The basis number of connected vertex-disjoint graphs

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Abstract

The basis number b (G) of a graph G is defined to be the smallest positive integer k such that G has a k-fold basis for its cycle space. We try to find an upper bound for $b(G_1 + G_2 + G_3 + G_4)$.
We prove that, if G_1, G_2, G_3 and G_4 are connected vertex-disjoint graphs and each has a spanning tree of vertex degree not more than 4, then $b(G_1 + G_2 + G_3 + G_4) \leq max\{4, b(G_1) + 1, b(G_2) + 2, b(G_3) + 2, b(G_4) + 1\}$.
The basis number of quadruple join of paths will be studied, where we prove that
$b(p_m + p_n + p_p + p_t) = 4, \ \ \forall m, t \geq 5 \ \ and \ \ n, p \geq 6.$

Keywords: Graph; Basis number; Connected vertex-disjoint graphs; Path

1. Introduction

In the literature of graph theory, the interest in the basic number of data has increased; we refer the reader back to the research [4-11]. In this research, we will assume that all the data that we encounter are finite, undirected, and simple data; for terms that are not defined (see [9] and [10]). Let G be a statement connected by its edges $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}, \ldots, \boldsymbol{e_q}$.. For every partial s set of edges of G, there is a vector $(\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}, \ldots, \boldsymbol{a_q})$ corresponds to S such that $\boldsymbol{a_i} = 1$ if $e_i \notin S$. These vectors are a vector space of dimension q on the field $\boldsymbol{Z_2}$, it is called the vector space associated with the statement G and is denoted by $⟦(\boldsymbol{Z_2})^q$. The vectors $(\boldsymbol{Z_2})^q$ corresponding to the circuits of G generate a partial vector space called **the circuit space** of G, denoted by $\xi(G)$. Each vector in $\xi(G)$ represents either a circuit in G or a union of circuits separated from each other relative to the edges. One of the well-known results in the statement theorem is that the dimension of the circuit space of a statement connected to $\xi(G)G$ is q-p+1, since p represents the number of vertices of G and Q the number of its edges. The way to find the base for the space of circuits $\xi(G)$is as follows:

- Let T be a generator tree in G; If e is an edge belonging to G-T then T+e contains only one circuit and let be C_e. That q-p+1 of the circuits C_e, where $e \in G - T$ forms the base of the circuit space $\xi(G)$.
- A base B in the space of circuits $\xi(G)$ is said to be **k-fold** if each edge of G does not appear more than k times in the circuits corresponding to the vectors in base B.
- The base number of the statement G is defined to be the smallest integer k such that $\xi(G)$ has a base with a k-fold; and is denoted by b(G). If B is a base for the space of circuits $\xi(G)$ and e is an edge of G, then **the fold of the edge e** in B is denoted by $f_B(e)$ is the number of circuits in B containing edge e.

2. Definition (1)

Let $G_1 = (X, E_1)$, $G_2 = (Y, E_2)$, $G_3 = (Z, E_3)$, $G_4 = (U, E_4)$ simple data and separate for warheads, the data connection $G_1$and G_2 is a data set of its vertices is $X \cup Y$ and the group of the IS $E_1 \cup E_2 \cup \{xy | x \in X, y \in Y\}$ is denoted by $G_1 + G_2$; as well as **the tripartite contact** data G_1, G_2 , G_3 is a data set of its vertices is $X \cup Y \cup Z$ And the sum of its edges is $E_1 \cup E_2 \cup E_3 \cup \{xy | x \in X, y \in Y\} \cup \{yz | y \in Y, z \in Z\}$ is denoted by $G_1 + G_2 + G_3$; similarly, we call **the**

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quadrilateral contact $G_1 + G_2 + G_3 + G_4$that the data set of its vertices is $X \cup Y \cup Z \cup U$ and the sum of its edges is $E_1 \cup E_2 \cup E_3 \cup E_4 \cup \{xy|x \in X, y \in Y\} \cup \{yz|y \in Y, z \in Z\} \cup \{zu|z \in Z, u \in U\}$
For example, $P_4 = K_1 + K_1 + K_1 + K_1$ and Figure 1 shows the quadrilateral contact of the complete statement K_2

