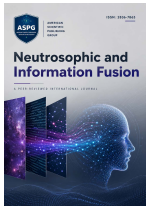
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FULL LENGTH ARTICLE

The Basis Number of Connected Vertex-Disjoint Graphs

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ABSTRACT

The basis number $b(G)$ of a graph G is defined to be the smallest positive integer k such that G has a k-fold basis for its cycle space. We seek an upper bound for $b(G_1+G_2+G_3+G_4)$. We prove that if G_1, G_2, G_3, and G_4 are connected vertex-disjoint graphs and each has a spanning tree of vertex degree at most 4, then

$$b(G_1+G_2+G_3+G_4) \le \max\{4, b(G_1)+1, b(G_2)+2, b(G_3)+2, b(G_4)+1\}.$$

The basis number of the quadruple join of paths is also studied. In particular, we prove that

$$b(P_m+P_n+P_p+P_t)=4, \qquad m,t\ge 5 \text{ and } n,p\ge 6.$$

Keywords: Graph ▪ Basis number ▪ Connected vertex-disjoint graphs ▪ Path

1. INTRODUCTION

Interest in the basis number of graphs has increased in the graph-theory literature; see, for example, Refs. [1, 2, 3, 4, 5, 6, 7, 8]. Throughout this paper, all graphs are assumed to be finite, undirected, and simple; undefined terminology follows Refs. [6, 7].

Let G be a connected graph with edges $e_1, e_2, \ldots, e_q$. Every subset S of the edges of G corresponds to a vector $(a_1, a_2, \ldots, a_q)$, where $a_i = 1$ if $e_i \in S$ and $a_i = 0$ otherwise. These vectors form the vector space $(\mathbb{Z}_2)^q$. The vectors corresponding to the cycles of G generate a subspace called the *cycle space* of G, denoted by $\mathscr{C}(G)$. Each vector in $\mathscr{C}(G)$ represents either a cycle or an edge-disjoint union of cycles.

It is well known that if G has p vertices and q edges, then

$$\dim \mathscr{C}(G) = q - p + 1. \tag{1}$$

A basis for $\mathscr{C}(G)$ can be found as follows. Let T be a spanning tree of G. If $e \in E(G)\setminus E(T)$, then $T+e$ contains a unique cycle, denoted by C_e. The $q-p+1$ cycles C_e, where $e \in E(G)\setminus E(T)$, form a basis for $\mathscr{C}(G)$.

A basis B of $\mathscr{C}(G)$ is called a k-fold basis if every edge of G occurs in at most k cycles of B. The *basis number* of G, denoted by $b(G)$, is the smallest integer k for which $\mathscr{C}(G)$ has a k-fold basis. If B is a basis for $\mathscr{C}(G)$ and $e \in E(G)$, the fold of e in B is

$$f_B(e) = \big|\{C \in B : e \in E(C)\}\big|. \tag{2}$$

2. PRELIMINARIES

Definition 2.1. Let $G_1 = (X, E_1)$ and $G_2 = (Y, E_2)$ be simple vertex-disjoint graphs. Their join $G_1 + G_2$ has vertex set $X \cup Y$ and edge set

$$E_1 \cup E_2 \cup \{xy : x \in X,\ y \in Y\}. \tag{3}$$

Similarly, for vertex-disjoint graphs $G_1 = (X, E_1)$, $G_2 = (Y, E_2)$, and $G_3 = (Z, E_3)$, the ternary join $G_1 + G_2 + G_3$ has

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vertex set $X \cup Y \cup Z$ and edge set

$$E_1 \cup E_2 \cup E_3 \cup \{xy : x \in X,\ y \in Y\} \cup \{yz : y \in Y,\ z \in Z\}. \quad (4)$$

The quadruple join $G_1 + G_2 + G_3 + G_4$, where $G_4 = (U, E_4)$, has vertex set $X \cup Y \cup Z \cup U$ and edge set

$$E_1 \cup E_2 \cup E_3 \cup E_4 \cup \{xy : x \in X,\ y \in Y\} \cup \{yz : y \in Y,\ z \in Z\} \cup \{zu : z \in Z,\ u \in U\}. \quad (5)$$

For example, $P_4 = K_1 + K_1 + K_1 + K_1$. Figure 1 shows the quadruple join $K_2 + K_2 + K_2 + K_2$. More generally, the join $G_1 + G_2 + \cdots + G_n$ is defined in the analogous way and is clearly connected. We use P_n to denote the path on n vertices.

