



Finding new similarities measures for Type-II Diophantine neutrosophic interval valued soft sets and its basic operations

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Abstract

The Type-II Diophantine neutrosophic interval valued soft set (Type-II DioNSIVSS) and related similarity measure are presented in this study. An extension of the neutrosophic interval valued soft set (NSIVSS) and the Diophantine fuzzy soft set is the Type-II DioNSIVSS. The suggested measure for Type-II DioNSIVSS assessment. We support a method of solving the problem using the Type-II soft set model. To demonstrate how they can be applied to successfully handle uncertainty-related challenges, illustrative examples are given.

Keywords: Type-II DioNSIVS set; PyNSIVS set; decision making problem

1 Introduction

Ever since Zadeh's fuzzy set (FS) demonstration, they have been incredibly popular in almost every scientific field. This suggests that while dealing with ambiguous situations, decision makers should consider membership degree (MD).¹ An intuitionistic fuzzy set (IFS) was defined by Atanassov.² An IFS can be identified by an MD and non-membership degree (NMD) that satisfy the condition that the sum of their MD and NMD be less than or equal to unity. On the other hand, we can have DM issues if the total of the MD and NMD for a given feature is more than unity. The concept of Pythagorean FS (PFS) was introduced by Yager.³ The condition that it not exceed unity indicates that the square summation of its MD and NMD has been extended from IFSs. FSs and IFS extensions and the notion of picture fuzzy sets.⁴ Cuong generated image fuzzy sets in 2015. Fuzzy picture sets could be helpful in managing human opinions with multiple possible responses, such as accept, abstain, refuse, and so on. Four types of voters exist in the human race: those who cast ballots for the candidate, those who abstain, those who cast ballots against the candidate, and those who choose not to. Voting is but one instance. Decision-makers have a greater range of options when assigning MD, NMD, and reluctance degrees using these sets. Cuong et al.,⁵ the authors of image FS logic, used three points: positive, neutral, and negative MD, with the total of these grades not going above 1. In the end, it is superior to IFS and PFS in a few specific situations. Since there are several application-related issues with this collection. Neutrosophy was founded by Smarandache⁶ in order to deal with the problem of unclear and conflicting information. Recently, NS logic and sets, a completely new theory, was proposed. Neutrosophy is the study of neutral cognition; the primary difference between IFS and FS is this neutrality. The neutrosophic (NS) set, which spans $[0, 1]$ in the NS set, is an additional generalization of the FS and IFS. Philosophically, it is well known that a classical set, like FS, IVFS, etc., can be generalized into an NS set. Jansi et al. have recently examined Pythagorean NS sets.⁷ The

possibility of a fuzzy soft set, as proposed by Alkhezaleh and associates,⁸ is a novel idea with useful applications. Palanikumar et al.⁹⁻¹² recently covered a wide range of algebraic structures and their applications. The most significant contribution of Molodtsov¹³ was the theory of soft sets. Soft sets fit the objectivity and complexity of decision-making in real-world circumstances better than earlier uncertain theories. Furthermore, it is imperative to conduct research on the integration of soft sets with other mathematical models. Maji proposed the notions of intuitionistic fuzzy soft set (IFSS)¹⁴ and fuzzy soft set (FSS).¹⁵ Many different DM problems are handled by these two ideas. In Yang,¹⁶ the fuzzy soft set was covered. In recent years, Peng¹⁷ has extended FSS to include Pythagorean fuzzy soft set. A set of MADM problems where the sum of the squares is equal to or less than one but the sum of the MD and NMD is larger than one were handled using this methodology. Covered in Majumdar¹⁸ transactions was generic FSS. We talked about the concept of a generalized interval valued fuzzy soft set in Shawkat et al.¹⁹ To demonstrate the feasibility of neutrosophic soft sets. Karaaslan proposed a DM method.²⁰ This article uses the soft set model to parameterize the Type-II DioNSIVSS, extending the idea of IVFSSs. Next, using this soft set model as a foundation, we will construct a similarity measure. After defining relations on Type-II DioNSIVS sets and examining their properties, an application is addressed involving decision making. Several researchers have investigated practical uses of the neutrosophic set.^{21,22}

The present study proposes at investigating the Type-II DioNSIVS set. The five sections listed below make up the article. Introduction is dealt with in the first part 1. Section 2 presents the main concepts. The Type-II DioNSIVFSS is conceptualized in Section 3. Using the Type-II DioNSIVFS approach, the similarity measure is set in Section 4. Give some numerical examples for the Type-II DioNSIVS set model in the scenario that you are evaluating. Section 5 deals with the conclusion of the paper.

2 Preliminaries

The generalized fuzzy soft set and the Pythagorean neutrosophic set are two well-known ideas in literature that are reviewed and briefly discussed in this section. Here, X be the universal set.

Definition 2.1.⁷ The Pythagorean neutrosophic interval valued (PyNSIVS) P in X is $\bar{P} = \{\mathfrak{R}_P(h), \mathfrak{S}_P(h), \mathfrak{U}_P(h) | h \in X\}$, where $\mathfrak{R}_P(h) = [\mathfrak{R}_P^l(h), \mathfrak{R}_P^u(h)]$ and $\mathfrak{S}_P(h) = [\mathfrak{S}_P^l(h), \mathfrak{S}_P^u(h)]$ and $\mathfrak{U}_P(h) = [\mathfrak{U}_P^l(h), \mathfrak{U}_P^u(h)]$ represent the degree of truth, indeterminacy and falsity membership of P , respectively. The map $\mathfrak{R}_P : X \rightarrow D[0, 1]$, $\mathfrak{S}_P : X \rightarrow D[0, 1]$, $\mathfrak{U}_P : X \rightarrow D[0, 1]$ and $0 \leq (\mathfrak{R}_P(h))^2 + (\mathfrak{S}_P(h))^2 + (\mathfrak{U}_P(h))^2 \leq 2$ and $0 \leq (\mathfrak{R}_P^u(h))^2 + (\mathfrak{S}_P^u(h))^2 + (\mathfrak{U}_P^u(h))^2 \leq 2$, where $\bar{P} = \langle [\mathfrak{R}_P^l, \mathfrak{R}_P^u], [\mathfrak{S}_P^l, \mathfrak{S}_P^u], [\mathfrak{U}_P^l, \mathfrak{U}_P^u] \rangle$ is called a Pythagorean neutrosophic interval valued number (PyNSIVN).

Definition 2.2.⁷ Let $\bar{\sigma}_1 = \langle \mathfrak{R}_{\bar{\sigma}_1}, \mathfrak{S}_{\bar{\sigma}_1}, \mathfrak{U}_{\bar{\sigma}_1} \rangle$, $\bar{\sigma}_2 = \langle \mathfrak{R}_{\bar{\sigma}_2}, \mathfrak{S}_{\bar{\sigma}_2}, \mathfrak{U}_{\bar{\sigma}_2} \rangle$ and $\bar{\sigma}_3 = \langle \mathfrak{R}_{\bar{\sigma}_3}, \mathfrak{S}_{\bar{\sigma}_3}, \mathfrak{U}_{\bar{\sigma}_3} \rangle$ are any three PyNSIVNs over (X, E) . Then

- (i) $\bar{\sigma}_1^c = \langle \mathfrak{U}_{\bar{\sigma}_1}, \mathfrak{S}_{\bar{\sigma}_1}, \mathfrak{R}_{\bar{\sigma}_1} \rangle$
- (ii) $\bar{\sigma}_1 \sqcup \bar{\sigma}_2 = \langle \max(\mathfrak{R}_{\bar{\sigma}_1}, \mathfrak{R}_{\bar{\sigma}_2}), \min(\mathfrak{S}_{\bar{\sigma}_1}, \mathfrak{S}_{\bar{\sigma}_2}), \min(\mathfrak{U}_{\bar{\sigma}_1}, \mathfrak{U}_{\bar{\sigma}_2}) \rangle$
- (iii) $\bar{\sigma}_1 \sqcap \bar{\sigma}_2 = \langle \min(\mathfrak{R}_{\bar{\sigma}_1}, \mathfrak{R}_{\bar{\sigma}_2}), \min(\mathfrak{S}_{\bar{\sigma}_1}, \mathfrak{S}_{\bar{\sigma}_2}), \max(\mathfrak{U}_{\bar{\sigma}_1}, \mathfrak{U}_{\bar{\sigma}_2}) \rangle$
- (iv) $\bar{\sigma}_1 \preceq \bar{\sigma}_2$ if and only if $\mathfrak{R}_{\bar{\sigma}_1} \preceq \mathfrak{R}_{\bar{\sigma}_2}$ and $\mathfrak{S}_{\bar{\sigma}_1} \preceq \mathfrak{S}_{\bar{\sigma}_2}$ and $\mathfrak{U}_{\bar{\sigma}_1} \succeq \mathfrak{U}_{\bar{\sigma}_2}$
- (v) $\bar{\sigma}_1 = \bar{\sigma}_2$ if and only if $\mathfrak{R}_{\bar{\sigma}_1} = \mathfrak{R}_{\bar{\sigma}_2}$ and $\mathfrak{S}_{\bar{\sigma}_1} = \mathfrak{S}_{\bar{\sigma}_2}$ and $\mathfrak{U}_{\bar{\sigma}_1} = \mathfrak{U}_{\bar{\sigma}_2}$.

Definition 2.3.¹⁹ Let $X = \{h_1, h_2, \dots, h_n\}$ and $E = \{b_1, b_2, \dots, b_m\}$ called the universal set and set of parameter and (X, E) is called as soft universe. The map $\bar{U} : E \rightarrow D(I)^X$ and $\bar{\ell}$ be an interval-valued fuzzy subset of E , ie. $\bar{\ell} : E \rightarrow I = D[0, 1]$. Let $\bar{U}_\ell : E \rightarrow D(I)^X \times D(I)$ and $\bar{U}_\ell(b) = (\bar{U}(b)(h), \bar{\ell}(b))$, $\forall h \in X$. Then $\bar{U}_\ell$ is called as generalized IVFSS (GIVFSS). For each parameter b^i , $\bar{U}_\ell(b^i) = (\bar{U}(b^i)(h), \bar{\ell}(b^i))$, $\forall h \in X$.

