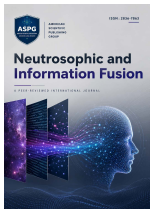
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FULL LENGTH ARTICLE

Finding the Complete List of Different K-Brackets for the Projective Plane $\mathrm{PG}(2,8)$

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ABSTRACT

A k-arc in a plane $\mathrm{PG}(2,q)$ is a set of k points such that every line in the plane intersects it in at most two points and there is a line intersecting it in exactly two points. A k-arc is complete if there is no $(k+1)$-arc containing it. This study is concerned with k-arcs, $k = 4,5,\ldots,10$, and with the classification of projectively distinct k-arcs and distinct arcs under collineation. Using a computer program, we prove that complete k-arcs occur only for $k = 6$ and $k = 10$.

Keywords: k-bracket ▪ Projection ▪ Projective plane ▪ k-arc

1. INTRODUCTION

The K-arc is called complete if it is not possible to have a containing $(k+1)$-arc. Hirschfeld [6] made a set of studies of K-arcs in the projective plane $\mathrm{PG}(2,q)$ defined over the Galois field $\mathrm{GF}(q)$ for values $q < 9$. Younis [10] also conducted the first study of the K-arcs in the projective plane $\mathrm{PG}(2,16)$. More recently, Coolsaet and Sticker [3, 4] classified the K-arcs in the projective plane $\mathrm{PG}(2,q)$ for $q = 23,25,27$. Chao and Kaneta [2] classified the k-arcs in the projective plane $\mathrm{PG}(2,q)$ for $23 \le q \le 29$.

It is worth noting that studying this topic required a substantial period of time to implement the computer programs used to find projectively distinct k-arcs and distinct K-arcs under collineation.

Example. Let $F(x) = x^3 + x + 1$ be an indecomposable polynomial over $\mathbb{Z}_2$. Then

$$\mathrm{GF}(8) = \left\{ \sum_{i=0}^{h-1} a_i w^i \;\middle|\; a_i \in \mathbb{Z}_2,\ w^3 + w + 1 = 0 \right\}, \qquad h = 3. \tag{1}$$

Table 1. The elements of $\mathrm{GF}(8)$

a_0	a_1	a_2	$a_0 + a_1 w + a_2 w^2$
0	0	0	0
0	1	0	w
0	1	1	$w + w^2$
0	0	1	w^2
1	0	0	1
1	1	0	$1 + w$
1	0	1	$1 + w^2$
1	1	1	$1 + w + w^2$

The elements of $\mathrm{GF}(8)$ are

$$\mathrm{GF}(8) = \{0, 1, w, w^2, w^3, w^4, w^5, w^6 : w^7 = 1\}, \tag{2}$$

where w is a primitive root.

(1-2) Definition [6]. If $F(x) = x^n - a_{n-1}x^{n-1} - \cdots - a_0$ is a monic polynomial, the companion matrix $C(F)$ is the $n \times n$

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matrix

$$C(F)=\begin{bmatrix}0&1&0&0&\cdots&0\\0&0&1&0&\cdots&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&0&\cdots&1\\a_0&a_1&a_2&a_3&\cdots&a_{n-1}\end{bmatrix}. \tag{3}$$

(1-3) Definition [6]. If α and α' are projective spaces, then the projection $S:\alpha\to\alpha'$ is a bijection represented by a nonsingular matrix T such that, if $P(x')=P(x)S$, then $tx'=xT$, where x and x' are coordinate vectors of the points $P(x)$ and $P(x')$, and $t\in K_0$.

(1-4) Principle of duality [6]. For any projective space $S=\mathrm{PG}(n,K)$, there is a space S^* called a dual space in which the points and primes correspond respectively to the primes and points in S.

(1-5) Fundamental theorem of projective geometry [6].

1. If $\{p_1,p_2,\ldots,p_{n+2}\}$ and $\{p'_1,p'_2,\ldots,p'_{n+2}\}$ are two sets of $n+2$ points in $\mathrm{PG}(n,k)$ such that no $n+1$ selected points from the same set fall in a prime, there is a unique projection S such that $p'_i=p_iS$.
2. Let $A=\mathrm{PG}(n,k)$. The function $S':A\to A$ is called a collineation if $S'=\sigma S$, where σ is an automorphism on A and S is a projection.

