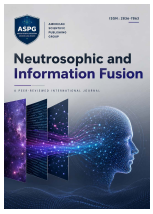
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FULL LENGTH ARTICLE

On the Bayesian Estimation of Parameters of SQDM

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ABSTRACT

This work is concerned with the problem of estimating parameters of spatial quadratic models by the Bayesian technique (SQDM). This technique involves prior information on the first and second moments of the parameters, where its estimation model is called the Bayesian quadratic unbiased estimator. The results of the estimation are compared with the estimates of minimum-norm quadratic unbiased estimators.

Keywords: Bayesian estimation ▪ Parameter ▪ Estimation ▪ Estimator

1. INTRODUCTION

Realistic data were obtained from the Center for Research of Dams and Water Resources at the University of Mosul. They represent the rise of groundwater levels in the Al-Qaim region in Iraq, as shown in Table 1. The appropriate heterogeneity model for these data was analyzed and reconciled as

$$\sigma_{ij} = \sigma^2 \exp\left[-\left(\frac{h_{ij}}{a}\right)^t\right], \qquad 0 \le t \le 2. \tag{1}$$

The estimated quasi-variogram function is

$$\gamma(h) = \frac{1}{2n(h)} \sum_{i=1}^{n(h)} [Z(x_i) - Z(x_i + h)]^2 . \tag{2}$$

Because the data are irregular, they were converted to a regular grid of data in the form of a 15×15 observation table. The experimental quasi-variogram function of this network was then calculated for four directions. The mean experimental quasi-variogram was plotted, as shown in Figure 1. The figure indicates that the heterogeneity does not depend on direction, but only on the displacement h. Because the quasi-variogram functions in the four directions are very similar, the quasi-variogram function may be considered isotropic.

Thus, the appropriate variogram or covariance model for these data is the model in (1), where σ_{ij} represents the covariance between observations i and j, σ^2 represents the total variance, and a represents the extent of the groundwater phenomenon in the studied area.

Model (1) is a nonlinear covariance model, but the variogram shown in Figure 1 is almost linear. Therefore, the model was converted to an approximate linear model. When $t = 2$, the linear approximation is

$$\sigma_{ij} = \begin{cases} \sigma_1^2 - \sigma_2^2 h_{ij}^2, & i \ne j, \\ \sigma_1^2 + \sigma_3^2, & i = j. \end{cases} \tag{3}$$

Here, σ_1^2 represents the variance of the observation effect, σ_2^2 represents the variance of the spatial effect of observation, and σ_3^2 is the nugget effect, an amount representing a weakening of the continuity of the phenomenon under study. The parameters σ_1^2, σ_2^2, and σ_3^2 must be estimated to obtain an approximate covariance model of the data for prediction of the spatial random process, which in this study represents groundwater and may represent minerals or environmental pollution in other studies.

In (3), $\sigma_1^2 + \sigma_3^2$ for $i = j$ represents the variation on the main diagonal of the covariance matrix. The covariance σ_{ij} for $i \ne$

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Table 1. Data on the rise of groundwater levels in the Al-Qaim region in Iraq.

$u(x)$	$v(x)$	$Z(x)$
25	125	220.04
125	125	220.54
220	125	219.56
325	125	221.26
25	75	220.28
125	75	219.81
225	75	219.3
325	75	219.92
25	25	220.45
125	25	220.96
240	25	220.87
325	25	223.04
0	150	220.0
350	0	223.3