Definition 2.4.⁸ Let $X = \{h_1, h_2, \dots, h_n\}$ and $E = \{b_1, b_2, \dots, b_m\}$ be the universe and set of parameter. Let $\bar{U} : E \rightarrow \bar{U}(X)$ and $\bar{\ell}$ be an IVF subset of E , ie. $\bar{\ell} : E \rightarrow \bar{U}(X)$. Let $\bar{U}_\ell : E \rightarrow \bar{U}(X) \times \bar{U}(X)$ and defined as $\bar{U}_\ell(b) = (\bar{U}(b)(h), \bar{\ell}(b)(h))$, $\forall h \in X$. Then $\bar{U}_\ell$ is called a possibility IVFSS (PIVFSS) on (X, E) . For b^i , $\bar{U}_\ell(b^i) = (\bar{U}(b^i)(h), \bar{\ell}(b^i)(h))$.

Hence, $\bar{U}_\ell(b^i) = \left\{ \left(\frac{h_1}{\bar{U}(b^i)(h_1)}, \bar{\ell}(b^i)(h_1) \right), \left(\frac{h_2}{\bar{U}(b^i)(h_2)}, \bar{\ell}(b^i)(h_2) \right), \dots, \left(\frac{h_n}{\bar{U}(b^i)(h_n)}, \bar{\ell}(b^i)(h_n) \right) \right\}$.

3 Type-II DioNSIVFS set

The idea of Type-II Diophantine neutrosophic interval valued soft sets is being started.

Definition 3.1. Let $X = \{h_1, h_2, \dots, h_n\}$ and $E = \{b_1, b_2, \dots, b_m\}$ be the universal and set of parameter, respectively and (X, E) is represent a soft universe. The map $\bar{U} : E \rightarrow S\bar{U}(X)$ and $\bar{p}$ is a interval-valued neutrosophic subset of E and $\bar{p} : E \rightarrow D[0, 1]$, $S\bar{U}(X)$ denotes the collection of all PyNSIV subsets of X . If $\bar{U}_p : E \rightarrow S\bar{U}(X) \times D[0, 1]$ and defined as $\bar{U}_p(b) = (\bar{U}(b)(h), \bar{p}(b))$, $h \in X$, then $\bar{U}_p$ is a Type-II DioNSIVSS on (X, E) . For each parameter e ,

$$\bar{U}_p(b^i) = \left\{ \frac{h_1}{(\Re_{\bar{U}(b)}(h_1), \Im_{\bar{U}(b)}(h_1), \mathfrak{U}_{\bar{U}(b)}(h_1)), (\rho_1, \varrho_1, \sigma_1)}, \dots, \frac{h_n}{(\Re_{\bar{U}(b)}(h_n), \Im_{\bar{U}(b)}(h_n), \mathfrak{U}_{\bar{U}(b)}(h_n)), (\rho_n, \varrho_n, \sigma_n)} \right\},$$

$$(\bar{p}_1(b^i), \bar{p}_2(b^i), \bar{p}_3(b^i)).$$

Example 3.2. Let $X = \{h_1, h_2, h_3\}$ and $E = \{b_1, b_2, b_3\}$. Consider $\bar{U}_p : E \rightarrow S\bar{U}(X) \times D[0, 1]$ is defined as

$$\bar{U}_p(b_1) = \left(\left\{ \frac{h_1}{([0.65, 0.75], [0.35, 0.40], [0.75, 0.90]), (0.45, 0.3, 0.2)}, \frac{h_2}{([0.80, 0.85], [0.35, 0.45], [0.70, 0.75]), (0.5, 0.4, 0.1)}, \frac{h_3}{([0.60, 0.65], [0.50, 0.55], [0.70, 0.80]), (0.35, 0.25, 0.3)} \right\}, ([0.70, 0.80], [0.70, 0.75], [0.40, 0.45]) \right);$$

$$\bar{U}_p(b_2) = \left(\left\{ \frac{h_1}{([0.60, 0.65], [0.40, 0.45], [0.80, 0.90]), (0.25, 0.35, 0.4)}, \frac{h_2}{([0.65, 0.75], [0.50, 0.60], [0.70, 0.80]), (0.3, 0.2, 0.5)}, \frac{h_3}{([0.60, 0.70], [0.40, 0.50], [0.75, 0.90]), (0.6, 0.2, 0.2)} \right\}, ([0.50, 0.60], [0.40, 0.50], [0.60, 0.70]) \right);$$

$$\bar{U}_p(b_3) = \left(\left\{ \frac{h_1}{([0.35, 0.40], [0.60, 0.65], [0.80, 0.90]), (0.4, 0.4, 0.2)}, \frac{h_2}{([0.45, 0.50], [0.70, 0.75], [0.65, 0.70]), (0.65, 0.25, 0.1)}, \frac{h_3}{([0.35, 0.40], [0.45, 0.55], [0.75, 0.80]), (0.4, 0.4, 0.2)} \right\}, ([0.55, 0.70], [0.50, 0.60], [0.60, 0.70]) \right)$$

Definition 3.3. Let $\bar{U}_p$ and $\bar{V}_q$ be the Type-II DioNSIVS sets on (X, E) . Now $\bar{U}_q$ is a subset of $\bar{V}_q$ if and only if

- (i) $\bar{U}(b)(h) \subseteq \bar{V}(b)(h)$ if $\Re_{\bar{U}(b)}(h) \preceq \Re_{\bar{V}(b)}(h)$, $\Im_{\bar{U}(b)}(h) \preceq \Im_{\bar{V}(b)}(h)$, $\mathfrak{U}_{\bar{U}(b)}(h) \succeq \mathfrak{U}_{\bar{V}(b)}(h)$,
- (ii) $\bar{p}(b) \preceq \bar{q}(b)$, $\forall b \in E$ and $\forall h \in X$.

4 Finding new type of similarity measure

This section provides the process for determining the similarity measure between Type-II DioNSIVS sets.

Definition 4.1. Let $\bar{P}_p$ and $\bar{Q}_q$ be the Type-II DioNSIVS sets on (X, E) . The similarity measure between two Type-II DioNSIVS sets $\bar{P}_p$ and $\bar{Q}_q$ is defined as $Sim(\bar{P}_p, \bar{Q}_q) = \varphi(\bar{P}, \bar{Q}) \cdot \psi(\bar{p}, \bar{q})$, where $\varphi(\bar{P}, \bar{Q}) =$

$$\frac{1}{m} \sum_{j=1}^m \left[\min \left\{ T_1^l(\bar{P}(b)(h^j), \bar{Q}(b)(h^j)), T_2^l(\bar{P}(b)(h^j), \bar{Q}(b)(h^j)), S^l(\bar{P}(b)(h^j), \bar{Q}(b)(h^j)) \right\}, \right. \\ \left. \max \left\{ T_1^u(\bar{P}(b)(h^j), \bar{Q}(b)(h^j)), T_2^u(\bar{P}(b)(h^j), \bar{Q}(b)(h^j)), S^u(\bar{P}(b)(h^j), \bar{Q}(b)(h^j)) \right\} \right]$$

and

$$\bar{T}_1(\bar{P}(b)(h^j), \bar{Q}(b)(h^j)) = \left[\frac{\sum_{i=1}^n (\rho_1 \Re_{P(b^i)}^l(h^j) \cdot \rho_2 \Re_{Q(b^i)}^l(h^j))}{\sum_{i=1}^n \left(1 - \sqrt{(1 - (\rho_1)^2 \Re_{P(b^i)}^{2l}(h^j)) \cdot (1 - (\rho_2)^2 \Re_{Q(b^i)}^{2l}(h^j))} \right)}, \frac{\sum_{i=1}^n (\rho_1 \Re_{P(b^i)}^u(h^j) \cdot \rho_2 \Re_{Q(b^i)}^u(h^j))}{\sum_{i=1}^n \left(1 - \sqrt{(1 - (\rho_1)^2 \Re_{P(b^i)}^{2u}(h^j)) \cdot (1 - (\rho_2)^2 \Re_{Q(b^i)}^{2u}(h^j))} \right)} \right]$$

$$\bar{T}_2(\bar{P}(b)(h^j), \bar{Q}(b)(h^j)) = \left[\frac{\sum_{i=1}^n ((\varrho_1)^2 \Im_{P(b^i)}^{2l}(h^j) \cdot (\varrho_2)^2 \Im_{Q(b^i)}^{2l}(h^j))}{\sum_{i=1}^n \left(1 - \sqrt{(1 - (\varrho_1)^4 \Im_{P(b^i)}^{4l}(h^j)) \cdot (1 - (\varrho_2)^4 \Im_{Q(b^i)}^{4l}(h^j))} \right)}, \frac{\sum_{i=1}^n ((\varrho_1)^2 \Im_{P(b^i)}^{2u}(h^j) \cdot (\varrho_2)^2 \Im_{Q(b^i)}^{2u}(h^j))}{\sum_{i=1}^n \left(1 - \sqrt{(1 - (\varrho_1)^4 \Im_{P(b^i)}^{4u}(h^j)) \cdot (1 - (\varrho_2)^4 \Im_{Q(b^i)}^{4u}(h^j))} \right)} \right]$$

$$\bar{S}(\bar{P}(b)(h^j), \bar{Q}(b)(h^j)) =$$

$$\left[1 - \sqrt{\left| \frac{\sum_{i=1}^n (\sigma_1^2 \mathfrak{U}_{P(b^i)}^{2l}(h^j) - (\sigma_2)^2 \mathfrak{U}_{Q(b^i)}^{2l}(h^j))}{\sum_{i=1}^n 1 + (\sigma_1^2 \mathfrak{U}_{P(b^i)}^{2u}(h^j) \cdot (\sigma_2)^2 \mathfrak{U}_{Q(b^i)}^{2u}(h^j))}, \frac{\sum_{i=1}^n (\sigma_1^2 \mathfrak{U}_{P(b^i)}^{2u}(h^j) - (\sigma_2)^2 \mathfrak{U}_{Q(b^i)}^{2l}(h^j))}{\sum_{i=1}^n 1 + (\sigma_1^2 \mathfrak{U}_{P(b^i)}^{2l}(h^j) \cdot (\sigma_2)^2 \mathfrak{U}_{Q(b^i)}^{2l}(h^j))} \right|} \right],$$

where $j = 1, 2, \dots, m$.