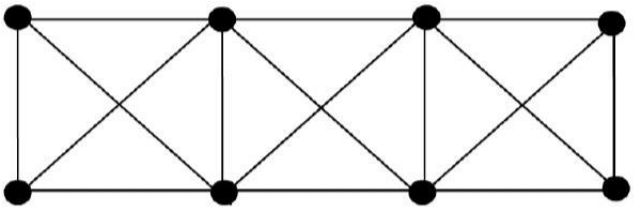


Figure 1. The quadruple join $K_2 + K_2 + K_2 + K_2$.

In 1937, MacLane [9] proved the first important theorem on the subject: a graph G is planar if and only if $b(G) \le 2$. Schmeichel and Banks [5, 10] determined the basis numbers of the complete graph K_n, the complete bipartite graph $K_{m,n}$, and the n-cube. The basis numbers of lexicographic products and Cartesian products were studied in Refs. [11, 12]. Ali [13] proved that if G_1 and G_2 are vertex-disjoint graphs and each has a spanning tree of maximum degree at most 4, then

$$b(G_1 + G_2) \le \max\{4, b(G_1) + 1, b(G_2) + 1\}. \quad (6)$$

The basis number of the ternary join was considered in Ref. [14]. Here we study the basis number of the quadruple join and establish

$$b(G_1 + G_2 + G_3 + G_4) \le \max\{4, b(G_1) + 1, b(G_2) + 2, b(G_3) + 2, b(G_4) + 1\}, \quad (7)$$

as well as

$$b(P_m + P_n + P_p + P_t) = 4 \quad (8)$$

whenever $m, t \ge 5$ and $n, p \ge 6$.

3. THE BASIS NUMBER OF $G_1 + G_2 + G_3 + G_4$

Let

$$X = \{x_1, x_2, \ldots, x_m\}, \quad Y = \{y_1, y_2, \ldots, y_n\},$$

$$Z = \{z_1, z_2, \ldots, z_p\}, \quad U = \{u_1, u_2, \ldots, u_t\}.$$

Let $K_{m,n}$, $K_{n,p}$, and $K_{p,t}$ be complete bipartite graphs whose independent sets are, respectively, X, Y; Y, Z; and Z, U. Schmeichel [10] proved that the following are 4-fold bases for their cycle spaces:

$$\mathscr{B}_r(K_{m,n}) = \{x_i y_j x_{i+1} y_{j+1} : 1 \le i \le m-1,\ 1 \le j \le n-1\}, \quad (9)$$

$$\mathscr{B}_r(K_{n,p}) = \{y_i z_j y_{i+1} z_{j+1} : 1 \le i \le n-1,\ 1 \le j \le p-1\}, \quad (10)$$

$$\mathscr{B}_r(K_{p,t}) = \{z_i u_j z_{i+1} u_{j+1} : 1 \le i \le p-1,\ 1 \le j \le t-1\}. \quad (11)$$

Consequently, $b(K_{m,n}), b(K_{n,p}), b(K_{p,t}) \le 4$ whenever $m, n, p, t \ge 5$.

Ali [13] also showed that the set of edges of fold 1 in these bases is

$$A_1 = \{x_1y_1, x_1y_n, x_my_1, x_my_n\} \cup \{y_1z_1, y_1z_p, y_nz_1, y_nz_p\} \cup \{z_1u_1, z_1u_t, z_pu_1, z_pu_t\}, \quad (12)$$

whereas the set of edges of fold 2 is

$$A_2 = \{x_1y_j, x_my_j : 2 \le j \le n-1\} \cup \{x_iy_1, x_iy_n : 2 \le i \le m-1\} \cup \{y_1z_j, y_nz_j : 2 \le j \le p-1\} \cup \{y_iz_1, y_iz_p : 2 \le i \le n-1\} \cup \{z_1u_j, z_pu_j : 2 \le j \le t-1\} \cup \{z_iu_1, z_iu_t : 2 \le i \le p-1\}. \quad (13)$$

No edge has fold 3 in these bases; all remaining edges have fold 4.

We now consider $b(G_1 + G_2 + G_3 + G_4)$ when the G_i are connected and each has a spanning tree of maximum degree at most 4. Let T_1, T_2, T_3, T_4 be spanning trees for G_1, G_2, G_3, G_4, respectively, such that

$$\deg_{T_i}(v) \le 4, \qquad v \in V(T_i), \quad i = 1, 2, 3, 4. \quad (14)$$

Every tree has at least two vertices of degree one, so we may label the vertices such that

$$\deg_{T_1}(x_1) = \deg_{T_1}(x_m) = \deg_{T_2}(y_1) = \deg_{T_2}(y_n) = \deg_{T_3}(z_1) = \deg_{T_3}(z_p) = \deg_{T_4}(u_1) = \deg_{T_4}(u_t) = 1. \quad (15)$$

