



On the Stability Analysis of the Fisher Equation Based on Some Numerical Galerkin Techniques

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Abstract

We studied the stability of the steady state solutions for Fisher Equation in two cases, the First one with constant amplitude and we show that the steady state solution $u_1 = 1$ is always stable under any condition, but the other two solutions $u_1 = 0$ and $u_1(x) = A \cos(n\pi X)$ are conditionally stable.

In the Second case, we studied the steady state solutions for various amplitude by using two Methods. The First is analytically by direct Method and the second is numerical method using Galerkin technique which shows the same results, that is the steady state solution $u_1 = 1$ is always stable under any conditions, but the other two solutions $u_1 = 0$ and $u_1(x) = A \cos(n\pi X)$ are conditionally stable.

Keywords: Galrekin techniques; Fisher's equation; Stability; Analysis; Numerical algorithm

1. Introduction

If any system (system) is physical, biological or other, if it is exposed to external influences (small disturbances) at some time and the behavior of the system is not significantly affected in all future times, then this system is in a stable state, for example, the solar system, it is known that if an additional small celestial body is introduced into this system, the original state will not be affected by the small perturbations caused by that body, so it can be said that the original state is stable, and similar questions about stability can be raised in every physical or natural problem[5]. The Fisher equation was developed by the scientist Fisher in 1937, when he proposed it as a mathematical model for studying the science of population movement and spread [2]. In (2000) Gourley studied [4] the transmission wave solutions of the nonlocal Fisher equation, a special case of the diffusion equations, In 2002, [9] Wiese & Puri developed perturbation expansions to obtain solutions to the problems of elementary values of two types of diffusion equations, the first is the Fisher equation, the second is the time-dependent Ginzburg-Landau equation, and in 2004 Fort & Mendez & Ortega studied [7] the role of delay time in modeling the extent of biological expansion of the Fisher equation, the research has proved that the delay time reduces the predicted speed of the advancing wave compared to the classical model of the Fisher equation i.e. (Fisher equation without delay time). In this research, the analysis of the Steady State Solutions of the Fisher equation will be studied in two cases, the first deals with the study of stability analysis in the case that the amplitude is constant, and the second deals with the study of stability analysis in the case that the amplitude is variable using the Galerkin numerical method in this case.

2. Mathematical Model

One of the nonlinear diffusion equations is the Fisher equation, which is formulated by:

$$\frac{\partial u(x,t)}{\partial t} = D \frac{\partial^2 u(x,t)}{\partial x^2} + h(u) \quad (1)$$

Where $h(u) = ru(1-u)$ is a nonlinear function that satisfies the following conditions:

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$$\begin{aligned} h(u) &\in C^1([0,1]) \\ h(0) &= h(1) = 0 \\ h'(0) &= r > 0 \\ h'(u) &< r \\ u &\in [0,1] \end{aligned}$$

$u(x,t)$ represents the population Density at time (t) and at location (D), (x) is the diffusion Coefficient, which is always positive, (r) represents the population growth Rate, which is also positive and increases if there is a huge increase in the population, it shows how the population is after a certain time[8].

The boundary and elementary conditions used to solve the Fisher equation in the period $x \in [0,1]$ are the Neumann boundary Conditions [3]:

$$\left. \begin{aligned} \frac{\partial u(0,t)}{\partial x} &= 0 \\ \frac{\partial u(1,t)}{\partial x} &= 0 \end{aligned} \right\} \dots(2)$$

$$u(x, 0) = f(x), \quad x \in [0,1]$$

For the purpose of finding the Fisher dimensionless equation we will know the dimensionless values of the following [10, 11]:

$$X = \frac{x}{L}, \quad T = \frac{D}{L^2}t$$

Where L represents the length of the period for x, that is:

$$0 \leq x \leq L$$

Using these transformations, we obtain:

$$\begin{aligned} \frac{D}{L^2} \frac{\partial u}{\partial T} &= \frac{D}{L^2} \frac{\partial^2 u}{\partial X^2} + ru - ru^2 \\ \Rightarrow \frac{\partial u}{\partial T} &= \frac{\partial^2 u}{\partial X^2} + \frac{rL^2}{D}u - \frac{rL^2}{D}u^2 \end{aligned} \quad (3)$$

$$\frac{\partial u(0,T)}{\partial X} = 0, \quad \frac{\partial u(1,T)}{\partial X} = 0 \quad (4)$$

Equation (3) with boundary conditions (4) represents the dimensionless formula of the Fisher equation.