$$\text{and } \psi(\bar{p}, \bar{q}) = 1 - \left[\frac{\sum_{i=1}^n \min \left\{ \left[p^l(b^i), q^l(b^i) \right] \right\}}{\sum_{i=1}^n (p^u(b^i) + q^u(b^i))} \right], \quad \frac{\sum_{i=1}^n \max \left\{ \left[p^u(b^i), q^u(b^i) \right] \right\}}{\sum_{i=1}^n (p^l(b^i) + q^l(b^i))} \right]$$

Theorem 4.2. Let $\bar{P}_p$, $\bar{Q}_q$ and $\bar{R}_r$ be the any three Type-II DioNSIVS sets over (X, E) . Prove that $\bar{P}_p \sqsubseteq \bar{Q}_q \sqsubseteq \bar{R}_r \implies \text{Sim}(\bar{P}_p, \bar{R}_r) \preceq \text{Sim}(\bar{Q}_q, \bar{R}_r)$.

Proof. For $j = 1, 2, \dots, m$

$$\begin{aligned} \bar{P}_p \sqsubseteq \bar{Q}_q &\implies \left[\rho_1 \mathfrak{R}_{P(b^i)}^l(\bar{h}^j), \rho_1 \mathfrak{R}_{P(b^i)}^u(\bar{h}^j) \right] \preceq \left[\rho_2 \mathfrak{R}_{Q(b^i)}^l(\bar{h}^j), \rho_2 \mathfrak{R}_{Q(b^i)}^u(\bar{h}^j) \right] \\ &\left[\varrho_1 \mathfrak{S}_{P(b^i)}^l(\bar{h}^j), \varrho_1 \mathfrak{S}_{P(b^i)}^u(\bar{h}^j) \right] \preceq \left[\varrho_2 \mathfrak{S}_{Q(b^i)}^l(\bar{h}^j), \varrho_2 \mathfrak{S}_{Q(b^i)}^u(\bar{h}^j) \right] \\ &\left[\sigma_1 \mathfrak{U}_{P(b^i)}^l(\bar{h}^j), \sigma_1 \mathfrak{U}_{P(b^i)}^u(\bar{h}^j) \right] \succeq \left[\sigma_2 \mathfrak{U}_{Q(b^i)}^l(\bar{h}^j), \sigma_2 \mathfrak{U}_{Q(b^i)}^u(\bar{h}^j) \right] \\ &\left[p^l(b^i), p^u(b^i) \right] \preceq \left[q^l(b^i), q^u(b^i) \right] \\ \bar{P}_p \sqsubseteq \bar{R}_r &\implies \left[\rho_1 \mathfrak{R}_{P(b^i)}^l(\bar{h}^j), \rho_1 \mathfrak{R}_{P(b^i)}^u(\bar{h}^j) \right] \preceq \left[\rho_3 \mathfrak{R}_{R(b^i)}^l(\bar{h}^j), \rho_3 \mathfrak{R}_{R(b^i)}^u(\bar{h}^j) \right] \\ &\left[\varrho_1 \mathfrak{S}_{P(b^i)}^l(\bar{h}^j), \varrho_1 \mathfrak{S}_{P(b^i)}^u(\bar{h}^j) \right] \preceq \left[\varrho_3 \mathfrak{S}_{R(b^i)}^l(\bar{h}^j), \varrho_3 \mathfrak{S}_{R(b^i)}^u(\bar{h}^j) \right] \\ &\left[\sigma_1 \mathfrak{U}_{P(b^i)}^l(\bar{h}^j), \sigma_1 \mathfrak{U}_{P(b^i)}^u(\bar{h}^j) \right] \succeq \left[\sigma_3 \mathfrak{U}_{R(b^i)}^l(\bar{h}^j), \sigma_3 \mathfrak{U}_{R(b^i)}^u(\bar{h}^j) \right] \\ &\left[p^l(b^i), p^u(b^i) \right] \preceq \left[r^l(b^i), r^u(b^i) \right] \\ \bar{Q}_q \sqsubseteq \bar{R}_r &\implies \left[\rho_2 \mathfrak{R}_{Q(b^i)}^l(\bar{h}^j), \rho_2 \mathfrak{R}_{Q(b^i)}^u(\bar{h}^j) \right] \preceq \left[\rho_3 \mathfrak{R}_{R(b^i)}^l(\bar{h}^j), \rho_3 \mathfrak{R}_{R(b^i)}^u(\bar{h}^j) \right] \\ &\left[\varrho_2 \mathfrak{S}_{Q(b^i)}^l(\bar{h}^j), \varrho_2 \mathfrak{S}_{Q(b^i)}^u(\bar{h}^j) \right] \preceq \left[\varrho_3 \mathfrak{S}_{R(b^i)}^l(\bar{h}^j), \varrho_3 \mathfrak{S}_{R(b^i)}^u(\bar{h}^j) \right] \\ &\left[\sigma_2 \mathfrak{U}_{Q(b^i)}^l(\bar{h}^j), \sigma_2 \mathfrak{U}_{Q(b^i)}^u(\bar{h}^j) \right] \succeq \left[\sigma_3 \mathfrak{U}_{R(b^i)}^l(\bar{h}^j), \sigma_3 \mathfrak{U}_{R(b^i)}^u(\bar{h}^j) \right] \\ &\left[q^l(b^i), q^u(b^i) \right] \preceq \left[r^l(b^i), r^u(b^i) \right] \end{aligned}$$

Clearly,

$$\begin{aligned} &\left[\left(\rho_1 \mathfrak{R}_{P(b^i)}^l(\bar{h}^j) \cdot \rho_3 \mathfrak{R}_{R(b^i)}^l(\bar{h}^j) \right), \left(\rho_1 \mathfrak{R}_{P(b^i)}^u(\bar{h}^j) \cdot \rho_3 \mathfrak{R}_{R(b^i)}^u(\bar{h}^j) \right) \right] \\ &\preceq \left[\left(\rho_2 \mathfrak{R}_{Q(b^i)}^l(\bar{h}^j) \cdot \rho_3 \mathfrak{R}_{R(b^i)}^l(\bar{h}^j) \right), \left(\rho_2 \mathfrak{R}_{Q(b^i)}^u(\bar{h}^j) \cdot \rho_3 \mathfrak{R}_{R(b^i)}^u(\bar{h}^j) \right) \right] \end{aligned}$$

implies that

$$\begin{aligned} &\left[\sum_{i=1}^n \left(\rho_1 \mathfrak{R}_{P(b^i)}^l(\bar{h}^j) \cdot \rho_3 \mathfrak{R}_{R(b^i)}^l(\bar{h}^j) \right), \sum_{i=1}^n \left(\rho_1 \mathfrak{R}_{P(b^i)}^u(\bar{h}^j) \cdot \rho_3 \mathfrak{R}_{R(b^i)}^u(\bar{h}^j) \right) \right] \\ &\preceq \left[\sum_{i=1}^n \left(\rho_2 \mathfrak{R}_{Q(b^i)}^l(\bar{h}^j) \cdot \rho_3 \mathfrak{R}_{R(b^i)}^l(\bar{h}^j) \right), \sum_{i=1}^n \left(\rho_2 \mathfrak{R}_{Q(b^i)}^u(\bar{h}^j) \cdot \rho_3 \mathfrak{R}_{R(b^i)}^u(\bar{h}^j) \right) \right] \quad (1) \end{aligned}$$

for $j = 1, 2, \dots, m$

Clearly,

$$\left[(\rho_1)^2 \mathfrak{R}_{P(b^i)}^{2l}(\bar{h}^j), (\rho_1)^2 \mathfrak{R}_{P(b^i)}^{2u}(\bar{h}^j) \right] \preceq \left[(\rho_2)^2 \mathfrak{R}_{Q(b^i)}^{2l}(\bar{h}^j), (\rho_2)^2 \mathfrak{R}_{Q(b^i)}^{2u}(\bar{h}^j) \right] \preceq \left[(\rho_3)^2 \mathfrak{R}_{R(b^i)}^{2l}(\bar{h}^j), (\rho_3)^2 \mathfrak{R}_{R(b^i)}^{2u}(\bar{h}^j) \right]$$

implies that

$$\begin{aligned} &\left[-(\rho_1)^2 \mathfrak{R}_{P(b^i)}^{2u}(\bar{h}^j), -(\rho_1)^2 \mathfrak{R}_{P(b^i)}^{2l}(\bar{h}^j) \right] \succeq \left[-(\rho_2)^2 \mathfrak{R}_{Q(b^i)}^{2u}(\bar{h}^j), -(\rho_2)^2 \mathfrak{R}_{Q(b^i)}^{2l}(\bar{h}^j) \right] \\ &\succeq \left[-(\rho_3)^2 \mathfrak{R}_{R(b^i)}^{2u}(\bar{h}^j), -(\rho_3)^2 \mathfrak{R}_{R(b^i)}^{2l}(\bar{h}^j) \right] \end{aligned}$$

and

$$\begin{aligned} & \left[1 - (\rho_1)^2 \Re_{P(b^i)}^{2l}(\hbar^j), 1 - (\rho_1)^2 \Re_{P(b^i)}^{2u}(\hbar^j) \right] \succeq \left[1 - (\rho_2)^2 \Re_{Q(b^i)}^{2l}(\hbar^j), 1 - (\rho_2)^2 \Re_{Q(b^i)}^{2u}(\hbar^j) \right] \\ & \succeq \left[1 - (\rho_3)^2 \Re_{R(b^i)}^{2l}(\hbar^j), 1 - (\rho_3)^2 \Re_{R(b^i)}^{2u}(\hbar^j) \right] \end{aligned}$$