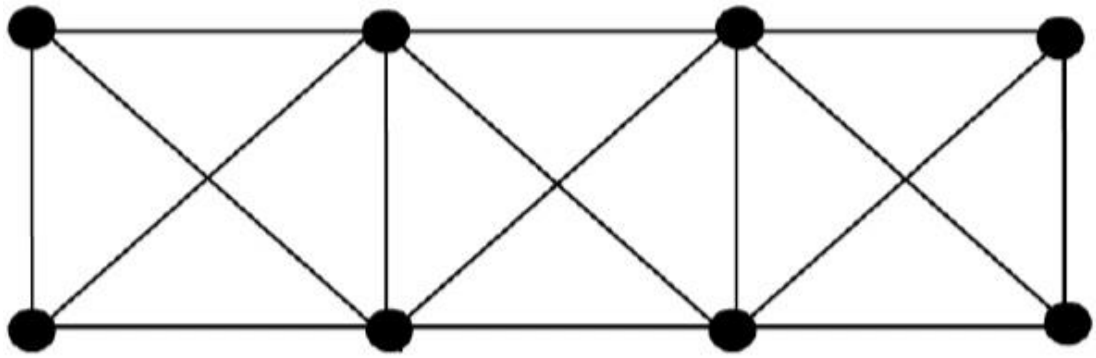

Figure 1. The quadrilateral contact of the complete statement K_2**.**

Form (1). $\boldsymbol{K_2 + K_2 + K_2 + K_2}$ And in a observational way we know the connection n of the data, $G_1 + G_2 + \ldots + G_n$; it is clear that this statement is connected. We will encode the path that has n vertices with the symbol P_n
In 1937,McClain [13]proved the first important theorem on the subject, which states that "a statement G is planar if and only if b(G)≤2 ".in 1981 and 1982, Schleicher ([8], [16]) found the base number of the perfect statement K_n, the perfect binary hash statement $K_{m,n}$ and a cube-n. The base number of the lexical novelty of the data was calculated in [1], as well as the base number of the Cartesian novelty of some data was studied in [2], and Ali studied [3] the base number of the connection of two data where he proved if $G_1$and $G_1$are two separate data for vertices and each of them has a generating forest with a degree not exceeding 4, then:
$b(G_1 + G_2) \le max\{4, b(G_1) + 1, b(G_2) + 1\}$.
In addition, in [15] the basic number of the tripartite contact of data was studied. In this research, we will study the fundamental number of the quadratic contact of data, where we proved:

$$b(G_1 + G_2 + G_3 + G_4) \le max\{4, b(G_1) + 1, b(G_2) + 2, b(G_3) + 2, b(G_4) + 1\}$$

And

$$b(P_m + P_n + P_P + P_t) = 4, \ \forall m, t \ge 5 \quad and \quad n, p \ge 6$$

The base number of $\boldsymbol{G_1 + G_2 + G_3 + G_4}$:

Suppose that $X = \{x_1, x_2, \ldots, x_m\}$, $Y = \{y_1, y_2, \ldots, y_n\}$, $Z = \{z_1, z_2, \ldots, z_p\}$, $U = \{u_1, u_2, \ldots, u_t\}$ we suppose that $K_{m,n}$, $K_{n,p}$, $K_{p,t}$are the binary hash data whose independent sets, respectively, are *Y,X* , *Y,Z*, *Z,U*. Schmeichel [16] proved that:
$B_r(K_{m,n}) = \{x_i y_j x_{i+1} y_{j+1} | i = 1,2, \ldots, m-1 \ and \ j = 1,2, \ldots, n-1\}$, ..(1)
$B_r(K_{n,p}) = \{y_i z_j y_{i+1} z_{j+1} | i = 1,2, \ldots, n-1 \ and \ j = 1,2, \ldots, p-1\}$, ..(2)
$B_r(K_{p,t}) = \{z_i u_j z_{i+1} u_{j+1} | i = 1,2, \ldots, p-1 \ and \ j = 1,2, \ldots, t-1\}$, ..(3)
Are the required bases for $\xi(K_{m,n})$, $\xi(K_{n,p})$, $\xi(K_{p,t})$ with 4 folds respectively; that is, $b(K_{m,n}) \le 4$, $b(K_{n,p}) \le 4$, $b(K_{p,t}) \le 4$ for each $m, n, p, t \ge 5$; Also, Ali [3] proved that:

$$A_1 = \{x_1y_1, x_1y_n, x_my_1, x_my_n\} \bigcup \{y_1z_1, y_1z_p, y_nz_1, y_nz_p\} \bigcup \{z_1u_1, z_1u_t, z_pu_1, z_pu_t\}$$

..(4)

