On The Bayesian Estimation of Parameters of SQDM

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Abstract

This work is concerned with the problem of estimating parameters of spatial quadratic models by Bayesian technique (SQDM). This technique involves the prior information of the first and second moment of the parameters, where its estimation model is called the Bayesian quadratic unbiased estimator. The results of the estimation are taken in compared with the estimates of minimum norm quadratic unbiased estimators.

Keywords: Bayesian estimation; Parameter; Estimation; Estimator

1. Introduction

Realistic data were obtained from the Center for research of dams and Water Resources at the University of Mosul, which represent the rise of groundwater levels for the al-Qaim region in Iraq as shown in Table 1 - the appropriate heterogeneity model for these data was analyzed and reconciled:

$$\sigma_{ij} = \sigma^2 exp - (h_{ij}/a)^t \quad 0 \leq t \leq 2 \qquad (1)$$

After being studied by the estimated quasi-variogram function

$$\gamma(h) = \frac{1}{2n(h)} \sum_{i=1}^{n} \left(Z(x_i) - Z(x_i + h)\right)^2 \qquad (2)$$

Where the data is irregular it was converted to a regular grid of data in the form of a 15.15 View table. Then we calculated the experimental quasi-variogram function of this network for the four directions. After that, the experimental quasi-variogram rate was plotted and the drawing is shown in Figure -1- from our observation of the said figure, we see that the heterogeneity does not depend on the direction, but only on the displacement h.
Because the quasi-variogram function of the four directions is very similar, it is therefore possible to say that the quasi-variogram function has the isotropic property.
Thus, we noticed that the appropriate variogram or covariance model for these data is Form -1- since:
σ_{ij}: represents the covariance between view i and view J.
σ^2: represents the total variance.
α: represents the extent of the groundwater phenomenon in the studied area.
We note that:
Model -1- is a non-linear covariance model, but the graph -1- of the variogram is almost linear, so we converted Model -1- to an approximate linear model when t=2, the linear approximation of Model -1- will be:

$$\left.\begin{aligned} \sigma_{ij} &= \sigma_1^2 - \sigma_2^2 h_{ij}^2 & i \neq j \\ &= \sigma_1^2 + \sigma_3^2 & i = j \end{aligned}\right\} \qquad (3)$$

σ_1^2 represents the contrast of the viewing effect.
σ_2^2 represents the contrast of the spatial effect of viewing.
σ_3^2 is a joke nugget Effect (an amount that represents a weakening of the continuity of the phenomenon under study).
$\sigma_1^2, \sigma_2^2, \sigma_3^2$ represent the parameters of variability that must be estimated to obtain an approximate covariance model

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of the data in order to use it for the prediction process about the spatial random process that in this study represents groundwater and may be minerals or environmental pollution in other studies.

Table 1: data on the rise of groundwater levels in the al-Qaim region in Iraq.

u(x)	v(x)	Z(x)
25	125	220.04
125	125	220.54
220	125	219.56
325	125	221.26
25	75	220.28
125	75	219.81
225	75	219.3
325	75	219.92
25	25	220.45
125	25	220.96
240	25	220.87
325	25	223.04
0	150	220.
350	0	223.3