and

$$\begin{aligned} & \left[\left((1 - (\rho_1)^2 \Re_{P(b^i)}^{2l}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2l}(\hbar^j)) \right), \left((1 - (\rho_1)^2 \Re_{P(b^i)}^{2u}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2u}(\hbar^j)) \right) \right] \\ & \succeq \left[\left((1 - (\rho_2)^2 \Re_{Q(b^i)}^{2l}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2l}(\hbar^j)) \right), \left((1 - (\rho_2)^2 \Re_{Q(b^i)}^{2u}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2u}(\hbar^j)) \right) \right] \end{aligned}$$

and

$$\begin{aligned} & \left[\sqrt{\left((1 - (\rho_1)^2 \Re_{P(b^i)}^{2l}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2l}(\hbar^j)) \right)}, \sqrt{\left((1 - (\rho_1)^2 \Re_{P(b^i)}^{2u}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2u}(\hbar^j)) \right)} \right] \\ & \succeq \left[\sqrt{\left((1 - (\rho_2)^2 \Re_{Q(b^i)}^{2l}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2l}(\hbar^j)) \right)}, \sqrt{\left((1 - (\rho_2)^2 \Re_{Q(b^i)}^{2u}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2u}(\hbar^j)) \right)} \right] \end{aligned}$$

and

$$\begin{aligned} & 1 - \left[\sqrt{\left((1 - (\rho_1)^2 \Re_{P(b^i)}^{2l}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2l}(\hbar^j)) \right)}, \sqrt{\left((1 - (\rho_1)^2 \Re_{P(b^i)}^{2u}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2u}(\hbar^j)) \right)} \right] \\ & \succeq 1 - \left[\sqrt{\left((1 - (\rho_2)^2 \Re_{Q(b^i)}^{2l}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2l}(\hbar^j)) \right)}, \sqrt{\left((1 - (\rho_2)^2 \Re_{Q(b^i)}^{2u}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2u}(\hbar^j)) \right)} \right] \end{aligned}$$

and

$$\begin{aligned} & \left[1 - \sqrt{\left((1 - (\rho_1)^2 \Re_{P(b^i)}^{2u}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2u}(\hbar^j)) \right)}, 1 - \sqrt{\left((1 - (\rho_1)^2 \Re_{P(b^i)}^{2l}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2l}(\hbar^j)) \right)} \right] \\ & \succeq \left[1 - \sqrt{\left((1 - (\rho_2)^2 \Re_{Q(b^i)}^{2u}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2u}(\hbar^j)) \right)}, 1 - \sqrt{\left((1 - (\rho_2)^2 \Re_{Q(b^i)}^{2l}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2l}(\hbar^j)) \right)} \right] \end{aligned}$$

and

$$\begin{aligned} & \left[\sum_{i=1}^n \left(1 - \sqrt{\left((1 - (\rho_1)^2 \Re_{P(b^i)}^{2u}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2u}(\hbar^j)) \right)} \right), \sum_{i=1}^n \left(1 - \sqrt{\left((1 - (\rho_1)^2 \Re_{P(b^i)}^{2l}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2l}(\hbar^j)) \right)} \right) \right] \\ & \succeq \left[\sum_{i=1}^n \left(1 - \sqrt{\left((1 - (\rho_2)^2 \Re_{Q(b^i)}^{2u}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2u}(\hbar^j)) \right)} \right), \sum_{i=1}^n \left(1 - \sqrt{\left((1 - (\rho_2)^2 \Re_{Q(b^i)}^{2l}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2l}(\hbar^j)) \right)} \right) \right] \quad (2) \end{aligned}$$

Equations 1 and 2,

$$\begin{aligned} & \frac{\left[\sum_{i=1}^n \left(\rho_1 \Re_{P(b^i)}^l(\hbar^j) \cdot \rho_3 \Re_{R(b^i)}^l(\hbar^j) \right), \sum_{i=1}^n \left(\rho_1 \Re_{P(b^i)}^u(\hbar^j) \cdot \rho_3 \Re_{R(b^i)}^u(\hbar^j) \right) \right]}{\left[\sum_{i=1}^n \left(1 - \sqrt{\left((1 - (\rho_1)^2 \Re_{P(b^i)}^{2u}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2u}(\hbar^j)) \right)} \right), \sum_{i=1}^n \left(1 - \sqrt{\left((1 - (\rho_1)^2 \Re_{P(b^i)}^{2l}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2l}(\hbar^j)) \right)} \right) \right]} \\ & \succeq \frac{\left[\sum_{i=1}^n \left(\rho_2 \Re_{Q(b^i)}^l(\hbar^j) \cdot \rho_3 \Re_{R(b^i)}^l(\hbar^j) \right), \sum_{i=1}^n \left(\rho_2 \Re_{Q(b^i)}^u(\hbar^j) \cdot \rho_3 \Re_{R(b^i)}^u(\hbar^j) \right) \right]}{\left[\sum_{i=1}^n \left(1 - \sqrt{\left((1 - (\rho_2)^2 \Re_{Q(b^i)}^{2u}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2u}(\hbar^j)) \right)} \right), \sum_{i=1}^n \left(1 - \sqrt{\left((1 - (\rho_2)^2 \Re_{Q(b^i)}^{2l}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2l}(\hbar^j)) \right)} \right) \right]} \end{aligned}$$

implies that

$$\begin{aligned} & \left[\frac{\left[\sum_{i=1}^n \left(\rho_1 \Re_{P(b^i)}^l(\hbar^j) \cdot \rho_3 \Re_{R(b^i)}^l(\hbar^j) \right) \right]}{\left[\sum_{i=1}^n \left(1 - \sqrt{\left((1 - (\rho_1)^2 \Re_{P(b^i)}^{2l}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2l}(\hbar^j)) \right)} \right) \right]}, \frac{\left[\sum_{i=1}^n \left(\rho_1 \Re_{P(b^i)}^u(\hbar^j) \cdot \rho_3 \Re_{R(b^i)}^u(\hbar^j) \right) \right]}{\left[\sum_{i=1}^n \left(1 - \sqrt{\left((1 - (\rho_1)^2 \Re_{P(b^i)}^{2u}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2u}(\hbar^j)) \right)} \right) \right]} \right] \\ & \succeq \left[\frac{\left[\sum_{i=1}^n \left(\rho_2 \Re_{Q(b^i)}^l(\hbar^j) \cdot \rho_3 \Re_{R(b^i)}^l(\hbar^j) \right) \right]}{\left[\sum_{i=1}^n \left(1 - \sqrt{\left((1 - (\rho_2)^2 \Re_{Q(b^i)}^{2l}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2l}(\hbar^j)) \right)} \right) \right]}, \frac{\left[\sum_{i=1}^n \left(\rho_2 \Re_{Q(b^i)}^u(\hbar^j) \cdot \rho_3 \Re_{R(b^i)}^u(\hbar^j) \right) \right]}{\left[\sum_{i=1}^n \left(1 - \sqrt{\left((1 - (\rho_2)^2 \Re_{Q(b^i)}^{2u}(\hbar^j)) \cdot (1 - (\rho_3)^2 \Re_{R(b^i)}^{2u}(\hbar^j)) \right)} \right) \right]} \right] \end{aligned}$$

Therefore

$$\overline{T_1}(\overline{P}(b)(\hbar^j), \overline{R}(b)(\hbar^j)) \preceq \overline{T_1}(\overline{Q}(b)(\hbar^j), \overline{R}(b)(\hbar^j)) \quad (3)$$

Clearly,

$$\begin{aligned} & \left[\left((\varrho_1)^2 \mathfrak{S}_{P(b^i)}^{2l}(\hbar^j) \cdot (\varrho_3)^2 \mathfrak{S}_{R(b^i)}^{2l}(\hbar^j) \right), \left((\varrho_1)^2 \mathfrak{S}_{P(b^i)}^{2u}(\hbar^j) \cdot (\varrho_3)^2 \mathfrak{S}_{R(b^i)}^{2u}(\hbar^j) \right) \right] \\ & \preceq \left[\left((\varrho_2)^2 \mathfrak{S}_{Q(b^i)}^{2l}(\hbar^j) \cdot (\varrho_3)^2 \mathfrak{S}_{R(b^i)}^{2l}(\hbar^j) \right), \left((\varrho_2)^2 \mathfrak{S}_{Q(b^i)}^{2u}(\hbar^j) \cdot (\varrho_3)^2 \mathfrak{S}_{R(b^i)}^{2u}(\hbar^j) \right) \right] \end{aligned}$$

implies that

$$\begin{aligned} & \left[\sum_{i=1}^n \left((\varrho_1)^2 \mathfrak{S}_{P(b^i)}^{2l}(\hbar^j) \cdot (\varrho_3)^2 \mathfrak{S}_{R(b^i)}^{2l}(\hbar^j) \right), \sum_{i=1}^n \left((\varrho_1)^2 \mathfrak{S}_{P(b^i)}^{2u}(\hbar^j) \cdot (\varrho_3)^2 \mathfrak{S}_{R(b^i)}^{2u}(\hbar^j) \right) \right] \\ & \preceq \left[\sum_{i=1}^n \left((\varrho_2)^2 \mathfrak{S}_{Q(b^i)}^{2l}(\hbar^j) \cdot (\varrho_3)^2 \mathfrak{S}_{R(b^i)}^{2l}(\hbar^j) \right), \sum_{i=1}^n \left((\varrho_2)^2 \mathfrak{S}_{Q(b^i)}^{2u}(\hbar^j) \cdot (\varrho_3)^2 \mathfrak{S}_{R(b^i)}^{2u}(\hbar^j) \right) \right] \quad (4) \end{aligned}$$