(1-6) Cyclic projectivity [6]. Let T be a projectivity in the plane $\mathrm{PG}(n,q)$. Then T is called a cyclic projectivity if all points of the space can be arranged in one cycle. For example,

$$T=\begin{bmatrix}0&1&0\\0&0&1\\1&0&\lambda^3\end{bmatrix} \tag{4}$$

is a cyclic projectivity, where λ is a primitive element of $\mathrm{GF}(8)$.

(1-7) Definition [6]. An (k,n)-arc in $\mathrm{PG}(n,q)$ is a set consisting of k points such that at most n of them are on a line, and exactly n of them are on some line. When $n=2$, the $(k,2)$-arc is denoted by a k-arc.

(1-8) Theorem [10]. Let $t(P)$ represent the number of unisecants through a point P of a k-arc K, and let T_i represent the number of i-secants. For a k-arc in the plane,

$$t(P)=q+2-k=t, \tag{5}$$

$$T_2=\frac{k(k-1)}{2}, \tag{6}$$

$$T_1=kt, \tag{7}$$

$$T_0=\frac{q(q-1)}{2}+\frac{t(t-1)}{2}. \tag{8}$$

(1-9) Definition [6]. An (k,n)-arc is said to be complete if there is no $(k+1,n)$-arc containing it.

(1-10) Definition [5]. A quadric is a variety of rank 2 in the space $\mathrm{PG}(n,q)$. If $R=V(F)$, where F is a quadratic form, then

$$\begin{aligned}F&=\sum_{i\le j}a_{ij}x_ix_j\\&=a_{00}x_0^2+a_{01}x_0x_1+a_{02}x_0x_2+a_{11}x_1^2+a_{12}x_1x_2+a_{22}x_2^2.\end{aligned} \tag{9}$$

(1-11) Definition [6]. A nonsingular quadric in the plane $\mathrm{PG}(2,q)$ is called a conic. A general form of the conic is $x_0^2+x_1x_2$, so the conic $C=V(x_0^2+x_1x_2)$ contains $q+1$ points.

2. CONSTRUCTION AND CLASSIFICATION OF K-ARCS IN $\mathrm{PG}(2,8)$

We construct the k-arcs for $4\le k\le 10$, find the projectively distinct arcs and the distinct arcs under collineation, and determine the action of the $\mathrm{PGL}(3,8)$ group on each arc.

2.1 The Plane $\mathrm{PG}(2,8)$

The plane $\mathrm{PG}(2,8)$ contains 73 points and 73 lines. Each line contains nine points, and each point lies on nine lines. If $\mathrm{GF}(8)$ is a Galois field of order 8, then its elements are

$$0,1,\lambda,\lambda^2,\lambda^3,\ldots,\lambda^6,\qquad \lambda^7=1, \tag{10}$$

and $\lambda^2+\lambda^3+1=0$. Thus,

$$C(f)=\begin{bmatrix}0&1&0\\0&0&1\\1&0&\lambda^3\end{bmatrix} \tag{11}$$

is the incidence matrix that rotates the 73 points through one cycle. Let $P_0=(1,0,0)$. Then

$$P_i=P_0C(f)^i,\qquad i=1,2,\ldots,72. \tag{12}$$

The resulting 73 points are listed in Table 2.

Let L be the line at infinity, $X_2=0$. Its points are

$$0,\ 1,\ 11,\ 20,\ 38,\ 43,\ 59,\ 67,\ 71. \tag{13}$$

Since the companion matrix $C(f)$ rotates all points of the plane in one cycle, and points and lines are dual, $C(f)$ also rotates the 73 lines in one cycle. The last line is

$$72,\ 0,\ 10,\ 19,\ 37,\ 42,\ 58,\ 66,\ 70. \tag{14}$$

2.2 Construction of k-Arcs, $k=4,5,\ldots,10$

Let P_0,P_1,P_2,P_3 be

$$\begin{aligned}P_0&=(1,0,0),&P_1&=(0,1,0),\\P_2&=(0,0,1),&P_3&=(1,1,1).\end{aligned} \tag{15}$$

These points form the source 4-arc. According to the fundamental theorem of projective geometry, every 4-arc is equivalent to $\{P_0,P_1,P_2,P_3\}$. Thus, the arcs for $k=5$ to $k=10$ can be constructed from this 4-arc using the following computer algorithm.