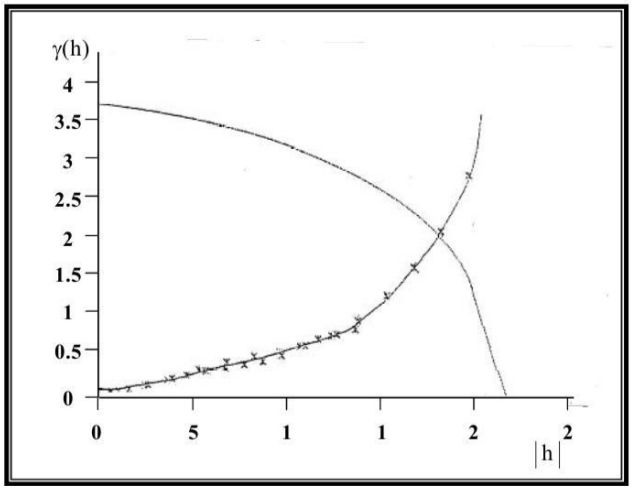


Figure 1. The rate curve of the quasi-variogram function.

j decreases as the square of the offset between observations increases; hence, (3) is called the quadratic decay model (QD).

The parameters σ_1^2, σ_2^2, and σ_3^2 were previously estimated by minimum-norm quadratic unbiased estimation (MINQUE). In this research, the parameters are estimated using the Bayesian method described below.

2. FORMULATION OF THE GENERAL LINEAR MODEL IN SPATIAL STATISTICS

Consider the locational variable $Z(x)$ defined in the domain D, which is a subset of Euclidean space $\mathbb{R}^2$ or $\mathbb{R}^3$, as follows:

$$Z(x) = \beta' \mathbf{f}(x) + e(x), \qquad \forall x \in D. \tag{4}$$

Here, $x = (u,v)$ in $\mathbb{R}^2$ or $x = (u,v,w)$ in $\mathbb{R}^3$; $Z(x)$ is the value of the random process at point x; $\mathbf{f}(x) = [f_1(x), f_2(x), \ldots, f_s(x)]'$ is a known vector of coordinate functions; β is an s-dimensional vector of unknown parameters; and $e(x)$ is a random error with zero expectation and finite variance.

Suppose that the variable $Z(x)$ satisfies the following hypotheses:

$$E[Z(x)] = \beta' \mathbf{f}(x), \tag{5}$$

$$E\left([Z(x+h) - Z(x)]^2\right) = 2\gamma(h), \quad \forall x, x+h \in D, \tag{6}$$

$$\operatorname{cov}[Z(x), Z(x+h)] = c(h), \quad \forall x, x+h \in D. \tag{7}$$

The function $2\gamma(h)$ is isotropic and homogeneous.

Suppose that n observations of the spatial variable, $Z(x_1), Z(x_2), \ldots, Z(x_n)$, are available at positions $x_1, x_2, \ldots, x_n$. Equation (4) may then be written as

$$\mathbf{Z} = \mathbf{F}\beta + \mathbf{e}, \tag{8}$$

where $\mathbf{Z} = [Z(x_1), Z(x_2), \ldots, Z(x_n)]'$ is the n-dimensional vector of observations, $\mathbf{F} = [\mathbf{f}(x_1), \mathbf{f}(x_2), \ldots, \mathbf{f}(x_n)]'$ is an $n \times s$ information matrix, $\beta = (\beta_1, \beta_2, \ldots, \beta_s)'$, and $\mathbf{e} = [e(x_1), e(x_2), \ldots, e(x_n)]'$ is the n-dimensional random-error vector.

Consider the following parametric covariance model:

$$C(h) = C(h, \theta), \qquad \forall h \in D, \tag{9}$$

with quasi-variogram function

$$\gamma(h) = \gamma(h, \theta), \qquad \forall h \in D. \tag{10}$$

The vector θ contains unknown parameters to be estimated from the data using the Bayesian estimation method. In light of (9), the covariance matrix is

$$E(\mathbf{e}\mathbf{e}') = \Sigma(\theta). \tag{11}$$

2.1 Bayesian Quadratic Unbiased Estimation

Consider the parametric heterogeneity model in the linear form

$$C(h;\theta) = \theta_1 u_1(h) + \theta_2 u_2(h) + \cdots + \theta_r u_r(h), \tag{12}$$

