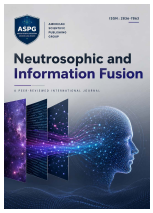
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FULL LENGTH ARTICLE

# On the Stability Analysis of the Fisher Equation Based on Some Numerical Galerkin Techniques

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## ABSTRACT

We studied the stability of the steady-state solutions for the Fisher equation in two cases. In the first case, with constant amplitude, we show that the steady-state solution $u_1 = 1$ is always stable under any condition, whereas the other two solutions, $u_1 = 0$ and $u_1(x) = A\cos(n\pi X)$, are conditionally stable. In the second case, we studied the steady-state solutions for variable amplitude using two methods. The first is an analytical direct method, and the second is a numerical method using the Galerkin technique. Both methods show the same results: the steady-state solution $u_1 = 1$ is always stable under any conditions, whereas the other two solutions, $u_1 = 0$ and $u_1(x) = A\cos(n\pi X)$, are conditionally stable.

**Keywords:** Galerkin techniques ▪ Fisher's equation ▪ Stability ▪ Analysis ▪ Numerical algorithm

## 1. INTRODUCTION

If any system is physical, biological, or otherwise, and it is exposed to external influences (small disturbances) at some time while its behaviour is not significantly affected at all future times, then this system is in a stable state. For example, consider the solar system. If an additional small celestial body is introduced into this system and the original state is not affected by the small perturbations caused by that body, the original state can be said to be stable. Similar questions about stability can be raised in every physical or natural problem [5].

The Fisher equation was developed by Fisher in 1937 as a mathematical model for studying population movement and spread [2]. In 2000, Gourley studied travelling-wave solutions of the nonlocal Fisher equation, a special case of diffusion equations [4]. In 2003, Puri and Wiese developed perturbation expansions to obtain solutions to eigenvalue problems for two types of diffusion equations: the Fisher equation and the time-dependent Ginzburg–Landau equation [9]. In 2004, Ortega-Cejas, Fort, and Mendez studied the role of delay time in modelling the extent of biological expansion for the Fisher equation [7]. Their research showed that delay time reduces the predicted speed of the advancing wave compared with the classical Fisher-equation model without delay.

In this research, the stability of the steady-state solutions of the Fisher equation is studied in two cases. The first deals with stability analysis when the amplitude is constant, and the second deals with stability analysis when the amplitude is variable, using the Galerkin numerical method.

## 2. MATHEMATICAL MODEL

One of the nonlinear diffusion equations is the Fisher equation, formulated as

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} + h(u). \tag{1}$$

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Here, $h(u) = ru(1-u)$ is a nonlinear function satisfying

$$h(u) \in C^1([0,1]), \qquad h(0) = h(1) = 0,$$
$$h'(0) = r > 0, \qquad h'(u) < r, \qquad u \in [0,1].$$

The function $u(x,t)$ represents population density at time $t$ and location $x$. The diffusion coefficient $D$ is always positive, and $r$ is the positive population growth rate [8].

The boundary and initial conditions used to solve the Fisher equation on $x \in [0,1]$ are the Neumann boundary conditions [3]:

$$\left.\begin{aligned} \frac{\partial u(0,t)}{\partial x} &= 0, \\ \frac{\partial u(1,t)}{\partial x} &= 0, \end{aligned}\right\} \qquad u(x,0) = f(x), \quad x \in [0,1]. \tag{2}$$

To obtain the dimensionless Fisher equation, introduce the following dimensionless variables [10, 11]:

$$X = \frac{x}{L}, \qquad T = \frac{D}{L^2}t,$$

where $L$ represents the length of the interval for $x$, so that $0 \le x \le L$. Using these transformations, we obtain

$$\begin{aligned} \frac{D}{L^2}\frac{\partial u}{\partial T} &= \frac{D}{L^2}\frac{\partial^2 u}{\partial X^2} + ru - ru^2, \\ \frac{\partial u}{\partial T} &= \frac{\partial^2 u}{\partial X^2} + \frac{rL^2}{D}u - \frac{rL^2}{D}u^2. \end{aligned} \tag{3}$$

The associated boundary conditions are

$$\frac{\partial u(0,T)}{\partial X} = 0, \qquad \frac{\partial u(1,T)}{\partial X} = 0. \tag{4}$$

Equation (3), with boundary conditions (4), represents the dimensionless form of the Fisher equation.