2.3 Classification Algorithm

1. Assign the 2-secants of the arc $K=\{P_0,P_1,P_2,P_3\}$.

Table 2. Points of PG$(2,8)$

i	Point	i	Point	i	Point
0	$(1,0,0)$	25	$(1,1,\lambda)$	50	$(1,\lambda,\lambda^4)$
1	$(0,1,0)$	26	$(1,\lambda^6,\lambda^5)$	51	$(1,\lambda^3,\lambda)$
2	$(0,0,1)$	27	$(1,\lambda^2,\lambda^4)$	52	$(1,\lambda^6,1)$
3	$(1,0,\lambda^3)$	28	$(1,\lambda^3,\lambda^6)$	53	$(1,1,\lambda^5)$
4	$(1,\lambda^4,\lambda^3)$	29	$(1,\lambda,\lambda)$	54	$(1,\lambda^2,1)$
5	$(1,\lambda^4,\lambda^4)$	30	$(1,\lambda^6,\lambda^2)$	55	$(1,1,1)$
6	$(1,\lambda^3,\lambda^2)$	31	$(1,\lambda^5,\lambda)$	56	$(1,1,\lambda^2)$
7	$(1,\lambda^5,\lambda^4)$	32	$(1,\lambda^6,\lambda)$	57	$(1,\lambda^5,\lambda^6)$
8	$(1,\lambda^3,\lambda^4)$	33	$(1,\lambda^6,\lambda^6)$	58	$(1,\lambda,\lambda^5)$
9	$(1,\lambda^3,\lambda^5)$	34	$(1,\lambda,\lambda^2)$	59	$(1,\lambda^2,0)$
10	$(1,\lambda^2,\lambda^6)$	35	$(1,\lambda^5,\lambda^5)$	60	$(0,1,\lambda^2)$
11	$(1,\lambda,0)$	36	$(1,\lambda^2,\lambda^2)$	61	$(1,0,\lambda^6)$
12	$(0,1,\lambda)$	37	$(1,\lambda^5,\lambda^2)$	62	$(1,\lambda,\lambda^3)$
13	$(1,0,\lambda^5)$	38	$(1,\lambda^5,0)$	63	$(1,\lambda^4,\lambda^6)$
14	$(1,\lambda^2,\lambda^3)$	39	$(0,1,\lambda^5)$	64	$(1,\lambda,\lambda^6)$
15	$(1,\lambda^4,\lambda^5)$	40	$(1,0,1)$	65	$(1,\lambda,1)$
16	$(1,\lambda^2,\lambda^5)$	41	$(1,1,\lambda^3)$	66	$(1,1,\lambda^4)$
17	$(1,\lambda^2,\lambda)$	42	$(1,\lambda^4,\lambda)$	67	$(1,\lambda^3,0)$
18	$(1,\lambda^6,\lambda^4)$	43	$(1,\lambda^6,0)$	68	$(0,1,\lambda^3)$
19	$(1,\lambda^3,1)$	44	$(0,1,\lambda^6)$	69	$(1,0,\lambda)$
20	$(1,1,0)$	45	$(1,0,\lambda^4)$	70	$(1,\lambda^6,\lambda^3)$
21	$(0,1,1)$	46	$(1,\lambda^3,\lambda^3)$	71	$(1,\lambda^4,0)$
22	$(1,0,\lambda^2)$	47	$(1,\lambda^4,\lambda^2)$	72	$(0,1,\lambda^4)$
23	$(1,\lambda^5,\lambda^3)$	48	$(1,\lambda^5,1)$		
24	$(1,\lambda^4,1)$	49	$(1,1,\lambda^6)$		

2. Find the points that do not lie on the bisecants.

3. Add these points individually to the arc to obtain all 5-arcs.

4. Find the projectively distinct arcs and the distinct arcs under collineation.

5. Find the groups that stabilize these arcs.

6. Repeat Steps 2–5 to find the 6-, 7-, 8-, 9-, and 10-arcs.

7. Determine the complete arcs at every step, if any.

2.4 Projectively Distinct and Collineation-Distinct K-Arcs

2.4.1 The 5-Arcs

To construct the 5-arcs, place the four source points on the lines of the plane, delete all points on the bisecants from the point set of PG$(2,8)$, and add each remaining point individually to the source points.