for $j = 1, 2, \dots, m$

Clearly,

$$\left[(\varrho_1)^4 \mathfrak{S}_{P(b^i)}^{4l}(\hbar^j), (\varrho_1)^4 \mathfrak{S}_{P(b^i)}^{4u}(\hbar^j) \right] \preceq \left[(\varrho_2)^4 \mathfrak{S}_{Q(b^i)}^{4l}(\hbar^j), (\varrho_2)^4 \mathfrak{S}_{Q(b^i)}^{4u}(\hbar^j) \right] \preceq \left[(\varrho_3)^4 \mathfrak{S}_{R(b^i)}^{4l}(\hbar^j), (\varrho_3)^4 \mathfrak{S}_{R(b^i)}^{4u}(\hbar^j) \right]$$

implies that

$$\begin{aligned} & \left[-(\varrho_1)^4 \mathfrak{S}_{P(b^i)}^{4u}(\hbar^j), -(\varrho_1)^4 \mathfrak{S}_{P(b^i)}^{4l}(\hbar^j) \right] \succeq \left[-(\varrho_2)^4 \mathfrak{S}_{Q(b^i)}^{4u}(\hbar^j), -(\varrho_2)^4 \mathfrak{S}_{Q(b^i)}^{4l}(\hbar^j) \right] \\ & \succeq \left[-(\varrho_3)^4 \mathfrak{S}_{R(b^i)}^{4u}(\hbar^j), -(\varrho_3)^4 \mathfrak{S}_{R(b^i)}^{4l}(\hbar^j) \right] \end{aligned}$$

and

$$\begin{aligned} & \left[1 - (\varrho_1)^4 \mathfrak{S}_{P(b^i)}^{4l}(\hbar^j), 1 - (\varrho_1)^4 \mathfrak{S}_{P(b^i)}^{4u}(\hbar^j) \right] \succeq \left[1 - (\varrho_2)^4 \mathfrak{S}_{Q(b^i)}^{4l}(\hbar^j), 1 - (\varrho_2)^4 \mathfrak{S}_{Q(b^i)}^{4u}(\hbar^j) \right] \\ & \succeq \left[1 - (\varrho_3)^4 \mathfrak{S}_{R(b^i)}^{4l}(\hbar^j), 1 - (\varrho_3)^4 \mathfrak{S}_{R(b^i)}^{4u}(\hbar^j) \right] \end{aligned}$$

and

$$\begin{aligned} & \left[\left((1 - (\varrho_1)^4 \mathfrak{S}_{P(b^i)}^{4l}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \mathfrak{S}_{R(b^i)}^{4l}(\hbar^j)) \right), \left((1 - (\varrho_1)^4 \mathfrak{S}_{P(b^i)}^{4u}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \mathfrak{S}_{R(b^i)}^{4u}(\hbar^j)) \right) \right] \\ & \succeq \left[\left((1 - (\varrho_2)^4 \mathfrak{S}_{Q(b^i)}^{4l}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \mathfrak{S}_{R(b^i)}^{4l}(\hbar^j)) \right), \left((1 - (\varrho_2)^4 \mathfrak{S}_{Q(b^i)}^{4u}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \mathfrak{S}_{R(b^i)}^{4u}(\hbar^j)) \right) \right] \end{aligned}$$

and

$$\begin{aligned} & \left[\sqrt{\left((1 - (\varrho_1)^4 \mathfrak{S}_{P(b^i)}^{4l}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \mathfrak{S}_{R(b^i)}^{4l}(\hbar^j)) \right)}, \sqrt{\left((1 - (\varrho_1)^4 \mathfrak{S}_{P(b^i)}^{4u}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \mathfrak{S}_{R(b^i)}^{4u}(\hbar^j)) \right)} \right] \\ & \succeq \left[\sqrt{\left((1 - (\varrho_2)^4 \mathfrak{S}_{Q(b^i)}^{4l}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \mathfrak{S}_{R(b^i)}^{4l}(\hbar^j)) \right)}, \sqrt{\left((1 - (\varrho_2)^4 \mathfrak{S}_{Q(b^i)}^{4u}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \mathfrak{S}_{R(b^i)}^{4u}(\hbar^j)) \right)} \right] \end{aligned}$$

and

$$\begin{aligned} & 1 - \left[\sqrt{\left((1 - (\varrho_1)^4 \mathfrak{S}_{P(b^i)}^{4l}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \mathfrak{S}_{R(b^i)}^{4l}(\hbar^j)) \right)}, \sqrt{\left((1 - (\varrho_1)^4 \mathfrak{S}_{P(b^i)}^{4u}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \mathfrak{S}_{R(b^i)}^{4u}(\hbar^j)) \right)} \right] \\ & \preceq 1 - \left[\sqrt{\left((1 - (\varrho_2)^4 \mathfrak{S}_{Q(b^i)}^{4l}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \mathfrak{S}_{R(b^i)}^{4l}(\hbar^j)) \right)}, \sqrt{\left((1 - (\varrho_2)^4 \mathfrak{S}_{Q(b^i)}^{4u}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \mathfrak{S}_{R(b^i)}^{4u}(\hbar^j)) \right)} \right] \end{aligned}$$

and

$$\begin{aligned} & \left[1 - \sqrt{\left((1 - (\varrho_1)^4 \Im_{P(b^i)}^{4u}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \Im_{R(b^i)}^{4u}(\hbar^j)) \right)}, 1 - \sqrt{\left((1 - (\varrho_1)^4 \Im_{P(b^i)}^{4l}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \Im_{R(b^i)}^{4l}(\hbar^j)) \right)} \right] \\ \preceq & \left[1 - \sqrt{\left((1 - (\varrho_2)^4 \Im_{Q(b^i)}^{4u}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \Im_{R(b^i)}^{4u}(\hbar^j)) \right)}, 1 - \sqrt{\left((1 - (\varrho_2)^4 \Im_{Q(b^i)}^{4l}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \Im_{R(b^i)}^{4l}(\hbar^j)) \right)} \right] \end{aligned}$$

and

$$\begin{aligned} & \left[\sum_{i=1}^n \left(1 - \sqrt{\left((1 - (\varrho_1)^4 \Im_{P(b^i)}^{4u}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \Im_{R(b^i)}^{4u}(\hbar^j)) \right)} \right), \sum_{i=1}^n \left(1 - \sqrt{\left((1 - (\varrho_1)^4 \Im_{P(b^i)}^{4l}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \Im_{R(b^i)}^{4l}(\hbar^j)) \right)} \right) \right] \\ \preceq & \left[\sum_{i=1}^n \left(1 - \sqrt{\left((1 - (\varrho_2)^4 \Im_{Q(b^i)}^{4u}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \Im_{R(b^i)}^{4u}(\hbar^j)) \right)} \right), \sum_{i=1}^n \left(1 - \sqrt{\left((1 - (\varrho_2)^4 \Im_{Q(b^i)}^{4l}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \Im_{R(b^i)}^{4l}(\hbar^j)) \right)} \right) \right] \quad (5) \end{aligned}$$

Equations 4 and 5,

$$\begin{aligned} & \frac{\left[\sum_{i=1}^n \left((\varrho_1)^2 \Im_{P(b^i)}^{2l}(\hbar^j) \cdot (\varrho_3)^2 \Im_{R(b^i)}^{2l}(\hbar^j) \right), \sum_{i=1}^n \left((\varrho_1)^2 \Im_{P(b^i)}^{2u}(\hbar^j) \cdot (\varrho_3)^2 \Im_{R(b^i)}^{2u}(\hbar^j) \right) \right]}{\left[\sum_{i=1}^n \left(1 - \sqrt{\left((1 - (\varrho_1)^4 \Im_{P(b^i)}^{4u}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \Im_{R(b^i)}^{4u}(\hbar^j)) \right)} \right), \sum_{i=1}^n \left(1 - \sqrt{\left((1 - (\varrho_1)^4 \Im_{P(b^i)}^{4l}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \Im_{R(b^i)}^{4l}(\hbar^j)) \right)} \right) \right]} \\ \preceq & \frac{\left[\sum_{i=1}^n \left((\varrho_2)^2 \Im_{Q(b^i)}^{2l}(\hbar^j) \cdot (\varrho_3)^2 \Im_{R(b^i)}^{2l}(\hbar^j) \right), \sum_{i=1}^n \left((\varrho_2)^2 \Im_{Q(b^i)}^{2u}(\hbar^j) \cdot (\varrho_3)^2 \Im_{R(b^i)}^{2u}(\hbar^j) \right) \right]}{\left[\sum_{i=1}^n \left(1 - \sqrt{\left((1 - (\varrho_2)^4 \Im_{Q(b^i)}^{4u}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \Im_{R(b^i)}^{4u}(\hbar^j)) \right)} \right), \sum_{i=1}^n \left(1 - \sqrt{\left((1 - (\varrho_2)^4 \Im_{Q(b^i)}^{4l}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \Im_{R(b^i)}^{4l}(\hbar^j)) \right)} \right) \right]} \end{aligned}$$

implies that

$$\begin{aligned} & \left[\frac{\left[\sum_{i=1}^n \left((\varrho_1)^2 \Im_{P(b^i)}^{2l}(\hbar^j) \cdot (\varrho_3)^2 \Im_{R(b^i)}^{2l}(\hbar^j) \right) \right]}{\left[\sum_{i=1}^n \left(1 - \sqrt{\left((1 - (\varrho_1)^4 \Im_{P(b^i)}^{4u}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \Im_{R(b^i)}^{4u}(\hbar^j)) \right)} \right) \right]}, \frac{\left[\sum_{i=1}^n \left((\varrho_1)^2 \Im_{P(b^i)}^{2u}(\hbar^j) \cdot (\varrho_3)^2 \Im_{R(b^i)}^{2u}(\hbar^j) \right) \right]}{\left[\sum_{i=1}^n \left(1 - \sqrt{\left((1 - (\varrho_1)^4 \Im_{P(b^i)}^{4l}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \Im_{R(b^i)}^{4l}(\hbar^j)) \right)} \right) \right]} \right] \\ \preceq & \left[\frac{\left[\sum_{i=1}^n \left((\varrho_2)^2 \Im_{Q(b^i)}^{2l}(\hbar^j) \cdot (\varrho_3)^2 \Im_{R(b^i)}^{2l}(\hbar^j) \right) \right]}{\left[\sum_{i=1}^n \left(1 - \sqrt{\left((1 - (\varrho_2)^4 \Im_{Q(b^i)}^{4u}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \Im_{R(b^i)}^{4u}(\hbar^j)) \right)} \right) \right]}, \frac{\left[\sum_{i=1}^n \left((\varrho_2)^2 \Im_{Q(b^i)}^{2u}(\hbar^j) \cdot (\varrho_3)^2 \Im_{R(b^i)}^{2u}(\hbar^j) \right) \right]}{\left[\sum_{i=1}^n \left(1 - \sqrt{\left((1 - (\varrho_2)^4 \Im_{Q(b^i)}^{4l}(\hbar^j)) \cdot (1 - (\varrho_3)^4 \Im_{R(b^i)}^{4l}(\hbar^j)) \right)} \right) \right]} \right] \end{aligned}$$