3. Stability Analysis

We assume that the system has been disturbed, and we assume that $u_1(X)$ represents the stability of the system and $u_2(X, T)$ represents the imposed state of disturbance [5]:

$$u(X, T) = u_1(X) + u_2(X, T) \quad (5)$$

Substituting (5) into Equation (3) we get:

$$\begin{aligned} \frac{\partial u_2}{\partial T} &= \left(\frac{\partial^2 u_1}{\partial X^2} + \frac{\partial^2 u_2}{\partial X^2} \right) + \frac{rL^2}{D}(u_1 + u_2) - \frac{rL^2}{D}(u_1 + u_2)^2 \\ \Rightarrow \frac{\partial u_2}{\partial T} &= \frac{\partial^2 u_1}{\partial X^2} + \frac{\partial^2 u_2}{\partial X^2} + \frac{rL^2}{D}u_1 + \frac{rL^2}{D}u_2 - \frac{rL^2}{D}u_1^2 - \frac{2rL^2}{D}u_1u_2 - \frac{rL^2}{D}u_2^2 \end{aligned}$$

The stability of the Fisher equation can be separated from the time stability as follows:

$$0 = \frac{\partial^2 u_1}{\partial X^2} + \frac{rL^2}{D}u_1 - \frac{rL^2}{D}u_1^2 \quad ..(6)$$

$$\frac{du_1}{dx} = 0 \text{ at } X = 0,1 \quad (7)$$

Where equation (6) with boundary conditions (7) represents the stability of Fisher Equation:

$$\frac{\partial u_2}{\partial T} = \frac{\partial^2 u_2}{\partial X^2} + \frac{rL^2}{D}u_2 - \frac{2rL^2}{D}u_1u_2 - \frac{rL^2}{D}u_2^2 \quad (8)$$

$$\frac{\partial u_2(X,T)}{\partial X} = 0 \text{ at } X = 0,1 \quad (9)$$

Where equation (8) with boundary conditions (9) represents the time state of the Fisher equation.

4. Uniform stability Solution

Equation (6) with boundary conditions (7) has two solutions: $u_1 = 1, u_1 = 0$, These are constant Solutions, since $u_1 = 0$ is also called the trivial solution [4].

To obtain the non-constant solution of equation (6) with boundary conditions, we resort to Linearized Stability Analysis of equation (6) [5, 10] to become:

$$\frac{d^2 u_1}{dx^2} = \frac{rL^2}{D}u_1 = 0 \quad (10)$$

Equation (10) is a linear homogeneous Ordinary Differential Equation and the general solution of this equation is:

$$u_1(X) = A \cos MX + B \sin MX \tag{11}$$

Where $M = \sqrt{\frac{rL^2}{D}}$, A , B are constants.

$$BM = 0 \tag{12}$$

$$-AM \sin(M) + BM \cos(M) = 0 \tag{13}$$

From equation (12), since $M = \sqrt{\frac{rL^2}{D}} > 0$, then $B = 0$.

Substituting the value of B in equation (13), we find that:

$$-AM \sin(M) = 0$$

Where $M > 0$

Either $A = 0$ and this is the trivial solution $u_1 = 0$

Or

$$\sin(M) = 0$$

$$\Rightarrow M = n\pi, \quad n = \pm 1, \pm 2, \pm 3, \dots$$

$$\Rightarrow \sqrt{\frac{rL^2}{D}} = n\pi, \quad n = \pm 1, \pm 2, \pm 3, \dots$$

$$\Rightarrow \frac{rL^2}{D} = (n\pi)^2, \quad n = \pm 1, \pm 2, \pm 3, \dots$$

So the Steady State Solution of the Fisher equation, which satisfies the boundary conditions (7), is:

$$u_1(X) = A \cos(n\pi X), \quad n = \pm 1, \pm 2, \pm 3, \dots \tag{14}$$

5. Stability analysis in the case of constant amplitude (perturbation state)

Equation (8) represents the time state of the Fisher equation and neglects $u_2^2(X, T)$ because it is a very small amount (under the assumption that the disturbance is small) [5].

