J_{5+i} &= \left(\frac{8+2i}{x}\right)J_{4+i} - J_{3+i} \quad \dots(3.6)
 \end{aligned}$$

When $n = 3$ in equation (3.5), we get:

$$\begin{aligned}
 2(3+i)J_{3+i} &= x(J_{2+i} + J_{4+i}) \\
 \left(\frac{6+2i}{x}\right)J_{3+i} &= (J_{2+i} + J_{4+i}) \\
 J_{4+i} &= \left(\frac{6+2i}{x}\right)J_{3+i} - J_{2+i} \quad \dots(3.7)
 \end{aligned}$$

When $n = 2$ in equation (3.5), we get:

$$\begin{aligned} 2(2+i)J_{2+i} &= x(J_{1+i} + J_{3+i}) \\ \left(\frac{4+2i}{x}\right)J_{2+i} &= (J_{1+i} + J_{3+i}) \\ J_{3+i} &= \left(\frac{4+2i}{x}\right)J_{2+i} - J_{1+i} \quad \dots(3.8) \end{aligned}$$

We substitute equation (3.7) and equation (3.8) into equation (3.6) and get:

$$\begin{aligned} J_{5+i} &= \left(\frac{8+2i}{x}\right)\left(\left(\frac{6+2i}{x}\right)\left[\left(\frac{4+2i}{x}\right)J_{2+i} - J_{1+i}\right] - J_{2+i}\right) - \left[\left(\frac{4+2i}{x}\right)J_{2+i} - J_{1+i}\right] \\ J_{5+i} &= \frac{44+28i}{x}\left(\frac{4+2i}{x}J_{2+i} - J_{1+i}\right) - \frac{8+2i}{x}J_{2+i} - \frac{4+2i}{x}J_{2+i} - J_{1+i} \\ J_{5+i} &= \frac{120+200i}{x^3}J_{2+i} - \frac{44+28i}{x^2}J_{1+i} - \frac{8+2i}{x}J_{2+i} - \frac{4+2i}{x}J_{2+i} + J_{1+i} \\ J_{5+i} &= \frac{120+200i}{x^3}J_{2+i} - \frac{44+28i}{x^2}J_{1+i} - \frac{12+4i}{x}J_{2+i} + J_{1+i} \\ J_{5+i} &= \left[1 - \frac{44+28i}{x^2}\right]J_{1+i} + \left[\frac{120+200i}{x^3} - \frac{12+4i}{x}\right]J_{2+i} \end{aligned}$$

Example (2): prove that:

$$\frac{d}{dx}[J_{n+i}^2(x) + J_{n+1+i}^2(x)] = 2\left[\frac{n+i}{x}J_{n+i}^2(x) - \frac{(n+1+i)}{x}J_{n+1+i}^2(x)\right]$$

Proof: from case (3.1), we get:

$$xJ'_{n+i}(x) = (n+i)J_{n+i}(x) - xJ_{n+1+i}(x) \quad \dots(3.9)$$

$$xJ'_{n+i}(x) = -(n+i)J_{n+i}(x) + xJ_{n-1+i}(x) \quad \dots(3.10)$$

We put the formula (3.10) instead of every n a $(n+1)$, we get:

$$xJ'_{n+1+i}(x) = -(n+1+i)J_{n+1+i}(x) + xJ_{n+i}(x) \quad \dots(3.11)$$

$$\frac{d}{dx}[J_{n+i}^2(x) + J_{n+1+i}^2(x)] = 2J_{n+i}(x)J'_{n+i}(x) + 2J_{n+1+i}(x)J'_{n+1+i}(x) \quad \dots(3.12)$$

The values of $J'_{n+i}(x)$ and $J'_{n+1+i}(x)$ in the formula (3.11), (3.9), respectively, we substitute them in (3.12) and we get:

$$\begin{aligned} \frac{d}{dx}[J_{n+i}^2(x) + J_{n+1+i}^2(x)] &= 2J_{n+i}(x)\left[\frac{n+i}{x}J_{n+i}(x) - J_{n+1+i}(x)\right] \\ &\quad + 2J_{n+1+i}(x)\left[\frac{-(n+1+i)}{x}J_{n+1+i}(x) + J_{n+i}(x)\right] \\ \frac{d}{dx}[J_{n+i}^2(x) + J_{n+1+i}^2(x)] &= \frac{2}{x}J_{n+i}(x)[(n+i)J_{n+i}(x) - xJ_{n+1+i}(x)] \\ &\quad + \frac{2}{x}J_{n+1+i}(x)[-(n+1+i)J_{n+1+i}(x) + xJ_{n+i}(x)] \\ &= 2\frac{n+i}{x}J_{n+i}^2(x) - 2J_{n+i}(x)J_{n+1+i}(x) - 2\frac{(n+1+i)}{x}J_{n+1+i}^2(x) + 2J_{n+1+i}(x)J_{n+i}(x) \\ &\quad 2\left[\frac{n+i}{x}J_{n+i}^2(x) - \frac{(n+1+i)}{x}J_{n+1+i}^2(x)\right] \end{aligned}$$

Example (3): if $n > -1$, prove that:

$$\int_0^x x^{-(n+i)}J_{n+1+i}dx = \frac{1}{2^{n+i}[(n+i+1)]} - x^{-(n+i)}J_{n+i}(x)$$

Proof: the fifth relation of case (3.3), we get:

$$\begin{aligned}
\frac{d}{dx}(x^{-(n+i)}J_{n+i}) &= -x^{-(n+i)}J_{n+1+i} \\
\int_0^x x^{-(n+i)}J_{n+1+i}(x) dx &= -[x^{-(n+i)}J_{n+i}(x)]_0^x \\
&= -x^{-(n+i)}J_{n+i} + \lim_{x \rightarrow 0} \frac{J_{n+i}(x)}{x^{n+i}}
\end{aligned}$$

By theorem (2.2) we obtain:

$$\begin{aligned}
&= -x^{-(n+i)}J_{n+i} + \frac{1}{2^{n+i}\Gamma(n+i+1)} \\
&= \frac{1}{2^{n+i}\Gamma(n+i+1)} - x^{-(n+i)}J_{n+i}(x)
\end{aligned}$$

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