## 3. STABILITY ANALYSIS

Assume that the system has been disturbed. Let $u_1(X)$ represent the steady state of the system and let $u_2(X,T)$ represent the imposed disturbance [5]:

$$u(X,T) = u_1(X) + u_2(X,T). \tag{5}$$

Substituting (5) into Equation (3) gives

$$\begin{aligned} \frac{\partial u_2}{\partial T} &= \left(\frac{d^2u_1}{dX^2} + \frac{\partial^2 u_2}{\partial X^2}\right) + \frac{rL^2}{D}(u_1+u_2) - \frac{rL^2}{D}(u_1+u_2)^2 \\ &= \frac{d^2u_1}{dX^2} + \frac{\partial^2 u_2}{\partial X^2} + \frac{rL^2}{D}u_1 + \frac{rL^2}{D}u_2 - \frac{rL^2}{D}u_1^2 \\ &\quad - \frac{2rL^2}{D}u_1u_2 - \frac{rL^2}{D}u_2^2. \end{aligned}$$

The steady and time-dependent parts of the Fisher equation can be separated as follows:

$$0 = \frac{d^2u_1}{dX^2} + \frac{rL^2}{D}u_1 - \frac{rL^2}{D}u_1^2, \tag{6}$$

$$\frac{du_1}{dX} = 0 \quad \text{at} \quad X = 0,1. \tag{7}$$

Equation (6), with boundary conditions (7), represents the steady state of the Fisher equation. The disturbance satisfies

$$\frac{\partial u_2}{\partial T} = \frac{\partial^2 u_2}{\partial X^2} + \frac{rL^2}{D}u_2 - \frac{2rL^2}{D}u_1u_2 - \frac{rL^2}{D}u_2^2, \tag{8}$$

$$\frac{\partial u_2(X,T)}{\partial X} = 0 \quad \text{at} \quad X = 0,1. \tag{9}$$

Equation (8), with boundary conditions (9), represents the time-dependent state of the Fisher equation.

## 4. UNIFORM STABILITY SOLUTION

Equation (6), with boundary conditions (7), has the two constant solutions $u_1 = 1$ and $u_1 = 0$; the latter is also called the trivial solution [4]. To obtain a non-constant solution, we use the linearized stability analysis of Equation (6) [5, 10], which becomes

$$\frac{d^2u_1}{dX^2} = \frac{rL^2}{D}u_1 = 0. \tag{10}$$

Equation (10) is a linear homogeneous ordinary differential equation, whose general solution is

$$u_1(X) = A\cos(MX) + B\sin(MX), \tag{11}$$

where $M = \sqrt{rL^2/D}$ and $A$ and $B$ are constants. Applying the boundary conditions gives

$$BM = 0, \tag{12}$$

$$-AM\sin(M) + BM\cos(M) = 0. \tag{13}$$

From Equation (12), since $M = \sqrt{rL^2/D} > 0$, it follows that $B = 0$. Substituting $B = 0$ into Equation (13) gives

$$-AM\sin(M) = 0, \qquad M > 0.$$

Either $A = 0$, which gives the trivial solution $u_1 = 0$, or

$$\begin{aligned} &\sin(M) = 0, \\ &\quad M = n\pi, \qquad n = \pm1, \pm2, \pm3, \ldots, \\ &\sqrt{\frac{rL^2}{D}} = n\pi, \qquad n = \pm1, \pm2, \pm3, \ldots, \\ &\sqrt{\frac{rL^2}{D}} = (n\pi)^2, \qquad n = \pm1, \pm2, \pm3, \ldots. \end{aligned}$$

Thus, the steady-state solution of the Fisher equation satisfying the boundary conditions (7) is

$$u_1(X) = A\cos(n\pi X), \qquad n = \pm1, \pm2, \pm3, \ldots. \tag{14}$$

## 5. STABILITY ANALYSIS FOR CONSTANT AMPLITUDE

Equation (8) represents the time-dependent state of the Fisher equation. Neglecting $u_2^2(X,T)$ because it is very small under the assumption of a small disturbance [5], the equation becomes

$$\frac{\partial u_2}{\partial T} = \frac{\partial^2 u_2}{\partial X^2} + \frac{rL^2}{D}u_2 - \frac{2rL^2}{D}u_1u_2. \tag{15}$$

Assume that

$$u_2(X,T) = Pe^{ik(x-ct)}. \tag{16}$$

Here, $k > 0$ represents the wavenumber, $P > 0$ represents the wave amplitude, and $c = c_1 + ic_2$ represents the complex wave speed. The sign of $c_2$ determines whether the disturbance grows or disappears: the system is unstable when $c_2 > 0$ and stable when $c_2 < 0$ [5].