Equation (8) becomes:

$$\frac{\partial u_2}{\partial T} = \frac{\partial^2 u_2}{\partial X^2} + \frac{rL^2}{D} u_2 - \frac{2rL^2}{D} u_1 u_2 \tag{15}$$

We assume that:

$$u_2(X, T) = P e^{ik(x-ct)} \tag{16}$$

Since $k > 0$ represents the wavenumber, $P > 0$ represents the wave amplitude, and in this case it has a constant value, either c represents the wave speed, which has a complex value ($c = c_1 + ic_2$), and the positive or negative value of c_2 in this case is what leads to the growth of the disturbance or its disappearance, respectively, when $c_2 > 0$, the system is unstable, and when $c_2 < 0$ the system is stable. [5]

Substituting hypothesis (16) into equation (15), we find that:

$$-ikcP e^{ik(x-ct)} = i^2 k^2 P e^{ik(x-ct)} + \frac{rL^2}{D} P e^{ik(x-ct)} - \frac{2rL^2}{D} P e^{ik(x-ct)} u_1(X)$$

The previous equation can be simplified to become:

$$-ikc = -k^2 + \frac{rL^2}{D} - \frac{2rL^2}{D} u_1(X)$$

$$-ik(c_1 + ic_2) = -k^2 + \frac{rL^2}{D} - \frac{2rL^2}{D} u_1(X)$$

$$\Rightarrow -ikc_1 + kc_2 = -k^2 + \frac{rL^2}{D} - \frac{2rL^2}{D} u_1(X)$$

After separating the real part and the imaginary part we get:

$$-ikc_1 = 0$$

$$kc_2 = -k^2 + \frac{rL^2}{D} - \frac{2rL^2}{D} u_1(X)$$

$$\Rightarrow c_2 = \frac{-k^2 D + rL^2 - 2rL^2 u_1(X)}{kD} \tag{18}$$

Since c_2 is the Amplification Factor that leads to the determination of the stability condition, and there are three timeless solutions $u_1(X)$ to the Fisher equation, so there are three cases of equation (16) are:

1. The first stable solution:

When $u_1 = 0$, equation (18) becomes:

$$c_2 = \frac{-k^2 D + rL^2}{kD} \tag{19}$$

Thus, the solution $u_1 = 0$ is stable if:

$$k^2 D > r L^2$$

Where $r > 0, D > 0, L^2 > 0$

2.The second stable solution:

When $u_1 = 1$, equation (18) becomes:

$$c_2 = \frac{-k^2 D - r L^2}{k D}$$

$$\Rightarrow c_2 = \frac{-[k^2 D + r L^2]}{k D}$$

Since $r > 0, D > 0, L^2 > 0, k > 0$, therefore $c_2 < 0$ is always such that the system in this case is always stable.

3.The third stable solution:

When $u_1(X) = A \cos n\pi X$ as in equation (14), equation (18) becomes:

$$c_2 = \frac{-k^2 D + r L^2 - 2r L^2 A \cos(n\pi X)}{k D} \tag{20}$$

To find the neutral Stability Curve, we put $c_2 = 0$, and take the lowest eigenvalue when $n = 1$, and from equation (20) we get:

$$k^2 D + r L^2 - 2r L^2 A \cos(n\pi X)$$

$$\Rightarrow k = \sqrt{\frac{r L^2 - 2r L^2 A \cos(\pi X)}{D}} \tag{21}$$

Where $r > 0, D > 0$ and $k > 0, L^2 > 0, A > 0$ and the starting values of X are:
 $0 \leq X \leq 1$

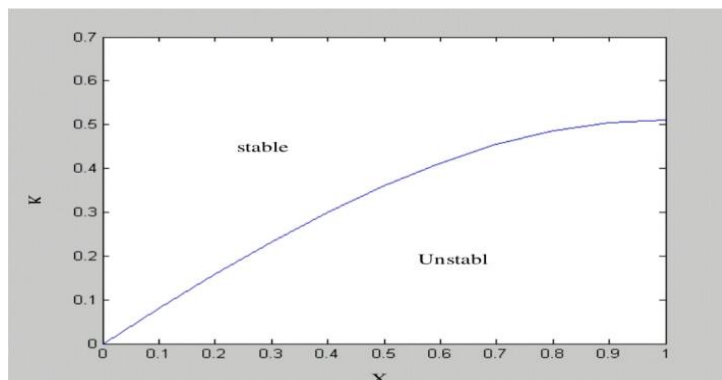


Figure 1. The equivalent stability curve of equation (21) when $D = 1, r = 1, L^2 = 1, A = 0.5$

6. Stability analysis in the case of variable amplitude (perturbation state)

In this section, the stability study in the case of variable capacitance will be carried out in two ways, the first is the stability study in an analytical way, and the second is the stability study in a numerical way using the Galerkin numerical method.