Are the sets of edges $K_{m,n}$ و , $K_{n,p}$, $K_{p,t}$with a 1-fold in the required base $B_r(K_{m,n})$, $B_r(K_{n,p})$, $B_r(K_{p,t})$ respectively and that:

$$A_2 = \left(\{x_1y_j, x_my_j | j = 2,3, \ldots, n-1\} \bigcup \{x_iy_1, x_iy_n | j = 2,3, \ldots, m-1\}\right)$$
$$\bigcup \left(\{y_1z_j, y_nz_j | j = 2,3, \ldots, p-1\} \bigcup \{y_jz_1, y_jz_p | j = 2,3, \ldots, n-1\}\right)$$
$$\bigcup \left(\{z_1u_j, z_pu_j | j = 2,3, \ldots, t-1\} \bigcup \{z_ju_1, z_ju_t | j = 2,3, \ldots, p-1\}\right)$$

….(5)

Is the set of edges $K_{m,n}$, $K_{n,p}$, $K_{p,t}$ with a fold of 2 in the required base $B_r(K_{m,n})$, $B_r(K_{n,p}$, $B_r(K_{p,t})$ respectively, there is no edge with a fold of 3 in B_r and the rest of the edges are bent 4.

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Now we will study the base number of the quadratic contact of the data, that is, $b(G_1+G_2+G_3+G_4)$ when G_1 , G_2 , G_3 , G_4 are connected statements and have a generating tree with a degree of not more than 4.
Let T_1,T_2, T_3, T_4 be generating trees for G_1 , G_2 , G_3 , G_4 in such an order that for i =1,2,3,4, then:

$$deg_{T_i}u \leq 4, \forall u \in V(T_i)$$

Since every tree has at least two vertices with degree one, we assume that:

$$deg_{T_1}(x_1) = deg_{T_1}(x_m) = deg_{T_2}(y_1) = deg_{T_2}(y_n) = deg_{T_3}(z_1) = deg_{T_3}(z_p) = deg_{T_4}(u_1) = deg_{T_4}(u_t) = 1$$

Now for each edge x_ix_j in the tree T_1 we will choose one triple circuit and only one by the formula $x_ix_jy_1$ or $x_ix_jy_n$ and this is in the following way:
- We divide the set of edges of T_1 into two partial sets, W_1 and W_1', so that each vertex of x in T_1 is the number of edges located on it that belong to the same partial set of not more than 2.
Since $deg_{T_1}(x) \leq 4$ for every x in T_1, this hash is possible. Now, let S_1 be the set of three circuits of T_1 defined in the following form:

$$S_1 = \{x_ix_jy \mid x_ix_j \in E(T_1), y = y_i \ if \ x_ix_j \in W_1 \ and \ y = y_n \ if \ x_ix_j \in W_1'$$

In the same way, we obtain the sets S_2, S_3 ,S_4 from triangular circuits of tree edges T_2,T_3, T_4 from the data G_2,G_3, G_4, respectively, where:
$S_2 = \{xy_iy_j, y_iy_jz \mid y_iy_j \in E(T_2), x = x_1, z = z_1 \ if \ y_iy_j \in W_2 \ and \ x = x_m, z = z_p \ if \ y_iy_j \in W_2'\}$
$S_3 = \{yz_iz_j, z_iz_ju \mid z_iz_j \in E(T_3), y = y_1, u = u_1 \ if \ z_iz_j \in W_3 \ and \ y = y_n, u = u_t \ if \ z_iz_j \in W_3'\}$ and

$$S_4 = \{zu_iu_j \mid u_iu_j \in E(T_4), z = z_1 \ if \ u_iu_j \in W_4 \ and \ z = z_p \ if \ u_iu_j \in W_4'\}$$

It is clear that $|S_1| = m-1$, $|S_2| = 2(n-1)$, $|S_3| = 2(p-1)$, $|S_4| = t-1$ and that
Each of the edges x_iy_1, x_iy_n for the values *i=1,2,…,m* appears at most in two circles in S_1.
And each of the edges y_jx_1, y_jx_m for the values *j=1,2,…,n* appears at most in two circles in S_2.
And each of the edges y_jz_1, y_jz_p for the values *j=1,2,…,n* appears at most in two circles in S_2.
And each of the edges z_jy_1, z_jy_n for the values *j=1,2,…,p* appears at most in two circles in S_3.
And each of the edges z_ju_1, z_ju_t for the values *j=1,2,…,p* appears at most in two circles in S_3.
Finally:
Each of the edges u_jz_1 ،u_jz_p for the values *j=1,2,…,t* appear at most in two circles in S_4
And now, we are ready to prove the basic theorem in this research.