The computer program shows that there is one projectively distinct 5-arc, with points

$$(1,0,0),\ (0,1,0),\ (0,0,1),\ (1,1,1),\ (1,\lambda^4,\lambda^3). \tag{16}$$

Its stabilizer group is $C_2 \times C_2$, generated by

$$\phi_1 : (X_0,X_1,X_2) \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \lambda^4 \\ 0 & \lambda^3 & 0 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}, \tag{17}$$

$$\phi_2 : (X_0,X_1,X_2) \longmapsto \begin{bmatrix} 1 & \lambda & \lambda^5 \\ 0 & \lambda^5 & \lambda^5 \\ 0 & \lambda^4 & \lambda^5 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}. \tag{18}$$

The transformation ϕ_1 fixes $(1,0,0)$ and divides the remaining arc points into the orbits

$$\{(0,1,0),(0,0,1)\}, \qquad \{(1,1,1),(1,\lambda^4,\lambda^3)\}. \tag{19}$$

Table 3. Projectively distinct 6-arcs

Arc	P_0	P_1	P_2	P_3	P_4	P_5	G	$\lvert G\rvert$
X_0	0	1	2	55	4	8	$C_2 \times C_2$	4
X_2	0	1	2	55	4	9	S_3	6
X_3	0	1	2	55	4	28	S_4	24
X_4	0	1	2	55	4	30	S_4	24
X_5	0	1	2	55	4	50	S_4	24

The transformation ϕ_2 fixes $(1,0,0)$ and divides the remaining arc points into the orbits

$$\{(0,1,0),(1,\lambda^4,\lambda^3)\}, \qquad \{(0,0,1),(1,1,1)\}. \tag{20}$$

The 5-arc is contained in the conic

$$V(X_0X_1 + \lambda^4 X_0X_2 + \lambda^6 X_1X_2). \tag{21}$$

Let T_0,T_1,T_2 be the numbers of i-secants of the 5-arc. By the theorem above,

$$t(P) = 5, \qquad T_0 = 38, \qquad T_1 = 25, \qquad T_2 = 10. \tag{22}$$

Since there is one projectively distinct 5-arc, there is also one 5-arc up to collineation.

2.4.2 The 6-Arcs

The same method gives five projectively distinct 6-arcs, shown in Table 3.

The arcs X_3,X_4,X_5 are complete. The stabilizer group of the

Table 4. Collineation-distinct 6-arcs

Arc	P_0	P_1	P_2	P_3	P_4	P_5
Y_1	0	1	2	55	4	8
Y_2	0	1	2	55	4	9
Y_3	0	1	2	55	4	28

X_1 arc is $C_2 \times C_2$, generated by

$$\phi_1 : (X_0, X_1, X_2) \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}, \tag{23}$$

$$\phi_2 : (X_0, X_1, X_2) \longmapsto \begin{bmatrix} 1 & \lambda^6 & \lambda^6 \\ 0 & \lambda^3 & \lambda^2 \\ 0 & \lambda^2 & \lambda^3 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}. \tag{24}$$

The transformation ϕ_1 fixes $(1,0,0)$ and $(1,1,1)$ and has the remaining orbits

$$\{(0,0,1),(0,1,0)\}, \qquad \{(1,\lambda^4,\lambda^3),(1,\lambda^3,\lambda^4)\}. \tag{25}$$

The transformation ϕ_2 also fixes $(1,0,0)$ and $(1,1,1)$ and has the remaining orbits

$$\{(0,1,0),(1,\lambda^4,\lambda^3)\}, \qquad \{(0,0,1),(1,\lambda^3,\lambda^4)\}. \tag{26}$$

The conic $V(X_0X_1 + \lambda^4 X_0X_2 + \lambda^6 X_1X_2)$ contains the five points $4, 55, 2, 1, 0$ of X_1.