where $u_i(h)$ is a correlation function and $u_i(0) = 1$ for $i = 1, 2, \ldots, r$. The vector $\theta = (\theta_1, \theta_2, \ldots, \theta_r)'$ contains the unknown parameters to be estimated. Since $\operatorname{var}[Z(x)] = \theta_1 + \theta_2 + \cdots + \theta_r$, these parameters form the covariance components, and the covariance matrix of Z under (8) is

$$\mathbf{U} = \theta_1 \mathbf{U}_1 + \theta_2 \mathbf{U}_2 + \cdots + \theta_r \mathbf{U}_r = \Sigma(\theta), \tag{13}$$

where $\mathbf{U}_i$, $i = 1, 2, \ldots, r$, are information matrices. Therefore,

$$\operatorname{var}(\mathbf{Z}) = \sum_{i=1}^{r} \theta_i \mathbf{U}_i = \Sigma(\theta) = \operatorname{var}(\mathbf{e}). \tag{14}$$

With $\mathbf{U}_i = \mathbf{D}_i \mathbf{D}_i'$, the preceding presentation gives the linear model

$$\mathbf{Z} = \mathbf{F}\beta + \mathbf{e}, \qquad E(\mathbf{Z}) = \mathbf{F}\beta, \qquad \operatorname{var}(\mathbf{Z}) = \sum_{i=1}^{r} \theta_i \mathbf{U}_i = \Sigma(\theta). \tag{15}$$

The matrix $\Sigma(\theta)$ is a special case of the model proposed by Kleffe and Pincus. Estimation of the parameters θ_i is formulated as estimation of the linear function

$$\alpha(\theta) = b_1\theta_1 + b_2\theta_2 + \cdots + b_r\theta_r = \mathbf{b}'\theta \tag{16}$$

by the quadratic form $\widehat{\alpha}(\mathbf{Z}) = \mathbf{Z}'\mathbf{C}\mathbf{Z}$, where $\mathbf{C}$ is an $n \times n$ symmetric matrix to be determined. The estimator $\widehat{\alpha}$ must satisfy the following conditions:

1. It is invariant under the translation $\mathbf{Z} \to \mathbf{Z} + \mathbf{F}\beta$, so that $\widehat{\alpha}(\mathbf{Z}) = \widehat{\alpha}(\mathbf{Z} + \mathbf{F}\beta)$.
2. It is unbiased.
3. It minimizes the Bayes risk function.

Assume that an a priori distribution $P(\theta)$ exists for θ. For the loss function $L(\alpha, \widehat{\alpha}) = (\widehat{\alpha} - \alpha)^2$, the corresponding risk function is

$$g(\alpha, \widehat{\alpha}) = E[L(\alpha, \widehat{\alpha})] = E(\widehat{\alpha} - \alpha)^2.$$

The Bayes risk is

$$\begin{aligned} B(\widehat{\alpha}) &= E_\theta[g(\alpha, \widehat{\alpha})] = E_\theta\left[E(\widehat{\alpha} - \alpha)^2\right] \\ &= \int_{\theta \in \Omega} E(\widehat{\alpha} - \alpha)^2 \, dP(\theta). \end{aligned} \tag{17}$$

The Bayes estimator $\widehat{\alpha}_\beta$ makes the expected, or posterior, risk as small as possible. In expanded form,

$$\begin{aligned} B(\widehat{\alpha}) = &\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P(Z_1, \ldots, Z_n) \\ &\times \left[\int_{\theta \in \Omega} L(\alpha, \widehat{\alpha}) P(\theta \mid Z_1, \ldots, Z_n) \, d\theta \right] dZ_1 \cdots dZ_n. \end{aligned}$$