Therefore

$$\overline{T}_2(\overline{P}(b)(\hbar^j), \overline{R}(b)(\hbar^j)) \preceq \overline{T}_2(\overline{Q}(b)(\hbar^j), \overline{R}(b)(\hbar^j)) \quad (6)$$

Clearly,

$$\left[\sigma_1^2 \Im_{P(b^i)}^{2l}(\hbar^j), \sigma_1^2 \Im_{P(b^i)}^{2u}(\hbar^j) \right] \succeq \left[(\sigma_2)^2 \Im_{Q(b^i)}^{2l}(\hbar^j), (\sigma_2)^2 \Im_{Q(b^i)}^{2u}(\hbar^j) \right] \succeq \left[(\sigma_3)^2 \Im_{R(b^i)}^{2l}(\hbar^j), (\sigma_3)^2 \Im_{R(b^i)}^{2u}(\hbar^j) \right]$$

and

$$\begin{aligned} & \left[\sigma_1^2 \Im_{P(b^i)}^{2l}(\hbar^j), \sigma_1^2 \Im_{P(b^i)}^{2u}(\hbar^j) \right] - \left[(\sigma_3)^2 \Im_{R(b^i)}^{2l}(\hbar^j), (\sigma_3)^2 \Im_{R(b^i)}^{2u}(\hbar^j) \right] \\ \succeq & \left[(\sigma_2)^2 \Im_{Q(b^i)}^{2l}(\hbar^j), (\sigma_2)^2 \Im_{Q(b^i)}^{2u}(\hbar^j) \right] - \left[(\sigma_3)^2 \Im_{R(b^i)}^{2l}(\hbar^j), (\sigma_3)^2 \Im_{R(b^i)}^{2u}(\hbar^j) \right] \end{aligned}$$

implies

$$\begin{aligned} & \left| \left[\sigma_1^2 \Im_{P(b^i)}^{2l}(\hbar^j), \sigma_1^2 \Im_{P(b^i)}^{2u}(\hbar^j) \right] - \left[(\sigma_3)^2 \Im_{R(b^i)}^{2l}(\hbar^j), (\sigma_3)^2 \Im_{R(b^i)}^{2u}(\hbar^j) \right] \right| \\ \succeq & \left| \left[(\sigma_2)^2 \Im_{Q(b^i)}^{2l}(\hbar^j), (\sigma_2)^2 \Im_{Q(b^i)}^{2u}(\hbar^j) \right] - \left[(\sigma_3)^2 \Im_{R(b^i)}^{2l}(\hbar^j), (\sigma_3)^2 \Im_{R(b^i)}^{2u}(\hbar^j) \right] \right| \end{aligned}$$

and

$$\begin{aligned} & \left| \left[\sigma_1^2 \Im_{P(b^i)}^{2l}(\hbar^j) - (\sigma_3)^2 \Im_{R(b^i)}^{2l}(\hbar^j), \sigma_1^2 \Im_{P(b^i)}^{2u}(\hbar^j) - (\sigma_3)^2 \Im_{R(b^i)}^{2u}(\hbar^j) \right] \right| \\ \succeq & \left| \left[(\sigma_2)^2 \Im_{Q(b^i)}^{2l}(\hbar^j) - (\sigma_3)^2 \Im_{R(b^i)}^{2l}(\hbar^j), (\sigma_2)^2 \Im_{Q(b^i)}^{2u}(\hbar^j) - (\sigma_3)^2 \Im_{R(b^i)}^{2u}(\hbar^j) \right] \right| \end{aligned}$$

Hence

$$\begin{aligned} & \left| \left[\sum_{i=1}^n \left(\sigma_1^2 \mathcal{U}_{P(b^i)}^{2l}(\hbar^j) - (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2u}(\hbar^j) \right), \sum_{i=1}^n \left(\sigma_1^2 \mathcal{U}_{P(b^i)}^{2u}(\hbar^j) - (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2l}(\hbar^j) \right) \right] \right| \\ \preceq & \left| \left[\sum_{i=1}^n \left((\sigma_2)^2 \mathcal{U}_{Q(b^i)}^{2l}(\hbar^j) - (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2u}(\hbar^j) \right), \sum_{i=1}^n \left((\sigma_2)^2 \mathcal{U}_{Q(b^i)}^{2u}(\hbar^j) - (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2l}(\hbar^j) \right) \right] \right| \quad (7) \end{aligned}$$

Also,

$$\begin{aligned} & \left[\left(\sigma_1^2 \mathcal{U}_{P(b^i)}^{2l}(\hbar^j) \cdot (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2l}(\hbar^j) \right), \left(\sigma_1^2 \mathcal{U}_{P(b^i)}^{2u}(\hbar^j) \cdot (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2u}(\hbar^j) \right) \right] \\ \succeq & \left[\left((\sigma_2)^2 \mathcal{U}_{Q(b^i)}^{2l}(\hbar^j) \cdot (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2l}(\hbar^j) \right), \left((\sigma_2)^2 \mathcal{U}_{Q(b^i)}^{2u}(\hbar^j) \cdot (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2u}(\hbar^j) \right) \right] \end{aligned}$$

implies that

$$\begin{aligned} & \left| \left[\left(\sigma_1^2 \mathcal{U}_{P(b^i)}^{2l}(\hbar^j) \cdot (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2l}(\hbar^j) \right), \left(\sigma_1^2 \mathcal{U}_{P(b^i)}^{2u}(\hbar^j) \cdot (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2u}(\hbar^j) \right) \right] \right| \\ \succeq & \left| \left[\left((\sigma_2)^2 \mathcal{U}_{Q(b^i)}^{2l}(\hbar^j) \cdot (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2l}(\hbar^j) \right), \left((\sigma_2)^2 \mathcal{U}_{Q(b^i)}^{2u}(\hbar^j) \cdot (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2u}(\hbar^j) \right) \right] \right| \\ & \left| \left[\sum_{i=1}^n 1 + \left(\sigma_1^2 \mathcal{U}_{P(b^i)}^{2l}(\hbar^j) \cdot (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2l}(\hbar^j) \right), \sum_{i=1}^n 1 + \left(\sigma_1^2 \mathcal{U}_{P(b^i)}^{2u}(\hbar^j) \cdot (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2u}(\hbar^j) \right) \right] \right| \\ \succeq & \left| \left[\sum_{i=1}^n 1 + \left((\sigma_2)^2 \mathcal{U}_{Q(b^i)}^{2l}(\hbar^j) \cdot (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2l}(\hbar^j) \right), \sum_{i=1}^n 1 + \left((\sigma_2)^2 \mathcal{U}_{Q(b^i)}^{2u}(\hbar^j) \cdot (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2u}(\hbar^j) \right) \right] \right| \quad (8) \end{aligned}$$

for $j = 1, 2, \dots, m$.

Equations 7 and 8, we get

$$\begin{aligned} & \left| \left[\sum_{i=1}^n \left(\sigma_1^2 \mathcal{U}_{P(b^i)}^{2l}(\hbar^j) - (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2u}(\hbar^j) \right), \sum_{i=1}^n \left(\sigma_1^2 \mathcal{U}_{P(b^i)}^{2u}(\hbar^j) - (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2l}(\hbar^j) \right) \right] \right| \\ & \left| \left[\sum_{i=1}^n 1 + \left(\sigma_1^2 \mathcal{U}_{P(b^i)}^{2l}(\hbar^j) \cdot (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2l}(\hbar^j) \right), \sum_{i=1}^n 1 + \left(\sigma_1^2 \mathcal{U}_{P(b^i)}^{2u}(\hbar^j) \cdot (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2u}(\hbar^j) \right) \right] \right| \\ \preceq & \left| \left[\sum_{i=1}^n \left((\sigma_2)^2 \mathcal{U}_{Q(b^i)}^{2l}(\hbar^j) - (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2u}(\hbar^j) \right), \sum_{i=1}^n \left((\sigma_2)^2 \mathcal{U}_{Q(b^i)}^{2u}(\hbar^j) - (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2l}(\hbar^j) \right) \right] \right| \\ & \left| \left[\sum_{i=1}^n 1 + \left((\sigma_2)^2 \mathcal{U}_{Q(b^i)}^{2l}(\hbar^j) \cdot (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2l}(\hbar^j) \right), \sum_{i=1}^n 1 + \left((\sigma_2)^2 \mathcal{U}_{Q(b^i)}^{2u}(\hbar^j) \cdot (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2u}(\hbar^j) \right) \right] \right| \end{aligned}$$