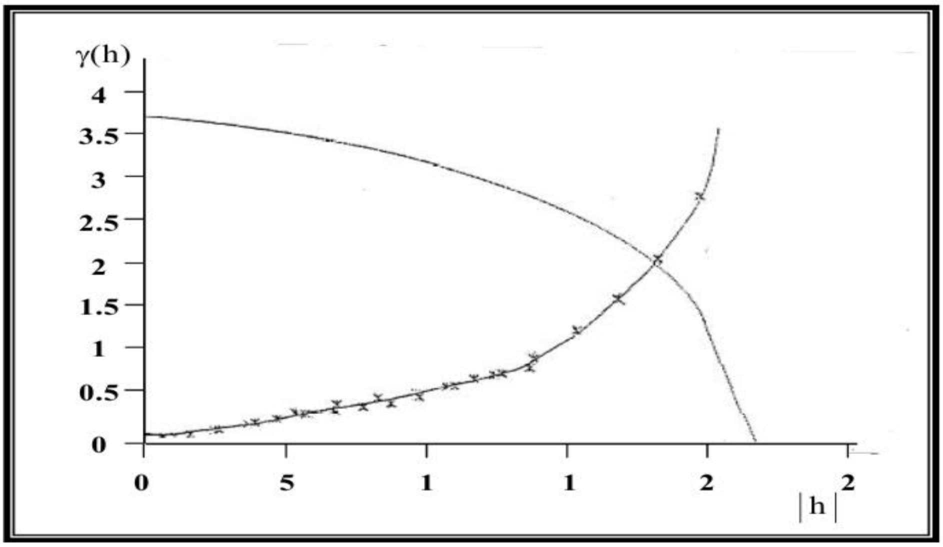

Figure 1. The rate curve of the quasi-variogram function

We observe in the form -3- when $\sigma_1^2 + \sigma_3^2$ $i = j$ represents the variation in the main diameter in the covariance matrix and that the covariance $(i \neq j)$ σ_{ij}decreases as the square of the offset between the views increases and that is why the form -3 - is called the quadratic decay Model (QD).
The parameters $\sigma_1^2, \sigma_2^2, \sigma_3^2$were estimated by the unbiased minimum Norm Quadratic Unbiased Estimation (MINQUE).
In this research, the parameters were estimated by the Bayes method, which will be explained in detail later.

2. Formulation of the general linear model in vacuum statistics

Consider the locational variable Z(x) defined in the Domain Region-D which is a partial set of Euclidean space $R^2$and R^3as follows:

$$Z(x) = \beta' f(x) + e(x); \quad \forall x \in D \qquad (4)$$

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That is, x=(u,v)if it is in R^2or x=(u,v,w) if it is in R^3.
Z(x): is the value of the random process in the random field at Point X.
$F(x) = [f_1(x), f_2(x), \dots, f_s(x)]`$: is a vector known from the coordinates
β: is a vector with an amplitude s of unknown parameters
e(x): a random vector with zero expectation and finite variance.
Suppose that the variable Z(x) fulfills the following hypotheses:

1. $E(Z(x)) = \beta' f(x)$ (5)
2. $E(Z(x+h) - Z(x))^2 = 2\gamma(h);\ \forall x, x+h \in D$ (6)

And $2\gamma(h)$ is isotropic uniform.

3. $cov(Z(x), Z(x+h)) = c(h);\quad \forall x, x+h \in D$ (7)

Suppose that we have n of the views of the spatial variable are:
$Z(x_1), Z(x_2), \dots, Z(x_n)$ at positions $x_1, x_2, \dots, x_n$
Then it is possible to write Form 4 as follows:
$Z = F\beta + e$ (8)
So $Z = (Z(x_1), Z(x_2), \dots, Z(x_n))'$is a vector with an amplitude of n of views
$F = (f(x_1), f(x_2), \dots, f(x_n))'$is an information matrix with capacity n.s
$\beta = (\beta_1, \beta_2, \dots, \beta_s)'$ is a vector with an amplitude s of unknown parameters
$e = (e(x_1), e(x_2), \dots, e(x_n))'$ the vector of random errors with amplitude n.
Consider the following parametric covariance model:
$C(h) = C(h, \theta);\quad \forall h \in D$ (9)
Represents the covariance function and that the quasi-variogram function $\gamma(h)$ be:
$\gamma(h) = \gamma(h, \theta);\ \forall h \in D$ (10)
Where θ is a vector of unknown parameters, and it is required in this research to estimate these parameters from the data by the Bayes estimation method, and in light of the covariance function 9, the covariance matrix is as follows:
$E(ee') = \sum(\theta)$ (11)

Bayesian Quadratic unbiased estimation:

We consider the parametric model of heterogeneity in the following linear form:
$C(h;\theta) = \theta_1 u_1(h) + \theta_2 u_2(h) + \dots + \theta_r u_r(h)$ (12)
Where $U_i(h)$is the correlation function and $U_i(0) = 1\forall\ \forall\ i = 1,2,\dots,r$
$\theta = (\theta_1, \theta_2, \dots, \theta_r)`$' is a vector of unknown and desired parameters to be estimated. And since $var(Z(x)) = \theta_1 + \theta_2 + \dots + \theta_r$ these parameters form the covariance compounds and the covariance matrix of the variable Z under the influence of formula (8) is written as follows:
$U = \theta_1 U_1 + \theta_2 U_2 + \dots + \theta_r U_r = \sum(\theta)$ (13)
Where U_i are information matrices i=1,2,..., r
Then the variance is:
$var(Z) = \theta_1 U_1 + \theta_2 U_2 + \dots + \theta_r U_r = \sum(\theta) = var(e) = \sum(\theta)$ (14)
Where $U_i = D_i D_i{}'$ equation 14 is the same as equation 13, then from the previous presentation we consider the following linear model:
$Z = F\beta + e$
$E(z) = F\beta, var(z) = \sum_{i=1}^{r} \theta_i U_i = \sum(\theta)$ (15)
That the Matrix $\sum(\theta)$ is a special case of the model proposed by kleffe and Pincus. The problem in estimating the parameters $\theta_i$is formed by estimating the linear function, which takes the following formula:
$\alpha(z) = b_1\theta_1 + b_2\theta_2 + \dots + b_r\theta_r = b'\theta$ (16)
By the binary form $\hat{\alpha}(Z) = Z'CZ$
Where C is a symmetric matrix with capacity n. N is required to be found and $\hat{\alpha}$ satisfies the following conditions:

1. $\hat{\alpha}$ is non-interchangeable (Invariant) for the transition $Z \to Z + F\beta$, so that:
   $\hat{\alpha}(Z) = \hat{\alpha}(Z + F\beta)$
2. $\hat{\alpha}$ unbiased.
3. Minimize the biz risk function.

Now we assume the existence of an initial distribution Function (Apriori Distribution Function) for the parameter θ, which is P(θ), so the loss function takes the following formula:
$L(\alpha, \hat{\alpha}) = (\hat{\alpha} - \alpha)^2$
And that the risk function will take the formula:

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$$g(\alpha,\hat{\alpha}) = E\big(L(\alpha,\hat{\alpha})\big) = E((\hat{\alpha}-\alpha)^2)$$

While the biz risk function $B(\hat{\alpha})$ is of the following form:

$B(\hat{\alpha}) = E_\theta(g(\alpha,\hat{\alpha})) = E_\theta(E(\hat{\alpha}-\alpha)^2)$

$= \int_{\theta\in\Omega} g(\alpha,\hat{\alpha})dP(\theta) = \int_{\theta\in\Omega} E(\hat{\alpha}-\alpha)^2 dP(\theta)$ (17)

While the biz estimator $\hat{\alpha}\beta$ is an estimator of the parameter θ, which makes the expected risk or the final risk (the Posterior Risk) as low as possible.

$$B(\hat{\alpha}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\ldots\int_{-\infty}^{\infty} P(Z_1,Z_2,\ldots,Z_n)\,[\int_{\theta\in\Omega} L(\alpha_1\hat{\alpha})P(\theta/Z_1,Z_2,\ldots,Z_n)d\theta]dZ_1,\ldots,dZ_n$$

The value of the estimator can be found by reducing the value of $E_\theta(g(\alpha,\hat{\alpha}))$and this is sufficient by reducing the value of the final biz risk $Q(\hat{\alpha},Z_1,Z_2,\ldots,Z_n)$that is, by reducing:

$Q(\hat{\alpha},Z_1,Z_2,\ldots,Z_n) = \int_{\theta\in\Omega} L(\alpha,\hat{\alpha})P(\theta/Z_1,Z_2,\ldots,Z_n)d\theta$ (18)

By solving the equation:

$$\frac{\partial Q(\hat{\alpha},Z_1,Z_2,\ldots,Z_n)}{\partial\hat{\alpha}} = 0$$

It is possible to find the value of the BIS estimator $\hat{\alpha}\beta$

When the initial distribution function for the parameters θ_i is available, the second moment of the parameters $\theta_i$takes the following formula:

$E(\theta_i\theta_j) = \int_{\theta\in\Omega}\theta_i\theta_j dP(\theta) = C_{ij};\ \forall_{ij}= 1,2,\ldots,r$ (19)

The sign of integration in relations 20 and 18 represents a multiple integral with the number of parameters in the vector $\theta = (\theta_1,\theta_2,\ldots,\theta_r)'$

Form 8 contains the vector of unknown parameters β, which we do not aim to estimate, and it is possible to get rid of it by multiplying equation 8 by the projection matrix M, so we have:

$MZ = MF\beta + Me$

$MZ = ME$

Now we assume that y=MZ where E(y)=0 and for the purpose of the proof we observe that:

$E(y) = E(MZ) = ME(Z) = MF\beta = 0$

And also be

$var(y) = var(MZ) = Mvar(Z)M'$

Where M=M'

$$var(y) = var(Z)M = M\sum(\theta)M = M(\sum_{i=1}^{r}\theta_iU_i)M = \sum_{i=1}^{r}\theta_iMU_iM = \sum_{i=1}^{r}\theta_iV_i$$

Where $V_i = MU_iM$ and as a result of the foregoing we get the following form:

$Y = Me, E(Y) = 0,\ var(Y) = \sum_{i=1}^{r}\theta_iV_i = V(\theta)$ (20)

We note that $\hat{\alpha} = Z'KZ$ is a Bayesian Quadratic unbiased estimate to $\alpha = b'\theta$ in Form 16 if and only if $K = MAM$ and $\hat{\alpha} = Y'AY$ are an unbiased binary estimate of the function α.

Estimation conditions for the binary form $Y'AY$:

1- <u>Impartiality:</u>

$E(Y'AY) = \alpha = b'\theta$

Proof:

$$E(Y'AY) = E(tr\,Y'AY) = E(tr\,AYY')$$
$$= trAE(YY')$$
$$= trAE(YY')$$
$$= trA\ var(y) = tr\ A\sum_{i=1}^{r}\theta_iV_i$$
$$= \sum_{i=1}^{r}\theta_i trAV_i$$

Is unbiased for α if and only if:

$$trAV_i = b_i;\ \ i = 1,2,\ldots,r$$

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2- Minimizing the biz risk function:

$$B(\hat{\alpha}) = \int_{\theta\in\Omega} E(\hat{\alpha}-\alpha)^2 dP(\theta) = E\theta(E(\hat{\alpha}-\alpha)^2)$$
$$= E\theta(var(\hat{\alpha})) = E\theta(var(Y'AY))$$
$$= E\theta(2trAV(\theta)AV(\theta))$$
$$= E\theta\left(2trA\sum_{i=1}^{r}\theta_i V_i\, A\sum_{j=1}^{r}\theta_j V_j\right)$$
$$= E\theta\left(2\sum_i\sum_j \theta_i\theta_j tr\, A\, V_i A V_j\right)$$
$$= 2\sum_i\sum_j E(\theta)(\theta_i\theta_j) tr\, A\, V_i A V_j$$
$$= 2\sum_i\sum_j C_{ij} tr\, A\, V_i A V_j \quad (22)$$

The initial information we need to estimate the feature vector $\theta = (\theta_1, \theta_2, \dots, \theta_r)'$ is the first moment E(θ) and the second moment $E(\theta\theta')$and the matrix of the second moment.

$$\left.\begin{array}{l} C = E(\theta\theta') = var(\theta) + E(\theta)E(\theta') \\ C = \sqrt{C}\sqrt{C} = RR \end{array}\right\} \quad (23)$$

Where $R = \sqrt{C}$ is the root of the matrix C

Matrix C can be represented in the following form:

$C = (C_{ij}) = \sum_{k=1}^{r} r_{ik} r_{kj})$ $\forall_{1,j} = 1,2,\dots,r$ (24)

Substituting 24 into the relation 22, we get the following relation:

$$B(\hat{\alpha}) = 2\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r} r_{ik} r_{kj} trAv_i Av_j$$
$$= 2\sum_{i=1}^{r} trA(\sum_{i=1}^{r} r_{ik} v_i)A(\sum_{j=1}^{r} r_{kj} v_j)$$
$$B(\hat{\alpha}) = 2\sum_{k=1}^{r} trAT_k AT_k \quad (25)$$

Where $T_k = \sum_{i=1}^{r} r_{ik} v_i$

Now we will use the Lagrange's method to solve the relation 25 according to the conditions of non-bias 21 to obtain a small maximum value. For that we assume that:

$N = 2\sum_{k=1}^{r} trAT_k AT_k + 4\sum_{i=1}^{r} \delta(trAv_i - b_i)$ (26)

Where δ_i represents the Lagrange multipliers and satisfies the conditions of non-bias 21

We derive relation 26 for the Lagrange's multipliers δ_i and equate the derivative to zero, so it is:

$\frac{\partial N}{\partial \delta_i} = trAv_i - b_i = 0$ $i = 1,2,\dots,r$ (27)

That is:

$trAv_i = b_i$ $i = 1,2,\dots,r$ (28)