3. Theorem (1)

If G_1 , G_2 , G_3 , G_4 are connected and separate data for the vertices and each of them has a generating tree with a degree of not more than 4, then:
$b(G_1+G_2+G_3+G_4) \leq max\{4, b(G_1)+1, b(G_2)+2, b(G_3)+2, b(G_4)+1\}$.

Proof:

Let $B_r(G_1)$, $B_r(G_2)$,$B_r(G_3)$, $B_r(G_4)$ be the required bases for the data G_1 , G_2 , G_3 , G_4 ,respectively; in addition, let $B_r(K_{m,n})$, $B_r(K_{n,p})$,$B_r(K_{p,t})$ be the required bases for the data $K_{m,n}$, $K_{n,p}$,$K_{p,t}$ are defined in (1), (2) ,(3) respectively.
Let:

$$B = B_r(G_1) \bigcup B_r(G_2) \bigcup B_r(G_3) \bigcup B_r(G_4) \bigcup B_r(K_{m,n}) \bigcup B_r(K_{n,p}) \bigcup B_r(K_{p,t}) \bigcup S_1 \bigcup S_2 \bigcup S_3 \bigcup S_4$$

Now we will prove that B is a norm of the space of circuits $\xi(G_1+G_2+G_3+G_4)$; it is obvious that:
$|B| = dim\xi(G_1) + dim\xi(G_2) + dim\xi(G_3) + dim\xi(G_4) + dim\xi(K_{m,n}) + dim\xi(K_{n,p}) + dim\xi(K_{p,t}) + (m-1) + 2(n-1) + 2(p-1) + (t-1) = (q_1-m+1) + (q_2-n+1) + (q_3-p+1) + (q_4-t+1) + mn - (m+n) + 1 + np - (n+p) + 1 + pt - (p+t) + 1 + m - 1 + 2n - 2 + 2p - 2 + t - 1 \quad = (q_1+q_2+q_3+q_4+mn+np+pt) - (m+n+p+t) + 1$

$$= dim\xi(G_1+G_2+G_3+G_4)$$

Where q_1 , q_2, q_3 , q_4 are the number of edges in G_1 , G_2 , G_3 , G_4 respectively.
We will prove that B is a linearly independent set of circuits; it is obvious that:

$$B_r(G_1) \bigcup B_r(G_2) \bigcup B_r(G_3) \bigcup B_r(G_4) \bigcup B_r(K_{m,n}) \bigcup B_r(K_{n,p}) \bigcup B_r(K_{p,t})$$

A linearly independent set and that both sets S_1, S_2, S_3, S_4 are independent because they represent the boundaries of regions in a planar statement; now if C is any circuit generated by circuits S_1, C has an edge of T_1; this edge does not

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exist in any linear combination of circuits S_2. Therefore, $S_1 \cup S_2$ independent; as well as circuits S_3 be independent of the $S_1 \cup S_2$ because each circuit of S_3 contain on the edge of the T_3; this edge is not present in any circuit of circuits $S_1 \cup S_2$ therefore $S_1 \cup S_2 \cup S_3$ circuits are linearly independent; and so the circuits S_4 independent circuits $S_1 \cup S_2 \cup S_3$ because any circuit in S_4 contain on the edge of T_4; this edge is not present in any linear structure of $S_1 \cup S_2 \cup S_3$ thus be $S_1 \cup S_2 \cup S_3 \cup S_4$ circuits are linearly independent. And for the proof of the set $S = S_1 \cup S_2 \cup S_3 \cup S_4$ is linearly independent of the set.

$$B' = B_r(G_1) \bigcup B_r(G_2) \bigcup B_r(G_3) \bigcup B_r(G_4) \bigcup B_r(K_{m,n}) \bigcup B_r(K_{n,p}) \bigcup B_r(K_{p,t})$$

Note that any circuit generated from S contains edges from G_i for the values of i =1,2,3,4 and also contains edges from $K_{j,k}$ for the values of j=m, n, p and k = n, p, t ; in contrast, any circuit generated from B' contains edges from G_i for the values of i =1,2,3,4 or edges from $K_{j,k}$for the values of j=m, n, p and k = n, p, t; because it is a discrete set for edges, that is:

$$\sum_{i=1}^{|S|} C_i \neq \sum_{j=1}^{|B\prime|} b_j \quad , C_i \in S \ \ and \ \ b_j \in B'$$

Thus, $B = S \cup B'$ is a linearly independent set and therefore a base for $\xi(G_1 + G_2 + G_3 + G_4)$.