implies that

$$\begin{aligned} & \left| \left[\frac{\sum_{i=1}^n \left(\sigma_1^2 \mathcal{U}_{P(b^i)}^{2l}(\hbar^j) - (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2u}(\hbar^j) \right)}{\sum_{i=1}^n 1 + \left(\sigma_1^2 \mathcal{U}_{P(b^i)}^{2l}(\hbar^j) \cdot (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2l}(\hbar^j) \right)}, \frac{\sum_{i=1}^n \left(\sigma_1^2 \mathcal{U}_{P(b^i)}^{2u}(\hbar^j) - (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2l}(\hbar^j) \right)}{\sum_{i=1}^n 1 + \left(\sigma_1^2 \mathcal{U}_{P(b^i)}^{2u}(\hbar^j) \cdot (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2u}(\hbar^j) \right)} \right] \right| \\ \succeq & \left| \left[\frac{\sum_{i=1}^n \left((\sigma_2)^2 \mathcal{U}_{Q(b^i)}^{2l}(\hbar^j) - (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2u}(\hbar^j) \right)}{\sum_{i=1}^n 1 + \left((\sigma_2)^2 \mathcal{U}_{Q(b^i)}^{2l}(\hbar^j) \cdot (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2l}(\hbar^j) \right)}, \frac{\sum_{i=1}^n \left((\sigma_2)^2 \mathcal{U}_{Q(b^i)}^{2u}(\hbar^j) - (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2l}(\hbar^j) \right)}{\sum_{i=1}^n 1 + \left((\sigma_2)^2 \mathcal{U}_{Q(b^i)}^{2u}(\hbar^j) \cdot (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2u}(\hbar^j) \right)} \right] \right| \end{aligned}$$

and

$$\begin{aligned} & \sqrt{\left| \left[\frac{\sum_{i=1}^n \left(\sigma_1^2 \mathcal{U}_{P(b^i)}^{2l}(\hbar^j) - (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2u}(\hbar^j) \right)}{\sum_{i=1}^n 1 + \left(\sigma_1^2 \mathcal{U}_{P(b^i)}^{2l}(\hbar^j) \cdot (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2l}(\hbar^j) \right)}, \frac{\sum_{i=1}^n \left(\sigma_1^2 \mathcal{U}_{P(b^i)}^{2u}(\hbar^j) - (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2l}(\hbar^j) \right)}{\sum_{i=1}^n 1 + \left(\sigma_1^2 \mathcal{U}_{P(b^i)}^{2u}(\hbar^j) \cdot (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2u}(\hbar^j) \right)} \right] \right|} \\ \succeq & \sqrt{\left| \left[\frac{\sum_{i=1}^n \left((\sigma_2)^2 \mathcal{U}_{Q(b^i)}^{2l}(\hbar^j) - (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2u}(\hbar^j) \right)}{\sum_{i=1}^n 1 + \left((\sigma_2)^2 \mathcal{U}_{Q(b^i)}^{2l}(\hbar^j) \cdot (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2l}(\hbar^j) \right)}, \frac{\sum_{i=1}^n \left((\sigma_2)^2 \mathcal{U}_{Q(b^i)}^{2u}(\hbar^j) - (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2l}(\hbar^j) \right)}{\sum_{i=1}^n 1 + \left((\sigma_2)^2 \mathcal{U}_{Q(b^i)}^{2u}(\hbar^j) \cdot (\sigma_3)^2 \mathcal{U}_{R(b^i)}^{2u}(\hbar^j) \right)} \right] \right|} \end{aligned}$$

and

$$\begin{aligned} & \left[1 - \sqrt{\left| \left[\frac{\sum_{i=1}^n \left(\sigma_1^2 \mathfrak{U}_{P(b^i)}^{2l}(\hbar^j) - (\sigma_3)^2 \mathfrak{U}_{R(b^i)}^{2u}(\hbar^j) \right)}{\sum_{i=1}^n 1 + \left(\sigma_1^2 \mathfrak{U}_{P(b^i)}^{2u}(\hbar^j) \cdot (\sigma_3)^2 \mathfrak{U}_{R(b^i)}^{2u}(\hbar^j) \right)}, \frac{\sum_{i=1}^n \left(\sigma_1^2 \mathfrak{U}_{P(b^i)}^{2u}(\hbar^j) - (\sigma_3)^2 \mathfrak{U}_{R(b^i)}^{2l}(\hbar^j) \right)}{\sum_{i=1}^n 1 + \left(\sigma_1^2 \mathfrak{U}_{P(b^i)}^{2l}(\hbar^j) \cdot (\sigma_3)^2 \mathfrak{U}_{R(b^i)}^{2l}(\hbar^j) \right)} \right] \right|} \right] \\ & \preceq \left[1 - \sqrt{\left| \left[\frac{\sum_{i=1}^n \left((\sigma_2)^2 \mathfrak{U}_{Q(b^i)}^{2l}(\hbar^j) - (\sigma_3)^2 \mathfrak{U}_{R(b^i)}^{2u}(\hbar^j) \right)}{\sum_{i=1}^n 1 + \left((\sigma_2)^2 \mathfrak{U}_{Q(b^i)}^{2u}(\hbar^j) \cdot (\sigma_3)^2 \mathfrak{U}_{R(b^i)}^{2u}(\hbar^j) \right)}, \frac{\sum_{i=1}^n \left((\sigma_2)^2 \mathfrak{U}_{Q(b^i)}^{2u}(\hbar^j) - (\sigma_3)^2 \mathfrak{U}_{R(b^i)}^{2l}(\hbar^j) \right)}{\sum_{i=1}^n 1 + \left((\sigma_2)^2 \mathfrak{U}_{Q(b^i)}^{2l}(\hbar^j) \cdot (\sigma_3)^2 \mathfrak{U}_{R(b^i)}^{2l}(\hbar^j) \right)} \right] \right|} \right] \end{aligned}$$

Therefore

$$\overline{S}(\overline{P}(b)(\hbar^j), \overline{R}(b)(\hbar^j)) \preceq \overline{S}(\overline{Q}(b)(\hbar^j), \overline{R}(b)(\hbar^j)) \quad (9)$$

Equations 3, 6 and 9, for each $j = 1, 2, \dots, m$.

$$\varphi(\overline{P}, \overline{R}) \preceq \varphi(\overline{Q}, \overline{R}) \quad (10)$$

Clearly

$$\left[p^l(b^i), p^u(b^i) \right] \preceq \left[q^l(b^i), q^u(b^i) \right] \preceq \left[r^l(b^i), r^u(b^i) \right]$$

and

$$\left[\min \left\{ [p^l(b^i), r^l(b^i)] \right\}, \max \left\{ [p^u(b^i), r^u(b^i)] \right\} \right] \preceq \left[\min \left\{ [q^l(b^i), r^l(b^i)] \right\}, \max \left\{ [q^u(b^i), r^u(b^i)] \right\} \right]$$

Hence

$$\begin{aligned} & \left[\sum_{i=1}^n \min \left\{ [p^l(b^i), r^l(b^i)] \right\}, \sum_{i=1}^n \max \left\{ [p^u(b^i), r^u(b^i)] \right\} \right] \\ & \preceq \left[\sum_{i=1}^n \min \left\{ [q^l(b^i), r^l(b^i)] \right\}, \sum_{i=1}^n \max \left\{ [q^u(b^i), r^u(b^i)] \right\} \right] \end{aligned} \quad (11)$$

Plso,

$$\begin{aligned} & \left[(p^l(b^i) + r^l(b^i)), (p^u(b^i) + r^u(b^i)) \right] \preceq \left[(q^l(b^i) + r^l(b^i)), (q^u(b^i) + r^u(b^i)) \right] \\ & \left[\sum_{i=1}^n (p^l(b^i) + r^l(b^i)), \sum_{i=1}^n (p^u(b^i) + r^u(b^i)) \right] \\ & \preceq \left[\sum_{i=1}^n (q^l(b^i) + r^l(b^i)), \sum_{i=1}^n (q^u(b^i) + r^u(b^i)) \right] \end{aligned} \quad (12)$$

Equations 11 and 12, we get

$$\begin{aligned} & \frac{\left[\sum_{i=1}^n \min \left\{ [p^l(b^i), r^l(b^i)] \right\}, \sum_{i=1}^n \max \left\{ [p^u(b^i), r^u(b^i)] \right\} \right]}{\left[\sum_{i=1}^n (p^l(b^i) + r^l(b^i)), \sum_{i=1}^n (p^u(b^i) + r^u(b^i)) \right]} \\ & \succeq \frac{\left[\sum_{i=1}^n \min \left\{ [q^l(b^i), r^l(b^i)] \right\}, \sum_{i=1}^n \max \left\{ [q^u(b^i), r^u(b^i)] \right\} \right]}{\left[\sum_{i=1}^n (q^l(b^i) + r^l(b^i)), \sum_{i=1}^n (q^u(b^i) + r^u(b^i)) \right]} \end{aligned}$$

implies that

$$\begin{aligned} & - \left[\frac{\sum_{i=1}^n \min \left\{ [p^l(b^i), r^l(b^i)] \right\}}{\sum_{i=1}^n (p^u(b^i) + r^u(b^i))}, \frac{\sum_{i=1}^n \max \left\{ [p^u(b^i), r^u(b^i)] \right\}}{\sum_{i=1}^n (p^l(b^i) + r^l(b^i))} \right] \\ & \succeq - \left[\frac{\sum_{i=1}^n \min \left\{ [q^l(b^i), r^l(b^i)] \right\}}{\sum_{i=1}^n (q^u(b^i) + r^u(b^i))}, \frac{\sum_{i=1}^n \max \left\{ [q^u(b^i), r^u(b^i)] \right\}}{\sum_{i=1}^n (q^l(b^i) + r^l(b^i))} \right] \end{aligned}$$

Thus,

$$1 - \left[\frac{\sum_{i=1}^n \min \left\{ \left[p^l(b^i), r^l(b^i) \right] \right\}}{\sum_{i=1}^n (p^u(b^i) + r^u(b^i))}, \frac{\sum_{i=1}^n \max \left\{ \left[p^u(b^i), r^u(b^i) \right] \right\}}{\sum_{i=1}^n (p^l(b^i) + r^l(b^i))} \right] \\ \succeq 1 - \left[\frac{\sum_{i=1}^n \min \left\{ \left[q^l(b^i), r^l(b^i) \right] \right\}}{\sum_{i=1}^n (q^u(b^i) + r^u(b^i))}, \frac{\sum_{i=1}^n \max \left\{ \left[q^u(b^i), r^u(b^i) \right] \right\}}{\sum_{i=1}^n (q^l(b^i) + r^l(b^i))} \right]$$

Hence

$$\psi(\bar{p}, \bar{r}) \succeq \psi(\bar{q}, \bar{r}). \quad (13)$$

Equations 10 and 13,

$$\varphi(\bar{P}, \bar{R}) \cdot \psi(\bar{p}, \bar{r}) \preceq \varphi(\bar{Q}, \bar{R}) \cdot \psi(\bar{q}, \bar{r}).$$

Hence $\text{Sim}(\bar{P}_p, \bar{R}_r) \preceq \text{Sim}(\bar{Q}_q, \bar{R}_r)$.