Equations 27 and 28 are in the form of Matrices and the unknowns in which the matrix A and the Lagrange's multiples δ_i where i=1,2,..., r and in order to solve these equations and find the unknowns, we convert these equations from the formulas of the matrix system into a system of linear equations and use for this the Kronecker product and the VEC Operation and:

$(\sum_{k=1}^{r} T_k \otimes T_k)vecA + \sum_{i=1}^{r} \delta_i vecV_i = 0$ (29)

$(vecV_i)'vecA = b_i$ $i = 1,2,\dots,r$ (30)

In relation 29 we assume that:

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$$w=\sum_{k=1}^{r}T_k\otimes T_k$$

Then the relation (29) becomes the following formula:

$w\ vecA+\sum_{i=1}^{r}\delta_i vecV_i=0$ (31)

We note that in general there are r parameters. That is, $\theta=(\theta_1,\theta_2,\dots,\theta_r)'$

Now it is possible to formulate the system of linear equations 30 and 31 in the following form:

$$\begin{pmatrix} vecv_1 & vecv_2 & \cdots & vecv_r & w \\ 0 & 0 & & 0 & vecv'_1 \\ 0 & 0 & & 0 & vecv'_2 \\ & & & & \vdots \\ 0 & 0 & \cdots & 0 & vecv'_r \end{pmatrix}\begin{pmatrix}\delta_1\\ \delta_2\\ \vdots\\ \delta_r\\ vecA\end{pmatrix}=\begin{pmatrix}0\\0\\ \vdots\\0\\b_1\\b_2\\ \vdots\\b_r\end{pmatrix} \qquad (32)$$

In the case we are considering in this paper, the number of parameters is r=3.

When r=3, the feature vector is given by the following formula $\theta=(\theta_1,\theta_2,\theta_3)'$

Then the system of equations 31 and 30 becomes as follows:

$w\ vecA+\sum_{i=1}^{3}\delta_i vecv_i=0$ (33)

$(vecv_i)'vecA=b_i$ i=1,2,3 (34)

Therefore, the system of linear equations 34 and 35 can be formulated in the following form:

$$\begin{pmatrix} vec\ v_1 & vec\ v_2 & vecv_3 & w \\ 0 & 0 & 0 & vec\ v'_1 \\ 0 & 0 & 0 & vec\ v'_2 \\ 0 & 0 & 0 & vecv'_3 \end{pmatrix}\begin{pmatrix}\delta'_1\\ \delta'_2\\ \delta'_3\\ vecA\end{pmatrix}=\begin{pmatrix}0\\0\\0\\b_1\\b_2\\b_3\end{pmatrix} \qquad (35)$$

Let G represent a matrix of Order 35 with a capacity of $(n^2+3).(n^2+3)$

O represents the vector of unknowns in order 35 and with an amplitude of $(n^2+3).1$

P represents a vector of constants in order 35 and with an amplitude of $(n^2+3).1$

Thus, the system of equations 35 can be formulated with the following relation:

GO=P (36)

The unknown in relation 36 is the vector O and if G is an invertible (non-anomalous) Non-singular matrix, then a solution of system 36 can be obtained in the following form:

$O=G^{-1}P$ (37)

By finding the values of the vector of unknowns O,which contains δ_i and vec A,it is possible to find the matrix A, through which it is possible to obtain an estimate of the three parameters $\theta_1,\theta_2,\theta_3$ of the binary form $\hat{\alpha}=Y'AY$ because it is an unbiased estimate of the linear function α.

Whereas the:

$\alpha=b_1\theta_1+b_2\theta_2+b_3\theta_3$ (38)

If and only if:

$trAv_i=b_i$ i=1,2,3

See the condition of non-bias for the binary form $Y'AY$ in the system of equations 38 we put $b_1=1,b_2=b_3=0$,we find $\hat{\theta}_1$, which is the estimate of the parameter θ_1, we also put $b_2=1,b_1=b_3=0$, we find $\hat{\theta}_2$, which is the estimate of the parameter θ_2 we also put $b_3=1,b_1=b_2=0$, we find $\hat{\theta}_3$, which is the estimate of the parameter θ_3. If we want another estimate, for example $\theta_1+\theta_2+\theta_3$, we put in relation 38 both $b_1=1,b_2=1,b_3=1$.