Substitution of Equation (16) into Equation (15) gives

$$-ikcPe^{ik(x-ct)} = -k^2Pe^{ik(x-ct)} + \frac{rL^2}{D}Pe^{ik(x-ct)} - \frac{2rL^2}{D}Pe^{ik(x-ct)}u_1(X).$$

After simplification,

$$-ikc = -k^2 + \frac{rL^2}{D} - \frac{2rL^2}{D}u_1(X),$$
$$-ik(c_1 + ic_2) = -k^2 + \frac{rL^2}{D} - \frac{2rL^2}{D}u_1(X),$$
$$-ikc_1 + kc_2 = -k^2 + \frac{rL^2}{D} - \frac{2rL^2}{D}u_1(X).$$

Separating the real and imaginary parts gives

$$-ikc_1 = 0, \qquad kc_2 = -k^2 + \frac{rL^2}{D} - \frac{2rL^2}{D}u_1(X).$$

The source article omits equation number 17; its numbering is retained here. Thus,

$$c_2 = \frac{-k^2D + rL^2 - 2rL^2u_1(X)}{kD}. \tag{18}$$

Since $c_2$ is the amplification factor that determines the stability condition, and there are three steady solutions $u_1(X)$ of the Fisher equation, Equation (18) yields three cases.

**1. The first stable solution:** When $u_1 = 0$, Equation (18) becomes

$$c_2 = \frac{-k^2D + rL^2}{kD}. \tag{19}$$

Thus, the solution $u_1 = 0$ is stable if

$$k^2D > rL^2,$$

where $r > 0$, $D > 0$, and $L^2 > 0$.

**2. The second stable solution:** When $u_1 = 1$, Equation (18) becomes

$$c_2 = \frac{-k^2D - rL^2}{kD} = -\frac{k^2D + rL^2}{kD}.$$

Since $r > 0$, $D > 0$, $L^2 > 0$, and $k > 0$, it follows that $c_2 < 0$; therefore, the system is always stable in this case.

**3. The third stable solution:** When $u_1(X) = A\cos(n\pi X)$, as in Equation (14), Equation (18) becomes

$$c_2 = \frac{-k^2D + rL^2 - 2rL^2A\cos(n\pi X)}{kD}. \tag{20}$$

To find the neutral stability curve, set $c_2 = 0$ and take the lowest eigenvalue, $n = 1$. Equation (20) then gives

$$k^2D + rL^2 - 2rL^2A\cos(n\pi X),$$

and hence

$$k = \sqrt{\frac{rL^2 - 2rL^2A\cos(\pi X)}{D}}. \tag{21}$$

Here, $r > 0$, $D > 0$, $k > 0$, $L^2 > 0$, $A > 0$, and $0 \le X \le 1$.

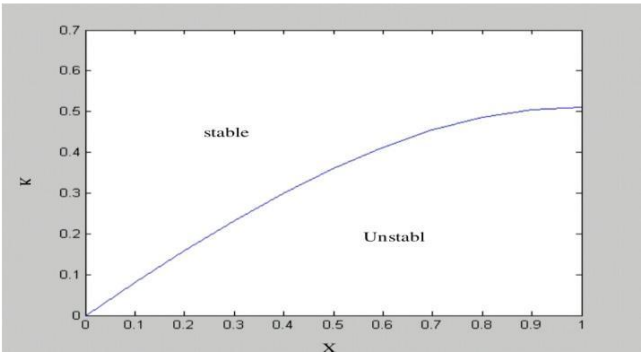



**Figure 1.** The equivalent stability curve of Equation (21) when $D = 1$, $r = 1$, $L^2 = 1$, and $A = 0.5$.

## 6. STABILITY ANALYSIS FOR VARIABLE AMPLITUDE

In this section, stability in the variable-amplitude case is studied in two ways: first analytically, and second numerically using the Galerkin method. Assume that

$$u_2(X,T) = F(X)e^{ik(x-ct)}. \tag{22}$$

Here, $k > 0$ is a real, dimensionless wavenumber, and $F(X)$ represents the variable wave amplitude [6]. Substituting Equation (22) into Equation (15) gives

$$-ikcF(X)e^{ik(x-ct)} = -k^2F(X)e^{ik(x-ct)} + i2kF'(X)e^{ik(x-ct)} + F''(X)e^{ik(x-ct)} + \frac{rL^2}{D}F(X)e^{ik(x-ct)} - \frac{2rL^2}{D}u_1(X)F(X)e^{ik(x-ct)}.$$

After simplification,

$$-ikcF(X) = -k^2F(X) + i2kF'(X) + F''(X) + \frac{rL^2}{D}F(X) - \frac{2rL^2}{D}u_1(X)F(X). \tag{23}$$