Partition $E(T_1)$ into W_1 and W_1' so that at most two edges incident with any vertex lie in either part. Such a partition exists because $\deg_{T_1}(x) \le 4$. Define

$$S_1 = \{x_ix_jy : x_ix_j \in E(T_1),\ y = y_1 \text{ if } x_ix_j \in W_1,\ y = y_n \text{ if } x_ix_j \in W_1'\}. \quad (16)$$

Using analogous partitions $W_i \cup W_i' = E(T_i)$, define

$$S_2 = \{xy_iy_j,\ y_iy_jz : y_iy_j \in E(T_2),\ x = x_1, z = z_1 \text{ if } y_iy_j \in W_2,\ x = x_m, z = z_p \text{ if } y_iy_j \in W_2'\}. \quad (17)$$

$$S_3 = \{yz_iz_j,\ z_iz_ju : z_iz_j \in E(T_3),\\ y = y_1, u = u_1 \text{ if } z_iz_j \in W_3,\\ y = y_n, u = u_t \text{ if } z_iz_j \in W_3'\}. \tag{18}$$

$$S_4 = \{zu_iu_j : u_iu_j \in E(T_4),\\ z = z_1 \text{ if } u_iu_j \in W_4,\\ z = z_p \text{ if } u_iu_j \in W_4'\}. \tag{19}$$

Then

$$|S_1| = m-1, \qquad |S_2| = 2(n-1),\\ |S_3| = 2(p-1), \qquad |S_4| = t-1. \tag{20}$$

Moreover, each edge x_iy_1 or x_iy_n occurs in at most two cycles of S_1; each edge y_jx_1 or y_jx_m, and each edge y_jz_1 or y_jz_p, occurs in at most two cycles of S_2; each edge z_jy_1 or z_jy_n, and each edge z_ju_1 or z_ju_t, occurs in at most two cycles of S_3; and each edge u_jz_1 or u_jz_p occurs in at most two cycles of S_4.

Theorem 3.1. *If G_1, G_2, G_3, G_4 are connected vertex-disjoint graphs and each has a spanning tree of maximum degree at most 4, then*

$$b(G_1+G_2+G_3+G_4)\\ \le \max\{4, b(G_1)+1, b(G_2)+2, b(G_3)+2, b(G_4)+1\}. \tag{21}$$

Proof. Let $\mathscr{B}_r(G_i)$ be a minimum-fold basis of $\mathscr{C}(G_i)$, $i = 1,2,3,4$, and let the bases of the three complete bipartite graphs be those in Eqs. (9)–(11). Put

$$B = \mathscr{B}_r(G_1)\cup\mathscr{B}_r(G_2)\cup\mathscr{B}_r(G_3)\cup\mathscr{B}_r(G_4)\\ \cup\mathscr{B}_r(K_{m,n})\cup\mathscr{B}_r(K_{n,p})\cup\mathscr{B}_r(K_{p,t})\\ \cup S_1\cup S_2\cup S_3\cup S_4. \tag{22}$$

First, the cardinality of B is

$$\begin{aligned}|B| &= \sum_{i=1}^{4}\dim\mathscr{C}(G_i)+\dim\mathscr{C}(K_{m,n})\\ &\quad+\dim\mathscr{C}(K_{n,p})+\dim\mathscr{C}(K_{p,t})\\ &\quad+(m-1)+2(n-1)+2(p-1)+(t-1)\\ &= (q_1-m+1)+(q_2-n+1)\\ &\quad+(q_3-p+1)+(q_4-t+1)\\ &\quad+mn-(m+n)+1+np-(n+p)+1\\ &\quad+pt-(p+t)+1+m-1+2n-2+2p-2+t-1\\ &= q_1+q_2+q_3+q_4+mn+np+pt\\ &\quad-(m+n+p+t)+1\\ &= \dim\mathscr{C}(G_1+G_2+G_3+G_4),\end{aligned} \tag{23}$$

where $q_i = |E(G_i)|$.

The union of the seven bases in the first two lines of Eq. (22) is linearly independent. Each S_i is independent because its cycles can be viewed as boundaries of regions in a planar subgraph. Furthermore, every nonzero cycle combination generated by S_i contains an edge of T_i that does not occur in any combination generated by the other S_j. Hence $S_1\cup S_2\cup S_3\cup S_4$ is linearly independent.

Let

$$B' = \bigcup_{i=1}^{4}\mathscr{B}_r(G_i)\cup\mathscr{B}_r(K_{m,n})\cup\mathscr{B}_r(K_{n,p})\cup\mathscr{B}_r(K_{p,t}). \tag{24}$$

Any nonzero combination generated by $S = S_1\cup S_2\cup S_3\cup S_4$ contains both an edge internal to some G_i and an edge of a corresponding complete bipartite graph, whereas a combination generated by B' decomposes within the individual constituent graphs. Thus

$$\sum_{i=1}^{|S|}C_i \ne \sum_{j=1}^{|B'|}b_j, \qquad C_i \in S, \quad b_j \in B', \tag{25}$$

for nontrivial combinations. Therefore, $B = S\cup B'$ is linearly independent. Together with Eq. (23), this proves that B is a basis of $\mathscr{C}(G_1+G_2+G_3+G_4)$.