Now we calculate the Tuck for each of the edges of the statement; from (5), The Tuck for each of the edges below:

$x_i y_1, x_i y_n \quad , i = 2,3, \dots, m-1,$
$x_1 y_j, x_m y_j \quad , j = 2,3, \dots, n-1,$
$y_1 z_j, y_n z_j \quad , j = 2,3, \dots, p-1,$
$y_j z_1, y_j z_p \quad , j = 2,3, \dots, n-1,$
$z_1 u_j, z_p u_j \quad , j = 2,3, \dots, t-1,$
$z_j u_1, z_j u_t \quad , j = 2,3, \dots, p-1.$

In the next $B_r(K_{m,n}) \cup B_r(K_{n,p}) \cup B_r(K_{p,t})$ is 2 and each edge appears at most at two of $S_1 \cup S_2 \cup S_3 \cup S_4$. Also, from (4) tuck to each of the edges below:

$$x_1y_1, x_1y_n, x_my_1, x_my_n, y_1z_1, y_1z_p, y_nz_1, y_nz_p, z_1u_1, z_1u_t, z_pu_1, z_pu_t$$

Is 1 in $B_r(K_{m,n}) \cup B_r(K_{n,p}) \cup B_r(K_{p,t})$ each edge appears at most at two of $S_1 \cup S_2 \cup S_3 \cup S_4$; therefore:

$$f_B(e) \leq 4 \, , \forall e \in K_{m,n} \bigcup K_{n,p} \bigcup K_{p,t}$$

In addition, it is easy to verify that:

$$f_B(e) \leq \begin{cases} b(G_1) + 1 \ , & \forall e \in G_1, \\ b(G_2) + 2 \ , & \forall e \in G_2, \\ b(G_3) + 2 \ , & \forall e \in G_3, \\ b(G_4) + 1 \ , & \forall e \in G_4, \end{cases}$$

So, the Tuck of base B is $max\{4, b(G_1) + 1, b(G_2) + 2, b(G_3) + 2, b(G_4) + 1\}$

Thus, the proof is done.

4. Result

For every positive integer $m, n, p, t \geq 3$, $b(P_m + P_n + P_p + P_t) \leq 4$, equality is achieved when $m, t \geq 5$and $n, p \geq 6$.

Proof:

From the theorem (1):

$b(P_m + P_n + P_p + P_t) \leq 4 \qquad \dots(6)$

It is easy to prove that $P_m + P_n + P_p + P_t$ is an uneven statement and therefore,according to McLean [13], $b(P_m + P_n + P_p + P_t) \leq 4$ for the values $m, n, p, t \geq 3$.

Suppose that $b(P_m + P_n + P_p + P_t) = 3$; let B_r be the required base for $P_m + P_n + P_p + P_t$, so:

$$|B_r| = |E(P_m + P_n + P_p + P_t)| - |V(P_m + P_n + P_p + P_t)| + 1$$

$$|B_r| = m - 1 + n - 1 + p - 1 + t - 1 + mn + np + pt - (m + n + p + t) + 1 = mn + np + pt - 3$$

Now the number of circuits of length 3 in the required base B_r is:

(m-1) + 2(n -1) + 2(p -1) + t -1= m+ 2n + 2 p + t - 6

And the rest of the circuits with a length of 4 or more, since:

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$$\sum_{i=1}^{dim\xi(G)} |C_i| \leq b.|E(G)|$$

Where $|C_i|$ stands for the length of the circuit C_i and b is the base number of the statement G, then:

$$4[(mn+np+pt-3)-(m+2n+2p+t-6)]+3(m+2n+2p+t-6) \leq 3[(m-1)+(n-1)+(p-1)+(t-1)+mn+np+pt]$$

That is;

$$mn+np+pt+6 \leq 4m+5n+5p+4t$$

Now, we can easily verify that the above inequality does not hold when m, t ≥5 and n, p ≥6, so that:

$b(p_m+p_n+p_p+p_t)=4,\ \forall m,t \geq 5\ \ and\ \ n,p \geq 6$ ..(7)

In addition, by connecting (6) with (7) we get equality.

Observation:

In a similar way to the proof of theorem 1, we prove that if $G_1, G_2, \dots, G_n$ is continuous and discrete data relative to the vertices and each of them has a generating tree with Degree Not exceeding 4, for each n ≥3

$$b(G_1+G_2+\dots+G_n) \leq max\{4, b(G_1)+1, b(G_n)+1, b(G_i)+2 | i=2,3,\dots,n-1\}$$

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