The stabilizer group of the arc X_2 is S_3, generated by

$$\phi_1 : (X_0, X_1, X_2) \longmapsto \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}, \tag{27}$$

$$\phi_2 : (X_0, X_1, X_2) \longmapsto \begin{bmatrix} 1 & \lambda^5 & 1 \\ \lambda^3 & 0 & 1 \\ \lambda^5 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}. \tag{28}$$

The transformation ϕ_1 has the three orbits

$$\begin{aligned} &\{(0,0,1),(1,0,0)\}, \qquad \{(0,1,0),(1,1,1)\}, \\ &\{(1,\lambda^3,\lambda^5),(1,\lambda^4,\lambda^3)\}. \end{aligned} \tag{29}$$

The transformation ϕ_2 has the two orbits

$$\begin{aligned} &\{(1,0,0),(1,\lambda^3,\lambda^5),(0,1,0)\}, \\ &\{(0,0,1),(1,1,1),(1,\lambda^4,\lambda^3)\}. \end{aligned} \tag{30}$$

The conic $V(X_0X_1 + \lambda^4 X_0X_2 + \lambda^6 X_1X_2)$ contains all six points of X_2. The stabilizer groups of X_3, X_4, X_5 are isomorphic to S_4, and none of these arcs is completely contained in one conic.

The collineation program gives the three arcs in Table 4. The arc Y_3 is complete.

2.4.3 The 7-Arcs

The method gives the two projectively distinct arcs in Table 5. The arcs Z_1, Z_2 are not complete. Hence, there is no complete 7-arc in PG$(2,8)$.

Table 5. Projectively distinct 7-arcs

Arc	P_0	P_1	P_2	P_3	P_4	P_5	P_6	G	$\lvert G\rvert$
Z_1	0	1	2	55	4	8	16	S_3	6
Z_2	0	1	2	55	4	9	10	D_7	14

Table 6. Projectively distinct 8-arcs

Arc	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	G	$\lvert G\rvert$
F_1	0	1	2	55	4	8	16	32	D_4	14
F_2	0	1	2	55	4	9	10	18	PGL$(2,16)$	56

The arc Z_1 has stabilizer group S_3, generated by

$$\phi_1 : (X_0, X_1, X_2) \longmapsto \begin{bmatrix} 0 & \lambda^5 & \lambda \\ 0 & \lambda^5 & \lambda^4 \\ 1 & \lambda^5 & \lambda^5 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}, \tag{31}$$

$$\phi_2 : (X_0, X_1, X_2) \longmapsto \begin{bmatrix} 1 & 0 & \lambda^4 \\ 0 & 0 & \lambda \\ 0 & \lambda^6 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}. \tag{32}$$

The transformation ϕ_1 fixes $(1,0,0)$ and has the two remaining orbits

$$\begin{aligned} &\{(1,1,1),(1,\lambda^2,\lambda^5),(1,\lambda^3,\lambda^4)\}, \\ &\{(0,1,0),(0,0,1),(1,\lambda^4,\lambda^3)\}. \end{aligned} \tag{33}$$

The transformation ϕ_2 fixes $(1,0,0)$ and has the three remaining orbits

$$\begin{aligned} &\{(1,\lambda^4,\lambda^3),(1,\lambda^2,\lambda^5)\}, \\ &\{(0,0,1),(1,\lambda^3,\lambda^4)\}, \\ &\{(0,1,0),(1,1,1)\}. \end{aligned} \tag{34}$$

The conic $V(X_0X_1 + \lambda^4 X_0X_2 + \lambda^6 X_1X_2)$ contains five points of Z_1 and two external points.

The arc Z_2 has stabilizer group D_7, generated by

$$\phi_1 : (X_0, X_1, X_2) \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \lambda^4 \\ 0 & \lambda^3 & 0 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}, \tag{35}$$

$$\phi_2 : (X_0, X_1, X_2) \longmapsto \begin{bmatrix} 0 & \lambda^6 & \lambda^4 \\ \lambda^4 & \lambda & 0 \\ 0 & \lambda^5 & 0 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}. \tag{36}$$

Their orders are 2 and 7, respectively. The transformation ϕ_1 fixes $(1,0,0)$ and has the three remaining orbits

$$\begin{aligned} &\{(1,\lambda^3,\lambda^5),(1,\lambda^2,\lambda^6)\}, \\ &\{(1,1,1),(1,\lambda^4,\lambda^3)\}, \\ &\{(0,1,0),(0,0,1)\}. \end{aligned} \tag{37}$$

The transformation ϕ_2 rotates all points of the 7-arc in one orbit. The conic $V(X_0X_1 + \lambda^4 X_0X_2 + \lambda^6 X_1X_2)$ contains all seven points of Z_2. Up to collineation, the same two arcs Z_1, Z_2 are obtained.