Using $c = c_1 + ic_2$ and separating the real and imaginary parts yields

$$-ikc_1F(X) = i2kF'(X),$$
$$kc_2F(X) = -k^2F(X) + F''(X) + \frac{rL^2}{D}F(X) - \frac{2rL^2}{D}u_1(X)F(X).$$

Therefore,

$$F''(X) + \left(\frac{rL^2}{D} - k^2 - kc_2 - \frac{2rL^2}{D}u_1(X)\right)F(X) = 0. \tag{24}$$

The boundary conditions (9) become

$$F'(X) = 0 \quad \text{at} \quad X = 0 \text{ and } X = 1. \tag{25}$$

### First: Analytical Solution

Equation (24) can be written as

$$F''(X) + \lambda F(X) = 0, \tag{26}$$

where

$$\lambda = \frac{rL^2}{D} - k^2 - kc_2 - \frac{2rL^2}{D}u_1(X). \tag{27}$$

The general solution of Equation (26) is

$$F(X) = a\cos(\sqrt{\lambda}X) + b\sin(\sqrt{\lambda}X). \tag{28}$$

Substituting the boundary conditions (25) gives

$$b\sqrt{\lambda} = 0, \tag{29}$$

$$-a\sqrt{\lambda}\sin\sqrt{\lambda} + b\sqrt{\lambda}\cos\sqrt{\lambda} = 0. \tag{30}$$

From Equation (29), since $\lambda > 0$, it follows that $b = 0$. Equation (30) therefore becomes

$$-a\sqrt{\lambda}\sin\sqrt{\lambda} = 0, \qquad \lambda > 0.$$

Either $a = 0$, which leads to the trivial solution, or

$$\sin\sqrt{\lambda} = 0 \quad \Longrightarrow \quad \lambda = (n\pi)^2, \qquad n = \pm 1, \pm 2, \pm 3, \ldots.$$

Using Equation (27),

$$\frac{rL^2}{D} - k^2 - kc_2 - \frac{2rL^2}{D}u_1(X) = (n\pi)^2,$$

which gives

$$c_2 = \frac{-k^2D - 2rL^2u_1(X) - (n\pi)^2D + rL^2}{kD}. \tag{31}$$

Since $c_2$ determines the stability condition and there are three steady solutions $u_1(X)$ of the Fisher equation, Equation (31) yields three cases.

**1. The first stable solution:** When $u_1 = 0$, Equation (31) becomes

$$c_2 = \frac{-k^2D - (n\pi)^2D + rL^2}{kD}. \tag{32}$$

Thus, the solution $u_1 = 0$ is stable if

$$k^2D - (n\pi)^2D > rL^2,$$

where $L^2 > 0$, $D > 0$, and $r > 0$.

**2. The second stable solution:** When $u_1 = 1$, Equation (31) becomes

$$c_2 = \frac{-k^2D - rL^2 - (n\pi)^2D}{kD} = -\frac{k^2D + rL^2 + (n\pi)^2D}{kD}.$$

Since $r > 0$, $D > 0$, $L^2 > 0$, $(n\pi)^2 > 0$, and $k > 0$, the solution $u_1 = 1$ is always stable.

**3. The third stable solution:** When $u_1(X) = A\cos(n\pi X)$, Equation (31) becomes

$$c_2 = \frac{-k^2D - rL^2A\cos(n\pi X) - (n\pi)^2D + rL^2}{kD}. \tag{33}$$

To find the neutral stability curve, set $c_2 = 0$ and take the lowest eigenvalue at $n = 1$. Equation (33) gives

$$k^2D = -rL^2A\cos(n\pi X) - (n\pi)^2D + rL^2,$$

and hence

$$k = \sqrt{\frac{rL^2}{D} - \frac{rL^2A\cos(\pi X)}{D} - 9.8696}. \tag{34}$$

Here, $r > 0$, $D > 0$, $k > 0$, $L^2 > 0$, $A > 0$, and $0 \le X \le 1$.