We assume that:

$$u_2(X, T) = F(X)e^{ik(x-ct)} \tag{22}$$

Since $k > 0$ is a real, dimensionless value of the wavelength, $F(X)$ represents the amplitude of the wave and in this case is a variable, i.e. a function with a function X [6].

Substituting equation (22) into equation (15), we obtain:

$$-ikcF(X)e^{ik(x-ct)}$$

$$= -k^2 F(X)e^{ik(x-ct)} + i2kF'(X)e^{ik(x-ct)} + F''(X)e^{ik(x-ct)} + \frac{rL^2}{D} F(X)e^{ik(x-ct)}$$

$$- \frac{2rL^2}{D} u_1(X)F(X)e^{ik(x-ct)}$$

After simplifying the amount, we get:

$$-ikcF(X) = -k^2 F(X) + i2kF'(X) + F''(X) + \frac{rL^2}{D} F(X) - \frac{2rL^2}{D} u_1(X)F(X) \tag{23}$$

$$\Rightarrow -ikc_1F(X) + kc_2F(X) = -k^2F(X) + i2kF'(X) + F''(X) + \frac{rL^2}{D}F(X) - \frac{2rL^2}{D}u_1(X)F(X)$$

After separating the real part and the imaginary part we get:

$$-ikc_1F(X) = i2kF'(X)$$

$$kc_2F(X) = -k^2F(X) + F''(X) + \frac{rL^2}{D}F(X) - \frac{2rL^2}{D}u_1(X)F(X)$$

$$\Rightarrow F''(X) + \left(\frac{rL^2}{D} - k^2 - kc_2 - \frac{2rL^2}{D}u_1(X)\right)F(X) = 0 \tag{24}$$

And the boundary conditions (9) become:

$$F'(X) = 0 \text{ at } X=0 \text{ and } X=1 \tag{25}$$

First: Analytical Solution:

Equation (24) can be written in the following form:

$$F''(X) + \lambda F(X) = 0 \tag{26}$$

Where

$$\lambda = \frac{rL^2}{D} - k^2 - kc_2 - \frac{2rL^2}{D}u_1(X) \tag{27}$$

The general solution of equation (26) is as follows:

$$F(X) = a \cos\sqrt{\lambda}X + b \sin\sqrt{\lambda}X \tag{28}$$

Substituting the boundary conditions (25) we can obtain:

$$b\sqrt{\lambda} = 0 \tag{29}$$

$$-a\sqrt{\lambda} \sin\sqrt{\lambda} + b\sqrt{\lambda} \cos\sqrt{\lambda} = 0 \tag{30}$$

From equation (29), since $\lambda > 0$, therefore $b=0$.

So equation (30) becomes:

$$-a\sqrt{\lambda} \sin\sqrt{\lambda} = 0$$

Where: $\lambda > 0$

Either $a = 0$, this leads to the trivial solution.

Or $\sin\sqrt{\lambda} = 0$

$$\Rightarrow \lambda = (n\pi)^2 \quad n = \pm 1, \pm 2, \pm 3, \dots$$

From equation (27) we get:

$$\begin{aligned} \frac{rL^2}{D} - k^2 - kc_2 - \frac{2rL^2}{D}u_1(X) &= (n\pi)^2 \\ \Rightarrow c_2 &= \frac{-k^2D - 2rL^2u_1(X) - (n\pi)^2D + rL^2}{kD} \end{aligned} \tag{31}$$

Since it is c_2 that leads to the determination of the stability condition, and there are three timeless solutions $u_1(X)$ of the Fisher equation, so there are three states of equation (31) are:

1. The first stable solution:

When $u_1 = 0$, equation (31) becomes:

$$c_2 = \frac{-k^2D - (n\pi)^2D + rL^2}{kD} \tag{32}$$

Thus, the solution $u_1 = 0$ is stable if:

$$k^2D - (n\pi)^2D > rL^2$$

Where

$$L^2 > 0, D > 0, r > 0$$

2. The second stable solution:

When $u_1 = 1$, equation (31) becomes:

$$\begin{aligned} c_2 &= \frac{-k^2D - rL^2 - (n\pi)^2D}{kD} \\ \Rightarrow c_2 &= \frac{-[k^2D + rL^2 + (n\pi)^2D]}{kD} \end{aligned}$$

Since $r > 0, D > 0, L^2 > 0, (n\pi)^2 > 0, k > 0$, so the solution $u_1 = 1$ is always stable.