Applied aspect: application of the unbiased quadratic Bayes estimator

In this research, the unbiased Bayes quadratic estimator method was applied to a set of spatial observations of the rise of groundwater levels in the al-Qaim area in Iraq, as obtained from the Center for research of dams and water resources at the University of Mosul, the researcher obtained estimates using the method of the smallest unbiased quadratic criterion and based on real data was $60\le\hat{\theta}_1\le\mathbf{70}$

And $0.5\le\hat{\theta}_2\le\mathbf{1.5}$ and $0.0\le\hat{\theta}_3\le\mathbf{0.1}$ in view of this, we considered the initial information about the parametersθ_1, θ_2 and θ_3 take the following regular distribution pattern:

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$P_1(\theta_1)\frac{1}{70-60} = 0.1$ $\quad 60 \leq \hat{\theta}_1 \leq \mathbf{70}$
$P_2(\theta_2)\frac{1}{1.5-0.5} = 1$ $\quad 0.5 \leq \hat{\theta}_2 \leq \mathbf{1.5}$
$P_3(\theta_3)\frac{1}{0.1-0.0} = 10$ $\quad 0.0 \leq \hat{\theta}_3 \leq \mathbf{0.1}$
$P_1(\theta_1,\theta_2,\theta_3) = P_1(\theta_1)P_2(\theta_2)P_3(\theta_3)$
The distribution of θ_1, θ_2 and $\theta_3$is regular in the intervals [0,0.1] , [0.5,1.5] , [60,70]
$cov(\theta_1,\theta_2,\theta_3) = \mathbf{0}$
$E(\theta_1)\frac{70+60}{2} = 65$
$E(\theta_2)\frac{1.5+0.5}{2} = 1.0$

$$E(\theta_3)\frac{0.1+0.0}{2} = 0.05$$

$$var(\theta_1) = \frac{(70-60)^2}{12} = 8.33$$

$$var(\theta_2) = \frac{1}{12} = 0.0833$$

$$var(\theta_3) = \frac{(0.1)^2}{12} = 0.000833$$

$$C = E(\theta)E(\theta') + var(\theta)$$

$$= \begin{pmatrix} 65 \\ 1 \\ 0.05 \end{pmatrix} (65 \quad 1 \quad 0.05) + \begin{pmatrix} 8.33 & 0 & 0 \\ 0 & 0.0833 & 0 \\ 0 & 0 & 0.000833 \end{pmatrix}$$

$$= \begin{pmatrix} 4225 & 65 & 3.25 \\ 65 & 1 & 0.05 \\ 3.25 & 0.05 & 0.0025 \end{pmatrix} + \begin{pmatrix} 8.33 & 0 & 0 \\ 0 & 0.0833 & 0 \\ 0 & 0 & 0.000833 \end{pmatrix}$$

$$C = \begin{pmatrix} 4233.33 & 65 & 3.25 \\ 65 & 1.0833 & 0.05 \\ 3.25 & 0.05 & 0.003333 \end{pmatrix}$$

Since the square root of the matrix C here is:

$$R = \begin{pmatrix} 65.056 & 0.99443 & 0.049918 \\ 0.99443 & 0.3073 & 0.00107 \\ 0.049918 & 0.00107 & 0.028989 \end{pmatrix}$$

We have obtained the value of the BAQUE estimate for the parameters $\theta_1, \theta_2, \theta_3$ as shown in the following table:

Table 2: Results of estimates by unbiased quadratic Bayes estimator

$\hat{\boldsymbol{\theta}}_1$	$\hat{\boldsymbol{\theta}}_2$	$\hat{\boldsymbol{\theta}}_3$
$0.897.10^{-2}$	$4.27.10^{-2}$	-352.69

As for the results obtained by the researcher using the method of the smallest unbiased quadratic criterion as shown in Table 3:

Table 3: Results of estimates by the smallest unbiased quadratic criterion

$\hat{\boldsymbol{\theta}}_1$	$\hat{\boldsymbol{\theta}}_2$	$\hat{\boldsymbol{\theta}}_3$
7.036.10	$5.827.10^{-2}$	$-7.345.10^{-4}$

3. Conclusion

Notably, the estimates are close to each other, indicating that the Bayes method can be used to estimate these parameters. We believe that if the initial information represented informative Priors ' information-rich probability distributions, the results would have been more accurate than the results obtained in Table 3.

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