The estimator can therefore be found by minimizing the posterior risk

$$Q(\widehat{\alpha}, Z_1, \ldots, Z_n) = \int_{\theta \in \Omega} L(\alpha, \widehat{\alpha}) P(\theta \mid Z_1, \ldots, Z_n) \, d\theta, \tag{18}$$

which is achieved by solving

$$\frac{\partial Q(\widehat{\alpha}, Z_1, \ldots, Z_n)}{\partial \widehat{\alpha}} = 0.$$

When the a priori distribution of the parameters θ_i is available, their second moments are

$$E(\theta_i \theta_j) = \int_{\theta \in \Omega} \theta_i \theta_j \, dP(\theta) = C_{ij}, \qquad i, j = 1, 2, \ldots, r. \tag{19}$$

The integration signs in (18) and (19) represent multiple integrals over the number of parameters in $\theta = (\theta_1, \theta_2, \ldots, \theta_r)'$.

Model (8) contains the nuisance parameter vector β. It can be eliminated by multiplying the equation by the projection matrix $\mathbf{M}$:

$$\mathbf{MZ} = \mathbf{MF}\beta + \mathbf{Me} = \mathbf{Me}.$$

Let $\mathbf{Y} = \mathbf{MZ}$. Then

$$E(\mathbf{Y}) = E(\mathbf{MZ}) = \mathbf{M}E(\mathbf{Z}) = \mathbf{MF}\beta = 0$$

and, because $\mathbf{M} = \mathbf{M}'$,

$$\begin{aligned} \operatorname{var}(\mathbf{Y}) &= \mathbf{M} \operatorname{var}(\mathbf{Z}) \mathbf{M}' = \mathbf{M}\Sigma(\theta)\mathbf{M} \\ &= \sum_{i=1}^{r} \theta_i \mathbf{M}\mathbf{U}_i\mathbf{M} = \sum_{i=1}^{r} \theta_i \mathbf{V}_i, \end{aligned}$$

where $\mathbf{V}_i = \mathbf{M}\mathbf{U}_i\mathbf{M}$. Consequently,

$$\mathbf{Y} = \mathbf{Me}, \qquad E(\mathbf{Y}) = 0, \qquad \operatorname{var}(\mathbf{Y}) = \sum_{i=1}^{r} \theta_i \mathbf{V}_i = \mathbf{V}(\theta). \tag{20}$$

Thus, $\widehat{\alpha} = \mathbf{Z}'\mathbf{K}\mathbf{Z}$ is a Bayesian quadratic unbiased estimate of $\alpha = \mathbf{b}'\theta$ if and only if $\mathbf{K} = \mathbf{MAM}$ and $\widehat{\alpha} = \mathbf{Y}'\mathbf{AY}$ is an unbiased quadratic estimate of α.

2.2 Estimation Conditions for the Quadratic Form $\mathbf{Y}'\mathbf{AY}$

1) Unbiasedness. The quadratic form must satisfy

$$E(\mathbf{Y}'\mathbf{AY}) = \alpha = \mathbf{b}'\theta.$$

Indeed,

$$\begin{aligned} E(\mathbf{Y}'\mathbf{AY}) &= E[\operatorname{tr}(\mathbf{AYY}')] = \operatorname{tr}[\mathbf{A}E(\mathbf{YY}')] \\ &= \operatorname{tr}[\mathbf{A} \operatorname{var}(\mathbf{Y})] = \operatorname{tr}\left(\mathbf{A} \sum_{i=1}^{r} \theta_i \mathbf{V}_i\right) \\ &= \sum_{i=1}^{r} \theta_i \operatorname{tr}(\mathbf{AV}_i). \end{aligned}$$