3. The third stable solution:

When $u_1(X) = A\cos(n\pi X)$, equation (31) becomes:

$$c_2 = \frac{-k^2D - rL^2A\cos(n\pi X) - (n\pi)^2D + rL^2}{kD} \tag{33}$$

To find the neutral stability curve, we put $c_2 = 0$ and take the lowest eigenvalue at $n=1$, and from equation (33) we get:

$$k^2D = -rL^2A\cos(n\pi X) - (n\pi)^2D + rL^2$$

$$\Rightarrow k = \sqrt{\frac{rL^2}{D} - \frac{rL^2 \cos(\pi X)}{D} - 9.8696} \tag{34}$$

Where $r > 0$, $D > 0$, $k > 0$, $L^2 > 0$, $A > 0$ and the starting values for X are $0 \leq X \leq 1$

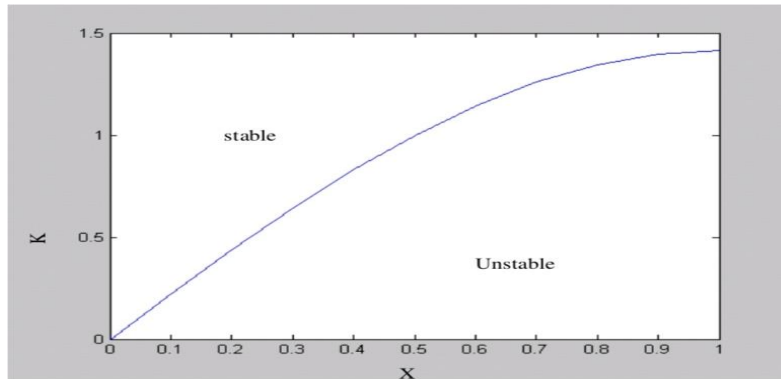


Figure 2. The equivalent stability curve of equation (34) When $A= 0.00652$, $L^2 = 1$, $r=10$, $D=1$

Second: Galerkin numerical technique

A numerical solution of equation (24) with boundary conditions (25) will be found using the Galerkin numerical method [1].

We assume that the solution to equation (26) is in the following form:

$$F(X) = \sum_{v=1}^{\infty} (A_v \cos \lambda_v X + B_v \sin \lambda_v X) \tag{35}$$

Taking one of the limits of the previous solution of equation (35), we obtain:

$$F(X) = A_v \cos \lambda_v X + B_v \sin \lambda_v X \tag{36}$$

Using the boundary conditions (25), we find that:

$$B_v \lambda_v = 0 \tag{37}$$

$$-A_v \lambda_v \sin \lambda_v + B_v \lambda_v \cos \lambda_v = 0 \tag{38}$$

From equation (37), since $\lambda_v \neq 0$, so $B_v = 0$, then equation (38) becomes:

$$-A_v \lambda_v \sin \lambda_v = 0$$

Since $\lambda_v \neq 0$, either $A_v = 0$ and this is the trivial solution.

Or

$$\sin \lambda_v = 0$$

$$\Rightarrow \lambda_v = n\pi, \quad n = 1, 2, 3, \dots$$

Since $B_v = 0$, then the solution (36) in order to satisfy the boundary conditions (25) becomes:

$$F(X) = A_v \cos \lambda_v X$$

Where:

$$\lambda_v = n\pi, \quad n = 1, 2, 3, \dots$$

Since solution (36) is one of the limits of the imposed solution (35), so solution (35), in order to achieve

The boundary conditions (25) must be in the following form:

$$F(X) = \sum_{v=1}^{\infty} A_v \cos \lambda_v X \tag{39}$$

$$\lambda_v = n\pi, \quad n = 1, 2, 3, \dots$$

Substituting equation (39) into equation (24), we obtain:

$$\sum_{v=1}^{\infty} (-\lambda_v^2 A_v \cos \lambda_v X) + \sum_{v=1}^{\infty} A_v \left(\frac{rL^2}{D} - k^2 - kc_2 - \frac{2rL^2}{D} u_1(X) \right) \cos \lambda_v X = 0$$