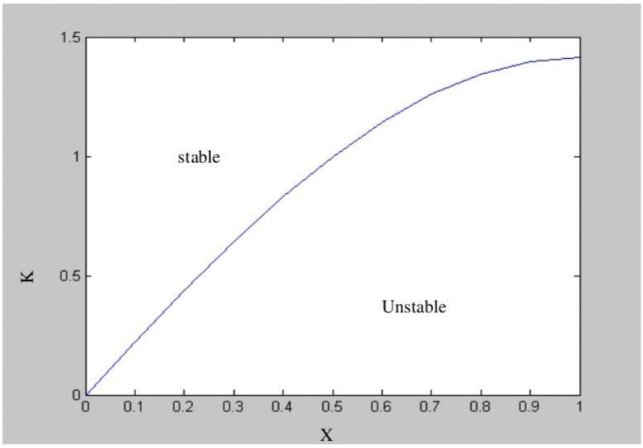



**Figure 2.** The equivalent stability curve of Equation (34) when $A = 0.00652$, $L^2 = 1$, $r = 10$, and $D = 1$.

### Second: Galerkin Numerical Technique

A numerical solution of Equation (24), with boundary conditions (25), is obtained using the Galerkin numerical method [1]. Assume that the solution of Equation (26) has the form

$$F(X) = \sum_{v=1}^{\infty} (A_v\cos\lambda_v X + B_v\sin\lambda_v X). \tag{35}$$

Taking one term of the series in Equation (35) gives

$$F(X) = A_v\cos\lambda_v X + B_v\sin\lambda_v X. \tag{36}$$

Using the boundary conditions (25), we find

$$B_v\lambda_v = 0, \tag{37}$$

$$-A_v\lambda_v\sin\lambda_v + B_v\lambda_v\cos\lambda_v = 0. \tag{38}$$

From Equation (37), since $\lambda_v \neq 0$, it follows that $B_v = 0$. Equation (38) then becomes

$$-A_v\lambda_v\sin\lambda_v = 0.$$

Since $\lambda_v \neq 0$, either $A_v = 0$, which is the trivial solution, or

$$\sin\lambda_v = 0 \quad \Longrightarrow \quad \lambda_v = n\pi, \qquad n = 1, 2, 3, \ldots.$$

Because $B_v = 0$, Equation (36) becomes

$$F(X) = A_v\cos\lambda_v X, \qquad \lambda_v = n\pi, \quad n = 1, 2, 3, \ldots.$$

Consequently, the assumed solution (35) satisfying the boundary conditions must have the form

$$F(X) = \sum_{v=1}^{\infty} A_v \cos \lambda_v X, \qquad \lambda_v = n\pi, \quad n = 1,2,3,\ldots. \tag{39}$$

Substituting Equation (39) into Equation (24) gives

$$\sum_{v=1}^{\infty} \left(-\lambda_v^2 A_v \cos \lambda_v X\right) + \sum_{v=1}^{\infty} A_v \left(\frac{rL^2}{D} - k^2 - kc_2 - \frac{2rL^2}{D} u_1(X)\right) \cos \lambda_v X = 0.$$

According to the Galerkin method, the finite approximation is not equal to zero but to an error ratio, called the residual:

$$\sum_{v=1}^{p} \left(-\lambda_v^2 A_v \cos \lambda_v X\right) + \sum_{v=1}^{p} A_v \left(\frac{rL^2}{D} - k^2 - kc_2 - \frac{2rL^2}{D} u_1(X)\right) \cos \lambda_v X = \varepsilon.$$

The Galerkin method seeks to minimize this residual. The integral, over the boundary interval, of the residual multiplied by a suitable function is therefore set equal to zero [1]:

$$\int_0^1 \left[ \sum_{v=1}^{p} \sum_{m=1}^{p} \left(-\lambda_v^2 A_v \cos \lambda_v X\right) \varphi_m(X) + \sum_{v=1}^{p} \sum_{m=1}^{p} \left(\frac{rL^2}{D} - k^2 - kc_2 - \frac{2rL^2}{D} u_1(X)\right) \times \left(A_v \cos \lambda_v X\right) \varphi_m(X) \right], dX = 0. \tag{40}$$

Here, $\varphi_m(X)$ is a suitable trigonometric function:

$$\varphi_m(X) = \sin \lambda_m X, \qquad \lambda_m = m\pi, \quad m = 1,2,3,\ldots.$$

From Equation (40),

$$\frac{rL^2}{D} - k^2 - kc_2 - \frac{2rL^2}{D} u_1(X) = \lambda_v^2,$$
$$\frac{rL^2}{D} - k^2 - kc_2 - \frac{2rL^2}{D} u_1(X) = (n\pi)^2.$$

Therefore,

$$c_2 = \frac{-k^2 D - 2rL^2 u_1(X) - (n\pi)^2 D + rL^2}{kD}. \tag{41}$$

Equation (41) is identical to the analytical solution (31). This indicates the efficiency of the numerical Galerkin method and confirms the analytical solution obtained for the stability conditions of the Fisher equation in the variable-amplitude case.

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