Example 4.3. Determine the similarity between the two Type-II DioNSIVS sets, namely P_p and Q_q . We choose $X = \{h_1, h_2, h_3\}$ and parameter $E = \{b_1, b_2, b_3\}$ can be defined below:

$P_p(b)$	b_1			b_2		
$P(b)(h_1)$	[0.85, 0.9]	[0.8, 0.9]	[0.75, 0.8], (0.4, 0.35, 0.25)	[0.8, 0.85]	[0.85, 0.9]	[0.85, 0.9], (0.5, 0.25, 0.2)
$P(b)(h_2)$	[0.9, 0.95]	[0.5, 0.55]	[0.7, 0.8], (0.2, 0.4, 0.2)	[0.75, 0.85]	[0.25, 0.3]	[0.65, 0.7], (0.25, 0.45, 0.25)
$P(b)(h_3)$	[0.75, 0.8]	[0.8, 0.85]	[0.3, 0.5], (0.35, 0.3, 0.2)	[0.8, 0.85]	[0.85, 0.9]	[0.4, 0.55], (0.3, 0.3, 0.4)
$p(e)$	([0.5, 0.55], [0.5, 0.6], [0.55, 0.6])			([0.4, 0.55], [0.6, 0.65], [0.35, 0.5])		

$P_p(b)$	b_3		
$P(b)(h_1)$	[0.9, 0.95]	[0.7, 0.75]	[0.9, 0.95], (0.35, 0.25, 0.35)
$P(b)(h_2)$	[0.7, 0.75]	[0.3, 0.5]	[0.6, 0.65], (0.5, 0.4, 0.1)
$P(b)(h_3)$	[0.75, 0.8]	[0.9, 0.95]	[0.5, 0.6], (0.45, 0.4, 0.15)
$p(e)$	([0.6, 0.65], [0.45, 0.55], [0.25, 0.4])		

$Q_q(b)$	b_1			b_2		
$Q(b)(h_1)$	[0.7, 0.85]	[0.75, 0.8]	[0.9, 0.95], (0.3, 0.5, 0.1)	[0.85, 0.9]	[0.7, 0.8]	[0.85, 0.9], (0.25, 0.3, 0.4)
$Q(b)(h_2)$	[0.65, 0.7]	[0.4, 0.5]	[0.65, 0.7], (0.45, 0.35, 0.1)	[0.7, 0.75]	[0.55, 0.7]	[0.55, 0.6], (0.4, 0.45, 0.1)
$Q(b)(h_3)$	[0.65, 0.7]	[0.8, 0.85]	[0.15, 0.2], (0.5, 0.25, 0.25)	[0.75, 0.85]	[0.75, 0.8]	[0.3, 0.35], (0.3, 0.5, 0.2)
$q(e)$	([0.45, 0.55], [0.55, 0.75], [0.45, 0.55])			([0.55, 0.6], [0.35, 0.6], [0.65, 0.7])		

$Q_q(b)$	b_3		
$Q(b)(h_1)$	[0.9, 0.95]	[0.6, 0.65]	[0.7, 0.8], (0.35, 0.25, 0.2)
$Q(b)(h_2)$	[0.85, 0.9]	[0.5, 0.65]	[0.7, 0.75], (0.2, 0.3, 0.25)
$Q(b)(h_3)$	[0.8, 0.9]	[0.7, 0.8]	[0.3, 0.4], (0.35, 0.45, 0.1)
$q(e)$	([0.65, 0.7], [0.45, 0.65], [0.55, 0.6])		

$P_p(b)$	b_1			b_2		
$P(b)(\hbar_1)$	[0.34, 0.36],	[0.28, 0.315],	[0.19, 0.2]	[0.4, 0.425],	[0.2125, 0.225],	[0.17, 0.18]
$P(b)(\hbar_2)$	[0.18, 0.19],	[0.2, 0.22],	[0.14, 0.16]	[0.188, 0.2125],	[0.1125, 0.135],	[0.16, 0.175]
$P(b)(\hbar_3)$	[0.263, 0.28],	[0.24, 0.255],	[0.06, 0.1]	[0.24, 0.255],	[0.255, 0.27],	[0.16, 0.22]
$p(e)$	([0.5, 0.55], [0.5, 0.6], [0.55, 0.6])			([0.4, 0.55], [0.6, 0.65], [0.35, 0.5])		

$P_p(b)$	b_3		
$P(b)(h_1)$	[0.315, 0.3325], [0.175, 0.1875], [0.32, 0.333]		
$P(b)(h_2)$	[0.35, 0.375], [0.12, 0.2], [0.06, 0.065]		
$P(b)(h_3)$	[0.338, 0.36], [0.36, 0.38], [0.08, 0.09]		
$p(e)$	([0.6, 0.65], [0.45, 0.55], [0.25, 0.4])		

$Q_q(b)$	b_1	b_2
$Q(b)(h_1)$	$[0.21, 0.255], [0.375, 0.4], [0.09, 0.095]$	$[0.213, 0.225], [0.21, 0.24], [0.34, 0.36]$
$Q(b)(h_2)$	$[0.293, 0.315], [0.14, 0.175], [0.07, 0.07]$	$[0.28, 0.3], [0.2475, 0.315], [0.06, 0.06]$
$Q(b)(h_3)$	$[0.325, 0.35], [0.2, 0.2125], [0.04, 0.05]$	$[0.225, 0.255], [0.375, 0.4], [0.06, 0.07]$
$q(e)$	$([0.45, 0.55], [0.55, 0.75], [0.45, 0.55])$	$([0.55, 0.6], [0.35, 0.6], [0.65, 0.7])$

$Q_q(b)$	b_3
$Q(b)(h_1)$	$[0.315, 0.3325], [0.15, 0.1625], [0.14, 0.16]$
$Q(b)(h_2)$	$[0.17, 0.18], [0.15, 0.195], [0.18, 0.188]$
$Q(b)(h_3)$	$[0.28, 0.315], [0.315, 0.36], [0.03, 0.04]$
$q(e)$	$([0.65, 0.7], [0.45, 0.65], [0.55, 0.6])$

Using Definition 4.1, $T_1(P(b)(h_1), Q(b)(h_1)) = [0.899496, 0.912756]$, $T_2(P(b)(h_1), Q(b)(h_1)) = [0.87304, 0.910389]$ and $S(P(b)(h_1), Q(b)(h_1)) = [0.885134, 0.982285]$.

Now,

$$\left[\begin{array}{l} \min \left\{ T_1^l(P(b)(h_1), Q(b)(h_1)), T_2^l(P(b)(h_1), Q(b)(h_1)), S^l(P(b)(h_1), Q(b)(h_1)) \right\}, \\ \max \left\{ T_1^u(P(b)(h_1), Q(b)(h_1)), T_2^u(P(b)(h_1), Q(b)(h_1)), S^u(P(b)(h_1), Q(b)(h_1)) \right\} \end{array} \right] = [0.873040, 0.982285].$$

Similarly, $T_1(P(b)(h_2), Q(b)(h_2)) = [0.851939, 0.852511]$,

$T_2(P(b)(h_2), Q(b)(h_2)) = [0.56949, 0.582262]$ and $S(P(b)(h_2), Q(b)(h_2)) = [0.913257, 0.955468]$.

Now,

$$\left[\begin{array}{l} \min \left\{ T_1^l(P(b)(h_2), Q(b)(h_2)), T_2^l(P(b)(h_2), Q(b)(h_2)), S^l(P(b)(h_2), Q(b)(h_2)) \right\}, \\ \max \left\{ T_1^u(P(b)(h_2), Q(b)(h_2)), T_2^u(P(b)(h_2), Q(b)(h_2)), S^u(P(b)(h_2), Q(b)(h_2)) \right\} \end{array} \right] = [0.56949, 0.955468].$$

Similarly, $T_1(P(b)(h_3), Q(b)(h_3)) = [0.982832, 0.986218]$,

$T_2(P(b)(h_3), Q(b)(h_3)) = [0.873791, 0.889109]$ and $S(P(b)(h_3), Q(b)(h_3)) = [0.857883, 0.907223]$.

Now,

$$\left[\begin{array}{l} \min \left\{ T_1^l(P(b)(h_3), Q(b)(h_3)), T_2^l(P(b)(h_3), Q(b)(h_3)), S^l(P(b)(h_3), Q(b)(h_3)) \right\}, \\ \max \left\{ T_1^u(P(b)(h_3), Q(b)(h_3)), T_2^u(P(b)(h_3), Q(b)(h_3)), S^u(P(b)(h_3), Q(b)(h_3)) \right\} \end{array} \right] = [0.857883, 0.986218].$$

Thus, $\varphi(\overline{P}, \overline{Q}) = [0.766804, 0.974657]$.

Also, $\psi(\overline{p}, \overline{q}) = [0.353488, 0.655367]$.

Hence, $\text{Sim}(\overline{P}_p, \overline{Q}_q) = [0.2642658, 0.630127]$.

5 Conclusion

This work's main goal is to present a Type-II DioNSIVSS and examine some of its characteristics. Examined is the application of the similarity metric between two Type-II DioNSIVSS. In the future, the theory of generalized neutrosophic cubic soft sets and generalized bipolar neutrosophic fuzzy soft sets will be used.

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