It is unbiased for α if and only if

$$\operatorname{tr}(\mathbf{AV}_i) = b_i, \qquad i = 1, 2, \ldots, r. \tag{21}$$

2) Minimization of the Bayes risk. Under the unbiasedness condition,

$$\begin{aligned} B(\widehat{\alpha}) &= \int_{\theta \in \Omega} E(\widehat{\alpha} - \alpha)^2 \, dP(\theta) \\ &= E_\theta\left[\operatorname{var}(\mathbf{Y}'\mathbf{AY})\right] \\ &= E_\theta[2 \operatorname{tr}\{\mathbf{AV}(\theta)\mathbf{AV}(\theta)\}] \\ &= 2 \sum_i \sum_j E(\theta_i \theta_j) \operatorname{tr}(\mathbf{AV}_i\mathbf{AV}_j) \\ &= 2 \sum_i \sum_j C_{ij} \operatorname{tr}(\mathbf{AV}_i\mathbf{AV}_j). \end{aligned} \tag{22}$$

The initial information needed to estimate $\theta = (\theta_1, \theta_2, \ldots, \theta_r)'$ consists of its first moment $E(\theta)$ and second-moment matrix $E(\theta\theta')$. Thus,

$$\begin{aligned} &\mathbf{C} = E(\theta\theta') = \operatorname{var}(\theta) + E(\theta)E(\theta'), \\ &\mathbf{C} = \sqrt{\mathbf{C}}\sqrt{\mathbf{C}} = \mathbf{RR}, \qquad \mathbf{R} = \sqrt{\mathbf{C}}. \end{aligned} \tag{23}$$

The matrix $\mathbf{C}$ can be represented as

$$C_{ij} = \sum_{k=1}^{r} r_{ik} r_{kj}, \qquad i, j = 1, 2, \ldots, r. \tag{24}$$

Substituting (24) into (22) gives

$$\begin{aligned} B(\widehat{\alpha}) &= 2 \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{k=1}^{r} r_{ik} r_{kj} \operatorname{tr}(\mathbf{AV}_i\mathbf{AV}_j) \\ &= 2 \sum_{k=1}^{r} \operatorname{tr}\left[\mathbf{A}\left(\sum_{i=1}^{r} r_{ik}\mathbf{V}_i\right)\mathbf{A}\left(\sum_{j=1}^{r} r_{kj}\mathbf{V}_j\right)\right] \\ &= 2 \sum_{k=1}^{r} \operatorname{tr}(\mathbf{AT}_k\mathbf{AT}_k), \end{aligned} \tag{25}$$

where $\mathbf{T}_k = \sum_{i=1}^{r} r_{ik}\mathbf{V}_i$.

Lagrange's method is now used to minimize (25) subject to the unbiasedness conditions in (21). Define

$$N = 2\sum_{k=1}^{r} \operatorname{tr}(\mathbf{A}\mathbf{T}_k\mathbf{A}\mathbf{T}_k) + 4\sum_{i=1}^{r} \delta_i \left[\operatorname{tr}(\mathbf{A}\mathbf{V}_i) - b_i\right], \tag{26}$$

where δ_i are Lagrange multipliers. Differentiating with respect to δ_i and equating the derivative to zero gives

$$\frac{\partial N}{\partial \delta_i} = \operatorname{tr}(\mathbf{A}\mathbf{V}_i) - b_i = 0, \qquad i = 1,2,\ldots,r, \tag{27}$$

or equivalently

$$\operatorname{tr}(\mathbf{A}\mathbf{V}_i) = b_i, \qquad i = 1,2,\ldots,r. \tag{28}$$

To solve for the unknown matrix $\mathbf{A}$ and multipliers δ_i, the matrix equations are converted to a system of linear equations using the Kronecker product and the vectorization operation:

$$\left(\sum_{k=1}^{r} \mathbf{T}_k \otimes \mathbf{T}_k\right) \operatorname{vec}\mathbf{A} + \sum_{i=1}^{r} \delta_i \operatorname{vec}\mathbf{V}_i = 0, \tag{29}$$

$$(\operatorname{vec}\mathbf{V}_i)' \operatorname{vec}\mathbf{A} = b_i, \qquad i = 1,2,\ldots,r. \tag{30}$$