It remains to calculate its fold. From Eq. (13), each edge in the following families has fold 2 in $\mathscr{B}_r(K_{m,n})\cup\mathscr{B}_r(K_{n,p})\cup\mathscr{B}_r(K_{p,t})$:

$$\begin{aligned}&x_iy_1, x_iy_n && (2\le i\le m-1),\\ &x_1y_j, x_my_j && (2\le j\le n-1),\\ &y_1z_j, y_nz_j && (2\le j\le p-1),\\ &y_jz_1, y_jz_p && (2\le j\le n-1),\\ &z_1u_j, z_pu_j && (2\le j\le t-1),\\ &z_ju_1, z_ju_t && (2\le j\le p-1).\end{aligned}$$

Every such edge occurs in at most two cycles of $S_1\cup S_2\cup S_3\cup S_4$. By Eq. (12), each corner edge has fold 1 in the three bipartite bases and again occurs at most twice in the S_i. All remaining bipartite edges have fold at most 4 and do not occur in any S_i. Consequently,

$$f_B(e) \le 4, \qquad e \in E(K_{m,n})\cup E(K_{n,p})\cup E(K_{p,t}). \tag{26}$$

For edges internal to the four original graphs,

$$f_B(e) \le \begin{cases} b(G_1)+1, & e\in E(G_1),\\ b(G_2)+2, & e\in E(G_2),\\ b(G_3)+2, & e\in E(G_3),\\ b(G_4)+1, & e\in E(G_4).\end{cases} \tag{27}$$

Thus the fold of B is at most

$$\max\{4, b(G_1)+1, b(G_2)+2, b(G_3)+2, b(G_4)+1\},$$

which completes the proof. □

4. QUADRUPLE JOINS OF PATHS

Result 4.1. *For all integers $m,n,p,t \ge 3$,*

$$b(P_m+P_n+P_p+P_t) \le 4. \tag{28}$$

Equality holds when $m,t \ge 5$ and $n,p \ge 6$.

Proof. The upper bound in Eq. (28) follows immediately from Theorem 3.1. The graph $P_m+P_n+P_p+P_t$ is nonplanar,

and hence MacLane's theorem [9] gives the lower bound $b(P_m+P_n+P_p+P_t)\geq 3$.

Suppose that $b(P_m+P_n+P_p+P_t)=3$, and let $\mathscr{B}_r$ be a 3-fold basis. Its size is

$$\begin{aligned}|\mathscr{B}_r| &= |E(P_m+P_n+P_p+P_t)| \\ &\quad -|V(P_m+P_n+P_p+P_t)|+1 \\ &= m-1+n-1+p-1+t-1 \\ &\quad +mn+np+pt-(m+n+p+t)+1 \\ &= mn+np+pt-3. \end{aligned} \tag{29}$$

The number of 3-cycles in this basis is

$$(m-1)+2(n-1)+2(p-1)+(t-1)=m+2n+2p+t-6. \tag{30}$$

All remaining basis cycles have length at least 4. For any graph G with basis cycles C_i and basis number b,

$$\sum_{i=1}^{\dim\mathscr{C}(G)} |C_i| \leq b\,|E(G)|. \tag{31}$$

Applying this inequality with $b=3$ yields

$$\begin{aligned}4\big[(mn+np+pt-3)-(m+2n+2p+t-6)\big] &\\ +3(m+2n+2p+t-6) &\\ \leq 3\big[(m-1)+(n-1)+(p-1)+(t-1) &\\ +mn+np+pt\big]. \end{aligned} \tag{32}$$

Equivalently,

$$mn+np+pt+6\leq 4m+5n+5p+4t. \tag{33}$$

Inequality (33) fails whenever $m,t\geq 5$ and $n,p\geq 6$. Therefore,

$$b(P_m+P_n+P_p+P_t)=4 \tag{34}$$

for these values, proving the result. □

Observation. The proof of Theorem 3.1 extends directly to n connected vertex-disjoint graphs $G_1,G_2,\ldots,G_n$, each having a spanning tree of maximum degree at most 4. For every $n\geq 3$,

$$\begin{aligned}b(G_1+G_2+\cdots+G_n) &\\ \leq \max\big(\{4,b(G_1)+1,b(G_n)+1\} &\\ \cup\{b(G_i)+2 : 2\leq i\leq n-1\}\big). \end{aligned} \tag{35}$$

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