2.4.4 The 8-Arcs

The two projectively distinct 8-arcs are listed in Table 6. The

Table 7. Projectively distinct 9-arcs

Arc	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	G	$\|G\|$
H_1	0	1	2	55	4	8	16	32	37	PGL(2,16)	56
H_2	0	1	2	55	4	9	10	18	34	PGO(3,16)	504

Table 8. Projectively distinct complete 10-arc

Arc	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	G	$\|G\|$
J	0	1	2	55	4	8	16	32	37	64	PGO(3,16)	504

arcs F_1, F_2 are not complete. Hence, there is no complete 8-arc in PG$(2,8)$.

The arc F_1 has stabilizer group D_4, generated by

$$\phi_1 : (X_0, X_1, X_2) \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \lambda^6 \\ 0 & \lambda & 0 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}, \tag{38}$$

$$\phi_2 : (X_0, X_1, X_2) \longmapsto \begin{bmatrix} 1 & 1 & \lambda^5 \\ 0 & \lambda^3 & \lambda^2 \\ 0 & \lambda^4 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}. \tag{39}$$

Their orders are 2 and 7, respectively. The transformation ϕ_1 fixes $(1,0,0)$ and $(1,\lambda^3,\lambda^4)$ and has the remaining orbits

$$\begin{aligned} &\{(0,1,0),(0,0,1)\}, \qquad \{(1,1,1),(1,\lambda^6,\lambda)\}, \\ &\{(1,\lambda^4,\lambda^3),(1,\lambda^2,\lambda^5)\}. \end{aligned} \tag{40}$$

The transformation ϕ_2 fixes $(1,0,0)$ and has the remaining orbit

$$\begin{aligned} &\{(0,1,0),(1,\lambda^3,\lambda^4),(0,0,1),(1,\lambda^4,\lambda^5), \\ &(1,\lambda^6,\lambda),(1,1,1),(1,\lambda^2,\lambda^5)\}. \end{aligned} \tag{41}$$

The conic $V(X_0X_1 + \lambda^4 X_0X_2 + \lambda^6 X_1X_2)$ contains five points of F_1 and three external points. The stabilizer group of F_2 is PGL$(2,16)$, and the same conic contains all points of F_2. Up to collineation, the same two arcs F_1, F_2 are obtained.

2.4.5 The 9-Arcs

The two projectively distinct 9-arcs are shown in Table 7. The arcs H_1, H_2 are incomplete. Hence, there is no complete 9-arc in PG$(2,8)$.

The conic

$$V(x_0^2 + x_0x_1 + x_0x_2 + \lambda^5 x_1x_2) \tag{42}$$

contains five points of H_1, with four points outside it. The conic

$$V(x_0x_1 + \lambda^4 x_0x_2 + \lambda^6 x_1x_2) \tag{43}$$

contains all nine points of H_2. Up to collineation, the same two arcs H_1, H_2 are obtained.

2.4.6 The 10-Arcs

The two projectively distinct 9-arcs are H_1 and H_2. Adding, one at a time, the points remaining after deleting the points on their bisecants produces only two 10-arcs. The computer program gives one complete arc, shown in Table 8.

3. RESULTS

3.1 Projectively Distinct k-Arcs

Table 9 gives the complete classification of projectively distinct arcs for $k = 5, \ldots, 10$. Here, N_k is the number of projectively distinct arcs and G_s is the stabilizer group of the k-arc. The computer calculations required 150 hours.

Table 9. Obtained results A

k	N_k	G_s	#
5	1	$C_2 \times C_2$	1
6	3	$C_2 \times C_2$	1
6	3	S_3	1
6	3	S_4	3
7	2	S_3	1
7	2	D_7	1
8	2	D_4	1
8	2	PGL(2,16)	1
9	2	PGL(2,16)	1
9	2	PGO(3,16)	1
10	1	PGL(2,16)	1

3.2 Collineation-Distinct k-Arcs

Table 10 gives the classification of k-arcs up to collineation. Here, N_k^* is the number of distinct arcs under collineation and G_s is the stabilizer group of the k-arc.

Table 10. Obtained results B

k	N_k^*	G_s	#
5	1	$C_2 \times C_2$	1
6	3	$C_2 \times C_2$	1
6	3	S_3	1
6	3	S_4	1
7	2	S_3	1
7	2	D_7	1
8	2	D_4	1
8	2	PGL(2,16)	1
9	2	PGL(2,16)	1
9	2	PGO(3,16)	1
10	1	PGL(2,16)	1

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