Let

$$\mathbf{w} = \sum_{k=1}^{r} \mathbf{T}_k \otimes \mathbf{T}_k.$$

Then (29) becomes

$$\mathbf{w}\operatorname{vec}\mathbf{A} + \sum_{i=1}^{r} \delta_i \operatorname{vec}\mathbf{V}_i = 0. \tag{31}$$

In general, there are r parameters, $\theta = (\theta_1, \theta_2, \ldots, \theta_r)'$. Equations (30) and (31) can be written as

$$\begin{pmatrix} \operatorname{vec}\mathbf{V}_1 & \operatorname{vec}\mathbf{V}_2 & \cdots & \operatorname{vec}\mathbf{V}_r & \mathbf{w} \\ 0 & 0 & \cdots & 0 & (\operatorname{vec}\mathbf{V}_1)' \\ 0 & 0 & \cdots & 0 & (\operatorname{vec}\mathbf{V}_2)' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & (\operatorname{vec}\mathbf{V}_r)' \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_r \\ \operatorname{vec}\mathbf{A} \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{n^2} \\ b_1 \\ b_2 \\ \vdots \\ b_r \end{pmatrix}. \tag{32}$$

For the case considered in this paper, $r = 3$ and $\theta = (\theta_1, \theta_2, \theta_3)'$. The system becomes

$$\mathbf{w}\operatorname{vec}\mathbf{A} + \sum_{i=1}^{3} \delta_i \operatorname{vec}\mathbf{V}_i = 0, \tag{33}$$

$$(\operatorname{vec}\mathbf{V}_i)' \operatorname{vec}\mathbf{A} = b_i, \qquad i = 1,2,3. \tag{34}$$

Hence,

$$\begin{pmatrix} \operatorname{vec}\mathbf{V}_1 & \operatorname{vec}\mathbf{V}_2 & \operatorname{vec}\mathbf{V}_3 & \mathbf{w} \\ 0 & 0 & 0 & (\operatorname{vec}\mathbf{V}_1)' \\ 0 & 0 & 0 & (\operatorname{vec}\mathbf{V}_2)' \\ 0 & 0 & 0 & (\operatorname{vec}\mathbf{V}_3)' \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \operatorname{vec}\mathbf{A} \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{n^2} \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}. \tag{35}$$

Let $\mathbf{G}$ represent the matrix in (35), with dimension $(n^2+3) \times (n^2+3)$; let $\mathbf{O}$ denote the $(n^2+3) \times 1$ vector of unknowns; and let $\mathbf{P}$ denote the corresponding vector of constants. The system can be written as

$$\mathbf{G}\mathbf{O} = \mathbf{P}. \tag{36}$$

If $\mathbf{G}$ is nonsingular, the solution is

$$\mathbf{O} = \mathbf{G}^{-1}\mathbf{P}. \tag{37}$$

The resulting vector $\mathbf{O}$ contains δ_i and $\operatorname{vec}\mathbf{A}$. Consequently, the matrix $\mathbf{A}$ can be found and used to estimate θ_1, θ_2, and θ_3 through the quadratic form $\widehat{\alpha} = \mathbf{Y}'\mathbf{A}\mathbf{Y}$, since it is an unbiased estimator of the linear function

$$\alpha = b_1\theta_1 + b_2\theta_2 + b_3\theta_3. \tag{38}$$

The unbiasedness condition is $\operatorname{tr}(\mathbf{A}\mathbf{V}_i) = b_i$ for $i = 1,2,3$. Setting $b_1 = 1$ and $b_2 = b_3 = 0$ gives $\widehat{\theta}_1$; setting $b_2 = 1$ and $b_1 = b_3 = 0$ gives $\widehat{\theta}_2$; and setting $b_3 = 1$ and $b_1 = b_2 = 0$ gives $\widehat{\theta}_3$. To estimate another linear combination, such as $\theta_1 + \theta_2 + \theta_3$, set $b_1 = b_2 = b_3 = 1$.