According to the Galerkin method, the approximation here is not equal to zero, but is the error ratio (Error), which is called the residual (Residual), that is:

$$\sum_{v=1}^p (-\lambda_v^2 A_v \cos \lambda_v X) + \sum_{v=1}^p A_v \left(\frac{rL^2}{D} - k^2 - kc_2 - \frac{2rL^2}{D} u_1(X) \right) \cos \lambda_v X = \epsilon$$

Since, according to the Galerkin method, we strive to make this error or residual (ϵ) as small as possible (Minimize the error), and the integration defined by the boundary conditions of the previous residual multiplied by a function is equal to Zero [1], therefore:

$$\int_0^1 \left[\sum_{v=1}^p \sum_{m=1}^p (-\lambda_v^2 A_v \cos \lambda_v X) \varphi_m(X) + \sum_{v=1}^p \sum_{m=1}^p \left(\frac{rL^2}{D} - k^2 - kc_2 - \frac{2rL^2}{D} u_1(X) \right) (A_v \cos \lambda_v X) \varphi_m(X) \right] dX = 0$$

Where $\varphi_m(X)$ is a suitable trigonometric function.

$$\varphi_m(X) = \sin \lambda_m X$$

$$\lambda_m = m\pi, \quad m = 1, 2, 3, \dots$$

From equation (40) we get:

$$\frac{rL^2}{D} - k^2 - kc_2 - \frac{2rL^2}{D} u_1(X) = \lambda_v^2$$

$$\Rightarrow \frac{rL^2}{D} - k^2 - kc_2 - \frac{2rL^2}{D} u_1(X) = (n\pi)^2$$

$$\Rightarrow c_2 = \frac{-k^2 D - 2rL^2 u_1(X) - (n\pi)^2 D + rL^2}{kD} \quad (41)$$

We note that equation (41) is identical to the analytical solution (31) and this indicates the efficiency of the numerical Galerkin method as well as proves the correctness of the analytical solution reached in finding the stability conditions of the Fisher equation in the case of variable capacitance.

References

- [1] Allaire, P. E. (1985) Basics of the Finite Element Method, Wm. C. Brown Publishers.
- [2] Fisher, R.A. (1937) "The Wave of Advance of Advantageous Genes", *Annals of Eugenics* 7, pp. 355–369
- [3] Fory's, U. and A. Marciniak-Czochra, (2003) "Logistic Equations in Tumour Growth Modelling", *Int. J. Appl. Math. Comput. Sci.*, Vol. 13, No. 3, pp. 317–325.
- [4] Gourley, S.A. (2000) "Travelling Front Solutions of a Nonlocal Fisher Equation", *J. Math. Biol.* 41, pp. 272–284.
- [5] Logan, J. D. (1987) Applied Mathematics, John Wiley and Sons.
- [6] Manaa, Saad A. and B.M. Ibrahim, (2004) "Stability Analysis for A Fully Developed Laminar Fluid Flow in A Rectangular Bend with Secondary Flow", *Raf. Jour. Sci.*, Vol. 15, No.1, Math & Stati., Special Issue, pp. 146-152.
- [7] Ortega-Cejas, V.; J. Fort, and V. Mendez, (2004) "The Role of The Delay Time in The Modeling of Biological Range Expansions", *J. Ecology*, 85 (1), pp. 258–264.
- [8] Pesin, Y. and A. Yurchenko, (2004) "Some Physical Models of The Reaction-Diffusion Equation and Coupled Map Lattices", *Russian Math. Surveys*, Vol. 8, No. 3, pp.177-218.
- [9] Puri, S. and Wiese, K. J. (2003) "Perturbative Linearization of Reaction-Diffusion Equations", *J. Phys.*, A36, pp.2043-2054.
- [10] Sasaki, T. (2004) "The Effect of Local Prevention in an Sis Model With Diffusion", *Discrete and Continuous Dynamical Systems Series B*, Vol. 4, No. 3, PP. 739–746
- [11] Smith, G. D. (1965) Numerical Solution of Partial Differential Equations, Oxford University Press