2.3 Application of the Unbiased Quadratic Bayes Estimator

The unbiased quadratic Bayes estimator was applied to spatial observations of the rise of groundwater levels in the Al-Qaim area in Iraq. The data were obtained from the Center for Research of Dams and Water Resources at the University of Mosul. Estimates previously obtained using the minimum-norm quadratic unbiased criterion and the real data yielded

$$60 \le \widehat{\theta}_1 \le 70, \qquad 0.5 \le \widehat{\theta}_2 \le 1.5, \qquad 0.0 \le \widehat{\theta}_3 \le 0.1.$$

Accordingly, the prior information for θ_1, θ_2, and θ_3 was represented by the following uniform distributions:

$$P_1(\theta_1) = \frac{1}{70-60} = 0.1, \qquad 60 \le \theta_1 \le 70,$$
$$P_2(\theta_2) = \frac{1}{1.5-0.5} = 1, \qquad 0.5 \le \theta_2 \le 1.5,$$
$$P_3(\theta_3) = \frac{1}{0.1-0.0} = 10, \qquad 0.0 \le \theta_3 \le 0.1,$$

with

$$P(\theta_1,\theta_2,\theta_3) = P_1(\theta_1)P_2(\theta_2)P_3(\theta_3), \qquad \operatorname{cov}(\theta_1,\theta_2,\theta_3) = 0.$$

The first moments and variances are

$$E(\theta_1) = \frac{70+60}{2} = 65, \qquad \operatorname{var}(\theta_1) = \frac{(70-60)^2}{12} = 8.33,$$
$$E(\theta_2) = \frac{1.5+0.5}{2} = 1.0, \qquad \operatorname{var}(\theta_2) = \frac{1}{12} = 0.0833,$$
$$E(\theta_3) = \frac{0.1+0.0}{2} = 0.05, \qquad \operatorname{var}(\theta_3) = \frac{(0.1)^2}{12} = 0.000833.$$

Therefore,

$$\begin{aligned} \mathbf{C} &= E(\theta)E(\theta') + \operatorname{var}(\theta) \\ &= \begin{pmatrix} 65 \\ 1 \\ 0.05 \end{pmatrix} \begin{pmatrix} 65 & 1 & 0.05 \end{pmatrix} + \begin{pmatrix} 8.33 & 0 & 0 \\ 0 & 0.0833 & 0 \\ 0 & 0 & 0.000833 \end{pmatrix} \\ &= \begin{pmatrix} 4233.33 & 65 & 3.25 \\ 65 & 1.0833 & 0.05 \\ 3.25 & 0.05 & 0.003333 \end{pmatrix}. \end{aligned}$$

The square root of $\mathbf{C}$ is

$$\mathbf{R} = \begin{pmatrix} 65.056 & 0.99443 & 0.049918 \\ 0.99443 & 0.3073 & 0.00107 \\ 0.049918 & 0.00107 & 0.028989 \end{pmatrix}.$$

The Bayesian quadratic unbiased estimator (BAQUE) results are shown in Table 2, while the minimum-norm quadratic unbiased estimator results are shown in Table 3.

Table 2. Results of estimates by the unbiased quadratic Bayes estimator.

$\widehat{\theta}_1$	$\widehat{\theta}_2$	$\widehat{\theta}_3$
0.897×10^{-2}	4.27×10^{-2}	-352.69

Table 3. Results of estimates by the minimum-norm quadratic unbiased criterion.

$\widehat{\theta}_1$	$\widehat{\theta}_2$	$\widehat{\theta}_3$
7.036×10	5.827×10^{-2}	-7.345×10^{-4}

3. CONCLUSION

The estimates are close to each other, indicating that the Bayesian method can be used to estimate these parameters. We believe that if the initial information were represented by informative priors—that is, information-rich probability distributions—the results would have been more accurate than those obtained in Table 3.

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